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NOTAS DE ALGEBRA Y ANALISIS

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CONTENTS

INTRODUCTION

There are several ways to define a product of distributions with values in distributions.

If $u,v \in \mathcal{D}'(R^m)$ then we write $(u.v)_H$ for the product, if it exists, given by definition 1 of the present work. See page 2 or [12], [1]. This definition is due to L. Hörmander and was generalized by W. Ambrose.

The products $(u.v)_M$ and $(u.v)_{MG}$ are described, if they exist, in definitions 4 and 2 respectively; see pages 14 and 2 or [14].

These definitions are due to J. Mikusinski.

Y. Hirata and H. Ogata gave a definition of product that corresponds to that given by our definition 3 (page 9) and, whenever it exists, we denote it by $(u.v)_{HO}$. See also [10].

Finally, the product that we denote by $(u.v)_{M(\Delta_n)}$ is defined in page 14. This is also due to J. Mikusinski ([14]).

The present work is divided in two parts: I and II. Part I is also divided in three sections: IA, IB and IC.

In IA we establish the relationships between the products defined above. These relationships are described in the follwing diagram. The arrows mean implication: if one product exists then ($\longrightarrow$) the other exists and they are equal.

The crossed arrow ($\not\leftarrow$) means that the implication does not hold.

$$\begin{array}{ccccccc} & \overset{2}{\not\leftarrow} & & \overset{4}{\not\leftarrow} & & \overset{6}{\not\leftarrow} & \\ (u.v)_H & \longrightarrow & (u.v)_{MG} & \longrightarrow & (u.v)_M & \longrightarrow & (u.v)_{M(\Delta_n)} \\ & 1 & \big\updownarrow 3 & 5 & & 7 & \\ & & (u.v)_{HO} & & & & \end{array}$$

J. Tysk prove $\xrightarrow{1}$ (see [20]). We give here a proof of $\xrightarrow{1}$ that is somewhat different.

J. Colombeau (see [3]), to prove $\xrightarrow{1}$, uses the three hypothesis that J. Tysk ([20]) needs but he alters one of them. We show that one hypothesis is superfluous (not the altered one) to prove the commutativity of the product (theorem 1', page 5).

The example $\overset{2}{\not\leftarrow}$ is also due to J. Tysk (see [20]) and we give a proof of it that follows essentially this author. See page 6.

The equivalence $\overset{3}{\longleftrightarrow}$ proved by R. Shiraishi and M. Itano ([16]) is included to make this work self-contained. The proof presented here follows essentially these authors. See page 9. Implications $\xrightarrow{5}$ and $\xrightarrow{7}$ are immediate.

The example $\overset{4}{\not\leftarrow}$ is the classic product $\delta.H$, H the Heaviside function and δ the Dirac measure (see page 14).

The example $\overset{6}{\not\leftarrow}$ is shown in page 15. We think that it is new.

In what follows of the work (parts IB, IC and part II) we use the $(u.v)_M$-definition of product with $u,v \in \mathcal{D}'(R^1)$.

Part IB is a generalization of the classic formula $(\delta.H)_M = \delta/2$: let v be a function of bounded variation over each compact interval contained in $(a,b) \subseteq R^1$, left continuous and defined at each point (see page 19). Let μ be a signed measure over the Borel sets of (a,b) (finite for each compact interval contained in $(a,b) \subseteq R^1$).

Then we have the following theorem due to P. Antosik and J. Ligeza ([2]):

THEOREM. *The product* $(\mu.v)_M$ *exists and it is a measure.*

We prove under the same conditions the formula:

$$(\mu.v)_M(\phi) = \int_a^b v(y).\phi(y)\, d\mu + \sum_{\substack{P \in \text{supp}.\phi \\ P \text{ point of discontinuity of } v}} \mu(P).\phi(P).\frac{\mathcal{S}(P)}{2}$$

where $\mathcal{S}(P) = [\lim\limits_{\substack{x \to P \\ x > P}} v(x) - \lim\limits_{\substack{x \to P \\ x < P}} v(x)]$, $\phi \in C_0^\infty(R^1)$.

In IC we prove that if $\{\delta_n\}_{n=1,2,...}$ is a sequence of $C_0^\infty(R^1)$-functions with the follwing properties:

i) $\delta_n \geq 0$ for all n,

ii) $\int \delta_n = 1$ for all n,

iii) supp. $\delta_n \to 0$ if $n \to \infty$,

(notice that $\delta_n \to \delta$ in $\mathcal{D}'(R^1)$) then

$$\lim_{n\to\infty} (\delta_n^{(j_1)}(y) + \tfrac{i}{\pi}((\tfrac{1}{x})^{(j_1)} * \delta_n)(y)). \ \dots \ .(\delta_n^{(j_s)}(y) + \tfrac{i}{\pi}((\tfrac{1}{x})^{(j_s)} * \delta_n)(y)) =$$

$$= \frac{j_1! \ \dots \ j_s!}{(j_1+\dots+j_s+s-1)!\ (i\pi)^{s-1}} .(\delta^{(j_1+\dots+j_s+s-1)} + \tfrac{i}{\pi}(\tfrac{1}{x})^{(j_1+\dots+j_s+s-1)}),$$

for s = 1, 2, ..., j_i = 0, 1, 2, ... Here $(\frac{1}{x})(\phi) = \text{p.v.} \int \frac{\phi(x)}{x} dx$ for $\phi \in C_0^\infty(R^1)$.

This formula is a particular case of a formula proved by S.E. Trione in R^m (see [19]). The proof given here is different.

We observe that the real part (or the imaginary part) of this limit with s = 2,

$j_1 = j_2$ is a formula due to B. Fisher ([6]) that contains as a particular case a well-known formula due to A. González Domínguez and R. Scarfiello ([8]). One also obtains in this way a formula due to J. Mikusinski ([15]) but we notice that this author uses sequences with different properties.

In part II we deal exclusively with certain distributions in R^1 which are known as pseudofunctions and denoted by x_+^λ, x_-^μ; $\lambda, \mu \in C$.

Their formal definitions can be found in the appendix. See page 70.

We are not only interested in products of the type $(x_+^\lambda . x_-^\mu)_M$ but also in the existence of certain balanced products

$$x_+^\lambda . x_-^\mu + x_+^\mu . x_-^\lambda$$

or

$$x_+^\lambda . x_-^\mu - x_+^\mu . x_-^\lambda .$$

for the definition of these balanced products see page 23 or 19.

The following theorem is the central result of part II. Almost all part II is devoted to its proof.

THEOREM 6 (page 65). *We have the following relations:*

a)
$$(x_+^{-r-1/2} . x_-^{-r-1/2})_M = \frac{(-1)^r}{(2r)!} \frac{\pi}{2} . S^{(2r)}$$

if $r = 1, 2, \ldots$.

b)
$$(x_+^{r-p} . x_-^{p-r-1})_M = \frac{(-1)^r \, \pi}{\text{sen } \pi p . 2} . S$$

if r *is integer,* $0 \leq \text{Re } p < 1$, $p \neq 0$.

c)
$$(x_+^\lambda . x_-^\mu)_M = 0$$

if $\text{Re } (\lambda+\mu) > -1$, *with* $\lambda, \mu \in C$.

d) The product $(x_+^\lambda . x_-^\mu)_M$ *does not exist outside the stated case (a, b, c).*

But we have:

e)
$$x_+^{-r-p} . x_-^{-m-q} + (-1)^{m+r} \, x_+^{-m-q} . x_-^{-r-p} = \frac{(-1)^r \, \pi}{\text{sen } \pi p . (m+r)!} . S^{(m+r)}$$

if m, r *are integers,* $0 \leq \text{Re } p < 1$, $p \neq 0$, $0 \leq \text{Re } q < 1$, $q \neq 0$, $1 - p - q = 0$, $r + m \geq 0$.

f)
$$x_+^\lambda . x_-^{-k} + (-1)^{k+1} \, x_+^{-k} . x_-^\lambda = 0$$

if $k = 2, 4, 6, \ldots$, $\text{Re } (\lambda-k) > -2$, $\text{Re } \lambda > -1$, $\lambda \in C$.

g)
$$x_+^{r-p} . x_-^{-q-r-1} - x_+^{-q-r-1} . x_-^{r-p} = 0$$

if r *is integer,* $0 \leq \text{Re } p < 1$, $p \neq 0$, $0 \leq \text{Re } q < 1$, $q \neq 0$, $-1 < \text{Re } (-p-q)$.

B. Fisher proved cases a), b), c) for real exponents (see [4],[5]).
S.E. Trione proved cases a), b), c) for complex exponents (see [19]).
We believe that the other cases (d), e), f) g)) are new. The most important of them is case d).
Theorem 6 is a summary of theorems 3, 4, and 5.
The only information not contained there is the one given by theorem 3 about the nonexistence of certain balanced products. See page 41.
Theorems 1 and 2 and lemmas 1, 2, and 3 are preparatory to the theorems that follow.
Theorem 0 (page 24) is due to B. Fisher ([4]). We give a proof of it for the sake of completeness.
There are repetitions in definitions and formulas. This is due to the fact that parts IA, IB, IC and part II are almost independent and can be read in any order.
The number of a formula is always on its left.
Pages have two numbers: one at the top to which references are always made and one at the bottom, which is used to separate part I from part II.

I want to express my gratitude to Profs. C. Segovia and R. Scarfiello for their comments, to Prof. E. Güichal for his support during the preparation of this work and very specially to Prof. S.E. Trione who carefully read and evaluated the manuscript.

PART I

IA. COMPARISON OF PRODUCTS. There are several ways to define a product of distributions with values distributions.

We shall use "^" to denote the Fourier transform ie. if $\phi \in C_0^\infty(R^m)$ then

$$\hat{\phi}(x) = \int \phi(t).e^{-i<x,t>}\, dt;$$

"$\hat{\hat{}}$" will denote the inverse Fourier transform

$$\hat{\hat{\phi}}(x) = (2\pi)^{-m}.\int \phi(t).e^{i<x,t>}\, dt;$$

"$^\vee$" will denote the reflection operator: $\check{\phi}(x) = \phi(-x)$.

Given two distributions $u,v \in \mathcal{D}'(R^m)$ we say that the product $(u.v)_H$ exists if for each $x \in R^m$ there exists an open neighbourhood of x, Ω_x, such that for each pair $\phi,\psi \in C_0^\infty(\Omega_x)$ we have:

i) $\widehat{(\phi u)}.\widehat{\widehat{(\psi v)}}$ is integrable,

ii) $\int |\widehat{(\phi u)}.\widehat{\widehat{(\psi v)}}|$ is a continuous function of ϕ, ie. $\int |\widehat{(\phi_i u)}.\widehat{\widehat{(\psi v)}}| \to 0$ if $\phi_i \to 0$ in $\mathcal{D}(\Omega_x)$, (see Hörmander [12], Ambrose [1]).

We define a distribution $w_x \in \mathcal{D}'(\Omega_x)$ in the following way,

$$w_x(\phi) := \int \widehat{(\phi u)}.\widehat{\widehat{(\psi v)}}$$

with $\psi \in C_0^\infty(\Omega_x)$ and such that $\psi \equiv 1$ in a neighbourhood of the support of ϕ.

w_x does not depend on ψ: take ψ_1, ψ_2 two functions with the same properties as ψ. Then

$$\int \widehat{(\phi u)}.\widehat{\widehat{(\psi_1 v)}} - \int \widehat{(\phi u)}.\widehat{\widehat{(\psi_2 v)}} = \int \widehat{(\phi u)}.\widehat{\widehat{((\psi_1 - \psi_2)v)}} =$$

$$= \lim_{n\to\infty} \int \hat{S}_n.\widehat{(\phi u)}.\widehat{\widehat{((\psi_1 - \psi_2)v)}}.\hat{\hat{S}}_n = \lim_{n\to\infty} (2\pi)^{-m}.\int \widehat{((\phi u) * S_n)}\widehat{\widehat{(((\psi_1 - \psi_2)v) * S_n)}} =$$

$$= \lim_{n\to\infty} (2\pi)^{-m}.\int ((\phi u) * S_n).(((\psi_1 - \psi_2)v) * S_n) = 0$$

where $\{S_n\}_{n=1,2,\dots}$ is a sequence of $C_0^\infty(R^m)$ functions such that $S_n \geq 0$, $\int S_n = 1$, supp. $S_n \to 0$ if $n \to \infty$.

In this way we get a family of distributions w_x associated with open sets Ω_x for each $x \in R^m$ with the following property

$$w_{x/\Omega_x \cap \Omega_y} = w_{y/\Omega_x \cap \Omega_y}, \text{ for all } x,y \in R^m.$$

It is easy to see that there is a unique distribution $w \in \mathcal{D}'(R^m)$ such that

$$w_{/\Omega_x} = w_x \quad \text{for each } x \in R^m.$$

DEFINITION 1. *If* u *and* v *satisfy the conditions i) and ii) then we define*

$$(u.v)_H := w.$$

The product given by definition 1 coincides with the usual one for functions in $\mathcal{S}$: let $\chi, \theta \in \mathcal{S}(R^m)$, $\phi, \psi \in C_0^\infty(R^m)$, $\psi \equiv 1$ in a neighbourhood of supp. ϕ. Then we have

$$\int \widehat{(\phi.\chi)}.\widehat{\widehat{(\psi.\theta)}} = \int \phi.\psi.\chi.\theta = (\chi.\theta)(\phi).$$

DEFINITION 2. *Let* $u,v \in \mathcal{D}'(R^m)$. *We say that the product* $(u.v)_{MG}$ *(generalized Mikusinski product, see [14]) exists if there is a* $w \in \mathcal{D}'(R^m)$ *such that for each pair of sequences* $\{d_n\}_{n=1,2,\ldots}$ *and* $\{e_n\}_{n=1,2,\ldots}$ *verifying*

i) $d_n \geq 0$, $e_n \geq 0$ *for all* n,

ii) $\int d_n = \int e_n = 1$ *for all* n,

iii) supp. $d_n \to 0$, supp. $e_n \to 0$ *if* $n \to \infty$,

and for each function $\phi \in C_0^\infty(R^m)$,

$$\lim_{n \to \infty} \int (u * d_n)(y).(v * e_n)(y).\phi(y)\, dy = w(\phi).$$

By definition $(u.v)_{MG} = w$.

The existence of this limit for each $\phi \in C_0^\infty(R^m)$ even for only one pair $\{d_n\}$, $\{e_n\}$, implies the existence of a unique $w \in \mathcal{D}'$ that satisfies the equality for that pair.

This product coincides with the usual one for continuous functions u,v, because $(u * S_n) \overset{\bullet}{\to} u$ and $(v * S_n) \overset{\bullet}{\to} v$ on compact sets.

We will see now a result due to J. Tysk ([20]):

THEOREM 1 (Tysk). *If* $(u.v)_H$ *exists then* $(u.v)_{MG}$ *exist and*

$$(u.v)_H = (u.v)_{MG}.$$

We shall need the following lemma.

LEMMA 1. *If* $u \in \mathcal{D}'(R^m)$ *and* $\phi, d \in C_0^\infty(R^m)$ *we have*

$$\widehat{[\phi.(u * d)]}(x) - \widehat{[(\phi.u) * d]}(x) =$$

$$= \sum_{j=1}^{m} \int_0^1 \int \widehat{[(\partial_j\phi)(z + w.t).u_z]}(x).w_j.d(w).e^{-i<w,x>}\, dw\, dt.$$

PROOF. We may suppose that **u** $\varepsilon\ \mathcal{E}'(R^m)$. Then

$$A(x) = \widehat{[\phi.(u * d)]}(x) - \widehat{[(\phi u) * d]}(x) = (2\pi)^{-m}[\hat{\phi} * \widehat{(u * d)}](x) - \widehat{(\phi u)}(x).\hat{d}(x) =$$

$$= (2\pi)^{-m}\{[\hat{\phi} * (\hat{u}.d)](x) - \hat{d}(x).(\hat{\phi} * \hat{u})(x)\} =$$

$$= (2\pi)^{-m}\{\int \hat{\phi}(x-y).\hat{u}(y).\hat{d}(y)\, dy - \int \hat{d}(x).\hat{u}(y).\hat{\phi}(x-y)\, dy\} =$$

$$= (2\pi)^{-m}\{\int \hat{u}(y).\hat{\phi}(x-y).(\hat{d}(y) - \hat{d}(x))\, dy\}.$$

Now

$$\hat{d}(y) - \hat{d}(x) = \int_0^1 \sum_{j=1}^{m} (\partial_j\hat{d})(x - t(x - y)).(y_j - x_j)\, dt =$$

$$= \int_0^1 f^{(1)}(t)\, dt = f(1) - f(0),$$

here $f(t) := \hat{d}(x - t.(x - y))$.

Then, we can write

$$A(x) = (2\pi)^{-m}.\sum_{j=1}^{m} \iint_0^1 \hat{u}(y).\hat{\phi}(x-y).(\partial_j\hat{d})(x - t(x - y))(y_j - x_j)\, dt\, dy =$$

$$= (2\pi)^{-m}.i.\sum_{j=1}^{m} \iint_0^1 \hat{u}(y).\widehat{(\partial_j\phi)}(x - y).(\partial_j\hat{d})(x - t(x - y))\, dt\, dy.$$

But

$$(\partial_j\hat{d})(x - t(x - y)) = (-i).\int \xi_j.d(\xi).e^{-i<\xi,x-t(x-y)>}\, d\xi =$$

$$= (-i).\int (-\xi'_j/t).d(-\xi'/t).e^{i<\xi'/t,x>}.e^{-i<\xi',x-y>}.\frac{1}{t^m}\, d\xi'.$$

Then

$$A(x) = (2\pi)^{-m}.\sum_{j=1}^{m} \iint_0^1 \hat{u}(y).\widehat{(\partial_j\phi)}(x - y).\frac{1}{t^m}.\widehat{[(s_j.d(s))(\tfrac{-\cdot}{t}).e^{i<\cdot,\frac{x}{t}>}]}(x-y)\, dt\, dy =$$

$$= (2\pi)^{-m}.\sum_{j=1}^{m} \iint_0^1 \hat{u}(y).\widehat{[(\partial_j\phi)(.) * \{(s_j.d(s))(\tfrac{-\cdot}{t}).e^{i<\cdot,\frac{x}{t}>}\}]}(x - y).\frac{1}{t^m}\, dt\, dy =$$

$$= \sum_{j=1}^{m} \int_0^1 ([(\partial_j\phi)(.) * (s_j.d(s))(\tfrac{-\cdot}{t}).e^{i<\cdot,\frac{x}{t}>}](z).u_z)(e^{-i<x,z>}).\frac{1}{t^m}\, dt =$$

$$= \sum_{j=1}^{m} \int_0^1 u_z(e^{-i<x,z>}.(\int (\partial_j\phi)(z - w).(s_j.d(s))(\frac{-w}{t}).e^{i<w,\frac{x}{t}>} dw)).\frac{1}{t^m} dt .$$

Since $V \otimes 1 = 1 \otimes V$, we have:

$$A(x) = \sum_{j=1}^{m} \int_0^1 (\int u_z(e^{-i<x,z>}.(\partial_j\phi)(z - w)).(s_j.d(s))(\frac{-w}{t}).\frac{e^{i<w,\frac{x}{t}>}}{t^m} dw) dt =$$

$$= \sum_{j=1}^{m} \int_0^1 (\int u_z(e^{-i<x,z>}.(\partial_j\phi)(z + w't)).(s_j.d(s))(w').e^{-i<w',x>} dw') dt =$$

$$= \sum_{j=1}^{m} \int_0^1 \int \widehat{[(\partial_j\phi)(z + wt).u_z]}(x).w_j.d(w).e^{-i<w,x>}dw\ dt, \qquad \text{QED.}$$

PROOF OF THEOREM 1. Suppose u,v are such that the product $(u.v)_H$ exists; let Ω_x be an open set containing x with the properties mentioned in definition 1.

Then the following equality holds with $\phi,\psi \in C_0^\infty(\Omega_x)$, $\psi \equiv 1$ in a neighborhood of supp. ϕ; d_n, e_n sequences as in definition 2:

$$\int \widehat{[\phi.(u * d_n)]}.\hat{\hat{[\psi.(v * e_n)]}} - \int \widehat{[(\phi u) * d_n]}.\hat{\hat{[(\psi v) * e_n]}} =$$

$$= \int (\widehat{[\phi.(u * d_n)]} - \widehat{[(\phi u) * d_n]}).(\hat{\hat{[\psi.(v * e_n)]}} - \hat{\hat{[(\psi v) * e_n]}}) +$$

$$+ \int (\widehat{[\phi.(u * d_n)]} - \widehat{[(\phi u) * d_n]}).\hat{\hat{[(\psi v) * e_n]}} + \int (\hat{\hat{[\psi.(v * e_n)]}} - \hat{\hat{[(\psi v) * e_n]}}).\widehat{[(\phi u) * d_n]}.$$

The first and last integral vanish if n is great enough. For example, for the last one we have:

$$\int (\hat{\hat{[\psi.(v * e_n)]}} - \hat{\hat{[(\psi v) * e_n]}}).\widehat{[(\phi u) * d_n]} =$$

$$= \int ([\psi.(v * e_n)] - [(\psi v) * e_n]).[(\phi u) * d_n]$$

and it is seen that $([\psi.(v * e_n)] - [(\psi v) * e_n])$ is equal to zero in a neighbourhood of supp. ϕ if n is great enough; therefore, for n great enough the integral vanishes.

Calling I_n the second integral and using lemma 1 we have:

$$I_n = \sum_{j=1}^{m} \iint_0^1 \int \widehat{[(\partial_j\phi)(z + wt)u_z]}(x).w_j d_n(w).e^{-i<w,x>}.\hat{\hat{[(\psi v) * e_n]}}(x)\ dw\ dt\ dx =$$

$$= (2\pi)^m \sum_{j=1}^{m} \int (\int_0^1 (\int \widehat{[(\partial_j\phi)(z + wt)u_z]}(x).\widehat{\widehat{(\psi v)}}(x).\hat{\hat{e}}_n(x).e^{-i<w,x>}dx)\ dt) w_j d_n(w)\ dw =$$

But

$$\left|\int [(\partial_j\phi)(z + \omega t)u_z]^{\wedge}(x)\widehat{\widehat{(\psi v)}}(x).\hat{\hat{e}}_n(x)\, e^{-i<\omega,x>}\, dx\right| \leq$$

$$\leq (2\pi)^{-m}\int \left|[(\partial_j\phi)(z + \omega t)u_z]^{\wedge}(x).\widehat{\widehat{(\psi v)}}(x)\right| dx.$$

By ii), definition 1, this last integral is a continuous function on $[0,1] \times \{|\omega| \leq \varepsilon'\}$ if ε' is small enough. Then, there exists $M < \infty$ such that

$$\left|\int [(\partial_j\phi)(z + \omega t)u_z]^{\wedge}(x).\widehat{\widehat{(\psi v)}}(x).\hat{\hat{e}}_n(x).e^{-i<\omega,x>}\, dx\right| \leq M$$

if $t \in [0,1]$, $|\omega| \leq \varepsilon'$. And

$$|I_n| \leq \sum_j \iint_0^1 M.|\omega_j.d_n(\omega)|\, dt\, d\omega \leq M.m.\varepsilon'.$$

if $n \geq n_1$, n_1 such that supp. $d_n(\omega) \subseteq \{|\omega| < \varepsilon'\}$. Then

$$\lim_{n\to\infty} \{\int [\phi.(u * d_n)]^{\wedge}.[\psi.(v * e_n)]^{\hat{\hat{}}} - \int [(\phi u) * d_n]^{\wedge}.[(\psi v) * e_n]^{\hat{\hat{}}}\} = 0.$$

Since $\hat{d}_n \to 1$, $\hat{\hat{e}}_n \to (2\pi)^{-m}$ uniformly on compact set and boundedly, we have

$$\lim_{n\to\infty} \int [(\phi u) * d_n]^{\wedge}.[(\psi v) * e_n]^{\hat{\hat{}}} = \lim_{n\to\infty} (2\pi)^m.\int \widehat{(\phi u)}.\hat{d}_n.\widehat{\widehat{(\psi v)}}.\hat{\hat{e}}_n = (u.v)_H(\phi)\ .$$

The theorem follows immediately by Parseval's formula:

$$\int [\phi.(u * d_n)]^{\wedge}.[\psi.(v * e_n)]^{\hat{\hat{}}} = \int \phi.(u * d_n).(v * e_n), \qquad \text{QED.}$$

Some authors (J. Tysk [20], Colombeau [3]) add the following third condition to those defining $(u.v)_H$:

iii) $\int \widehat{(\phi u)}.\widehat{\widehat{(\psi v)}} = \int \widehat{(\phi v)}.\widehat{\widehat{(\psi u)}}$ for each $\phi,\psi \in C_0^\infty(\Omega_x)$.

The product becomes commutative, if one defines as in definition 1, a distribution $\omega'_x \in \mathcal{D}'(\Omega_x)$ by

$$\omega'_x(\phi) := \int \widehat{(\phi v)}.\widehat{\widehat{(\psi u)}}$$

with $\psi \in C_0^\infty(\Omega_x)$ and such that $\psi \equiv 1$ in a neighbourhood of supp. ϕ. Notice that definition 1 is **not** symmetric, even when added this third condition. But to define ω'_x it is only necessary iii) with $\psi \equiv 1$ in a neighbourhood of supp. ϕ and then it is a consequence of i) and ii).

THEOREM 1'. *If* u *and* v *satisfy i) and ii) of definition 1 then*

$$\int \widehat{(\phi u)}.\widehat{\widehat{(\psi v)}} = \int \widehat{(\phi v)}.\widehat{\widehat{(\psi u)}}$$

for every $\phi, \psi \in C_0^\infty(\Omega_x)$, $\Psi \equiv 1$ *in a neighbourhood of supp.* ϕ.

PROOF. From theorem 1 we get

$$\int \widehat{(\phi u)}.\widehat{\widehat{(\psi v)}} = \lim_{n \to \infty} \int \phi.(u * d_n).(v * e'_n).$$

Since the limit is independent of $\{d_n\}_{n=1,2,\ldots}$, $\{e'_n\}_{n=1,2,\ldots}$ we have

$$\lim_{n \to \infty} \int \phi.(u * d_n).(v * e'_n) = \lim_{n \to \infty} v(\phi.(u * d_n)) = \lim_{n \to \infty} (\phi v)(\psi.(u * d_n)).$$

If $\{e_n\}_{n=1,2,\ldots}$ is a special sequence such that

$$|(\phi v)(\psi.(u * d_n)) - \int [(\phi v) * e_n].[\psi.(u * d_n)]| \leq 1/n$$

then

$$\lim_{n \to \infty} (\phi v)(\psi.(u * d_n)) =$$

$$= \lim_{n \to \infty} \int [(\phi v) * e_n\hat{]}.\{[\psi.(u * d_n)\hat{\hat{]}} - [(\psi u) * d_n\hat{\hat{]}}\} + \int \widehat{(\phi v)}.\widehat{\widehat{(\psi u)}}.$$

But

$$\int [(\phi v) * e_n\hat{]}.\{[\psi.(u * d_n)\hat{\hat{]}} - [(\psi u) * d_n\hat{\hat{]}}\} =$$

$$= \int [(\phi v) * e_n\hat{\hat{]}}.\{[\psi.(u * d_n)\hat{]} - [(\psi u) * d_n\hat{]}\} = I_n.$$

I_n is like the I_n in the proof of theorem 1 after changing ϕ by ψ. And it tends to zero because in the proof of this fact it was not used that $\psi \equiv 1$ in a neighbourhood of supp. ϕ, QED.

The counterexample that follows (also due to J. Tysk [20]) shows that definitions 1 and 2 are not equivalent.

Suppose we have $f: R \to C$ such that f is continuous, of compact support, $f \in C^\infty((-\infty,0) \cup (0,+\infty))$ and that

$$(*) \qquad \int |\widehat{\psi f}| = +\infty$$

for each $\psi \in C_0^\infty(R)$ such that $\psi \equiv 1$ in a neighbourhood of zero. In that case (S,f) can not be multiplied as stated in definition 1 but, if $\phi \in C_0^\infty(R)$,

$$\lim_{n \to \infty} \int (S * e_n)(x).(f * d_n)(x).\phi(x)\, dx = \phi(0).f(0).$$

Then we have

$$(S.f)_{MG} = f(0).S.$$

We will show now that such an f exists.

Let g be a $C_o^\infty(R)$ function such that $g \geq 0$, g is no identically zero, increasing in $(-\infty,0]$ and decreasing in $[0,+\infty)$. Let a_n, b_n be two decreasing sequences of positive numbers such that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0$. We set

$$f(x) := \sum_{k=0}^{\infty} (-1)^k . b_k . g(\frac{x}{a_k}).$$

The f(x) function is continuous and of compact support because by the criterion for alternating series we have that

$$|\sum_{k=n}^{m} (-1)^k . b_k . g(\frac{x}{a_k})| \leq b_n . g(\frac{x}{a_n}) \leq b_n . \max g.$$

It is clear also that $f \in C^\infty((-\infty,0) \cup (0,+\infty))$, and that

$$\hat{f}(x) = \sum_{k=0}^{\infty} (-1)^k . b_k . a_k . \hat{g}(a_k . x),$$

being the series uniformly convergent.

We will prove now that there exists two sequences a_n, b_n that make $\int |\hat{f}| = +\infty$ (more than that, (*) holds).

We call C_k, $k \in Z$, the following subsets of R:

$$C_k = \{x \mid r^{2k-1} \leq |x| \leq r^{2k+1}\}$$

where $r > 1$. It is obvious that $\bigcup_{k \in Z} C_k = R - \{0\}$.

For k = 0, 1, 2, ... we have

$$\int_{C_k} |\hat{f}| \geq \int_{C_k} |b_k . a_k . \hat{g}(a_k . t)|\, dt - \int_{C_k} |\sum_{\substack{j=0 \\ j \neq k}}^{+\infty} (-1)^j . b_j . a_j . \hat{g}(a_j . t)|\, dt \geq$$

$$\geq \int_{C_k} |b_k . a_k . \hat{g}(a_k . t)|\, dt - \sum_{\substack{j=0 \\ j \neq k}}^{+\infty} \int_{C_k} |b_j . a_j . \hat{g}(a_j . t)|\, dt.$$

And therefore

$$\text{(1-A)} \quad \int |\hat{f}| \geq \sum_{k=0}^{+\infty} \int_{C_k} |\hat{f}| \geq \sum_{k=0}^{+\infty} [\int_{C_k} |b_k . a_k . \hat{g}(a_k . t)|\, dt - \sum_{\substack{j=0 \\ j \neq k}}^{+\infty} \int_{C_k} |b_j . a_j . \hat{g}(a_j . t)|\, dt].$$

Take $a_k = r^{-2k}$. Rewriting (1-A) we have:

$$\int |\hat{f}| \geq \sum_{k=0}^{+\infty} [b_k . \int_{C_k} r^{-2k} . |\hat{g}(r^{-2k} . t)|\, dt - \sum_{\substack{j=0 \\ j \neq k}}^{+\infty} b_j . \int_{C_k} r^{-2j} |\hat{g}(r^{-2j} t)|\, dt] \geq$$

$$\geq \sum_{k=0}^{+\infty} [b_k . \int_{C_o} |\hat{g}(t)|\, dt - \sum_{\substack{j=0 \\ j \neq k}}^{+\infty} b_j . \int_{C_{k-j}} |\hat{g}(t)|\, dt].$$

Setting $c_i := \int_{C_i} |\hat{g}(t)|\, dt$, we can write:

(2-A) $$\int |f| \geq \sum_{k=0}^{+\infty} [b_k . c_o - \sum_{\substack{j=0 \\ j\neq k}}^{+\infty} b_j . c_{k-j}].$$

Set $b_j = \dfrac{1}{j+1}$ and an r such that $c_k \leqq c_o . 4^{-|k|}$ for every $k \in Z$. This last inequality can be obtained because $\hat{g} \in S(R)$. We know that there exists a constant M such that

$$|\hat{g}(x)| \leq \frac{M}{x^2}, \quad |\hat{g}(x)| \leq M.$$

And if $k > 0$

$$\int_{C_k} |\hat{g}| \leqq 2.\int_{r^{2k-1}}^{r^{2k+1}} \frac{M}{x^2}\, dx = 2.M. \frac{(r^2 - 1)}{r^{2k+1}} \leqq 2.M.r^{-2k+1} \leqq 2.M.r^{-k}$$

and if $k < 0$

$$\int_{C_k} |\hat{g}| \leqq 2.\int_{r^{2k-1}}^{r^{2k+1}} M\, dx \leqq 2.M.r^{2k-1}.(r^2 - 1) \leqq 2.M.r^{2k+1} \leqq 2.M.r^{k}.$$

Then we have for each $r > 1$ that

$$\int_{C_k} |\hat{g}| \leqq 2.M.r^{-|k|}, \quad k \in Z,\ k \neq 0.$$

From here we inmediately obtain

$$c_k \leqq c_o . 4^{-|k|} \text{ for every } k \in Z.$$

To finish the proof, observe that the following estimates hold:

(3-A) $$\sum_{j=k+1}^{+\infty} \frac{4^{k-j}}{j+1} \leqq \frac{1}{k+2} . [\frac{1}{4} + \frac{1}{4^2} + \dots] = \frac{1}{3.(k+2)} \leqq \frac{1}{3.(k+1)} .$$

(4-A) $$\sum_{j=0}^{k-1} \frac{4^{j-k}}{j+1} \leqq \frac{9}{16} \cdot \frac{1}{(k+1)} .$$

The inequality (4-A) follows easily by induction: for $k = 1, 2$ it is true and if $k' \geqq 2$ then

$$\sum_{j=0}^{(k'+1)-1} \frac{4^{j-(k'+1)}}{j+1} = \frac{1}{4} \cdot \sum_{j=0}^{k'} \frac{4^{j-k'}}{j+1} \leqq \frac{1}{4} . [\frac{9}{16.(k'+1)} + \frac{1}{(k'+1)}] =$$

$$= \frac{25}{64.(k'+1)} \leqq \frac{9}{16} \cdot \frac{1}{(k'+2)}.$$

From (2-A), (3-A) and (4-A) we have

$$\int |\hat{f}| \geqq \sum_{k=0}^{+\infty} [\frac{c_o}{k+1} - \sum_{\substack{j=0 \\ j\neq k}}^{+\infty} \frac{c_{k-j}}{j+1}] \geqq c_o . \sum_{k=0}^{+\infty} [\frac{1}{k+1} - \sum_{\substack{j=0 \\ j\neq k}}^{+\infty} \frac{4^{-|k-j|}}{j+1}] \geqq$$

$$\geq \frac{5}{48} \cdot c_0 \cdot \sum_{k=0}^{+\infty} \frac{1}{k+1} \, , \qquad \text{QED.}$$

DEFINITION 3 (Y. Hirata and H. Ogata, [10]). *Given two distributions* $u,v \in \mathcal{D}'(R^m)$, *we say that the product* $(u.v)_{HO}$ *exists, if there exists a distribution* $w \in \mathcal{D}'(R^m)$ *such that for each sequence* $\{S_n\}_{n=1,2,...} \in C_0^\infty(R^m)$ *with the properties:*

i) $S_n \geq 0$ *for all* n.

ii) $\int S_n = 1$ *for all* n.

iii) $\text{supp. } S_n \to 0$ *if* $n \to \infty$

and each $C_0^\infty(R^m)$ *function,* ϕ*, we have*

$$\lim_{n \to \infty} v((u * S_n)(y).\phi(y)) = w(\phi),$$

and define $(u.v)_{HO} := w$.

THEOREM 2 (R. Shiraishi and M. Itano, [16]). $(u.v)_{HO}$ *exists if and only if* $(u.v)_{MG}$ *exists, and the products are equal.*

Notice that if $(u.v)_{HO}$ exists then $(v.u)_{HO}$ exists and

$$(u.v)_{HO} = (v.u)_{HO} = (u.v)_{MG} \, .$$

For the proof of theorem 2 we need some auxiliary lemmas.

LEMMA 2. *Let* $u,v \in \mathcal{D}'(R^m)$ *be such that the product* $(u.v)_{HO}$ *exists. Then for each compact set* $K \subseteq R^m$ *there exist constants* k_0, M, L *and a closed ball* $K' \subseteq R^m$ *centered at zero such that for each function* $\phi \in C_0^\infty(R^m)$ *with* $\text{supp } \phi \subseteq K$ *and* $\max_{x \in R^m} |D^k\phi(x)| \leq M$ *if* $|k| \leq k_0$ *and each* $C_0^\infty(R^m)$ *function,* $\tilde{S}$*, such that*

$$\int |\tilde{S}| = 1, \quad \text{supp. } \tilde{S} \subseteq K',$$

it holds that

$$|v((u * \tilde{S})(x).\phi(x))| \leq L.$$

PROOF. Let $u,v \in \mathcal{D}'(R^m)$ be as in the lemma. If it were false we could find a compact set K and two sequences of functions S_n, ϕ_n such that $\phi_n \to 0$ in $\mathcal{D}(R^m)$ ($\text{supp } \phi_n \subseteq K$ for all n), $S_n \to z_0.S$ in $\mathcal{D}'(R^m)$ (z_0 a complex number, $|z_0| \leq 1$) the supports of S_n tending to zero, $\int |S_n| = 1$ for all n and verifying for all n the inequality

$$|v((u * S_n)(x).\phi_n(x))| > 1.$$

If we proved that there exists a subsequence such that

(5-A) $$(u * S_{n_j}).v \to z_o.(u.v)_{HO} \text{ in } \mathcal{D}'(R^m)$$

holds, then by Banach-Steinhaus theorem we would have that $\lim_{n_j \to \infty} ((u * S_{n_j}).v)(\phi_{n_j}) = 0$, which is absurd.

Call $S_n = S_n^I + iS_n^{II}$, then choose $\{\Delta_n\}_{n=1,2,\ldots}$ $(\Delta_n \in C_o^\infty(R^m)$, $\Delta_n \geq 0$, $\int \Delta_n = 1$, supp. $\Delta_n \to 0)$ such that

$$S_n^I - (S_n^I * \Delta_n) \to 0 \text{ in } \mathcal{D}(R^m).$$

Then

$$(u * S_n^I).v - (u * S_n^I * \Delta_n).v \to 0 \text{ in } \mathcal{D}'(R^m),$$

because $u * (S_n^I - S_n^I * \Delta_n) \to 0$ in $\mathcal{E}(R^m)$.

But if $S_n^I := S_{n_+}^I - S_{n_-}^I$ (where $S_{n_+}^I$ = supp. $(S_n^I, 0)$) we have

$$(u * S_n^I * \Delta_n).v = (u * (S_{n_+}^I * \Delta_n)).v - (u * (S_{n_-}^I * \Delta_n)).v$$

and we can choose subsequences of $(S_{n_+}^I * \Delta_n)$ and $(S_{n_-}^I * \Delta_n)$ (that, for convenience, we continue writing them in the same way) that have the properties i), iii) of definition 3 and that verify

$$\lim_{n \to \infty} \int (S_{n_+}^I * \Delta_n) = \alpha_+,$$

$$\lim_{n \to \infty} \int (S_{n_-}^I * \Delta_n) = \alpha_-,$$

where $\alpha_+ - \alpha_- = \text{Re } z_o$. Then, since $(u.v)_{HO}$ exists we have

$$(u * S_n^I * \Delta_n).v = (u * (S_{n_+}^I * \Delta_n)).v - (u * (S_{n_-}^I * \Delta_n)).v \to (\text{Re } z_o).(u.v)_{HO}$$

in $\mathcal{D}'(R^m)$.

Dealing in the same way with S_n^{II}, (5-A) follows. QED.

LEMMA 3. *If* $u \in \mathcal{D}'(R^m)$, $\psi \in \mathcal{D}(R^m)$, $L > 0$, *then for each* K *compact there exists a polynomial* $P_k(y)$ *with positive coefficients such that*

$$|(u * \psi(.).e^{-i<j,.>.\frac{\pi}{L}})(x)| \leq P_k(|j|)$$

for every $x \in K$ *and every* $j = (j_1, \ldots, j_m)$; $j_i \in N \cup \{0\}$.

PROOF. It follows inmediately from the fact that a distribution is locally the derivative of some order of a continuous function. QED.

LEMMA 4. *Let* $u, v \in \mathcal{D}'(R^m)$ *be such that* $(u.v)_{HO}$ *exists, then* $((\alpha u).v)_{HO}$ *exists if* $\alpha \in C_0^\infty(R^m)$, *and*

$$\alpha.(u.v)_{HO} = ((\alpha u).v)_{HO}.$$

PROOF. We assume that $\alpha \in C_0^\infty(R^m)$ is such that supp. $\alpha \subseteq \overset{\circ}{Q}$, where $Q \subseteq R^m$ is a cube of side L centered at zero. Then we have in Q

$$\alpha(x) = \sum_j c_j . e^{i<j,x>.\frac{\pi}{L}} \text{ in } Q,$$

with $\sum_j |c_j| \, (1 + |j|)^l \leq K(l)$ for each l.

Now, if $\phi \in C_0^\infty(R^m)$ and Q is great enough (i.e. supp. $\phi \subseteq \overset{\circ}{Q}$), we have

$$((\alpha u) * S_n)(x).\phi(x) = u_y(\alpha(y).S_n(x-y)).\phi(x) =$$

$$= u_y(\sum_j c_j.e^{i<j,y>.\frac{\pi}{L}}.S_n(x-y)).\phi(x) = \sum_j c_j.u_y(e^{i<j,y>.\frac{\pi}{L}}.S_n(x-y)).\phi(x) =$$

$$= \sum_j c_j.(u * S_n(.).e^{-i<j,.>.\frac{\pi}{L}})(x).\phi(x).e^{i<j,x>.\frac{\pi}{L}} .$$

By lemma 3, we have for fixed n that

$$\sum_j |c_j.(u * S_n(.).e^{-i<j,.>.\frac{\pi}{L}})(x).\phi(x).e^{i<j,x>.\frac{\pi}{L}}| \leq \sum_j |c_j|.P_k(|j|). \max_{x \in K} |\phi(x)|$$

where supp $\phi \subseteq K$ compact. A similar reasoning can be done for the derivatives fo the factors in the series. Therefore we can write

$$\sum_{|j| \leq j_0} c_j.(u * S_n(.).e^{-i<j,.>.\frac{\pi}{L}})(x).\phi(x).e^{i<j,x>.\frac{\pi}{L}} \xrightarrow[\substack{\text{in } \mathcal{D}(R^m) \\ \text{if } j_0 \to \infty}]{} ((\alpha u) * S_n)(x).\phi(x).$$

And then

$$v(((\alpha u) * S_n)(x).\phi(x)) = \sum_j c_j.v((u * S_n(.).e^{-i<j,.>.\frac{\pi}{L}})(x).\phi(x).e^{i<j,x>.\frac{\pi}{L}}).$$

Now by lemma 2 we have

$$|v((u * S_n(.).e^{-i<j,.>.\frac{\pi}{L}})(x).\phi(x).e^{i<j,x>.\frac{\pi}{L}})| \leq P'(|j|)$$

for every j and every $n \geq n_0$ (n_0 great enough) and where P' is a polynomial with positive coefficients.

Then if $n \geq n_0$

$$\sum_j |c_j . v((u * S_n(.).e^{-i<j,.>.\frac{\pi}{L}})(x).\phi(x).e^{i<j,x>.\frac{\pi}{L}})| \leq \sum_j c_j . P'(|j|).$$

Because $(u.v)_{HO}$ exists we can write (*)

$$\lim_{n \to \infty} ((\alpha u) * S_n).v)(\phi) = \lim_{n \to \infty} \sum_j c_j . v((u * S_n(.).e^{-i<j,.>.\frac{\pi}{L}})(x).\phi(x).e^{i<j,x>.\frac{\pi}{L}}) =$$

$$= \sum_j c_j . \lim_{n \to \infty} v((u * S_n(.).e^{-i<j,.>.\frac{\pi}{L}})(x).\phi(x).e^{i<j,x>.\frac{\pi}{L}}) =$$

$$= \sum_j c_j .(u.v)_{HO}(\phi(x).e^{i<j,x>.\frac{\pi}{L}}) = (u.v)_{HO}(\phi(x). \sum_j c_j . e^{i<j,x>.\frac{\pi}{L}}) =$$

$$= (u.v)_{HO}(\phi(x).\alpha(x)).$$

QED.

PROOF OF THEOREM 2. It is obvious that the existence of $(u.v)_{MG}$ implies the existence of $(u.v)_{HO}$:

$$((u * S_n).v)(\phi) = \lim_{m \to \infty} ((u * S_n).(v * \Delta_m))(\phi)$$

and taking a subsequence of $\{\Delta_m\}_{m=1,2,...}$ (linked to n) we have

$$\lim_{n \to \infty} ((u * S_n).v)(\phi) = \lim_{n \to \infty} ((u * S_n) . (v * \Delta_{m_n}))(\phi) = (u.v)_{MG}(\phi).$$

We will prove now that the existence of $(u.v)_{HO}$ implies the existence of $(u.v)_{MG}$ (and therefore they are equal).

If $\phi \in C_0^\infty(R^m)$, writing again

$$\phi(x) = \sum_j c_j . e^{i<j,x>.\frac{\pi}{L}}, \quad x \in Q,$$

with $\sum_j |c_j|.(1 + |j|)^l \leq K(l)$, for each l.

Choosing $\alpha \equiv 1$ in a neighbourhood of the support of ϕ, for $n \geq n_0$, $m \geq m_0$ we have that

---

(*) Suppose the existence of $(u.v)_{HO}$. If $\Delta_n = \Delta_n^R + i\Delta_n^C$ is a sequence of $C_0^\infty(R^m)$ functions such that:

a) $\Delta_n^R \geq 0$ for all n.

b) $\lim_{n \to \infty} \int \Delta_n^R = 1$, $\lim_{n \to \infty} \int |\Delta_n^C| = 0$.

c) supp. $\Delta_n^R \to 0$, supp. $\Delta_n^C \to 0$.

Then from lemma 2 we get: $\lim_{n \to \infty} (u * \Delta_n).v = (u.v)_{HO}$ (in $\mathcal{D}'(R^m)$).

$$((u * S_n).(v * \Delta_m))(\phi) = (((\alpha u) * S_n).(v * \Delta_m))(\phi) =$$

$$= ((\alpha u) * S_n)(\phi(x).(v * \Delta_m)) = ((\alpha u) * S_n)(\sum_j c_j.e^{i<j,x>.\frac{\pi}{L}}(v * \Delta_m)(x)) =$$

$$= \sum_j c_j.((\alpha u) * S_n)(e^{i<j,x>.\frac{\pi}{L}}.(v * \Delta_m)(x)) =$$

$$= \sum_j c_j.((\alpha u) * S_n)(v_y(\Delta_m(x-y).e^{i<j,x-y>.\frac{\pi}{L}}.e^{i<j,y>.\frac{\pi}{L}})) =$$

$$= \sum_j c_j.((\alpha u) * S_n)(((e^{i<j,\bullet>.\frac{\pi}{L}}.v_\bullet) * \Delta_m(\bullet).e^{i<j,\bullet>.\frac{\pi}{L}})(x)) =$$

$$= \sum_j c_j.(((\alpha u) * S_n) * ((e^{i<j,\bullet>.\frac{\pi}{L}}.v_\bullet) * \Delta_m(\bullet).e^{i<j,\bullet>.\frac{\pi}{L}})^{\vee})(0) =$$

$$= \sum_j c_j.(((\alpha u) * S_n * \check{\Delta}_m(\bullet).e^{-i<j,\bullet>.\frac{\pi}{L}}) * (e^{i<j,\bullet>.\frac{\pi}{L}}.v_\bullet)^{\vee})(0) =$$

$$= \sum_j c_j.(((\alpha u) * S_n * \check{\Delta}_m(\bullet).e^{-i<j,\bullet>.\frac{\pi}{L}}).v)(e^{i<j,x>.\frac{\pi}{L}}.\beta(x))$$

where $\beta \in C_0^\infty(R^m)$ and $\beta \equiv 1$ in a neighbourhood of supp. α.

By lemma 4 we know that $((\alpha u).v)_{HO}$ exists.

By lemma 2 we have then

$$|(((\alpha u) * S_n * \check{\Delta}_m(.).e^{-i<j,.>.\frac{\pi}{L}}).v)(e^{i<j,x>.\frac{\pi}{L}}.\beta(x))| \leq P''(|j|)$$

for every j and $n \geq n_1$, $m \geq m_1$, where P" is a polynomial with positive coefficients.

Then we have by lemma 4 and the last observation that

$$\lim_{n,m \to \infty} ((u * S_n).(v * \Delta_m))(\phi) =$$

$$= \lim_{n,m \to \infty} \sum_j c_j.(((\alpha u) * S_n * \check{\Delta}_m(.).e^{-i<j,.>.\frac{\pi}{L}}).v)(e^{i<j,x>.\frac{\pi}{L}}.\beta(x)) =$$

$$= \sum_j c_j. \lim_{n,m \to \infty} (((\alpha u) * S_n * \check{\Delta}_m(.).e^{-i<j,.>.\frac{\pi}{L}}).v)(e^{i<j,x>.\frac{\pi}{L}}.\beta(x)) =$$

$$= \sum_j c_j.((\alpha u).v)_{HO}(e^{i<j,x>.\frac{\pi}{L}}\ \beta(x)) = \sum_j c_j.(u.v)_{HO}(\alpha(x).e^{i<j,x>.\frac{\pi}{L}}) =$$

$$= (u.v)_{HO}((\sum_j c_j.e^{i<j,x>.\frac{\pi}{L}}).\alpha(x)) = (u.v)_{HO}(\phi), \qquad \text{QED.}$$

Definition 2 can be changed slightly.

DEFINITION 4. *Given* $u, v \in \mathcal{D}'(R^m)$ *we call their product* $(u.v)_M$, *if it exists, the distribution* $w \in \mathcal{D}'(R^m)$ *such that for each sequence* $\{S_n\}_{n=1,2,\ldots}$ *with the pro properties*

i) $S_n \geqq 0$ *for all* n.

ii) $\int S_n = 1$ *for all* n.

iii) $\operatorname{supp} S_n \to 0$, *if* $n \to \infty$,

and for each function $\phi \in C_0^\infty(R^m)$, *the following limit*

$$\lim_{n \to \infty} \int (u * S_n)(x).(v * S_n)(x).\phi(x)\, dx = w(\phi)$$

holds (see Mikusinski, [14]).

It is clear that if the product $(u.v)_{MG}$ exists then the product $(u.v)_M$ also exists and, of course, we have,

$$(u.v)_M = (u.v)_{MG} .$$

The converse is not true, because if

$$H(x) := \begin{cases} 1 & x \geqq 0, \\ 0 & x < 0, \end{cases}$$

and $\phi \in C_0^\infty(R^1)$ we have

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} (H * S_n)(x).S_n(x).\phi(x)\, dx = \lim_{n \to \infty} \int_{-\infty}^{+\infty} (H * S_n)(x).(H * S_n)^{(1)}(x).\phi(x)\, dx =$$

$$= \lim_{n \to \infty} \int_{-\infty}^{+\infty} \left[\frac{(H * S_n)^2(x)}{2}\right]^{(1)}.\phi(x)\, dx = \lim_{n \to \infty} \int_{-\infty}^{+\infty} -\frac{1}{2}(H * S_n)^2(x).\phi^{(1)}(x)\, dx = \frac{\phi(0)}{2} .$$

Then $(H.S)_M = S/2$ but it is clear that $(H.S)_{MG}$ does not exist.

The following diagram shows the relationship among the products

$$\begin{array}{ccccc} (u.v)_H & \overset{\not\leftarrow}{\longrightarrow} & (u.v)_{MG} & \overset{\not\leftarrow}{\longrightarrow} & (u.v)_M \\ & & \updownarrow & & \\ & & (u.v)_{HO} & & \end{array}$$

Some authors restrict in definition 4 the sequences $\{S_n\}_{n=1,2,\ldots}$, taking a fixed one with the properties i), ii), iii) of that definition. We will call this product $(u.v)_{M(\Delta_n)}$ where $\{\Delta_n\}_{n=1,2,\ldots}$ is this particular sequence.

The following is an example of two distributions $u, v \in \mathcal{D}'(R)$ and $\{\Delta_n\}_{n=1,2,\ldots}$

such that $(u.v)_{\mathcal{M}(\Delta_n)}$ exists but $(u.v)_{\mathcal{M}}$ does not exist.

We have (see appendix for the definitions of x_+^{λ}, x_-^{μ}):

$$(x_+^{-\frac{1}{2}} * \Delta_n)(y) = \langle x_+^{-\frac{1}{2}}, \Delta_n(y-x)\rangle = \int_0^{+\infty} x^{-\frac{1}{2}}.\Delta_n(y-x)\, dx = \int_{-\infty}^{y} (y - t)^{-\frac{1}{2}}.\Delta_n(t)\, dt$$

and also (here i is the imaginary unit)

$$(x_-^{-\frac{1}{2}+i} * \Delta_n)(y) = \langle x_-^{-\frac{1}{2}+i}, \Delta_n(y-x)\rangle = \langle x_+^{-\frac{1}{2}+i}, \Delta_n(y+x)\rangle =$$

$$= \int_0^{+\infty} x^{-\frac{1}{2}+i}.\Delta_n(y+x)\, dx = \int_y^{+\infty} (s - y)^{-\frac{1}{2}+i}.\Delta_n(s)\, ds.$$

Then if $\phi \in C_0^{\infty}(R)$ we have

$$\int (x_+^{-\frac{1}{2}} * \Delta_n)(y).(x_-^{-\frac{1}{2}+i} * \Delta_n)(y).\phi(y)\, dy =$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{y} (y - t)^{-\frac{1}{2}}.\Delta_n(t)\, dt\right).\left(\int_y^{+\infty} (s - y)^{-\frac{1}{2}+i}.\Delta_n(s)\, ds\right).\phi(y)\, dy =$$

$$= \int_{-\infty}^{+\infty}\left(\int_t^{+\infty}\left(\int_t^{s} (y - t)^{-\frac{1}{2}}.(s - y)^{-\frac{1}{2}+i}.\phi(y)\, dy\right).\Delta_n(s)\, ds\right).\Delta_n(t)\, dt.$$

Take $y = t + (s - t).v$. We have then

$$\int_t^{s} (y - t)^{-\frac{1}{2}}.(s - y)^{-\frac{1}{2}+i}.\phi(y)\, dy =$$

$$= \left(\int_0^1 v^{-\frac{1}{2}}.(1 - v)^{-\frac{1}{2}+i}.\phi(t+(s-t).v)\, dv\right).(s - t)^{i} = G(s,t).(s - t)^{i}$$

with $G(0,0) = B(\frac{1}{2}, \frac{1}{2} + i).\phi(0)$; $G(s,t) \in C^{\infty}(R^2)$; $B(.,.)$ is the Beta function.

Then

$$\int (x_+^{-\frac{1}{2}} * \Delta_n)(y).(x_-^{-\frac{1}{2}+i} * \Delta_n)(y).\phi(y)\, dy =$$

$$= \int_{-\infty}^{+\infty}\left(\int_t^{+\infty} G(s,t).(s - t)^{i}.\Delta_n(s)\, ds\right).\Delta_n(t)\, dt.$$

Let $\rho \in C_0^{\infty}(R)$, ρ an even function, $\rho \geq 0$, $\int\rho = 1$ and $\{\beta_n\}_{n=1,2,...}$ an increasing sequence of positive real numbers such that $\lim_{n \to \infty} \beta_n = +\infty$.

We set

$$\Delta_n(x) := \beta_n.\rho(x.\beta_n).$$

We can then write

$$\int_{-\infty}^{+\infty}\left(\int_{t}^{+\infty} G(s,t).(s-t)^{i}.\beta_n.\rho(s.\beta_n)\, ds\right).\beta_n.\rho(t.\beta_n)\, dt =$$

$$= \left[\int_{-\infty}^{+\infty}\left(\int_{t'}^{+\infty} G\left(\frac{s'}{\beta_n}, \frac{t'}{\beta_n}\right).(s'-t')^{i}.\rho(s')\, ds'\right).\rho(t')\, dt'\right].\frac{1}{\beta_n^{i}} .$$

But

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty}\left(\int_{t'}^{+\infty} G\left(\frac{s'}{\beta_n}, \frac{t'}{\beta_n}\right).(s'-t')^{i}.\rho(s')\, ds'\right).\rho(t')\, dt' =$$

$$= \phi(0).B\left(\tfrac{1}{2}, \tfrac{1}{2}+i\right).\int_{-\infty}^{+\infty}\left(\int_{t'}^{+\infty} (s'-t')^{i}.\rho(s')\, ds'\right).\rho(t')\, dt' =$$

$$= \frac{\phi(0)}{2}.B\left(\tfrac{1}{2}, \tfrac{1}{2}+i\right).\iint |s'-t'|^{i}\rho(s').\rho(t')\, ds'\, dt'$$

and there exists ρ such that

$$\iint |s'-t'|^{i}.\rho(s').\rho(t')\, ds'\, dt' \neq 0$$

(see theorem 1 and lemma 1 of §2 of part II).
Then, there exists $c \neq 0$ such that

$$\lim_{n\to\infty} \int (x_+^{-\frac{1}{2}} * \Delta_n)(y).(x_-^{-\frac{1}{2}+i} * \Delta_n)(y).\phi(y)\, dy = c.\phi(0).\lim_{n\to\infty}\frac{1}{\beta_n^{i}} .$$

If we choose $\beta_n = e^{2\pi n-\theta}$ we shall have $\lim_{n\to\infty}\frac{1}{\beta_n^{i}} = e^{i\theta}$, θ arbitrary. Therefore, given θ there exist sequences $\{\Delta_n(\theta)\}_{n=1,2,\ldots}$ such that

$$(x_+^{-\frac{1}{2}} . x_-^{-\frac{1}{2}+i})_{M(\Delta_n(\theta))} = c.e^{i\theta}.S.$$

This shows, that in this case the M-product does not exist.

IB. A PARTICULAR PRODUCT. Let (a,b) be an open interval $\subseteq R$. Let μ be a signed measure over the Borel sets of (a,b) ie. for each closed interval $[c,d]$, $a < c \leq d < b$, there exists a Borel set $A \subseteq [c,d]$ such that for each Borel set $B \subseteq A$ we have $0 \leq \mu(B) < +\infty$ and for each Borel set $C \subseteq \{[c,d] - A\}$, $-\infty < \mu(C) \leq 0$ (Hahn's theorem).
Let $v(x)$ be a function of bounded variation in each compact interval contained in (a,b). In what follows we suppose $v(x)$ continuous from the left and defined at each point (see the note at the end of this section).

THEOREM (P. Antosik and J. Ligeza, [2]): *The product* $(\mu.\nu)_M$ *exists (definition 4, IA) and we have*

$$(\mu.\nu)_M(\phi) = \int_a^b \nu(y).\phi(y)\, d\mu + \sum_{\substack{P \in \text{supp.}\, \phi \\ \nu(x) \text{ at } P \text{ is discontinous}}} \mu(P).\phi(P).\frac{\$(P)}{2}.$$

where $\$(P) = [\lim_{\substack{x \to P \\ x > P}} \nu(x) - \lim_{\substack{x \to P \\ x < P}} \nu(x)]$, $\phi \in C_o^\infty(R)$.

PROOF. Let $\phi \in C_o^\infty(R)$ and $[e,f] \subseteq (a,b)$ such that supp.$\phi \subseteq [e+\varepsilon\, ,\, f-\varepsilon]$. Decompose $\nu(x)$ on $[e,f]$ in the following way

$$\nu(x) = \nu_c(x) + \$_c(x) + \$_d(x),$$

where ν_c is continuous in $[e,f]$; $\$_c$, $\$_d$ are increasing and decreasing jump functions, continuous from the left. They can be written as

$$\$_c(x) = \sum C_j.H_{\xi_j}(x),$$

$$\$_d(x) = -\sum D_i.H_{\xi_i'}(x)$$

where

$$H_{\xi_j}(x) = H(x-\xi_j)$$

and

$$H(x) \equiv \begin{cases} 1 & x > 0, \\ 0 & x \leq 0, \end{cases}$$

and where ξ_j, ξ_i' are a denumerable number of points in $[e,f]$, $C_j \geq 0$, $D_i \geq 0$, $\sum C_j < \infty$, $\sum D_i < \infty$.

Then, if I is the distribution

$$I(\phi) = \int \phi,$$

we have

$$\int (\mu * S_n)(x).(\nu * S_n)(x).\phi(x)\, dx = (\mu_y \otimes \nu_x \otimes I_s)(\phi(s).S_n(s-x).S_n(s-y)) =$$

$$= \mu_y(\int \phi(s).(\nu * S_n)(s).S_n(s-y)\, ds) =$$

$$= \mu_y(\int \phi(s).(\nu_c * S_n)(s).S_n(s-y)\, ds) + \mu_y(\int \phi(s).(\$_c * S_n)(s).S_n(s-y)\, ds) -$$

$$- \mu_y(\int \phi(s).((-\$_d) * S_n)(s).S_n(s-y)\, ds).$$

It suffices to study the limits:

$$\lim_{n\to\infty} \mu_y\left(\int \phi(s).(v_c * S_n)(s).S_n(s-y)\, ds\right),$$

$$\lim_{n\to\infty} \mu_y\left(\int \phi(s).(\mathfrak{s}_c * S_n)(s).S_n(s-y)\, ds\right).$$

It is clear that

$$\int \phi(s).(v_c * S_n)(s).S_n(s-y)\, ds \xrightarrow[\text{in } [e,f]]{o} \phi(y).v_c(y)$$

then

$$\lim_{n\to\infty} \mu_y\left(\int \phi(s).(v_c * S_n)(s).S_n(s-y)\, ds\right) = \mu(\phi.v_c).$$

We call

$$\mathfrak{s}_c^m(x) := \sum_{j=1,\ldots,m} C_j.H_{\xi_j}(x).$$

Then $|\mathfrak{s}_c(x) - \mathfrak{s}_c^m(x)| \leq \varepsilon'$, if m is great enough.

Now, if the following limit exists

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} \mu_y\left(\int \phi(s).(\mathfrak{s}_c^m * S_n)(s).S_n(s-y)\, ds\right)\right) = \alpha$$

then we have

$$\lim_{n\to\infty} \mu_y\left(\int \phi(s).(\mathfrak{s}_c * S_n)(s).S_n(s-y)\, ds\right) = \alpha.$$

Let us calculate

$$\mu_y\left(\int \phi(s).(\mathfrak{s}_c^m * S_n)(s).S_n(s-y)\, ds\right) =$$

$$= \sum_{j=1,\ldots,m} C_j.\mu_y\left(\int \phi(s).(H_{\xi_j} * S_n)(s).S_n(s-y)\, ds\right).$$

But $\left|\int \phi(s).(H_{\xi_j} * S_n)(s).S_n(s-y)\, ds\right| \leq \max |\phi|$, for all n, all y, j = 1, ..., m,

and

$$\lim_{n\to\infty} \int \phi(s).(H_{\xi_j} * S_n)(s).S_n(s-y)\, ds = \begin{cases} 0 & \text{if } y < \xi_j, \\ \phi(y) & \text{if } y > \xi_j, \end{cases}$$

and also, since we have $(H_{\xi_j} * S_n)^{(1)}(s) = S_n(s-\xi_j)$, we can write

$$\lim_{n\to\infty} \int \phi(s).(H_{\xi_j} * S_n)(s).S_n(s-\xi_j)\, ds = \lim_{n\to\infty} \int \phi(s)\left[\frac{(H_{\xi_j} * S_n)^2(s)\ (1)}{2}\right] ds =$$

$$= \lim_{n \to \infty} -\frac{1}{2} \cdot \int (H_{\xi_j} * S_n)^2(s).\phi^{(1)}(s)\, ds = \frac{\phi(\xi_j)}{2},$$

By Lebesgue's theorem we get

$$\lim_{n \to \infty} \mu_y(\int \phi(s).(H_{\xi_j} * S_n)(s).\ S_n(s-y)\, ds) = \mu_y(\phi(y).H_{\xi_j}(y)) + \mu(\xi_j).\frac{\phi(\xi_j)}{2},$$

and therefore

$$\lim_{m \to \infty} (\lim_{n \to \infty} \mu_y(\int \phi(s).(\mathcal{S}_c^m * S_n)(s).S_n(s-y)\, ds)) =$$

$$= \lim_{m \to \infty} (\sum_{j=1,\dots,m} C_j.[\mu_y(\phi(y).H_{\xi_j}(y)) + \mu(\xi_j).\ \frac{\phi(\xi_j)}{2}]) =$$

$$= \lim_{m \to \infty} [\mu_y(\phi(y).\sum_{j=1,\dots,m} C_j.H_{\xi_j}(y)) + \sum_{j=1,\dots,m} \mu(\xi_j).\phi(\xi_j).\ \frac{C_j}{2}] =$$

$$= \mu_y(\phi(y).\mathcal{S}_c(y)) + \sum_j \mu(\xi_j).\phi(\xi_j).\ \frac{C_j}{2}.$$

Then

$$\lim_{n \to \infty} \int (\mu * S_n)(x).(v * S_n)(x).\phi(x)\, dx =$$

$$= \mu(\phi.v_c) + \mu(\phi.\mathcal{S}_c) + \sum_j \mu(\xi_j).\phi(\xi_j).\ \frac{C_j}{2} - \mu(-\phi.\mathcal{S}_d) - \sum_i \mu(\xi_i').\phi(\xi_i')\ \frac{D_i}{2}$$

and the theorem follows. QED.

Note. If v' is any function of bounded variation then $(\mu.v')_M = (\mu.v)_M$, where v is the unique function of bounded variation which is left continuous defined at each point and such that $v = v'$ almost everywhere. The equality is due to the fact that

$$(v' * S_n)(s) = (v * S_n)(s).$$

IC. A SPECIAL FORMULA.

DEFINITION 1. *Given a polynomial,* $P(x_1,\dots,x_m)$, *in* m *variables and* $m + 1$ *distributions* $U_1, \dots, U_m$, $w \in \mathcal{D}'(R)$, *we say that*

$$P(U_1, \dots, U_m) = w$$

if

$$\lim_{n \to \infty} \int P((U_1 * S_n)(x),\dots,(U_m * S_n)(x)).\phi(x)\, dx = W(\phi)$$

holds for every $\phi \in C_0^\infty(R)$ *and every sequence* $\{S_n\}_{n=1,2,\dots}$ *of* $C_0^\infty(R)$ *functions such that*

i) $S_n \geq 0$ *for all* n,

ii) $\int S_n = 1$ *for all* n,

iii) supp. $S_n \to 0$, *if* $n \to \infty$.

A particular case of this is the product given in definition 4, IA.

We want to prove the following formula

(1-C) $$(S^{(j_1)} + \frac{i}{\pi}(\frac{1}{x})^{(j_1)}). \dots .(S^{(j_s)} + \frac{i}{\pi}(\frac{1}{x})^{(j_s)}) =$$

$$= \frac{j_1! \dots j_s!}{(j_1+\dots+j_s+s-1)!.(i\pi)^{s-1}} .(S^{(j_1+\dots+j_s+s-1)} + \frac{i}{\pi}(\frac{1}{x})^{(j_1+\dots+j_s+s-1)}),$$

with $s = 1, 2, \dots, j_i \in N \cup \{0\}$. Here $(\frac{1}{x})(\phi) = -\int \ln|x|.\phi^{(1)}(x)\, dx =$

$= \text{p.v.}\int \frac{\phi(x)}{x} dx$, for every $\phi \in C_0^\infty(R)$.

We will need some auxiliary lemmas.

LEMMA 1. *Let* $U,V \in C^\infty(R)$ *be such that there exists a polynomial* P *with positive coefficients such that*

$$|U(x)| \leq P(|x|), \quad |V(x)| \leq P(|x|)$$

and $U.\psi \in \mathcal{S}(R)$ *if* $\psi \in \mathcal{S}(R)$ *(this means* $U \in O_M(R)$*). Assume* $\hat{U},\hat{V}$ *are functions with supports in* $[a,\infty)$, $a \in R$, *that are continouos except at one point at most and that there also exists a polynomial* P' *with positive coefficients such that*

$$|\hat{U}(x)| \leq P'(|x|), \quad |\hat{V}(x)| \leq P'(|x|).$$

Then we have

$$\hat{U} * \hat{V} = \widehat{(U.V)}.2\pi.$$

PROOF. By hypothesis U.V is of polynomial growth. It is $C^\infty(R)$ and $\hat{U} * \hat{V}$ is continuous, with support in a halfline and of polynomial growth too.

Then if $\phi \in C_0^\infty(R)$,

$$\int (\hat{U} * \hat{V})(y).\phi(y)\, dy = \int \phi(y).(\int \hat{U}(y-x).\hat{V}(x)\, dx)\, dy =$$

$$= \int \hat{V}(x).(\int \hat{U}(y-x).\phi(y)\, dy)\, dx = \int \hat{V}(x).(\int \hat{U}(y).\phi(y+x)\, dy)\, dx =$$

$$= \int \hat{V}(x).(\int U(y).\hat{\phi}(y).e^{i<x,y>}\, dy)\, dx = \int \hat{V}(x).\check{\widehat{U\hat{\phi}}}(x)\, dx = 2\pi.\int V.U.\hat{\phi}, \text{ QED.}$$

LEMMA 2.

$$\widehat{S^{(m)} + \frac{i}{\pi}\left(\frac{1}{x}\right)^{(m)}} = 2.i^m.x^m.H(x),$$

if m ε N ∪ {0} *where* H(x) *is the Heavyside function, ie.*

$$H(x) \equiv \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

PROOF. Although this is a well-known formula we include a proof of it (cf. [Schwartz II, P. 115]).

Let $\psi \in \mathcal{S}(R)$. Then

$$\widehat{\left(\frac{1}{x}\right)}(\psi) = \text{p.v.} \int \frac{\hat{\Psi}(x)}{x}\, dx =$$

$$= \lim_{n\to\infty} \left\{\int_{-n}^{-\frac{1}{n}} \frac{1}{x}.\left(\int e^{-i<x,\xi>}.\psi(\xi)\, d\xi\right) dx + \int_{1/n}^{n} \frac{1}{x}.\left(\int e^{-i<x,\xi>}.\psi(\xi)\, d\xi\right) dx\right\} =$$

$$= \lim_{n\to\infty} \left\{\int \psi(\xi).\left(\int_{-n}^{-1/n} \frac{e^{-i<x,\xi>}}{x}\, dx + \int_{1/n}^{n} \frac{e^{-i<x,\xi>}}{x}\, dx\right) d\xi\right\} =$$

$$= \lim_{n\to\infty} \left\{\int \psi(\xi).\left(-2i.\int_{1/n}^{n} \frac{\text{sen }(x.\xi)}{x}\, dx\right) d\xi\right\} = \lim_{n\to\infty} \left\{\int \psi(\xi).\left(-2i.\int_{\xi/n}^{n\xi} \frac{\text{sen } u}{u}\, du\right) d\xi\right\}.$$

But $\int_0^{+\infty} \frac{\text{sen } u}{u}\, du = \pi/2$, then we have

$$\widehat{\left(\frac{1}{x}\right)} = i\pi.(1 - 2.H(x)).$$

The case m = 0 follows if one observes that

$$\hat{S}(\psi) = \int \psi(\xi)\, d\xi$$

and therefore

$$\widehat{S + \frac{i}{\pi}\left(\frac{1}{x}\right)}(\psi) = \int 2.H(\xi).\psi(\xi)\, d\xi.$$

The general formula follows from the fact that

$$\widehat{T^{(m)}} = (ix)^m \hat{T},$$

QED.

PROOF OF FORMULA 1-C. We call

$$A_k^n(x) := \left(S_n^{(j_k)} + \frac{i}{\pi}\left(\left(\frac{1}{x}\right)^{(j_k)} * S_n\right)\right), \text{ with } k = 1, \ldots, s.$$

The $A_i^n(x)$ satisfy the conditions in lemma 1. We can write, using lemmas 1 and 2,

$$(2\pi)^{s-1}.(A_1^n. \dots .A_s^n)^{\wedge} = \hat{A}_1^n * \dots * \hat{A}_s^n =$$

$$= 2^s.i^{j_1+\dots+j_s}.(H(x).x^{j_1}.\hat{S}_n(x) * \dots * H(x).x^{j_s}.\hat{S}_n(x)).$$

Then, (cf. Schwartz, II. p.30]):

$$\lim_{n \to \infty} (2\pi)^{s-1}.(A_1^n. \dots .A_s^n)^{\wedge} = 2^s.i^{j_1+\dots+j_s}.(H(x).x^{j_1} * \dots * H(x).x^{j_s}) =$$

$$= 2^s.i^{j_1+\dots+j_s}.\frac{j_1! \dots j_s!}{(j_1 + \dots + j_s + s - 1)!}.H(x).x^{j_1+\dots+j_s+s-1} =$$

$$= 2^{s-1}.i^{1-s}.\frac{j_1! \dots j_s!}{(j_1 + \dots + j_s + s - 1)!}.(S^{(j_1+\dots+j_s+s-1)} + \frac{i}{\pi}(\frac{1}{x})^{(j_1+\dots+j_s+s-1)})^{\wedge}$$

and from this, the formula follows, QED.

APPLICATIONS. From formula 1-C we get

$$\mathrm{Re}\ (\mathrm{Im\ resp.})(S_n^{(j_1)} + \frac{i}{\pi}((\frac{1}{x})^{(j_1)} * S_n)). \dots .(S_n^{(j_s)} + \frac{i}{\pi}((\frac{1}{x})^{(j_s)} * S_n)) \to$$

$$\mathrm{Re}(\mathrm{Im\ resp.})\ \frac{j_1! \dots j_s!}{(j_1+\dots+j_s+s-1)!.(i\pi)^{s-1}}.(S^{(j_1+\dots+j_s+s-1)} + \frac{i}{\pi}(\frac{1}{x})^{(j_1+\dots+j_s+s-1)})$$

in $\mathcal{D}'(R)$ if $n \to \infty$.

From this several known formulas are deduced. Taking $s = 2$, $j_1 = j_2 = r$ ($r \in \mathbb{N} \cup \{0\}$):

$$S.(\frac{1}{x}) = - \frac{S^{(1)}}{2},$$ (A. González Domínguez and R. Scarfiello, [8]).

$$S^{(r)}.(\frac{1}{x})^{(r)} = - \frac{(r!)^2}{2.(2r + 1)!}.S^{(2r+1)},$$ (B. Fisher, [6]).

$$[S^2 - \frac{1}{\pi^2}.(\frac{1}{x})^2] = \frac{1}{\pi^2}.(\frac{1}{x}),$$ (J. Mikusinski, [15]).

$$[(S^{(r)})^2 - \frac{1}{\pi^2}.((\frac{1}{x})^{(r)})^2] = \frac{(r!)^2}{(2r + 1)!.\pi^2}.(\frac{1}{x})^{(2r+1)},$$

(S.E. Trione, [19]).

These formulas and several others due to B. Fisher ([4], [5], [6], [7]) have been generalized to higher demensions by S.E. Trione ([19]).

PART II

THE PRODUCT $x_+^\lambda . x_-^\mu$

1. INTRODUCTION. Let $u,v \in \mathcal{D}'(R^n)$. We say that the product u.v exists if for each $\phi \in C_o^\infty(R^n)$ and each sequence $\{S_n\}_{n=1,2,...}$ the limit

$$w(\phi) = \lim_{n \to \infty} \int (u * S_n)(y).(v * S_n)(y).\phi(y)\, dy$$

exists. $\{S_n\}_{n=1,2,...}$ is an arbitrary sequence of functions of $C_o^\infty(R^n)$ with the following properties:

$$\text{(I)} \qquad \begin{cases} \text{i)} & S_n \geqq 0 \text{ for all } n, \\ \text{ii)} & \int S_n = 1 \text{ for all } n, \\ \text{iii)} & \text{supp. } S_n \to 0 \text{ if } n \to \infty \end{cases}$$

It is clear that $S_n \to S$ in the distributional sense if $n \to \infty$. Here $S(\phi) = \phi(0)$ for all $\phi \in C_o^\infty(R^n)$.

The mentioned limit does not depend on the sequence $\{S_n\}_{n=1,2,...}$ and defines a distribution $w \in \mathcal{D}'(R^n)$.

More generally, consider a family $u_i \in \mathcal{D}'(R^n)$, $i = 1, \ldots, m$. Assume P is a polynomial in m variables. We say that a distribution $w \in \mathcal{D}'(R^n)$ is equal to $P(u_1,\ldots,u_m)$ if

$$w(\phi) = \lim_{n \to \infty} \int P((u_1 * S_n)(y),\ldots,(u_m * S_n)(y)).\phi(y)\, dy$$

for every function $\phi \in C_o^\infty(R^n)$ and for every sequence $\{S_n\}_{n=1,2,...}$ with properties (I).

Notice that the existence of $P(u_1,\ldots,u_m)$ does not imply the existence of $Q(u_1,\ldots,u_m)$ for any monomial Q that appears in the polynomial.

We shall use the following notation

$$P((u_1 * S_n)(y), \ldots , (u_m * S_n)(y)) = ((P(u_1,\ldots,u_m)) * S_n)(y).$$

We restrict ourselves to work in R^1 and with special distributions: the seudofunctions.

We recall their definition (here $\phi \in C_o^\infty(R^1)$ and $\check{\phi}(x) = \phi(-x)$):

$$
\text{(II)} \begin{cases}
\text{i)} & \langle x_+^\lambda, \phi\rangle = \int_0^{+\infty} x^\lambda . \phi(x)\, dx, \text{ if Re } \lambda > -1, \\
\text{ii)} & \langle (x_+^\lambda)^{(1)}, \phi\rangle = \lambda . \langle x_+^{\lambda-1}, \phi\rangle, \text{ if } \lambda \neq 0, -1, \ldots , \\
\text{iii)} & \langle x_+^{-k}, \phi\rangle = \\
 & = - \frac{1}{(k-1)!} \int_0^{+\infty} \log |x| . \phi^{(k)}(x)\, dx + \frac{(-1)^{k-1}}{(k-1)!} \left(\sum_{j=1}^{k-1} \frac{1}{j}\right) S^{(k-1)}(\phi), \\
 & \text{if } k = 1, 2, \ldots . \\
\text{iv)} & \langle x_+^\lambda, \phi\rangle = \langle x_-^\lambda, \check{\phi}\rangle, \text{ for all } \lambda \in C.
\end{cases}
$$

The aim of this part of the monograph is to decide for each pair $\lambda, \mu \in C$, whether the product

$$x_+^\lambda . x_-^\mu$$

exists or not. Moreover, for certain pairs λ, μ for which the product does not exist we show that $x_+^\lambda . x_-^\mu + x_+^\mu . x_-^\lambda$ or $x_+^\lambda . x_-^\mu - x_+^\mu . x_-^\lambda$ exists. This is the content of theorem 6 which also includes theorems 3, 4, 5.

Next theorem 1, and lemmas 1, 2, 3, are of an auxiliary nature.

The following result will be used in the future.

THEOREM 0 (B. Fisher). *Let* $f, g \in \mathcal{D}'(R^1)$ *be such that:* $f^{(r)}$ *is a* $L_{p,loc}(R^1)$ *function,* $G \in L_{q,loc}(R^1)$, $G^{(r)} = g$, $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Then the product $f.g$ *exists and*

$$f.g = \sum_{i=0}^{r} \binom{r}{i} . (-1)^i . [G.f^{(i)}]^{(r-i)} .$$

PROOF. If f, g, G are as stated then the following identity holds ($\{S_n\}_{n=1,2,\ldots}$ as in (I)):

$$(f * S_n)(y).(g * S_n)(y) = \sum_{i=0}^{r} \binom{r}{i} . (-1)^i . [(G * S_n)(y).(f^{(i)} * S_n)(y)]^{(r-i)} .$$

Besides we have $f^{(i)} \in L_{p,loc}(R^1)$ for $1 \leq i \leq r$.

If we show that

$$(G * S_n)(y).(f^{(i)} * S_n)(y) \to G.f^{(i)}, \text{ if } n \to \infty$$

holds in the sense of distributions, the theorem will follow.

Here $G.f^{(i)}$ is the ordinary product, which is defined since $G \in L_{q,loc}(R^1)$ and $f^{(i)} \in L_{p,loc}(R^1)$. We may assume without loss of generality that G and f are of compact support.

We have

$$\| (G * S_n)(y).(f^{(i)} * S_n)(y) - G.f^{(i)}\|_1 \leq$$

$$\leq \|(G * S_n)(y).[(f^{(i)} * S_n)(y) - f^{(i)}(y)]\|_1 + \|[(G * S_n)(y) - G(y)].f^{(i)}(y)\|_1$$

and from this, using Hölder's inequality and Lebesgue's theorem, we get

$$\|(G * S_n)(y).(f^{(i)} * S_n)(y) - G.f^{(i)}\|_1 \to 0 \quad \text{if } n \to \infty, \qquad \text{QED.}$$

If $\phi \in C_0^\infty(R^1)$, $\hat{\phi}$ is its Fourier transform:

$$\hat{\phi}(x) = \int \phi(t).e^{-i<x,t>}\, dt$$

and $\hat{\hat{\phi}}$ its inverse Fourier transform:

$$\hat{\hat{\phi}}(x) = (2\pi)^{-1}.\int \phi(t).e^{i<x,t>}\, dt.$$

Besides $\check{\phi}(x) := \phi(-x)$. $\Gamma(.)$ and $B(.,.)$ will denote the Gamma and the Beta function respectively.

Notice that if the product of two distributions $u,v \in \mathcal{D}'(R^1)$ exists then the derivate of the product also exists in the sense given before

$$(u.v)^{(1)} = u^{(1)}.v + u.v^{(1)}.$$

Also by reasons of simmetry we have that: $x_+^\lambda.x_-^\mu$ exists if and only if $x_+^\mu.x_-^\lambda$ exists.

Because of this, if $x_+^\lambda.x_-^\mu + x_+^\mu.x_-^\lambda$ does not exist or $x_+^\lambda.x_-^\mu - x_+^\mu.x_-^\lambda$ does not exist, then $x_+^\lambda.x_-^\mu$ does not exist.

2. LIMITS OF CERTAIN INTEGRALS.

THEOREM 1. *Let* $\rho(x) \in C_0^\infty(R^1)$ *and such that*

i) $\rho \geq 0$,

ii) supp. $\rho \subseteq [-1/2,1/2]$,

iii) $\int \rho = 1$,

iv) $\rho = \check{\rho}$,

and let β_n *be a sequence of positive real numbers such that* $1 \leq \beta_1$, $\beta_n \leq \beta_{n+1}$ *for each* n *and* $\lim_{n \to \infty} \beta_n = +\infty$.

Define $S_n(x)$ *as* $\beta_n.\rho(\beta_n.x) = S_n(x)$ *for each* n.

Then if m, k, $\alpha, \gamma = 0, 1, 2, \ldots, -1 <$ Re β *and* $\{S_n\}_{n=1,2,\ldots}$ *is a sequence* *as described above, then*

a) $$\lim_{n \to \infty} \frac{\iint \log^{\alpha} |s - t|.|s - t|^{\beta}.(s - t)^{\gamma}.s_n^{(m)}(t).s_n^{(k)}(s)\, ds\, dt}{\beta_n^{m+k-\beta-\gamma}.\log^{\alpha}\ \beta_n} =$$

$$= (-1)^{m+\alpha}.\int |u|^{\beta}.u^{\gamma}.(\rho * \rho)^{(m+k)}(u)\, du.$$

b) $$\lim_{n \to \infty} \frac{\iint \log^{\alpha} |s - t|.|s - t|^{\beta}.(s - t)^{\gamma}.t.s_n^{(m)}(t).s_n^{(k)}(s)\, ds\, dt}{\beta_n^{m+k-\beta-\gamma-1}.\log^{\alpha}\ \beta_n} =$$

$$= \frac{(-1)^{m+\alpha}.(k - m - \beta - \gamma - 1)}{2.(\beta + \gamma + 1)}.\int |u|^{\beta}.u^{\gamma+1}.(\rho * \rho)^{(m+k)}(u)\, du.$$

c) if besides $\alpha \geq 1$ *and* $m + k > \gamma$ *then*

$$\lim_{n \to \infty} \frac{\iint \log^{\alpha} |s - t|.(s - t)^{\gamma}.s_n^{(m)}(t).s_n^{(k)}(s)\, ds\, dt}{\beta_n^{m+k-\gamma}\ .\log^{\alpha-1}\ \beta_n} =$$

$$= (-1)^{m+\alpha+1}.\alpha.\int \log |u|.u^{\gamma}.(\rho * \rho)^{(m+k)}(u)\, du.$$

d) if $\alpha \geq 1$ *and* $m + k > \gamma + 1$ *then*

$$\lim_{n \to \infty} \frac{\iint \log^{\alpha} |s - t|.(s - t)^{\gamma}.t.s_n^{(m)}(t).s_n^{(k)}(s)\, ds\, dt}{\beta_n^{m+k-\gamma-1}.\log^{\alpha-1}\ \beta_n} =$$

$$= (-1)^{m+\alpha}.\ \frac{\alpha}{2}\ .(\frac{\gamma + 1 + m - k}{\gamma + 1}).\int \log |u|.u^{\gamma+1}.(\rho * \rho)^{(m+k)}(u)\, du.$$

PROOF. If $\phi \in C_0^{\infty}(R^1)$ then

$$(t.\phi^{(m)}(t) * \phi^{(k)}(t))(x) = \widehat{\widehat{(t.\phi^{(m)}(t) * \phi^{(k)}(t))}}(x) =$$

$$= \widehat{\widehat{(\widehat{t.\phi^{(m)}(t)}(s).\widehat{\phi^{(k)}(t)}(s))}}(x) = -i^{m+k-1}.\widehat{\widehat{(s^k.\hat{\phi}(s).(s^m.\hat{\phi}(s))^{(1)})}}(x) =$$

$$= -i^{m+k-1}.\widehat{\widehat{(m.s^{k+m-1}.\hat{\phi}^2(s) + \frac{s^{m+k}}{2}.(\hat{\phi}^2(s))^{(1)})}}(x) =$$

$$= -m.(\phi * \phi)^{(k+m-1)}(x) - \frac{1}{2i}.((\widehat{((\phi * \phi))^{(1)}}\hat{\ }))^{(m+k)}(x) =$$

$$= -m.(\phi * \phi)^{(k+m-1)}(x) + \frac{1}{2}.(x.(\phi * \phi)(x))^{(m+k)} =$$

$$= \frac{(k - m)}{2} .(\phi * \phi)^{(k+m-1)}(x) + \frac{1}{2} .x.(\phi * \phi)^{(m+k)}(x).$$

So we have

(1) $(t.\phi^{(m)}(t) * \phi^{(k)}(t))(x) = \frac{(k - m)}{2} .(\phi * \phi)^{(k+m-1)}(x) + \frac{x}{2}.(\phi * \phi)^{(m+k)}(x).$

Now let us consider case a). With a linear change of variables ($s' = \beta_n.s$, $t' = \beta_n.t$) we get

$$\iint \log^{\alpha} |s - t|.|s - t|^{\beta}.(s - t)^{\gamma}.S_n^{(m)}(t).S_n^{(k)}(s)\, ds\, dt =$$

$$= \beta_n^{m+k-\beta-\gamma}.\iint \log^{\alpha} \left|\frac{s' - t'}{\beta_n}\right| .|s' - t'|^{\beta}.(s' - t')^{\gamma}.\rho^{(m)}(t').\rho^{(k)}(s')\, ds'\, dt'.$$

Then

$$\lim_{n \to \infty} \frac{\iint \log^{\alpha} |s - t|.|s - t|^{\beta}.(s - t)^{\gamma}.S_n^{(m)}(t).S_n^{(k)}(s)\, ds\, dt}{\beta_n^{m+k-\beta-\gamma}.\log^{\alpha} \beta_n} =$$

$$= (-1)^{\alpha}.\iint |s - t|^{\beta}.(s - t)^{\gamma}.\rho^{(m)}(t).\rho^{(k)}(s)\, ds\, dt =$$

$$= (-1)^{\alpha+m}.\int |u|^{\beta}.u^{\gamma}.(\rho * \rho)^{(m+k)}(u)\, du$$

and this proves case a).

For case b), after making the same change of variables we get

$$\iint \log^{\alpha} |s - t|.|s - t|^{\beta}.(s - t)^{\gamma}.t.S_n^{(m)}(t).S_n^{(k)}(s)\, ds\, dt =$$

$$= \beta^{m+k-\beta-\gamma-1}.\iint \log^{\alpha} \left|\frac{s' - t'}{\beta_n}\right|.|s' - t'|^{\beta}.(s' - t')^{\gamma}.t'.\rho^{(m)}(t').\rho^{(k)}(s')\, ds'\, dt'.$$

Then

$$L = \lim_{n \to \infty} \frac{\iint \log^{\alpha} |s - t|.|s - t|^{\beta}.(s - t)^{\gamma}.t.S_n^{(m)}(t).S_n^{(k)}(s)\, ds\, dt}{\beta_n^{m+k-\beta-\gamma-1}.\log^{\alpha} \beta_n} =$$

$$= (-1)^{\alpha}.\iint |s - t|^{\beta}.(s - t)^{\gamma}.t.\rho^{(m)}(t).\rho^{(k)}(s)\, ds\, dt =$$

$$= (-1)^{\alpha}.\int (|t|^{\beta}.t^{\gamma} * t.\rho^{(m)}(t))(s).\rho^{(k)}(s)\, ds =$$

$$= (-1)^{m+k+\gamma}.\int |u|^{\beta}.u^{\gamma}.(t.\rho^{(m)}(t) * \rho^{(k)}(t))(u)\, du.$$

And utilizing (1) we obtain

$$L = (-1)^{\alpha+k+\gamma}.\int |u|^{\beta}.u^{\gamma}.[\frac{(k-m)}{2}.(\rho * \rho)^{(k+m-1)}(u) + \frac{u}{2}.(\rho * \rho)^{(k+m)}(u)]\ du.$$

If $k + m + \gamma$ is even then $L = 0$.

If $k + m + \gamma$ is odd, integrating by parts the integral in the first term, (notice that $m + \alpha = \alpha + k + \gamma + 1 + \dot{2}$) we obtain

$$L = \frac{(-1)^{m+\alpha}.(k - m - \beta - \gamma - 1)}{2.(\beta + \gamma + 1)}.\int |u|^{\beta}.u^{\gamma+1}.(\rho * \rho)^{(m+k)}(u)\ du.$$

Since the integrand is an odd or even function according as $\gamma + m + k + 1$ is an odd or even number the last equality holds in any case, and b) follows.

Consider the case c). We obtain as before

$$\iint \log^{\alpha} |s - t|.(s - t)^{\gamma}.S_n^{(m)}(t).S_n^{(k)}(s)\ ds\ dt =$$

$$= \beta_n^{m+k-\gamma}.\iint \log^{\alpha} \left|\frac{s' - t'}{\beta_n}\right| .(s' - t')^{\gamma}.\rho^{(m)}(t').\rho^{(k)}(s')\ ds'\ dt'.$$

Since $\alpha \geqq 1$ and $\gamma < m + k$ we have

$$\iint (s' - t')^{\gamma}.\rho^{(m)}(t').\rho^{(k)}(s')\ ds'\ dt' = 0$$

and it follows that

$$\lim_{n \to \infty} \frac{\iint \log^{\alpha} |s - t|.(s - t)^{\gamma}.S_n^{(m)}(t).S_n^{(k)}(s)\ ds\ dt}{\beta_n^{m+k-\gamma}.\log^{\alpha-1} \beta_n} =$$

$$= \alpha.(-1)^{\alpha-1}.\iint \log |s - t|.(s - t)^{\gamma}.\rho^{(m)}(t).\rho^{(k)}(s)\ ds\ dt =$$

$$= \alpha.(-1)^{\alpha+m+1}.\int \log |u|.u^{\gamma}.(\rho * \rho)^{(m+k)}(u)\ du.$$

Consider the case d). We obtain as before

$$\iint \log^{\alpha} |s - t|.(s - t)^{\gamma}.t.S_n^{(m)}(t).S_n^{(k)}(s)\ ds\ dt =$$

$$= \beta_n^{m+k-\gamma-1}.\iint \log^{\alpha} \left|\frac{s' - t'}{\beta_n}\right| .(s' - t')^{\gamma}.t'.\rho^{(m)}(t').\rho^{(k)}(s')\ ds'\ dt'.$$

Then since $\alpha \geqq 1$ and $m + k > \gamma + 1$ we get:

$$\iint (s' - t')^{\gamma}.t'.\rho^{(m)}(t').\rho^{(k)}(s')\ ds'\ dt' = 0$$

and it follows that

$$M = \lim_{n \to \infty} \frac{\iint \log^{\alpha} |s - t|.(s - t)^{\gamma}.t.S_n^{(m)}(t).S_n^{(k)}(s)\, ds\, dt}{\beta_n^{m+k-\gamma-1}.\log^{\alpha-1} \beta_n} =$$

$$= (-1)^{\alpha-1}.\alpha.\iint \log |s - t|.(s - t)^{\gamma}.t.\rho^{(m)}(t).\rho^{(k)}(s)\, ds\, dt =$$

$$= (-1)^{\alpha+k+\gamma-1}.\alpha.\int \log |u|.u^{\gamma}.(t.\rho^{(m)}(t) * \rho^{(k)}(t))(u)\, du = .$$

but using (1) we obtain that it is equal to

$$= (-1)^{\alpha+k+\gamma-1}.\alpha.\int \log |u|.u^{\gamma}.[\frac{(k - m)}{2}.(\rho * \rho)^{(m+k-1)}(u) + \frac{u}{2}.(\rho * \rho)^{(m+k)}(u)]\, du.$$

Integrating by parts the first integral and adding it to the second one, we get (observe that $\log |x|.x^{\gamma} = \frac{d}{dx}(\frac{x^{\gamma+1}}{(\gamma + 1)}.\log |x| - \frac{x^{\gamma+1}}{(\gamma + 1)^2})$),

$$M = (-1)^{\alpha+k+\gamma-1}.\frac{\alpha}{2}.\frac{(m - k + \gamma + 1)}{(\gamma + 1)}.\int \log |u|.u^{\gamma+1}.(\rho * \rho)^{(m+k)}(u)\, du.$$

The integrand is odd if $\gamma + 1 + m + k$ is odd, thus we get

$$M = (-1)^{\alpha+m}.\frac{\alpha}{2}.\frac{(m - k + \gamma + 1)}{(\gamma + 1)}.\int \log |u|.u^{\gamma+1}.(\rho * \rho)^{(m+k)}(u)\, du, \text{ QED.}$$

The integrals that appear in the righthand sides of the formulae a) - d) are different from zero for particular functions ρ. To show this we prove the following lemma.

LEMMA 1. *a) If* $m, k, \gamma = 0, 1, 2, \ldots$, $\gamma + m + k$ *is even,* $-1 < \text{Re}\,\beta$ *and* ($\beta = 0, 1, 2, \ldots$; $\beta + \gamma \geq m + k$) *or* ($\beta$ *not a positive integer or zero) or* (β *is odd,* $\beta + \gamma < m + k$) *then there exists* $\rho \in C_0^{\infty}(R^1)$, $\rho \geq 0$, $\rho = \check{\rho}$, $\int \rho = 1$, $\text{supp}\, \rho \subseteq [-1/2, 1/2]$ *such that*

$$\int |u|^{\beta}.u^{\gamma}.(\rho * \rho)^{(m+k)}(u)\, du \neq 0.$$

b) If $m, k, \gamma = 0, 1, 2, \ldots$ *and* $\gamma + m + k$ *is an even number then there exists* ρ *as in a) such that*

$$\int \log |u|.u^{\gamma}.(\rho * \rho)^{(m+k)}(u)\, du \neq 0.$$

PROOF. The proof is immediate for case a) if $\beta = 0, 1, 2, \ldots$, $\beta + \gamma \geq m + k$. If β is odd, $\beta + \gamma < m + k$ then

$$\int |u|^{\beta}.u^{\gamma}.(\rho * \rho)^{(m+k)}(u)\, du = 2.\int_0^{+\infty} u^{\gamma+\beta}.(\rho * \rho)^{(m+k)}(u)\, du =$$

$$= 2.(-1)^{\beta+\gamma}.(\beta + \gamma)!.\int_0^{+\infty} (\rho * \rho)^{(m+k-\beta-\gamma)}(u)\, du =$$

$$= 2.(-1)^{\beta+\gamma+1}.(\beta + \gamma)!.(\rho * \rho)^{(m+k-\beta-\gamma-1)}(0)$$

and since $m + k - \beta - \gamma - 1$ is even, we have

$$(\rho * \rho)^{(m+k-\beta-\gamma-1)}(0) \neq 0.$$

Suppose β is neither a positive integer nor zero. Then

$$\int |u|^{\beta}.u^{\gamma}.(\rho * \rho)^{(m+k)}(u)\ du = 2.\int_0^{+\infty} u^{\beta+\gamma}.(\rho * \rho)^{(m+k)}(u)\ du =$$

$$= 2.(-1)^{m+k}.\langle (x_+^{\gamma+\beta})^{(m+k)},(\rho * \rho)\rangle.$$

Proceeding in an analogous way in case b), we get

$$\int \log |u|.u^{\gamma}.(\rho * \rho)^{(m+k)}(u)\ du = 2.\int_0^{+\infty} \log |u|.u^{\gamma}.(\rho * \rho)^{(m+k)}(u)\ du =$$

$$= 2.(-1)^{m+k}.\langle (\log |x|.x^{\gamma})_+^{(m+k)},(\rho * \rho)\rangle.$$

In the last expression appears the (m+k)-th derivate of the following distribution

$$(\log |x|.x^{\gamma})_+ := \begin{cases} \log |x|.x^{\gamma} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let $0 < r < 1$. We can choose ε , $0 < \varepsilon < r$, such that there exists a sequence of even functions S_n with the following properties:

1) $S_n \geq 0$ for all n,

2) $\int S_n = 1$ for all n,

3) supp $S_n \subseteq [-\varepsilon,\varepsilon]$ for all n,

4) supp $S_n \to 0$ if $n \to \infty$.

We define

$$\Delta_n(x) := [S_n(x) + S_n(x+r) + S_n(x-r)].1/3.$$

Then

1) $\Delta_n \geq 0$ for all n,

2) $\int \Delta_n = 1$ for all n,

3) supp. $\Delta_n \subseteq [-1/2,1/2]$ for all n,

4) Δ_n is even.

Now

$$[3.(\Delta_n * \Delta_n)(x) - (S_n * S_n)(x)] =$$

$= \frac{2}{3} .[(S_n * S_n)(x+r) + (S_n * S_n)(x-r)] + \frac{1}{3} .[(S_n * S_n)(x+2r) + (S_n * S_n)(x-2r)].$

Since $(S_n * S_n)(x) \to S_0$ if $n \to \infty$, we have

$$\lim_{n \to \infty} [3.(\Delta_n * \Delta_n)(x) - (S_n * S_n)(x)] = \frac{1}{3} .S_{-2r} + \frac{2}{3} .S_{-r} + \frac{2}{3} .S_r + \frac{1}{3} .S_{2r},$$

in the sense of distributions.

Also $[3.(\Delta_n * \Delta_n)(x) - (S_n * S_n)(x)]$ has support in $\{(-\infty,-r/2) \cup (r/2,+\infty)\}$ if $\varepsilon < r$, ε small enough.

We define in an analogous way

$$\overline{\Delta_n}(x) := [\frac{S_n}{2}(x+r) + S_n(x) + \frac{S_n}{2}(x-r)].1/2.$$

We also have

1) $\overline{\Delta_n} \geq 0$ for all n,

2) $\int \overline{\Delta_n} = 1$ for all n,

3) supp. $\overline{\Delta_n} \subseteq [-1/2,1/2]$ for all n,

4) $\overline{\Delta_n}$ is even.

And

$$\lim_{n \to \infty} [8.(\overline{\Delta_n} * \overline{\Delta_n})(x) - 3.(S_n * S_n)(x)] = \frac{1}{2} .S_{-2r} + 2.S_{-r} + 2.S_r + \frac{1}{2} .S_{2r}.$$

Besides

$$\text{supp. } [8.(\overline{\Delta_n} * \overline{\Delta_n})(x) - 3.(S_n * S_n)(x)] \subseteq \{(-\infty,-r/2) \cup (r/2,+\infty)\}$$

if ε is small enough, $\varepsilon < r$.

The distributions $(x_+^{\gamma+\beta})^{(m+k)}$ and $(\log |x|.x^{\gamma})_+^{(m+k)}$ are continuous functions in $R^1\setminus\{0\}$.

Then, in general, if $f(x)$ is continuous in $R^1\setminus\{0\}$, $f(x) = 0$ for $x < 0$, we get

$$\lim_{n \to \infty} \langle f,3(\Delta_n * \Delta_n) - (S_n * S_n)\rangle = \frac{1}{3} [f(r).2 + f(2.r)],$$

$$\lim_{n \to \infty} \langle f,8.(\overline{\Delta_n} * \overline{\Delta_n}) - 3.(S_n * S_n)\rangle = [f(r).2 + \frac{1}{2} .f(2.r)].$$

Then if we can choose $r > 0$, small enough, such that $f(2.r) \neq 0$, then $[f(r).2 + f(r.2)]$ and $[f(r).2 + \frac{1}{2} .f(r.2)]$ cannot vanish simultaneously. Therefore, there exists n_0 such that $\langle f,(\Delta_{n_0} * \Delta_{n_0})\rangle \neq 0$ or $\langle f,(S_{n_0} * S_{n_0})\rangle \neq 0$ or $\langle f,(\overline{\Delta_{n_0}} * \overline{\Delta_{n_0}})\rangle \neq 0$.

Our lemma follows from this observation. QED.

3. SOME AUXILIARY FORMULAE.

THEOREM 2. *If* $S_n \in C_0^\infty(R^1)$, $S_n \geq 0$, $\int S_n = 1$ *and* supp. $S_n \subseteq [-1/2,1/2]$ *then*

a)
$$\int_{-\infty}^{+\infty} ([x_+^{-r}.x_-^{-m} + (-1)^{m+r+1}.x_+^{-m}.x_-^{-r}] * S_n)(y).y^l.\psi(y)\, dy =$$

$$= \frac{(-1)^{r-1}}{(r-1)!(m-1)!} .[\iint f(s,t).S_n^{(m)}(t).S_n^{(r)}(s)\, ds\, dt +$$

$$+ (\sum_{j=1}^{m-1} \frac{1}{j}).\iint \log |s-t|.S_n^{(r)}(t).S_n^{(m-1)}(s).s^l.\psi(s)\, ds\, dt -$$

$$- (\sum_{j=1}^{r-1} \frac{1}{j}).\iint \log |s-t|.S_n^{(m)}(t).S_n^{(r-1)}(s).s^l.\psi(s)\, ds\, dt]$$

where m *and* r *are positive integers,* l *is a nonnegative integer,* $\psi(y) \in C_0^\infty(R^1)$ *and*

$$f(s,t) := \int_t^s \log |y-t|.\log |s-y|.y^l.\psi(y)\, dy.$$

b)
$$\int_{-\infty}^{+\infty} ([x_+^{\lambda}.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^{\lambda}] * S_n)(y).y^l.\psi(y)\, dy =$$

$$= \frac{1}{(k-1)!} .(\sum_{j=1}^{k-1} \frac{1}{j}).\iint |t-s|^{\lambda}.S_n(t).S_n^{(k-1)}(s).s^l.\psi(s)\, ds\, dt +$$

$$+ (-\frac{1}{(k-1)!}).\iint f(s,t).S_n^{(k)}(s).S_n(t)\, ds\, dt$$

if Re $\lambda > -1$, k *is a positive integer,* l *is a nonnegative integer,* $\psi(y) \in C_0^\infty(R^1)$ *and*

$$f(s,t) := \int_t^s |y-t|^{\lambda}.\log |s-y|.y^l.\psi(y)\, dy.$$

c)
$$\int_{-\infty}^{+\infty} ([x_+^{-k},x_-^{-m-q} + (-1)^{k+m+1}.x_+^{-m-q}.x_-^{-k}] * S_n)(y).y^l.\psi(y)\, dy =$$

$$= \frac{(-1)^{k-1}.\Gamma(q)}{(k-1)!\ \Gamma(q+m)} .\iint f(s,t).S_n^{(m)}(s).S_n^{(k)}(t)\, ds\, dt +$$

$$+ \frac{(-1)^{k-1}.\Gamma(q)}{(k-1)!\ \Gamma(q+m)} .(\sum_{j=1}^{k-1} \frac{1}{j}).\iint |s-t|^{-q}.S_n^{(m)}(t).S_n^{(k-1)}(s).s^l.\psi(s)\, ds\, dt$$

for l,m *nonnegative integers,* k *a positive integer,* $0 \leq$ Re $q < 1$, $q \neq 0$, $\psi(y) \in C_0^\infty(R^1)$ *and*

$$f(s,t) := \int_t^s \log|y - t|.|s - y|^{-q}.y^l.\psi(y)\,dy.$$

d)

d1)

$$\int_{-\infty}^{+\infty} ([x_+^{-r-p}.x_-^{-m-q}] * S_n)(y).y^l.\psi(y)\,dy =$$

$$= \frac{(-1)^r.\Gamma(p).\Gamma(q)}{\Gamma(p+r).\Gamma(q+m)} .$$

$$\sum_{i=0}^{l} \binom{l}{i}.\int_{-1}^{1} \left(\int_t^1 |s - t|^{1-p-q}.s^{l-i}.K^l_{i,p,q}(s,t).S_n^{(m)}(s)\,ds\right).t^i.S_n^{(r)}(t)\,dt.$$

d2)

$$\int_{-\infty}^{+\infty} ([x_+^{-r-p}.x_-^{-m-q} + (-1)^{m+r}.x_-^{-r-p}.x_+^{-m-q}] * S_n)(y).y^l.\psi(y)\,dy =$$

$$= \frac{(-1)^r.\Gamma(p).\Gamma(q)}{\Gamma(p+r).\Gamma(q+m)} . \sum_{i=0}^{l} \binom{l}{i}.\iint |s - t|^{1-p-q}.s^{l-i}.t^i.K^l_{i,p,q}(s,t).S_n^{(m)}(s).S_n^{(r)}(t)\,ds\,dt$$

in both cases, $l,r,m = 0, 1, 2, \ldots;\ 0 \leq \mathrm{Re}\, p < 1,\ p \neq 0,\ 0 \leq \mathrm{Re}\, q < 1,\ q \neq 0$, $\psi(y) \in C_0^\infty(R^1)$ *and*

$$K^l_{i,p,q}(s,t) := \int_0^1 v^{l-i-p}.(1 - v)^{i-q}.\psi(t+(s-t).v)\,dv.$$

e)

$$\int_{-\infty}^{+\infty} ([x_+^r.x_-^{-m-q} + (-1)^{r+m+1}.x_+^{-m-q}.x_-^r] * S_n).y^l.\psi(y)\,dy =$$

$$= \frac{\Gamma(q)}{\Gamma(q+m)} . \iint f(s,t).S_n^{(m)}(s).S_n(t)\,ds\,dt$$

for $l,m,r = 0, 1, 2, \ldots;\ 0 \leq \mathrm{Re}\, q < 1,\ q \neq 0,\ \psi(y) \in C_0^\infty(R^1)$ *and*

$$f(s,t) := \int_t^s (y - t)^r.|s - y|^{-q}.y^l.\psi(y)\,dy.$$

PROOF. We recall that if $\phi \in C_0^\infty(R^1)$ then

i) $\langle x_+^{-k},\phi\rangle = -\frac{1}{(k-1)!}.\int_0^{+\infty} \log|x|.\phi^{(k)}(x)\,dx + \frac{(-1)^{k-1}}{(k-1)!}.\left(\sum_{j=1}^{k-1} \frac{1}{j}\right).S_0^{(k-1)}(\phi)$

if $k = 1, 2, \ldots$.

ii) $$\langle x_+^\lambda,\phi\rangle = \int_0^{+\infty} x^\lambda.\phi(x)\,dx, \qquad \text{if } \mathrm{Re}\,\lambda > -1.$$

iii) $$\langle x_+^{-r-p},\phi\rangle = \frac{\Gamma(p)}{\Gamma(p+r)}.\langle x_+^{-p},\phi^{(r)}\rangle$$

if $0 \leq \mathrm{Re}\, p < 1$, $p \neq 0$ and $r = 0, 1, 2, \ldots$.

iv) $$\langle x_+^\lambda, \phi\rangle = \langle x_-^\lambda, \check{\phi}\rangle \quad \text{for each } \lambda \in C.$$

Let k be a positive integer, then

$$(x_+^{-k} * S_n)(y) = \langle x_+^{-k}, S_n(y-x)\rangle =$$

$$= -\frac{1}{(k-1)!}\cdot\int_0^{+\infty}\log|x|.\frac{\partial^k}{\partial x^k}(S_n(y-x))\,dx + \frac{(-1)^{k-1}}{(k-1)!}.\left(\sum_{j=1}^{k-1}\frac{1}{j}\right).S_o^{(k-1)}(S_n(y-.)) =$$

$$= -\frac{1}{(k-1)!}.(-1)^k.\int_0^{+\infty}\log|x|.S_n^{(k)}(y-x)\,dx + \frac{(-1)^{k-1}}{(k-1)!}.\left(\sum_{j=1}^{k-1}\frac{1}{j}\right).S_n^{(k-1)}(y) =$$

$$= \frac{(-1)^{k-1}}{(k-1)!}.\int_{-\infty}^{y}\log|y-t|.S_n^{(k)}(t)\,dt + \frac{(-1)^{k-1}}{(k-1)!}.\left(\sum_{j=1}^{k-1}\frac{1}{j}\right).S_n^{(k-1)}(y).$$

Therefore

(1) $$(x_+^{-k} * S_n)(y) =$$

$$= \frac{(-1)^{k-1}}{(k-1)!}.\int_{-1}^{y}\log|y-t|.S_n^{(k)}(t)\,dt + \frac{(-1)^{k-1}}{(k-1)!}.\left(\sum_{j=1}^{k-1}\frac{1}{j}\right).S_n^{(k-1)}(y).$$

Also

$$(x_-^{-k} * S_n)(y) = \langle x_-^{-k}, S_n(y-x)\rangle = \langle x_+^{-k}, S_n(y+x)\rangle =$$

$$= -\frac{1}{(k-1)!}.\int_0^{+\infty}\log|x|.\frac{\partial^k}{\partial x^k}(S_n(y+x))\,dx + \frac{(-1)^{k-1}}{(k-1)!}.\left(\sum_{j=1}^{k-1}\frac{1}{j}\right).S_o^{(k-1)}(S_n(y+x)) =$$

$$= -\frac{1}{(k-1)!}.\int_0^{+\infty}\log|x|.S_n^{(k)}(y+x)\,dx + \frac{1}{(k-1)!}.\left(\sum_{j=1}^{k-1}\frac{1}{j}\right).S_n^{(k-1)}(y) =$$

$$= -\frac{1}{(k-1)!}.\int_y^{+\infty}\log|s-y|.S_n^{(k)}(s)\,ds + \frac{1}{(k-1)!}.\left(\sum_{j=1}^{k-1}\frac{1}{j}\right).S_n^{(k-1)}(y).$$

Therefore

(2) $$(x_-^{-k} * S_n)(y) =$$

$$= -\frac{1}{(k-1)!}.\int_y^{1}\log|s-y|.S_n^{(k)}(s)\,ds + \frac{1}{(k-1)!}.\left(\sum_{j=1}^{k-1}\frac{1}{j}\right).S_n^{(k-1)}(y).$$

Let $r = 0, 1, 2, \ldots$, $0 \leq \mathrm{Re}\, p < 1$, $p \neq 0$, then proceeding in the same way, we obtain

$$(x_+^{-r-p} * S_n)(y) = \langle x_+^{-r-p}, S_n(y-x)\rangle = \frac{\Gamma(p)}{\Gamma(p+r)}.\int_0^{+\infty} x^{-p}.\frac{\partial^r}{\partial x^r}(S_n(y-x))\,dx =$$

$$= \frac{(-1)^r.\Gamma(p)}{\Gamma(p+r)} . \int_0^{+\infty} x^{-p}.s_n^{(r)}(y-x)\ dx = \frac{(-1)^r.\Gamma(p)}{\Gamma(p+r)} . \int_{-\infty}^{y} (y-t)^{-p}.s_n^{(r)}(t)\ dt.$$

Therefore

(3) $$(x_+^{-r-p} * s_n)(y) = \frac{(-1)^r.\Gamma(p)}{\Gamma(p+r)} . \int_{-1}^{y} (y-t)^{-p}.s_n^{(r)}(t)\ dt.$$

Also, if $m = 0, 1, 2, \dots,\ q \neq 0,\ 0 \leq \text{Re}\ q < 1$, then

$$(x_-^{-m-q} * s_n)(y) = \langle x_-^{-m-q}, s_n(y-x)\rangle = \langle x_+^{-m-q}, s_n(y+x)\rangle =$$

$$= \frac{\Gamma(q)}{\Gamma(q+m)} . \int_0^{+\infty} x^{-q}.\frac{\partial^m}{\partial x^m}(s_n(y+x))\ dx = \frac{\Gamma(q)}{\Gamma(q+m)} . \int_0^{+\infty} x^{-q}.s_n^{(m)}(y+x)\ dx =$$

$$= \frac{\Gamma(q)}{\Gamma(q+m)} . \int_y^{+\infty} (s-y)^{-q}.s_n^{(m)}(s)\ ds.$$

Therefore

(4) $$(x_-^{-m-q} * s_n)(y) = \frac{\Gamma(q)}{\Gamma(q+m)} . \int_y^{1} (s-y)^{-q}.s_n^{(m)}(s)\ ds.$$

Finally, if $\text{Re}\ \lambda > -1$

(5) $$(x_+^{\lambda} * s_n)(y) = \int_0^{+\infty} x^{\lambda}.s_n(y-x)\ dx = \int_{-1}^{y} (y-t)^{\lambda}.s_n(t)\ dt$$

and also

(6) $$(x_-^{\lambda} * s_n)(y) = \langle x_-^{\lambda}, s_n(y-x)\rangle = \langle x_+^{\lambda}, s_n(y+x)\rangle =$$

$$= \int_0^{+\infty} x^{\lambda}.s_n(y+x)\ dx = \int_y^{1} (s-y)^{\lambda}.s_n(s)\ ds.$$

Let us consider the case a). From formulaes (1) and (2) and from the fact that $\text{supp.}([x_+^{-r}.x_-^{-m}] * s_n)(y) \subseteq [-1,1]$, we get for $r,m = 1, 2, \dots$ that

$$\int_{-\infty}^{+\infty} ([x_+^{-r}.x_-^{-m}] * s_n)(y).y^l.\psi(y)\ dy =$$

$$= \int_{-1}^{1} [\frac{(-1)^{r-1}}{(r-1)!} . \int_{-1}^{y} \log|y-t|.s_n^{(r)}(t)\ dt + \frac{(-1)^{r-1}}{(r-1)!} . (\sum_{j=1}^{r-1} \frac{1}{j}).s_n^{(r-1)}(y)].$$

$$.[-\frac{1}{(m-1)!} . \int_y^1 \log|s-y|.s_n^{(m)}(s)\ ds + \frac{1}{(m-1)!} . (\sum_{j=1}^{m-1} \frac{1}{j}).s_n^{(m-1)}(y)].y^l.\psi(y)\ dy =$$

$$= \frac{(-1)^r}{(r-1)!(m-1)!} . \int_{-1}^{1} [\int_{-1}^{y} \log|y-t|.s_n^{(r)}(t)\ dt].[\int_y^1 \log|s-y|.s_n^{(m)}(s)\ ds].y^l.\psi(y)\ dy +$$

$$+ \frac{(-1)^{r-1}}{(r-1)!(m-1)!} . (\sum_{j=1}^{m-1} \frac{1}{j}) \int_{-1}^{1} [\int_{-1}^{y} \log|y-t|.s_n^{(r)}(t)\ dt].s_n^{(m-1)}(y).y^l.\psi(y)\ \ dy +$$

$$+ \frac{(-1)^r}{(r-1)!(m-1)!} \cdot \left(\sum_{j=1}^{r-1} \frac{1}{j}\right) \cdot \int_{-1}^{1} \left[\int_{y}^{1} \log|s-y| \cdot S_n^{(m)}(s)\, ds\right] \cdot S_n^{(r-1)}(y) \cdot y^l \cdot \psi(y)\, dy +$$

$$+ \frac{(-1)^{r-1}}{(r-1)!(m-1)!} \cdot \left(\sum_{j=1}^{r-1} \frac{1}{j}\right) \cdot \left(\sum_{j=1}^{m-1} \frac{1}{j}\right) \cdot \int_{-1}^{1} S_n^{(r-1)}(y) \cdot S_n^{(m-1)}(y) \cdot y^l \cdot \psi(y)\, dy =$$

If in the first integral, we integrate first in y, then in s and finally in t, it follows that

$$(7) = \frac{(-1)^r}{(r-1)!(m-1)!} \cdot \int_{-1}^{1} \left(\int_{t}^{1} \left(\int_{t}^{s} \log|y-t| \cdot \log|s-y| \cdot y^l \cdot \psi(y) dy\right) \cdot S_n^{(m)}(s) ds\right) \cdot S_n^{(r)}(t)\, dt +$$

$$+ \frac{(-1)^{r-1}}{(r-1)!(m-1)!} \cdot \left(\sum_{j=1}^{m-1} \frac{1}{j}\right) \cdot \int_{-1}^{1} \left(\int_{-1}^{y} \log|y-t| \cdot S_n^{(r)}(t)\, dt\right) \cdot S_n^{(m-1)}(y) \cdot y^l \cdot \psi(y)\, dy +$$

$$+ \frac{(-1)^r}{(r-1)!(m-1)!} \cdot \left(\sum_{j=1}^{r-1} \frac{1}{j}\right) \cdot \int_{-1}^{1} \left(\int_{y}^{1} \log|s-y| \cdot S_n^{(m)}(s)\, ds\right) \cdot S_n^{(r-1)}(y) \cdot y^l \cdot \psi(y)\, dy +$$

$$+ \frac{(-1)^{r-1}}{(r-1)!(m-1)!} \cdot \left(\sum_{j=1}^{m-1} \frac{1}{j}\right) \cdot \left(\sum_{j=1}^{r-1} \frac{1}{j}\right) \cdot \int_{-1}^{1} S_n^{(r-1)}(y) \cdot S_n^{(m-1)}(y) \cdot y^l \cdot \psi(y)\, dy.$$

Set

$$f(s,t) := \int_{t}^{s} \log|y - t| \cdot \log|s - y| \cdot y^l \cdot \psi(y)\, dy$$

then $f(s,t) = - f(t,s)$. The first integral in (7) can be written as follows.

$$(8) \qquad \int_{-1}^{1} \left(\int_{t}^{1} f(s,t) \cdot S_n^{(m)}(s)\, ds\right) \cdot S_n^{(r)}(t)\, dt =$$

$$= \int_{-1}^{1} \left(\int_{-1}^{s} f(s,t) \cdot S_n^{(r)}(t)\, dt\right) \cdot S_n^{(m)}(s)\, ds = \int_{-1}^{1} \left(\int_{-1}^{t} f(t,s) \cdot S_n^{(r)}(s)\, ds\right) \cdot S_n^{(m)}(t)\, dt =$$

$$= -\int_{-1}^{1} \left(\int_{-1}^{t} f(s,t) \cdot S_n^{(r)}(s)\, ds\right) \cdot S_n^{(m)}(t)\, dt.$$

Finally, using formula (7) we get

$$\int_{-\infty}^{+\infty} \left(\left[x_+^{-r} \cdot x_-^{-m} + (-1)^{m+r+1} \cdot x_+^{-m} \cdot x_-^{-r}\right] * S_n\right)(y) \cdot y^l \cdot \psi(y)\, dy =$$

$$= \frac{(-1)^r}{(r - 1)!(m - 1)!} \cdot \int_{-1}^{1} \left(\int_{t}^{1} f(s,t) \cdot S_n^{(m)}(s)\, ds\right) \cdot S_n^{(r)}(t)\, dt +$$

$$+ \frac{(-1)^{r-1}}{(r - 1)!(m - 1)!} \cdot \left(\sum_{j=1}^{m-1} \frac{1}{j}\right) \cdot \int_{-1}^{1} \left(\int_{-1}^{s} \log|s - t| \cdot S_n^{(r)}(t)\, dt\right) \cdot S_n^{(m-1)}(s) \cdot s^l \cdot \psi(s)\, ds +$$

$$+ \frac{(-1)^r}{(r - 1)!(m - 1)!} \cdot \left(\sum_{j=1}^{r-1} \frac{1}{j}\right) \cdot \int_{-1}^{1} \left(\int_{t}^{1} \log|s - t| \cdot S_n^{(m)}(s)\, ds\right) \cdot S_n^{(r-1)}(t) \cdot t^l \cdot \psi(t)\, dt +$$

$$+\frac{(-1)^{r-1}}{(r-1)!(m-1)!}\cdot\left(\sum_{j=1}^{m-1}\frac{1}{j}\right).\left(\sum_{j=1}^{r-1}\frac{1}{j}\right)\int_{-1}^{1} S_n^{(r-1)}(y).S_n^{(m-1)}(y).y^1.\psi(y)\,dy\,+$$

$$+\frac{(-1)^m.(-1)^{m+r+1}}{(m-1)!(r-1)!}\cdot\int_{-1}^{1}\left(\int_t^1 f(s,t).S_n^{(r)}(s)\,ds\right).S_n^{(m)}(t)\,dt\,+$$

$$+\frac{(-1)^{m-1}.(-1)^{(m+r+1}}{(m-1)!(r-1)!}.\left(\sum_{j=1}^{r-1}\frac{1}{j}\right).\int_{-1}^{1}\left(\int_{-1}^{s}\log|s-t|.S_n^{(m)}(t)\,dt\right).S_n^{(r-1)}(s).s^1.\psi(s)\,ds\,+$$

$$+\frac{(-1)^m.(-1)^{m+r+1}}{(m-1)!(r-1)!}\cdot\left(\sum_{j=1}^{m-1}\frac{1}{j}\right).\int_{-1}^{1}\left(\int_t^1\log|s-t|.S_n^{(r)}(s)\,ds\right).S_n^{(m-1)}(t).t^1.\psi(t)\,dt\,+$$

$$+\frac{(-1)^{m-1}.(-1)^{m+r+1}}{(m-1)!\,(r-1)!}\cdot\left(\sum_{j=1}^{r-1}\frac{1}{j}\right).\left(\sum_{j=1}^{m-1}\frac{1}{j}\right).\int_{-1}^{1}S_n^{(m-1)}(y).S_n^{(r-1)}(y).y^1.\psi(y)\,dy.$$

We group the integrals together: the first with the fifth, the second with the seventh, the third with the sixth and the fourth with the eighth. Using formula (8) in the first with the fifth case a) follows.

Let us consider the case b). With formulae (1) and (6) and from the fact that supp. $([x_+^{-k}.x_-^{\lambda}] * S_n)(y) \subseteq [-1,1]$ we have, for Re $\lambda > -1$, $k = 1, 2, \ldots$,

(9) $$\int_{-\infty}^{+\infty}([x_+^{-k}.x_-^{\lambda}] * S_n)(y).y^1.\psi(y)\,dy =$$

$$=\int_{-1}^{1}\left(\int_y^1(s-y)^{\lambda}.S_n(s)\,ds\right).\left(\frac{(-1)^{k-1}}{(k-1)!}\cdot\int_{-1}^{y}\log\;|y-t|.S_n^{(k)}(t)\,dt\,+\right.$$

$$\left.+\frac{(-1)^{k-1}}{(k-1)!}\cdot\left(\sum_{j=1}^{k-1}\frac{1}{j}\right).S_n^{(k-1)}(y)\right).y^1.\psi(y)\,dy =$$

$$=\frac{(-1)^{k-1}}{(k-1)!}\cdot\int_{-1}^{1}\left(\int_t^1\left(\int_t^s\log|y-t|.(s-y)^{\lambda}.y^1.\psi(y)\,dy\right).S_n(s)\,ds\right).S_n^{(k)}(t)\,dt\,+$$

$$+\frac{(-1)^{k-1}}{(k-1)!}\cdot\left(\sum_{j=1}^{k-1}\frac{1}{j}\right).\int_{-1}^{1}\left(\int_y^1(t-y)^{\lambda}.S_n(t)\,dt\right).S_n^{(k-1)}(y).y^1.\psi(y)\,dy.$$

Using formula (2) and (5) and with the same hypothesis for λ and k, we get

(10) $$\int_{-\infty}^{+\infty}([x_+^{\lambda}.x_-^{-k}] * S_n)(y).y^1.\psi(y)\,dy =$$

$$=\int_{-1}^{1}\left(\int_{-1}^{y}(y-t)^{\lambda}.S_n(t)\,dt\right).\left(-\frac{1}{(k-1)!}\cdot\int_y^1\log|s-y|.S_n^{(k)}(s)\,ds\,+\right.$$

$$+ \frac{1}{(k-1)!} . (\sum_{j=1}^{k-1} \frac{1}{j}).S_n^{(k-1)}(y)).y^1.\psi(y)\ dy =$$

$$= - \frac{1}{(k-1)!}.\int_{-1}^{1}(\int_{t}^{1}(\int_{t}^{s} (y-t)^{\lambda}.\log\ |s-y|.y^1.\psi(y)\ dy).S_n^{(k)}(s)\ ds).S_n(t)\ dt +$$

$$+ \frac{1}{(k-1)!} .(\sum_{j=1}^{k-1} \frac{1}{j}).\int_{-1}^{1}(\int_{-1}^{y} (y-t)^{\lambda}.S_n(t)\ dt).S_n^{(k-1)}(y).y^1.\psi(y)\ dy.$$

Then, using formulae (10) and (9) and the following equality

$$\int_{-1}^{1}(\int_{t}^{1}(\int_{t}^{s} \log|y-t|.(s-y)^{\lambda}.y^1.\psi(y)\ dy)\ \ .S_n(s)\ ds).S_n^{(k)}(t)\ dt =$$

$$= \int_{-1}^{1}(\int_{-1}^{s}(\int_{t}^{s} \log\ |y-t|.(s-y)^{\lambda}.y^1.\psi(y)\ dy).S_n^{(k)}(t)\ dt).S_n(s)\ ds =$$

$$= \int_{-1}^{1}(\int_{-1}^{t}(\int_{s}^{t} \log\ |y-s|.(t-y)^{\lambda}.y^1.\psi(y)\ dy).S^{(k)}(s)\ ds).S_n(t)\ dt$$

we get

$$\int_{-\infty}^{+\infty} ([x_+^{\lambda}.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^{\lambda}] * S_n)(y).y^1.\psi(y)\ dy =$$

$$= - \frac{1}{(k-1)!} .\int_{-1}^{1}(\int_{t}^{1}(\int_{t}^{s}(y-t)^{\lambda}.\log|s-y|.y^1.\psi(y)\ dy).S_n^{(k)}(s)\ ds).S_n(t)\ dt +$$

$$+ \frac{1}{(k-1)!} .(\sum_{j=1}^{k-1} \frac{1}{j}).\int_{-1}^{1}(\int_{-1}^{y} (y-t)^{\lambda}.S_n(t)\ dt).S_n^{(k-1)}(y).y^1.\psi(y)\ dy +$$

$$+ \frac{1}{(k-1)!} .\int_{-1}^{1}(\int_{t}^{1}(\int_{t}^{s}\log|y-t|.(s-y)^{\lambda}.y^1.\psi(y)\ dy).S_n(s)\ ds).S_n^{(k)}(t)\ dt +$$

$$+ \frac{1}{(k-1)!} .(\sum_{j=1}^{k-1} \frac{1}{j}).\int_{-1}^{1}(\int_{y}^{1}(t-y)^{\lambda}.S_n(t)\ dt).S_n^{(k-1)}(y).y^1.\psi(y)\ dy =$$

$$= \frac{1}{(k-1)!} .(\sum_{j=1}^{k-1} \frac{1}{j}).\iint |y-t|^{\lambda}.S_n(t).S_n^{(k-1)}(y).y^1.\psi(y)\ dy\ dt +$$

$$- \frac{1}{(k-1)!} .\iint (\int_{t}^{s}|y-t|^{\lambda}.\log|s-y|.y^1.\psi(y)\ dy).S_n^{(k)}(s).S_n(t)\ ds\ dt$$

and from this b) follows.

Case c). From formulae (1), (2), (3) and (4), with $k = 1, 2, \ldots, m = 0, 1, 2, \ldots,$ $0 \leq \mathrm{Re}\ q < 1$, $q \neq 0$, we get:

$$(11) \qquad \int_{-\infty}^{+\infty}([x_+^{-k}.x_-^{-m-q} + (-1)^{k+m+1} x_+^{-m-q}.x_-^{-k}] * S_n)(y).y^1.\psi(y)\ dy =$$

$$= \frac{(-1)^{k-1}}{(k-1)!} \cdot \frac{\Gamma(q)}{\Gamma(q+m)} \cdot \int_{-1}^{1} [\int_{-1}^{y} \log |y - t| . s_n^{(k)}(t)\, dt].$$

$$.[\int_{y}^{1} (s - y)^{-q} . s_n^{(m)}(s)\, ds] . y^l . \psi(y)\, dy +$$

$$+ \frac{(-1)^{k-1}}{(k-1)!} . (\sum_{j=1}^{k-1} \frac{1}{j}) . \frac{\Gamma(q)}{\Gamma(q+m)} . \int_{-1}^{1} [\int_{y}^{1} (s - y)^{-q} . s_n^{(m)}(s)\, ds] . s_n^{(k-1)}(y) . y^l . \psi(y)\, dy +$$

$$+ \frac{(-1)^{k}}{(k-1)!} . \frac{\Gamma(q)}{\Gamma(q+m)} . \int_{-1}^{1} [\int_{-1}^{y} (y - t)^{-q} . s_n^{(m)}(t)\, dt].$$

$$.[\int_{y}^{1} \log |s - y| . s_n^{(k)}(s)\, ds] . y^l . \psi(y)\, dy +$$

$$+ \frac{(-1)^{k-1}}{(k-1)!} . (\sum_{j=1}^{k-1} \frac{1}{j}) . \frac{\Gamma(q)}{\Gamma(q+m)} . \int_{-1}^{1} [\int_{-1}^{y} (y - t)^{-q} . s_n^{(m)}(t)\, dt] . s_n^{(k-1)}(y) . y^l . \psi(y)\, dy =$$

$$= \frac{(-1)^{k-1} . \Gamma(q)}{(k-1)!\Gamma(q+m)} . \int_{-1}^{1} (\int_{t}^{1} (\int_{t}^{s} \log|y - t| . (s - y)^{-q} . y^l . \psi(y)\, dy) . s_n^{(m)}(s)\, ds) . s_n^{(k)}(t)\, dt +$$

$$+ \frac{(-1)^{k} . \Gamma(q)}{(k-1)!\Gamma(q+m)} . \int_{-1}^{1} (\int_{-1}^{s} (\int_{t}^{s} \log|s - y| . (y - t)^{-q} . y^l . \psi(y)\, dy) . s_n^{(m)}(t)\, dt) . s_n^{(k)}(s)\, ds +$$

$$+ \frac{(-1)^{k-1} . \Gamma(q)}{(k-1)! . \Gamma(q+m)} . (\sum_{j=1}^{k-1} \frac{1}{j}) . \iint |y - t|^{-q} . s_n^{(m)}(t) . s_n^{(k-1)}(y) . y^l . \psi(y)\, dy\, dt.$$

Now

$$\int_{-1}^{1} (\int_{-1}^{s} (\int_{t}^{s} \log|s - y| . (y - t)^{-q} . y^l . \psi(y)\, dy) . s_n^{(m)}(t)\, dt) . s_n^{(k)}(s)\, ds =$$

$$= \int_{-1}^{1} (\int_{-1}^{t} (\int_{s}^{t} \log|t - y| . (y - s)^{-q} . y^l . \psi(y)\, dy) . s_n^{(m)}(s)\, ds) . s_n^{(k)}(t)\, dt.$$

Then case c) follows immediately from formula (11) and the last equality.

Case d). From formulae (3) and (4) we have

$$(12) \qquad H = \int_{-\infty}^{+\infty} ([x_+^{-r-p} . x_-^{-m-q}] * s_n)(y) . y^l . \psi(y)\, dy =$$

$$= \frac{(-1)^r \Gamma(p) . \Gamma(q)}{\Gamma(p+r) . \Gamma(q+m)} . \int_{-1}^{1} [\int_{-1}^{y} (y - t)^{-p} . s_n^{(r)}(t)\, dt] . [\int_{y}^{1} (s - y)^{-q} . s_n^{(m)}(s)\, ds] . y^l . \psi(y)\, dy =$$

$$= \frac{(-1)^r \Gamma(p) . \Gamma(q)}{\Gamma(p+r) . \Gamma(q+m)} . \int_{-1}^{1} (\int_{t}^{1} (\int_{t}^{s} (y - t)^{-p} . (s - y)^{-q} . y^l . \psi(y)\, dy) . s_n^{(m)}(s)\, ds) . s_n^{(r)}(t)\, dt.$$

Since $1 \geq s > t \geq -1$, we have

(13) $$\int_t^s (y-t)^{-p}.(s-y)^{-q}.y^l.\psi(y)\,dy =$$

$$=\int_0^1 (s-t)^{-p}.v^{-p}.(s-t)^{-q}.(1-v)^{-q}.(s.v+(1-v).t)^l.\psi(t+(s-t).v).(s-t)\,dv =$$

$$= (s-t)^{1-p-q}.\sum_{i=0}^{l} \binom{l}{i}.s^{l-i}.t^i.K^l_{i,p,q}(s,t).$$

where $K^l_{i,p,q}(s,t) := \int_0^1 v^{l-i-p}.(1-v)^{i-q}.\psi(t+(s-t).v)\,dv.$

Then by formula (12) we get d1):

(14) $$H = \frac{(-1)^r.\Gamma(q).\Gamma(p)}{\Gamma(q+m).\Gamma(p+r)} . \sum_{i=0}^{l} \binom{l}{i}.$$

$$.\int_{-1}^1 \left(\int_t^1 |s-t|^{1-p-q}.s^{l-i}.K^l_{i,p,q}(s,t).S_n^{(m)}(s)\,ds\right).t^i.S_n^{(r)}(t)\,dt.$$

Observe that $K^l_{i,p,q}(s,t) = K^l_{l-i,q,p}(t,s)$ and that $K^l_{i,p,q}(s,t) \in C^\infty(R^2)$.

Now

$$\int_{-1}^1 \left(\int_t^1 |s-t|^{1-p-q}.s^{l-i}.K^l_{i,p,q}(s,t).S_n^{(m)}(s)\,ds\right).t^i.S_n^{(r)}(t)dt =$$

$$= \int_{-1}^1 \left(\int_{-1}^s |s-t|^{1-p-q}.t^i.K^l_{i,p,q}(s,t).S_n^{(r)}(t)\,dt\right).s^{l-i}.S_n^{(m)}(s)\,ds =$$

$$= \int_{-1}^1 \left(\int_{-1}^t |s-t|^{1-p-q}.s^i.K^l_{i,p,q}(t,s).S_n^{(r)}(s)\,ds\right).t^{l-i}.S_n^{(m)}(t)\,dt =$$

$$= \int_{-1}^1 \left(\int_{-1}^t |s-t|^{1-p-q}.s^i.K^l_{l-i,q,p}(s,t).S_n^{(r)}(s)\,ds\right).t^{l-i}.S_n^{(m)}(t)\,dt.$$

Then formula (14) is equal to

(15) $$\frac{(-1)^r.\Gamma(q).\Gamma(p)}{\Gamma(q+m).\Gamma(p+r)} . \sum_{i=0}^{l} \binom{l}{i}.$$

$$.\int_{-1}^1 \left(\int_{-1}^t |s-t|^{1-p-q}.s^i.K^l_{l-i,q,p}(s,t).S_n^{(r)}(s)\,ds\right).t^{l-i}.S_n^{(m)}(t)\,dt =$$

$$= \frac{(-1)^r.\Gamma(q).\Gamma(p)}{\Gamma(q+m).\Gamma(p+r)} . \sum_{j=0}^{l} \binom{l}{j}.$$

$$.\int_{-1}^1 \left(\int_{-1}^t |s-t|^{1-p-q}.s^{l-j}.K^l_{j,q,p}(s,t).S_n^{(r)}(s)\,ds\right).t^j.S_n^{(m)}(t)\,dt.$$

Therefore case d2) follows adding formula (14) and formula (15) multiplied by $(-1)^{m+r}$ after interchanging m and r, p and q.

Case e). Suppose r = 0, 1, 2, ... From formulae (4) and (5) we have

$$(16) \qquad \int ([x_+^r . x_-^{-m-q}] * s_n)(y).y^1.\psi(y)\ dy =$$

$$= \int (\int_{-1}^{y} (y - t)^r . s_n(t)\ dt).(\frac{\Gamma(q)}{\Gamma(q+m)} . \int_y^1 (s - y)^{-q}.s_n^{(m)}(s)\ ds).y^1.\psi(y)\ dy =$$

$$= \frac{\Gamma(q)}{\Gamma(q+m)} . \int_{-1}^{1} (\int_t^1 (\int_t^s (y - t)^r . |s - y|^{-q}.y^1.\psi(y)\ dy).s_n^{(m)}(s)\ ds).s_n(t)\ dt$$

and also from formulae (3) and (6) it follows that

$$(17) \qquad \int ([x_+^{-m-q}.x_-^r] * s_n)(y).y^1.\psi(y)\ dy =$$

$$= \int (\frac{(-1)^m.\Gamma(q)}{\Gamma(q+m)} . \int_{-1}^{y} (y - t)^{-q}.s_n^{(m)}(t)\ dt).(\int_y^1 (s - y)^r . s_n(s)\ ds).y^1.\psi(y)\ dy =$$

$$= \frac{(-1)^m.\Gamma(q)}{\Gamma(q+m)} . \int_{-1}^{1} (\int_t^1 (\int_t^s |y - t|^{-q}.(s - y)^r.y^1.\psi(y)\ dy).s_n(s)\ ds).s_n^{(m)}(t)\ dt =$$

$$= \frac{(-1)^m.\Gamma(q)}{\Gamma(q+m)} . \int_{-1}^{1} (\int_{-1}^{s} (\int_t^s |y - t|^{-q}.(s - y)^r.y^1.\psi(y)\ dy).s_n^{(m)}(t)\ dt).s_n(s)\ ds =$$

$$= \frac{(-1)^{m+r+1}.\Gamma(q)}{\Gamma(q+m)} . \int_{-1}^{1} (\int_{-1}^{t} (\int_t^s |y - s|^{-q}.(y - t)^r.y^1.\psi(y)\ dy).s_n^{(m)}(s)\ ds).s_n(t)\ dt\ .$$

Combining formulae (16) and (17) we get case e). QED.

4. POLYNOMIALS THAT DO NOT DEFINE A DISTRIBUTION WHEN THEY ARE CALCULATED OVER CERTAIN PSEUDOFUNCTIONS. We will show the nonexistence of the product $x_+^\lambda.x_-^\mu$ for certain values of λ and μ. This will be done in a stronger version. This section will be completed in §6.

THEOREM 3.

a)   *a1)* $x_+^{-r}.x_-^{-m} + (-1)^{r+m+1}.x_+^{-m}.x_-^{-r}$ *does not exist if* m,r *are positive integers,* $m \neq r$.

a2) $x_+^{-r}.x_-^{-r}$ *does not exist if* r *is a positive integer.*

b) $x_+^\lambda.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^\lambda$ *does not exist if* $k = 1, 2, \ldots$, $\mathrm{Re}\ \lambda > -1$, *(*k *odd and* $-1 \geq \mathrm{Re}\ (\lambda - k)$*) or (*k *even and* $-2 \geq \mathrm{Re}\ (\lambda - k)$*).*

c) $x_+^{-k}.x_-^{-m-q} + (-1)^{m+k+1}.x_+^{-m-q}.x_-^{-k}$ *does not exist if* $k = 1, 2, \ldots$, $m = 0, 1, 2, \ldots$,

$0 \leq \text{Re } q < 1,\ q \neq 0.$

d) d1) $x_+^{-r-p}.x_-^{-m-q} + (-1)^{m+r}.x_+^{-m-q}.x_-^{-r-p}$ *does not exist if* $r,m = 0, 1, 2, \ldots,$

$0 \leq \text{Re } p < 1,\ p \neq 0,\ 0 \leq \text{Re } q < 1,\ q \neq 0,\ 1 - p - q \neq 0,$ *(*$r + m$ *even and* $-1 \geq \text{Re }(-m-q-r-p)$*) or (*$r + m$ *odd and* $-2 \geq \text{Re }(-m-q-r-p)$*).*

d2) $x_+^{r-p}.x_-^{-m-q} + (-1)^{m-r}.x_+^{-m-q}.x_-^{r-p}$ *does not exist if* $0 \leq \text{Re } p < 1,\ p \neq 0,$

$0 \leq \text{Re } q < 1,\ q \neq 0,\ r,m = 0, 1, 2, \ldots,\ 1 - p - q \neq 0,$ *(*$m - r$ *even and* $-1 \geq \text{Re }(-m-q+r-p)$*) or (*$m - r$ *odd and* $-2 \geq \text{Re }(-m-q+r-p)$*).*

e) $x_+^{r}.x_-^{-m-q} + (-1)^{m+r+1}.x_+^{-m-q}.x_-^{r}$ *does not exist if* $0 \leq \text{Re } q < 1,\ q \neq 0;$

$r,m = 0, 1, 2, \ldots, -1 \geq \text{Re }(r-m-q).$

PROOF. We prove the theorem exhibiting a sequence $\{S_n\}_{n=1,2,\ldots}$ with the properties mentioned in theorem 1 and such that

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} ([x_+^{\lambda}.x_-^{\mu} + x_+^{\mu}.x_-^{\lambda}] * S_n)(y)\, dy$$

does not exist or

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} ([x_+^{\lambda}.x_-^{\mu} - x_+^{\mu}.x_-^{\lambda}] * S_n)(y).y\, dy.$$

does not exist.

This proves that $x_+^{\lambda}.x_-^{\mu} + x_+^{\mu}.x_-^{\lambda}$ or $x_+^{\lambda}.x_-^{\mu} - x_+^{\mu}.x_-^{\lambda}$ does not exist since the supports of the functions $([x_+^{\lambda}.x_-^{\mu} \pm x_+^{\mu}.x_-^{\lambda}] * S_n)(y)$ are contained in $[-\varepsilon,\varepsilon]$.

We shall need some auxiliary formulae. Putting $y - t = (s - t).v$ we have that $(s - y) = (s - t).(1 - v)$ and therefore

(1)
$$\int_t^s \log|y - t|.\log|s - y|.y\, dy =$$

$$= \int_0^1 (\log v + \log|s - t|).(\log(1 - v) + \log|s - t|).((s - t).v + t).(s - t)\, dv =$$

$$= (s - t)^2.\int_0^1 v.\log v.\log(1 - v)\, dv + t.(s - t).\int_0^1 \log v.\log(1 - v)\, dv -$$

$$- \log|s - t|.(s - t)^2 - 2.\log|s - t|.t.(s - t) +$$

$$+ \frac{1}{2}.(s - t)^2.\log^2|s - t| + t.(s - t).\log^2|s - t|.$$

Also by the same change of variables we get

(2)
$$\int_t^s \log|y - t|.\log|s - y|\, dy =$$

$$= \int_0^1 (\log v + \log |s - t|).(\log (1 - v) + \log |s - t|).(s - t)\, dv =$$

$$= (s - t).\int_0^1 \log v.\log (1 - v)\, dv - 2(s - t).\log |s - t| + \log^2 |s - t|.(s - t).$$

If $\lambda \in C$ and Re $\lambda > -1$, we get in the same fashion

(3)
$$\int_t^s |y - t|^\lambda.\log |s - y|.y\, dy =$$

$$= \int_0^1 |s - t|^\lambda.v^\lambda.(\log |s - t| + \log (1 - v)).((s - t).v + t).(s - t)\, dv =$$

$$= |s - t|^\lambda.(s - t)^2.\log|s - t|.\frac{1}{(\lambda + 2)} + |s - t|^\lambda.(s - t).t.\log|s - t|.\frac{1}{(\lambda + 1)} +$$

$$+ |s - t|^\lambda.(s - t)^2.\int_0^1 v^{\lambda+1}.\log(1 - v)\, dv + |s - t|^\lambda.(s - t).t.\int_0^1 v^\lambda.\log(1 - v)\, dv.$$

Finally for the same values of λ we get

(4)
$$\int_t^s |y - t|^\lambda.\log |s - y|\, dy =$$

$$= \int_0^1 |s - t|^\lambda.v^\lambda.(\log |s - t| + \log (1 - v)).(s - t)\, dv =$$

$$= (s - t).|s - t|^\lambda.\log|s - t|.\frac{1}{(\lambda + 1)} + (s - t).|s - t|^\lambda.\int_0^1 v^\lambda.\log (1 - v)\, dv.$$

If $r = 0, 1, 2, \ldots$ and $0 \leqq \text{Re } q < 1$, $q \neq 0$ we have

(5)
$$\int_t^s (y - t)^r.|s - y|^{-q}.y\, dy =$$

$$= \int_0^1 (s - t)^r.v^r.|s - t|^{-q}.(1 - v)^{-q}.((s - t).v + t).(s - t)\, dv =$$

$$= (s - t)^{r+2}.|s - t|^{-q}.B(r+2,1-q) + t.(s - t)^{r+1}.|s - t|^{-q}.B(r+1,1-q)$$

and also (same values for r and q):

(6)
$$\int_t^s (y - t)^r.|s - y|^{-q}\, dy =$$

$$= \int_0^1 (s - t)^r.v^r.|s - t|^{-q}.(1 - v)^{-q}.(s - t)\, dv =$$

$$= (s - t)^{r+1}.|s - t|^{-q}.B(r+1,1-q).$$

Consider case a1)

If $r + m$ is odd, take a sequence $\{S_n\}_{n=1,2,\ldots}$ as in theorem 1. We have by theorem 2 a), theorem 1 a), c) and formula (2) that

(7) $$\lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{-r}.x_-^{-m} + (-1)^{m+r+1}.x_+^{-m}.x_-^{-r}] * S_n)(y)\, dy}{\beta_n^{m+r-1}.\log \beta_n} =$$

$$= - \frac{2}{(r-1)!.(m-1)!} . \int \log |u|.u.(\rho * \rho)^{(m+r)}(u)\, du.$$

Verifying that $\int \log |u|.u.(\rho * \rho)^{(m+r)}(u)\, du \neq 0$ for some ρ (this by lemma 1 b)) we get case a1) for $m + r$ odd. This will be, as we said, our pattern of proof. If $r \neq m$, $r + m$ is even, take again a sequence $\{S_n\}_{n=1,2,\ldots}$ as in theorem 1. We get by theorem 2 a), theorem 1 c), d) and formula (1) that

(8) $$\lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{-r}.x_-^{-m} + (-1)^{m+r+1}.x_+^{-m}.x_-^{-r}] * S_n)(y).y\, dy}{\beta_n^{m+r-2}.\log \beta_n} =$$

$$= \frac{(-1)^{r-1}}{(r-1)!.(m-1)!} .[\frac{1}{2} .(-1)^{m+3}.2.\int \log |u|.u^2.(\rho * \rho)^{(m+r)}(u)\, du +$$

$$+ (-1)^{m+2}.(\frac{2+m-r}{2}).\int \log |u|.u^2.(\rho * \rho)^{(m+r)}(u)\, du] =$$

$$= \frac{1}{(r-1)!.(m-1)!} . \frac{(r-m)}{2} . \int \log |u|.u^2.(\rho * \rho)^{(m+r)}(u)\, du.$$

Then by lemma 1 b) we obtain case a 1).

The product $x_+^{-r}.x_-^{-r}$ will be considered later. Let us see case b).

Let $\{S_n\}_{n=1,2,\ldots}$ be a sequence as in theorem 1. We have by theorem 2 b), theorem 1 a) and formula (4) that

(9) $$\lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{\lambda}.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^{\lambda}] * S_n)(y)\, dy}{\beta_n^{k-1-\lambda}.\log \beta_n} =$$

$$= \frac{1}{(k-1)!.(\lambda+1)} . \int |u|^{\lambda}.\ u.(\rho * \rho)^{(k)}(u)\, du.$$

If Re $(k-1-\lambda) \geq 0$, k is odd and λ different from a nonnegative integer or $\lambda = k - 1$ or λ is odd and $k - 1 > \lambda$ (this implies by lemma 1 a) that $\int |u|^{\lambda}.u.(\rho * \rho)^{(k)}(u)\, du \neq 0$ for some ρ). We have that

$$x_+^{\lambda}.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^{\lambda}$$

does not exist.

We need to know what happens if k is odd; λ is even and less than k - 1. Notice that if λ is even then $|s - t|^{\lambda} = (s - t)^{\lambda}$ and therefore we can combine theorem 2b), theorem 1 a), c) and formula (4) to get

$$(10) \qquad \lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{\lambda}.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^{\lambda}] * S_n)(y)\, dy}{\beta_n^{k-1-\lambda}} =$$

$$= \frac{1}{(k-1)!} . (\sum_{j=1}^{k-1} \frac{1}{j}) . \int u^{\lambda}.(\rho * \rho)^{(k-1)}(u)\, du +$$

$$+ (-\frac{1}{(k-1)!}).[\frac{1}{(\lambda + 1)} . (\int \log |u| . u^{\lambda+1}.(\rho * \rho)^{(k)}(u)\, du) +$$

$$+ (\int_0^1 v^{\lambda}.\log (1 - v)\, dv).(\int u^{\lambda+1}.(\rho * \rho)^{(k)}(u)\, du)] =$$

$$= - \frac{1}{(k-1)!.(\lambda + 1)} . (\int \log |u| . u^{\lambda+1}.(\rho * \rho)^{(k)}(u)\, du)$$

since from $k - 1 - \lambda > 0$, after integrating by parts we obtain

$\int u^{\lambda+1}.(\rho * \rho)^{(k)}(u)\, du = 0$.

From lemma 1 b) it follows that $\int \log |u| . u^{\lambda+1}.(\rho * \rho)^{(k)}(u)\, du \neq 0$ for some ρ, (because $k + 1 + \lambda$ is even).

Summarizing: if Re $(\lambda - k) \leq -1$, k is odd the product

$$x_+^{\lambda}.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^{\lambda}$$

does not exist.

We have by theorem 2 b), theorem 1 a), b) and formula (3) that

$$(11) \qquad \lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{\lambda}.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^{\lambda}] * S_n)(y).y\, dy}{\beta_n^{k-2-\lambda}.\log \beta_n} =$$

$$= (-\frac{1}{(k-1)!}).[\frac{1}{(\lambda + 2)} . (-1) . \int |u|^{\lambda}.u^2.(\rho * \rho)^{(k)}(u)\, du +$$

$$+ \frac{1}{(\lambda + 1)} . \frac{(-1)}{2} . \frac{(k - \lambda - 2)}{(\lambda + 2)} . \int |u|^{\lambda}.u^2.(\rho * \rho)^{(k)}(u)\, du] =$$

$$= \frac{1}{(k-1)!} . \frac{(\lambda + k)}{(\lambda + 1).(\lambda + 2).2} . \int |u|^{\lambda}.u^2.(\rho * \rho)^{(k)}(u)\, du.$$

Then if Re $\lambda > -1$, Re $(k - 2 - \lambda) \geq 0$ and λ not a positve integer nor zero, or $\lambda + 2 = k$, or λ is odd and $k > \lambda + 2$ we get (again by lemma 1 a)) that

$$x_+^{\lambda}.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^{\lambda}$$

does not exist.

If k is even and λ is even, $\lambda < k - 2$ (this is the remaining case) by theorems 1 a), b), c), d), theorem 2 b) and formula (3) (remember $|s - t|^{\lambda} = (s - t)^{\lambda}$) we get

$$(12) \qquad \lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{\lambda}.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^{\lambda}] * s_n)(y).y\, dy}{\beta_n^{k-2-\lambda}} =$$

$$= \frac{1}{(k-1)!}.(\sum_{j=1}^{k-1} \frac{1}{j}).(\frac{(k+\lambda)}{2.(\lambda+1)}.\int u^{\lambda+1}.(\rho * \rho)^{(k-1)}(u)\, du) +$$

$$+ (-\frac{1}{(k-1)!}).[(\int_0^1 v^{\lambda}.\log(1-v)\, dv).\frac{1}{2}.\frac{(k-\lambda-2)}{(\lambda+2)}.(\int u^{\lambda+2}.(\rho * \rho)^{(k)}(u)\, du) +$$

$$+ (\int_0^1 v^{\lambda+1}.\log(1-v)\, dv).(\int u^{\lambda+2}.(\rho * \rho)^{(k)}(u)\, du) +$$

$$+ \frac{1}{(\lambda+1)}.(-1).\frac{1}{2}.(\frac{2+\lambda-k)}{2+\lambda}.\int \log|u|.u^{2+\lambda}.(\rho * \rho)^{(k)}(u)\, du +$$

$$+ \frac{1}{(\lambda+2)}.\int \log|u|.u^{2+\lambda}.(\rho * \rho)^{(k)}(u)\, du] =$$

$$= (-\frac{1}{(k-1)!}).\frac{(\lambda+k)}{(\lambda+1).(\lambda+2).2}.\int \log|u|.u^{2+\lambda}.(\rho * \rho)^{(k)}(u)\, du.$$

The lemma 1 b) completes our proof.

This proves part b).

We prove now part c).

Let k + m be odd, then by theorem 2 c), theorem 1 a) and formula (4) we have that

$$(13) \qquad \lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{-k}.x_-^{-m-q} + (-1)^{m+k+1}.x_+^{-m-q}.x_-^{-k}] * s_n)(y)\, dy}{\beta_n^{m+k-1+q}.\log \beta_n} =$$

$$= -\frac{\Gamma(q)}{(k-1)!.(q-1).\Gamma(q+m)}.\int |u|^{-q}.u.(\rho * \rho)^{(m+k)}(u)\, du.$$

By lemma 1 a) we know that $\int |u|^{-q}.u.(\rho * \rho)^{(m+k)}(u)\, du \neq 0$ for some ρ. Then we have proved that if k + m is odd then

$$x_+^{-k}.x_-^{-m-q} + (-1)^{m+k+1}.x_+^{-m-q}.x_-^{-k}$$

does not exist.

Let k + m be even. By theorem 2 c), theorem 1 a), b) and formula (3) it follows that

$$(14)\qquad \lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{-k}.x_-^{-m-q} + (-1)^{m+k+1}.x_+^{-m-q}.x_-^{-k}] * S_n)(y).y\,dy}{\beta_n^{m+k-2+q}.\log\beta_n} =$$

$$= \frac{(m+q-k).\Gamma(q)}{(k-1)!.2.(q-2).(q-1).\Gamma(q+m)} . \int |u|^{-q}.u^2.(\rho * \rho)^{(m+k)}(u)\,du.$$

By lemma 1 a) we have that $\int |u|^{-q}.u^2.(\rho * \rho)^{(m+k)}(u)\,du \neq 0$ for some ρ. This proves part c).

We consider now case d1). By theorem 2 d2), theorem 1 a), we get

$$(15)\qquad \lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{-r-p}.x_-^{-m-q} + (-1)^{m+r}.x_+^{-r-p}.x_-^{-m-q}] * S_n)(y)\,dy}{\beta_n^{m+r-1+p+q}} =$$

$$= \frac{\Gamma(p).\Gamma(q).B(1-p,1-q)}{\Gamma(p+r).\Gamma(q+m)} . \int |u|^{1-p-q}.(\rho * \rho)^{(r+m)}(u)\,du$$

(observe that the last integral is zero if $r + m$ is odd).

Let $r + m$ be even; then by lemma 1 a) we get $\int |u|^{1-p-q}.(\rho * \rho)^{(r+m)}(u)\,du \neq 0$ for some ρ because $1 - p - q \neq 0$.

Consider the subcase $-1 \geq \mathrm{Re}\,(-m-q-r-p)$. If $-1 > \mathrm{Re}\,(-m-q-r-p)$ then the numerator of (15) tends to ∞. If $-1 = \mathrm{Re}\,(-m-q-r-p)$ the hypothesis imply that $m+r-1+p+q$ is purely imaginary and β_n can be chosen in such a way that $\beta_{2h}^{m+r-1+p+q} = i$ and $\beta_{2h+1}^{m+r-1+p+q} = 1$. Hence the numerator of (15) does not have a limit either.

By theorem 2 d2), theorem 1 b), it follows, if $m + r$ is odd, that

$$(16)\qquad \lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{-r-p}.x_-^{-m-q} + (-1)^{m+r}.x_-^{-r-p}.x_+^{-m-q}] * S_n)(y).y\,dy}{\beta_n^{m+r+p+q-2}} =$$

$$= \frac{\Gamma(q).\Gamma(p).B(1-p,1-q).(m+q-r-p)}{\Gamma(q+m).\Gamma(p+r).2.(2-p-q)} . \int |u|^{1-p-q}.u.(\rho * \rho)^{(m+r)}(u)\,du =$$

$$= \frac{\Gamma(q).\Gamma(p).B(1-p,1-q).(r+p-m-q)}{\Gamma(q+m).\Gamma(p+r).2} . \int |u|^{1-p-q}.(\rho * \rho)^{(m+r-1)}(u)\,du.$$

We have $\int |u|^{1-p-q}.(\rho * \rho)^{(m+r-1)}(u)\,du \neq 0$ because $1 - p - q \neq 0$. Also $\mathrm{Re}\,(m + r + p + q - 2) \geq 0$. If $\mathrm{Re}\,(m + r + p + q - 2) > 0$ then $\beta_n^{m+r+p+q-2} \to \infty$ and the numerator does not converge. If $\mathrm{Re}\,(m + r + p + q - 2) = 0$ then $m + r + p + q - 2$ has to be an purely imaginary number since $m + r + p + q - 2 \neq 0$. Then we choose again β_n as before.

We consider next case d2). To prove this case we need an auxiliary formula. Let

$u_n(y)$ be defined as

$$u_n(y) := ([x_+^{r-p}.x_-^{-m-q} + (-1)^{m-r}.x_+^{-m-q}.x_-^{r-p}] * s_n(y)$$

with $r,m = 0, 1, 2, \ldots,\ 0 \leq \operatorname{Re} p < 1,\ p \neq 0,\ 0 \leq \operatorname{Re} q < 1,\ q \neq 0$.
We obtain after differentiation of $u_n(y)$ the following

$$(17)\quad \frac{u_n^{(1)}(y)}{(m+q)} - \frac{(r-p)}{(m+q)}.([x_+^{(r-1)-p}.x_-^{-m-q} + (-1)^{m-(r-1)}.x_+^{-m-q}.x_-^{(r-1)-p}] * s_n(y) =$$

$$= ([x_+^{r-p}.x_-^{-(m+1)-q} + (-1)^{(m+1)-r}.x_+^{-(m+1)-q}.x_-^{r-p}] * s_n)(y).$$

Let us prove that

$$(18)\quad \lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{r-p}.x_-^{-m-q} + (-1)^{m-r}.x_-^{-m-q}.x_-^{r-p}] * s_n)(y)\, dy}{\beta_n^{m-r+p+q-1}} =$$

$$= \frac{\Gamma(p).\Gamma(q).B(1-p,1-q)}{\Gamma(p-r).\Gamma(q+m)}.\int |u|^{1-p-q}.(\rho * \rho)^{(m-r)}(u)\, du$$

if $-1 \geq \operatorname{Re}(r-p-m-q)$, $m - r$ is even, $1 - p - q \neq 0$, $m,r = 0, 1, 2, \ldots,$
$0 \leq \operatorname{Re} p < 1,\ p \neq 0,\ 0 \leq \operatorname{Re} q < 1,\ q \neq 0$.
We prove (18) by induction using formula (17). Integrating (17) we get:

$$\frac{(p-r)}{(m+q)}.\int ([x_+^{(r-1)-p}.x_-^{-m-q} + (-1)^{m-(r-1)}.x_+^{-m-q}.x_-^{(r-1)-p}] * s_n)(y)\, dy =$$

$$= \int (x_+^{r-p}.x_-^{-(m+1)-q} + (-1)^{(m+1)-r}.x_+^{-(m+1)-q}.x_-^{r-p}] * s_n)(y)\, dy.$$

Now, formula (18) for $r = 0$ follows directly from formula (15).
If we assume that formula (18) is valid for $m = 0, 1, \ldots,\ 0 \leq r \leq r_0 - 1\ (r_0 \geq 1)$,
$-1 \geq \operatorname{Re}(r-p-m-q)$, $1 - p - q \neq 0$, $m - r$ even, then from the above identity we obtain

$$\frac{(p-r_0)}{(m+q)} \cdot \frac{\Gamma(p).\Gamma(q).B(1-p,1-q)}{\Gamma(p-r_0+1).\ \Gamma(q+m)}.\int |u|^{1-p-q}.(\rho * \rho)^{(m+1-r_0)}(u)\, du =$$

$$= \lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{r_0-p}.x_-^{-(m+1)-q} + (-1)^{(m+1)-r_0}.x_+^{-(m+1)-q}.x_-^{r_0-p}] * s_n)(y)\, dy}{\beta_n^{m-r_0+p+q}}$$

and formula (18) is proved.
In a similar way it can be proved that

$$(19)\quad \lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{r-p}.x_-^{-m-q} + (-1)^{m-r}.x_+^{-m-q}.x_-^{r-p}] * s_n)(y).y\, dy}{\beta_n^{m-r+p+q-2}} =$$

$$= \frac{(p - r - m - q).B(1-p,1-q).\Gamma(q).\Gamma(p)}{2.\Gamma(q+m).\Gamma(p-r)} . \int |u|^{1-p-q}.(\rho * \rho)^{(m-r-1)}(u)\, du$$

with $-2 \geq \mathrm{Re}\,(r-p-m-q)$, $1 - p - q \neq 0$, $m - r$ odd, $m,r = 0, 1, 2, \ldots,$
$0 \leq \mathrm{Re}\, q < 1$, $q \neq 0$, $0 \leq \mathrm{Re}\, p < 1$, $p \neq 0$.

Let us prove (19). Multiplying (17) by y and integrating, we get:

(20)
$$- \frac{1}{(m + q)} . \int u_n(y)\, dy +$$

$$+ \frac{(p - r)}{(m + q)} . \int ([x_+^{(r-1)-p}.x_-^{-m-q} + (-1)^{m-(r-1)}.x_+^{-m-q}.x_-^{(r-1)-p}] * S_n)(y).y\, dy =$$

$$= \int ([x_+^{r-p}.x_-^{-(m+1)-q} + (-1)^{(m+1)-r}.x_+^{-(m+1)-q}.x_-^{r-p}] * S_n)(y).y\, dy.$$

Formula (19) is valid if $r = 0$ (a particular case of formula (16)).
We use formula (20) to prove by induction formula (19).
We suppose that formula (19) is true for $m - r$ odd, $0 \leq r \leq r_0 - 1$ ($r_0 \geq 1$),
$1 - p - q \neq 0$, $-2 \geq \mathrm{Re}\,(r-m-p-q)$; from formulae (20) and (18) we have

$$\lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{r_0-p}.x_-^{-(m+1)-q} + (-1)^{(m+1)-r_0}.x_+^{-(m+1)-q}.x_-^{r_0-p}] * S_n)(y).y\, dy}{\beta_n^{m-r_0+p+q-1}} =$$

$$= - \frac{1}{(m + q)} . \frac{\Gamma(q).\Gamma(p).B(1-p,1-q)}{\Gamma(q+m).\Gamma(p-r_0)} . \int |u|^{1-p-q}.(\rho * \rho)^{(m-r_0)}(u)\, du +$$

$$+ \frac{(p - r_0)}{(m + q)} . \frac{(p-m-q-(r_0-1)).B(1-p,1-q).\Gamma(p).\Gamma(q)}{2.\Gamma(q+m).\Gamma(p-r_0+1)} . \int |u|^{1-p-q}.(\rho*\rho)^{(m-(r_0-1)-1)}(u)\, du =$$

$$= \frac{(p - (m + 1) - q - r_0).B(1-p,1-q).\Gamma(p).\Gamma(q)}{2.\Gamma(q+m+1).\Gamma(p-r_0)} . \int |u|^{1-p-q}.(\rho * \rho)^{(m-r_0)}(u)\, du$$

and formula (19) follows.
From (18) and (19), d2) follows using lemma 1.
Now we consider case e).
By formula (6), theorem 2 e) and theorem 1 a), we have:

(21)
$$\lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{r}.x_-^{-m-q} + (-1)^{(m+r+1)}.x_+^{-m-q}.x_-^{r}] * S_n)(y)\, dy}{\beta_n^{m+q-r-1}} =$$

$$= \frac{\Gamma(q).B(r+1,1-q)}{\Gamma(q+m)} . \int |u|^{-q}.u^{r+1}.(\rho * \rho)^{(m)}(u)\, du.$$

By formula (5), theorem 2 c) and theorem 1 a), b) we have:

$$(22)\qquad \lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} ([x_+^r.x_-^{-m-q} + (-1)^{m+r+1}.x_+^{-m-q}.x_-^r] * S_n)(y).y\ dy}{\beta_n^{m+q-r-2}} =$$

$$= \frac{(r+m+q).B(r+1,1-q).\Gamma(q)}{2.(r+2-q).\Gamma(q+m)} . \int |u|^{-q}.u^{r+2}.(\rho * \rho)^{(m)}(u)\ du.$$

By lemma 1 a) and formulae (21) and (22) we get case e) immediately. In fact from (21) the subcase m + r odd, $-1 \geq \mathrm{Re}\,(r-m-q)$ is obtained. If m + r is even we have to use (22). The preceding inequality implies that $\mathrm{Re}\,(m-r) \geq 1 - \mathrm{Re}\, q > 0$ and since m - r is even, $m - r \geq 2 \geq 2 - \mathrm{Re}\, q$.

Here again the exponent of β_n is purely imaginary or has positive real part.

We now deal with the product $x_+^{-r}.x_-^{-r}$ (case a2)).

By formula (7) in the proof of theorem 2 we get

$$\int_{-\infty}^{+\infty} ([x_+^{-r}.x_-^{-r}] * S_n)(y)\ dy =$$

$$= \frac{(-1)^r}{((r-1)!)^2} .[\int_{-1}^{1} (\int_{t}^{1} (\int_{t}^{s} \log|y-t|.\log|s-y|\ dy).S_n^{(r)}(s)\ ds).S_n^{(r)}(t)\ dt +$$

$$+ (-\sum_{j=1}^{r-1} \frac{1}{j}).\int_{-1}^{1} (\int_{-1}^{s} \log|s-t|.S_n^{(r)}(t)\ dt).S_n^{(r-1)}(s)\ ds +$$

$$+ (\sum_{j=1}^{r-1} \frac{1}{j}).\int_{-1}^{1} (\int_{t}^{1} \log|s-t|.S_n^{(r)}(s)\ ds).S_n^{(r-1)}(t)\ dt -$$

$$- (\sum_{j=1}^{r-1} \frac{1}{j})^2 .\int_{-1}^{1} (S_n^{(r-1)}(y))^2\ dy].$$

Let $\{S_n\}_{n=1,2,\ldots}$ be a sequence as in theorem 1. Then

$$\int (S_n^{(r-1)}(y))^2\ dy = (-1)^{r-1}.\beta_n^{2r-1}.(\rho * \rho)^{(2r-2)}(0),$$

$$\int_{-1}^{1} (\int_{-1}^{s} \log|s-t|.S_n^{(r)}(t)\ dt).S_n^{(r-1)}(s)\ ds =$$

$$= \beta_n^{2r-1}.\int_{-1}^{1} (\int_{-1}^{s} \log(\frac{|s-t|}{\beta_n}).\rho^{(r)}(t)\ dt).\rho^{(r-1)}(s)\ ds,$$

and finally

$$\int_{-1}^{1} (\int_{t}^{1} (\int_{t}^{s} \log|y-t|.\log|s-y|\ dy).S_n^{(r)}(s)\ ds).S_n^{(r)}(t)\ dt =$$

$$= \beta_n^{2r-1}.\int_{-1}^{1} (\int_{t}^{1} (\int_{t}^{s} \log(\frac{|y-t|}{\beta_n}).\log(\frac{|s-y|}{\beta_n})\ dy).\rho^{(r)}(s)\ ds).\rho^{(r)}(t)\ dt.$$

We can write

$$\lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{-r}.x_-^{-r}] * S_n)(y)\ dy}{\beta_n^{2r-1}.\log^2 \beta_n} =$$

$$= \frac{(-1)^r}{((r-1)!)^2}.(\int_{-1}^{1}(\int_t^1 (s-t).\rho^{(r)}(s)\ ds).\rho^{(r)}(t)\ dt = \frac{(\rho * \rho)^{(2r-2)}(0)}{((r-1)!)^2}$$

because the last three factors of formula (7) of theorem 2 are $o(\beta_n^{2r-1}.\log^2 \beta_n)$, QED.

5. EXISTENCE OF CERTAIN PRODUCTS.

LEMMA 2. *If* γ, m, k, $\alpha = 0, 1, 2, \ldots$, Re $\lambda > -1$, $0 <$ Re $(\lambda+\gamma+1-m-k)$, $m + k + \gamma$ *is an odd number and* $\{S_n\}_{n=1,2,\ldots} \subseteq C_0^\infty(R^1)$ *is a sequence of functions as in the introduction (ie. with properties (I)) we have:*

$$\lim_{n \to \infty} \iint (s-t)^\gamma.|s-t|^\lambda.\log^\alpha|s-t|.\phi(s,t).S_n^{(m)}(s).S_n^{(k)}(t)\ ds\ dt = 0$$

if $\phi(s,t) \in C^\infty(R^2)$.

PROOF. Since $0 <$ Re $(\lambda+\gamma+1-m-k)$ we can integrate by parts in

$$\iint (s-t)^\gamma.|s-t|^\lambda.\log^\alpha|s-t|.\phi(s,t).S_n^{(m)}(s).S_n^{(k)}(t)\ ds\ dt$$

and we can write it as a sum of integrals of the form

$$\iint g(s,t).S_n(s).S_n(t)\ ds\ dt$$

where $g \in L_{1,loc}(R^2)$. The main term is

$$\iint \frac{\partial^{m+k}[(s-t)^\gamma.|s-t|^\lambda.\log^\alpha|s-t|]}{\partial^m s.\partial^k t}.\phi(s,t).S_n(s).S_n(t)\ ds\ dt.$$

The others are of the form

$$\iint g(s,t).S_n(s).S_n(t)\ ds\ dt$$

With g continuous and $g(0,0) = 0$ and therefore

$$\lim_{n \to \infty} \iint g(s,t).S_n(s).S_n(t)\ ds\ dt = 0.$$

The main term can be written as follows

$$A_n = \iint \frac{\partial^{m+k}[(s-t)^\gamma.|s-t|^\lambda.\log^\alpha|s-t|]}{\partial^m s\, \partial^k t}.\phi(s,t).S_n(s).S_n(t)\ ds\ dt =$$

$$= (-1)^k. \iint f^{(m+k)}(s-t).\phi(s,t).S_n(s).S_n(t)\ ds\ dt$$

where $f(x) = x^{\gamma}.|x|^{\lambda}.\log^{\alpha}|x|$. Since $\gamma + m + k$ is odd we have that $f^{(m+k)}(s-t) = -f^{(m+k)}(t-s)$. So

$$A_n = (-1)^k.\iint f^{(m+k)}(t-s).\phi(t,s).S_n(t).S_n(s)\,ds\,dt =$$

$$= (-1)^{k+1}.\iint f^{(m+k)}(s-t).\phi(s,t).S_n(t).S_n(s)\,ds\,dt.$$

Therefore we can write

$$A_n = \frac{(-1)^k}{2}.\iint f^{(m+k)}(s-t).[\phi(s,t)-\phi(t,s)].S_n(s).S_n(t)\,ds\,dt.$$

It is easy to show that there exists a $\psi \in C^{\infty}(R^2)$ such that

$$\phi(s,t) - \phi(t,s) = (s-t).\psi(s,t).$$

(In fact,

$$\phi(s,t) - \phi(t,s) = \int_s^t \frac{d}{dy}(\phi(s,y) - \phi(y,s))\,dy =$$

$$= (t-s).\int_0^1 [\partial_2\phi(s,s+u(t-s)) - \partial_1\phi(s+u(t-s),s)]\,du).$$

But $x.f^{(m+k)}(x)$ is a continuous function that vanishes at zero. Then $A_n \to 0$ if $n \to \infty$. QED.

THEOREM 4 (existence).

a) $$x_+^{\lambda}.x_-^{\mu} = 0 \quad \textit{if}\ \mathrm{Re}\,(\lambda+\mu) > -1;\quad \lambda,\mu \in C.$$

b) b1) $$x_+^{-r-p}.x_-^{-m-q} + (-1)^{m+r}.x_+^{-m-q}.x_-^{-r-p} = \frac{(-1)^r\,\pi}{\mathrm{sen}\,\pi p.(m+r)!}.S^{(r+m)}$$

if $1-p-q=0$, $0 \leq \mathrm{Re}\,p < 1$, $p \neq 0$, $0 \leq \mathrm{Re}\,q < 1$, $q \neq 0$; $r,m = 0, 1, 2, \ldots$.

b2) $$x_+^{r-p}.x_-^{p-r-1} = \frac{(-1)^r.\pi}{\mathrm{sen}\,\pi p.2}.S$$

if $r = 0, \pm1, \pm2, \ldots$, $0 \leq \mathrm{Re}\ p < 1$, $p \neq 0$

c) c1) $$x_+^{\lambda}.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^{\lambda} = 0$$

if $k = 2, 4, 6, \ldots$, $\mathrm{Re}\,(\lambda-k) > -2$, $\mathrm{Re}\,\lambda > -1$ ($\lambda \in C$).

c2) $$x_+^{-r-p}.x_-^{-m-q} + (-1)^{m+r}.x_+^{-m-q}.x_-^{-r-p} = 0$$

if $0 \leq \mathrm{Re}\,p < 1$, $p \neq 0$, $0 \leq \mathrm{Re}\,q < 1$, $q \neq 0$; $r,m = 0, 1$, $r+m = 1$, $-1 < \mathrm{Re}\,(-p-q)$.

d) d1) $$x_+^{r-p}.x_-^{-m-q} + (-1)^{m-r}.x_+^{-m-q}.x_-^{r-p} = \frac{(-1)^r.\pi}{\mathrm{sen}\,\pi p.(m-r)!}.S^{(m-r)}$$

if $m,r = 0, 1, 2, \ldots$, $0 \leq \mathrm{Re}\,p < 1$, $p \neq 0$, $0 \leq \mathrm{Re}\,q < 1$, $q \neq 0$, $0 \leq r \leq m$,

$1 - p - q = 0.$

d2) $$x_+^{r-1-p}.x_-^{-r-q} - x_+^{-r-q}.x_-^{r-1-p} = 0$$

if $r = 0, 1, \ldots, 0 \leq \text{Re } p < 1, p \neq 0, 0 \leq \text{Re } q < 1, q \neq 0, -1 < \text{Re } (-p-q).$

PROOF. Case a). Notice that

$$\int ([x_+^\lambda.x_-^\mu] * \overset{\vee}{s}_n)(y).\phi(y)\ dy = \int ([x_-^\lambda.x_+^\mu] * \overset{\vee}{s}_n)(y).\phi(y)\ dy$$

where $\phi \in C_0^\infty(R^1)$, $\overset{\vee}{\phi}(y) = \phi(-y)$.

If we prove that $x_+^\lambda.x_-^\mu = 0$ then also $x_+^\mu.x_-^\lambda = 0$.

Suppose first that Re $(\lambda+\mu) > -1$, Re $\lambda > -1$, Re $\mu > -1$. Then x_+^λ and x_-^μ are functions and we can choose p and q: $1 < p,q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, such that in a neighbourhood U of zero one has: $x_+^\lambda \in L_p(U)$ and $x_-^\mu \in L_q(U)$. Then by theorem 0 (see introduction) it follows that $x_+^\lambda.x_-^\mu = 0$.

Suppose now Re $(\lambda+\mu) > -1$, Re $\lambda \leq -1$, and that λ is not a negative integer. Let m be an integer such that Re $(\lambda+m) > -1$ and Re $(\lambda+m-1) \leq 1$.

From the definition of $x_+^{\lambda+m}$ we get

$$<d^m x_+^{\lambda+m},\phi> = (\lambda + m). \ \ldots \ .(\lambda + 1).<x_+^\lambda,\phi>.$$

Also we have Re $(\mu-m) > -1$ and therefore

$$<d^m x_-^\mu,\phi> = \mu. \ \ldots \ .(\mu - m + 1).(-1)^m.<x_-^{\mu-m},\phi>.$$

We have as before constants p,q, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ such that in a neighbourhood U of zero $x_-^{\mu-m} \in L_p(U)$, $x_+^{\lambda+m} \in L_q(U)$. Using again theorem 0 we get $x_+^\lambda.x_-^\mu = 0$.

To complete case a) we have to show that $x_+^{-k}.x_-^\mu = 0$ if $k = 1, 2, \ldots,$ Re $(\mu-k) > -1$. But

$$<d^k x_-^\mu,\phi> = \mu. \ \ldots \ .(\mu - k + 1).(-1)^k.<x_-^{\mu-k},\phi>$$

so the derivative of order k of $x_-^\mu \in L_q(U)$ for some q such that $1 < q < \infty$.

Also x_+^{-k} is the kth-derivate of a $L_p(U)$ function for any p, $1 < p < q$. We take p such that $\frac{1}{p} + \frac{1}{q} = 1$ and apply theorem 0. This establishes case a).

Consider case b1). The idea of the proof is the following: a function in $C_0^\infty(R^1)$ can be written as

(1) $$\phi(y) = \phi(0) + y.\phi^{(1)}(0) + \dots + \frac{y^{r+m}.\phi^{(r+m)}(0)}{(r+m)!} + y^{r+m+1}.\psi(y)$$

where $\psi(y) \in C^{\infty}(R^1)$.

Then we calculate the limit on each term of the sum. Observe that this can be done because the support of

$$([x_+^{-r-p}.x_-^{-m-q} + (-1)^{m+r}.x_+^{-m-q}.x_-^{-r-p}] * S_n)(y)$$

tends to zero if n tends to infinity and therefore we can calculate the limit over $C^{\infty}(R^1)$-functions instead of $C_0^{\infty}(R^1)$-functions.

Let $0 \leq l \leq m+r$. By formula d2) of theorem 2 we have for $1 - p - q = 0$:

$$\int_{-\infty}^{+\infty} ([x_+^{-r-p}.x_-^{-m-q} + (-1)^{m+r}.x_+^{-m-q}.x_-^{-r-p}] * S_n)(y).y^l \, dy =$$

$$= \frac{(-1)^r.\Gamma(q).\Gamma(p)}{\Gamma(q+m).\Gamma(p+r)} . \sum_{i=0}^{l} \binom{l}{i} K_{i,p,q}^{l} . \iint s^{l-i}.t^i.S_n^{(m)}(s).S_n^{(r)}(t) \, ds \, dt$$

where $K_{i,p,q}^{l} = B(l-i-p+1, i-q+1)$. Besides

$$\iint s^{l-i}.t^i.S_n^{(m)}(s).S_n^{(r)}(t) \, ds \, dt =$$

$$= (\int s^{l-i}.S_n^{(m)}(s) \, ds).(\int t^i.S_n^{(r)}(t) \, dt).$$

But

(2) $$\lim_{n \to \infty} \int s^I.S^{(J)}(s) \, ds = \begin{cases} 0 & \text{if } I \neq J, \\ (-1)^J.J! & \text{if } I = J. \end{cases}$$

So

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} ([x_+^{-r-p}.x_-^{-m-q} + (-1)^{m+r}.x_+^{-m-q}.x_-^{-r-p}] * S_n)(y).y^l \, dy =$$

$$= \begin{cases} 0 & 0 \leq l < m + r \\ (-1)^m.\Gamma(q).\Gamma(p) \quad (*) & \text{if } l = m + r. \end{cases}$$

It remains to evaluate next limit. First,

(*) Here $1-p-q = 0$ and $\Gamma(p).\Gamma(q) = \Gamma(1-p).\Gamma(p) = \frac{\pi}{\text{sen } \pi p}$.

(3) $\lim_{n\to\infty}\int_{-\infty}^{+\infty}([x_+^{-r-p}.x_-^{-m-q}+(-1)^{m+r}.x_+^{-m-q}.x_-^{-r-p}]*S_n)(y).y^{r+m+1}.\psi(y)\,dy =$

$$= \lim_{n\to\infty}\frac{(-1)^r.\Gamma(q).\Gamma(p)}{\Gamma(q+m).\Gamma(p+r)}.\sum_{i=0}^{r+m+1}\binom{r+m+1}{i}.\iint s^{r+m+1-i}.t^i.K_{i,p,q}^{r+m+1}(s,t).S_n^{(m)}(s).S_n^{(r)}(t)\,ds\,dt$$

by formula d2 theorem 2.

Integrating by parts it follows that the integral

$$\iint s^{r+m+1-i}.t^i.K_{i,p,q}^{r+m+1}(s,t).S_n^{(m)}(s).S_n^{(r)}(t)\,ds\,dt$$

is a sum of terms of the form

$$\iint g(s,t).S_n(s).S_n(t)\,ds\,dt$$

with $g(.,.)\in C^\infty(R^2)$, $g(0,0)=0$. We get then

$$\lim_{n\to\infty}\iint g(s,t).S_n(s).S_n(t)\,ds\,dt = 0.$$

So, limit (3) is equal to zero.

Consider case b2) of the theorem. Here again we write for $\phi\in C_0^\infty(R^1)$,

(4) $$\phi(y) = \phi(0) + y.\psi(y)$$

with $\psi\in C^\infty(R^1)$.

If $0 < \mathrm{Re}\,p < 1$ then by formula d1) of theorem 2 it follows that

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}([x_+^{-p}.x_-^{p-1}]*S_n)(y)\,dy =$$

$$= \lim_{n\to\infty}\int_{-1}^{1}\left(\int_t^1 K_{0,p,1-p}^{0}(s,t).S_n(s)\,ds\right).S_n(t)\,dt =$$

$$= \lim_{n\to\infty} B(1-p,p).\int_{-1}^{1}\left(\int_t^1 S_n(s)\,ds\right).S_n(t)\,dt =$$

Since $\int_t^1 S_n(s)\,ds = 1 - \int_{-1}^{t} S_n(s)\,ds = 1 - H_n(t)$, our limit can be written as

$$= \lim_{n\to\infty} B(1-p,p).\int_{-1}^{1}(1-H_n(t)).H_n^{(1)}(t)\,dt =$$

$$= \lim_{n\to\infty} B(1-p,p).\left(1-\int_{-1}^{1}\frac{(H_n^2)(1)}{2}\,dt\right) = \frac{1}{2}.\Gamma(1-p).\Gamma(p) = \frac{\pi}{2.\mathrm{sen}\,\pi p}.$$

Now by the same formula d1) of theorem 2 we obtain for $\psi\in C^\infty(R^1)$:

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}([x_+^{-p}.x_-^{p-1}]*S_n)(y).y.\psi(y)\,dy =$$

$$= \lim_{n\to\infty} \left(\int_{-1}^{1}\left(\int_{t}^{1} s.g_1(s,t).S_n(s)\, ds\right)S_n(t)\, dt + \int_{-1}^{1}\left(\int_{t}^{1} g_2(s,t).S_n(s)\, ds\right).t.S_n(t)\, dt\right)$$

with $g_1, g_2 \in C^\infty(R^2)$.

But these integrals tend to zero because $S_n(s).S_n(t)$ is an approximation of S in R^2 and $s.g_1(s,t)$ is a continuous function that vanishes at the origin. We have proved that

$$x_+^{-p}.x_-^{p-1} = \frac{\pi}{2.\text{sen}\ \pi p}.S$$

if $0 < \text{Re}\ p < 1$.

If p is purely imaginary, by formula d1) of theorem 2, with $r = 0$, $p = p$, $m = 1$, $q = -p$, we obtain

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} ([x_+^{-p}.x_-^{p-1}] * S_n)(y)\, dy =$$

$$= \lim_{n\to\infty} \frac{1}{-p}.\int_{-1}^{1}\left(\int_{t}^{1}(s-t).K^{0}_{0,p,-p}(s,t).S_n^{(1)}(s)\, ds\right).S_n(t)\, dt =$$

$$= \lim_{n\to\infty} \frac{B(1-p,1+p)}{-p}.\int_{-1}^{1}\left(\int_{t}^{1}(s-t).S_n^{(1)}(s)\, ds\right).S_n(t)\, dt =$$

$$= \lim_{n\to\infty} \Gamma(1-p).\Gamma(p).\int_{-1}^{1}(1 - H_n(t)).H_n^{(1)}(t)\, dt = \frac{\pi}{2.\text{sen}\ \pi p}.$$

Finally, from the same d1)

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} ([x_+^{-p}.x_-^{p-1}] * S_n)(y).y.\psi(y)\, dy =$$

$$= \lim_{n\to\infty} \frac{1}{-p}.\left[\int_{-1}^{1}\left(\int_{t}^{1}(s-t).s.g_1(s,t).S_n^{(1)}(s)\, ds\right).S_n(t)\, dt + \right.$$

$$\left. + \int_{-1}^{1}\left(\int_{t}^{1}(s-t).g_2(s,t).S_n^{(1)}(s)\, ds\right).t.S_n(t)\, dt\right]$$

with $g_1, g_2 \in C^\infty(R^2)$.

It is easily seen after integrating by parts that these integrals tend to zero. So, if $0 \leq \text{Re}\ p < 1$, $p \neq 0$:

(5) $$x_+^{-p}.x_-^{p-1} = \frac{\pi}{2.\text{sen}\ \pi p}.\dot{S}.$$

From the case a) of this theorem, we know that

$$x_+^{-\beta+1}.x_-^{\beta-1} = 0, \quad \beta \notin Z.$$

We have, taking derivatives,

(6) $$x_+^{-\beta}.x_-^{\beta-1} + x_+^{-\beta+1}.x_-^{\beta-2} = 0.$$

Then, from (6) and (5) with $\beta = p$ we get

$$x_+^{1-p}.x_-^{p-2} = -\frac{\pi}{2.\text{sen }\pi p}.S.$$

If $\beta = 1 + p$ from (6) and (5) we obtain

(7) $$x_+^{-1-p}.x_-^{p} = \frac{\pi}{2.\text{sen }\pi p}.S.$$

Repeating this process we obtain case b2).

To prove c) we need an auxiliary formula.

If Re $\lambda > -1$, $\psi \in C^\infty(R^1)$ we have that

(8) $$\int_t^s |y - t|^\lambda.\log|s - y|.y.\psi(y)\,dy =$$

$$= \int_0^1 |s - t|^\lambda.(\log|s - t| + \log(1-v)).v^\lambda.((s - t).v + t).\psi((s-t).v+t).(s - t)\,dv =$$

$$= (s - t)^2.|s - t|^\lambda.\log|s - t|.\phi_1(s,t) + (s - t).|s - t|^\lambda.t.\log|s - t|.\phi_2(s,t) +$$

$$+ (s - t)^2.|s - t|^\lambda.\phi_3(s,t) + (s - t).|s - t|^\lambda.t.\phi_4(s,t)$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are functions in $C^\infty(R^2)$.

We split case c) in two subcases. The idea of proof is the same for both and consists in decomposing $\phi \in C_0^\infty(R^1)$ into the sum

(9) $$\phi(y) = \phi(0) + y.\psi(y)$$

where $\psi \in C^\infty(R^1)$.

From theorem 2 b) and formula (4) of §4 we get

$$\int_{-\infty}^{+\infty} ([x_+^\lambda.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^\lambda] * S_n)(y)\,dy =$$

$$= \frac{1}{(k-1)!}.(\sum_{j=1}^{k-1}\frac{1}{j}).\iint |t - s|^\lambda.S_n(t).S_n^{(k-1)}(s)\,ds\,dt +$$

$$+ (-\frac{1}{(k-1)!}).[\frac{1}{(\lambda+1)}.\iint (s - t).|s - t|^\lambda.\log|s - t|.S_n^{(k)}(s).S_n(t)\,ds\,dt +$$

$$+ (\int_0^1 v^\lambda.\log(1 - v)\,dv).(\iint (s - t).|s - t|^\lambda.S_n^{(k)}(s).S_n(t)\,ds\,dt)].$$

It tends to zero by lemma 2 of this section.

By theorem 2 b) and formula (8) of this section we have

$$\int_{-\infty}^{+\infty} ([x_+^\lambda.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^\lambda] * S_n)(y).y.\psi(y)\,dy =$$

$$= \frac{1}{(k-1)!} \cdot \left(\sum_{j=1}^{k-1} \frac{1}{j}\right) \cdot \iint |t - s|^{\lambda} . S_n(t) . S_n^{(k-1)}(s) . s . \psi(s) \, ds \, dt \, +$$

$$+ \left(- \frac{1}{(k-1)!}\right) . \left[\iint (s - t)^2 . |s - t|^{\lambda} . \log|s - t| . \phi_1(s,t) . S_n^{(k)}(s) . S_n(t) \, ds \, dt \, + \right.$$

$$+ \iint (s - t) . |s - t|^{\lambda} . t . \log|s - t| . \phi_2(s,t) . S^{(k)}(s) . S_n(t) \, ds \, dt \, +$$

$$+ \iint (\ddot{s} - t)^2 . |s - t|^{\lambda} . \phi_3(s,t) . S_n^{(k)}(s) . S_n(t) \, ds \, dt \, +$$

$$\left. + \iint (s - t) . |s - t|^{\lambda} . t . \phi_4(s,t) . S_n^{(k)}(s) . S_n(t) \, ds \, dt \right].$$

The integrals involved tend to zero by lemma 2. Observe that to apply lemma 2 it is necessary to incorporate a factor (s - t) (or s or t) to the function ϕ (or ψ) to satisfy the hypothesis of lemma 2.

We consider case c2). By theorem 2 d2) and lemma 2 it follows that

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} ([x_+^{-r-p} . x_-^{-m-q} + (-1)^{m+r} . x_+^{-m-q} . x_-^{-r-p}] * S_n)(y) \, dy = 0,$$

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} ([x_+^{-r-p} . x_-^{-m-q} + (-1)^{m+r} . x_+^{-m-q} . x_-^{-r-p}] * S_n)(y) . y . \psi(y) \, dy = 0$$

for m, r, p, q as pointed out.

Case d). We recall that if $r,m = 0, 1, 2, \ldots$, $0 \leq \mathrm{Re}\, p < 1$, $p \neq 0$, $0 \leq \mathrm{Re}\, q < 1$, $q \neq 0$ and

$$u_n(y) := ([x_+^{r-p} . x_-^{-m-q} + (-1)^{m-r} . x_+^{-m-q} . x_-^{r-p}] * S_n(y)$$

then the following identity holds (formula (17) of §4)

$$(10) \quad \frac{u_n^{(1)}(y)}{(m+q)} + \frac{(p - r)}{(m+q)} . ([x_+^{(r-1)-p} . x_-^{-m-q} + (-1)^{m-(r-1)} . x_+^{-m-q} . x_-^{(r-1)-p}] * S_n)(y) =$$

$$= ([x_+^{r-p} . x_-^{-(m+1)-q} + (-1)^{(m+1)-r} . x_+^{-(m+1)-q} . x_-^{r-p}] * S_n)(y).$$

With the aid of this formula we prove d1), d2) of the theorem.

We show first d2). It follows by induction using c2) and a) of this theorem and (10).

In fact, by c2) we have if $-1 < \mathrm{Re}\,(-p-q)$, $0 \leq \mathrm{Re}\, p < 1$, $p \neq 0$, $0 \leq \mathrm{Re}\, q < 1$, $q \neq 0$ that

$$x_+^{-p} . x_-^{-1-q} - x_+^{-1-q} . x_-^{-p} = 0.$$

From formula (10) with r = 1, m = 1, p = p, q = q it follows that $u_n(y) \to 0$ in $\mathcal{D}'(R^1)$ (theorem 4 a)) and therefore that $u_n^{(1)}(y) \to 0$ in $\mathcal{D}'(R^1)$. This means that

$$x_+^{1-p}.x_-^{-2-q} - x_+^{-2-q}.x_-^{1-p} = 0.$$

Repeating the process we obtain d2).

We now prove d1). We want to show that the following formula is true if r, m, p, q satisfy the indicated conditions:

$$x_+^{r-p}.x_-^{-m-q} + (-1)^{m+(-r)}.x_+^{-m-q}.x_-^{r-p} = (-1)^r.\frac{\Gamma(q).\Gamma(p)}{(m-r)!}.S^{(m-r)}.$$

The formula is true if $r = m$ (case b2) of this theorem) or if $r = 0$ (case b1)). Suppose the formula true for (r_o,m) and (r_o-1,m) with $r_o \geqq 1$, $0 \leqq r_o \leqq m$.

We get by formula (10);

$$(-1)^{r_o}.\frac{\Gamma(q).\Gamma(q)}{(m-r_o)!}.\frac{1}{(m+q)}.S^{(m-r_o+1)} + \frac{(p-r_o)}{(m+q)}.(-1)^{r_o-1}.\frac{\Gamma(q).\Gamma(q)}{(m-r_o+1)}S^{(m-r_o+1)} =$$

$$= \frac{(-1)^{r_o}.\Gamma(q).\Gamma(p)}{(m+1-r_o)!}.S^{(m+1-r_o)} =$$

$$= \lim_{n\to\infty}([x_+^{r_o-p}.x_-^{-(m+1)-q} + (-1)^{(m+1)-r_o}.x_+^{-(m+1)-q}.x_-^{r_o-p}] * S_n)(y).$$

Therefore the formula holds for $(r_o,m+1)$, and d1) follows. QED.

6. NONEXISTENCE. In this section we deal with the problem of nonexistence of a product of the type $x_+^\lambda.x_-^\mu$ but for which there exists a balanced product, ie. $x_+^\lambda.x_-^\mu \pm x_+^\mu.x_-^\lambda$ (cf. §4).

LEMMA 3. *Let* $\phi \in C_o^\infty(R^1)$, *even,* supp. $\phi \subseteq [-1/2,1/2]$, *then*

a) if $m + r \geqq 1$

$$\int(\int_t^1 \phi^{(m)}(s)\,ds).\phi^{(r)}(t)\,dt = (-1)^m.(\phi * \phi)^{(m+r-1)}(0).$$

b) if $m + r$ *is even and* $m + r \geqq 2$

$$\int(\int_t^1 s.\phi^{(m)}(s)\,ds).\phi^{(r)}(t)\,dt = (-1)^m.\frac{m-r+1}{2}.(\phi * \phi)^{(m+r-2)}(0).$$

PROOF. a) is immediate.

b) Suppose $m \geqq 1$. Integrating by parts and using a) we get

(1)
$$\int(\int_t^1 s.\phi^{(m)}(s)\,ds).\phi^{(r)}(t)\,dt =$$

$$= \int(-t.\phi^{(m-1)}(t) - \int_t^1 \phi^{(m-1)}(s)\,ds).\phi^{(r)}(t)\,dt =$$

$$= -\int t.\phi^{(m-1)}(t).\phi^{(r)}(t)\,dt - \int\left(\int_t^1 \phi^{(m-1)}(s)\,ds\right).\phi^{(r)}(t)\,dt =$$

$$= -\int t.\phi^{(m-1)}(t).\phi^{(r)}(t)\,dt - (-1)^{m-1}.(\phi * \phi)^{(m+r-2)}(0).$$

We prove next an auxiliary formula. Let α and β be nonnegative integer numbers. Suppose $\alpha + \beta$ odd and $\alpha \geq \beta$. Then

$$0 = \int\{[t.\phi^{(\alpha-1)}(t).\phi^{(\beta)}(t)]^{(1)} - [t.\phi^{(\alpha-2)}(t).\phi^{(\beta+1)}(t)]^{(1)} + + (-1)^{\frac{\alpha-1-\beta}{2}}.[t.\phi^{(\frac{\alpha+\beta-1}{2})}(t).\phi^{(\frac{\alpha+\beta-1}{2})}(t)]^{(1)}\}\,dt =$$

$$= \int\{[\phi^{(\alpha-1)}(t).\phi^{(\beta)}(t) + t.\phi^{(\alpha)}(t).\phi^{(\beta)}(t) + t.\phi^{(\alpha-1)}(t).\phi^{(\beta+1)}(t)] -$$

$$- [\phi^{(\alpha-2)}(t).\phi^{(\beta+1)}(t) + t.\phi^{(\alpha-1)}(t).\phi^{(\beta+1)}(t) + t.\phi^{(\alpha-2)}(t).\phi^{(\beta+2)}(t)] +$$

$$+ (-1)^{\frac{\alpha-1-\beta}{2}}.[\phi^{(\frac{\alpha+\beta-1}{2})}(t).\phi^{(\frac{\alpha+\beta-1}{2})}(t) + t.\phi^{(\frac{\alpha+\beta+1}{2})}(t).\phi^{(\frac{\alpha+\beta-1}{2})}(t).2]\}\,dt =$$

$$= \int\{\phi^{(\alpha-1)}(t).\phi^{(\beta)}(t) + t.\phi^{(\alpha)}(t).\phi^{(\beta)}(t) - \phi^{(\alpha-2)}(t).\phi^{(\beta+1)}(t) +$$

$$+ (-1)^{\frac{\alpha-1-\beta}{2}}.\phi^{(\frac{\alpha+\beta-1}{2})}(t).\phi^{(\frac{\alpha+\beta-1}{2})}(t) + (-1)^{\frac{\alpha-1-\beta}{2}}.t.\phi^{(\frac{\alpha+\beta-1}{2})}(t).\phi^{(\frac{\alpha+\beta+1}{2})}(t)\}\,dt =$$

Due to the fact that $\phi^{(\frac{\alpha+\beta+1}{2})}(t).\phi^{(\frac{\alpha+\beta-1}{2})}(t) = \dfrac{([\phi^{(\frac{\alpha+\beta-1}{2})}(t)]^2)^{(1)}}{2}$ and

$$\int \phi^{(\alpha-1)}(t).\phi^{(\beta)}(t)\,dt = -\int\phi^{(\alpha-2)}(t).\phi^{(\beta+1)}(t)\,dt = \ldots =$$

$$= (-1)^{\frac{\alpha-1-\beta}{2}}.\int \phi^{(\frac{\alpha+\beta-1}{2})}(t).\phi^{(\frac{\alpha+\beta-1}{2})}(t)\,dt$$

it follows that

$$0 = \int t.\phi^{(\alpha)}(t).\phi^{(\beta)}(t)\,dt + \frac{(\alpha - \beta + 1)}{2}.\int \phi^{(\alpha-1)}(t).\phi^{(\beta)}(t)\,dt +$$

$$+ (-1)^{\frac{\alpha-\beta-1}{2}}.\int t.\frac{([\phi^{(\frac{\alpha+\beta-1}{2})}(t)]^2)^{(1)}}{2}\,dt.$$

Finally, integrating by parts this last integral we get:

(2)
$$\int t.\phi^{(\alpha)}(t).\phi^{(\beta)}(t)\,dt = \frac{(\beta - \alpha)}{2}.\int\phi^{(\alpha-1)}(t).\phi^{(\beta)}(t)\,dt =$$

$$= \frac{(\beta - \alpha)}{2}.(-1)^{\alpha-1}.(\phi * \phi)^{(\beta+\alpha-1)}(0).$$

We arrived to the following identity (see (1)):

(3)
$$\int\left(\int_t^1 s.\phi^{(m)}(s)\,ds\right).\phi^{(r)}(t)\,dt =$$

$$= -\int t.\phi^{(m-1)}(t).\phi^{(r)}(t)\ dt - (-1)^{m-1}.(\phi * \phi)^{(m+r-2)}(0) = (\text{by } (2)) =$$

$$= (-1)^{m}.\ \frac{m - r + 1}{2}\ .(\phi * \phi)^{(m+r-2)}(0).$$

This is formula b) for $m \geq 1$. It is easy to verify that this formula is true also if $m = 0$. QED.

THEOREM 5.

a) a1) $x_+^{-r-p}.x_-^{-m-q}$ *does not exist if* $r,m = 0, 1, 2, \ldots, 0 \leq \text{Re } p < 1, p \neq 0,$

$0 \leq \text{Re } q < 1, q \neq 0, 1 - p - q = 0, m + r \geq 1, (r \neq m \text{ or } (r = m, p \neq q)).$

a2) $x_+^{r-p}.x_-^{-m-q}$ *does not exist if* $r,m = 0, 1, 2, \ldots, 0 \leq \text{Re } p < 1, p \neq 0,$

$0 \leq \text{Re } q < 1, q \neq 0, 1 - p - q = 0, m - r \geq 1.$

b) $x_+^{r-p}.x_-^{-r-1-q}$ *does not exist if* r *is an integer,* $0 \leq \text{Re } p < 1, p \neq 0,$

$0 \leq \text{Re } q < 1, q \neq 0, -2 < \text{Re } (-p-q-1) \leq -1, p + q \neq 0.$

c) $x_+^{\lambda}.x_-^{-k}$ *does not exist if* $k = 2, 4, 6, \ldots, \text{Re } \lambda > -1, -2 < \text{Re } (\lambda-k) \leq -1.$

d) $x_+^{-k}.x_-^{k-1}$ *does not exist if* $k = 0, 1, 2, \ldots$.

PROOF. Case a1). We recall that if $\{S_n\}_{n=1,2,\ldots}$ is a sequence as in theorem 1 then

$$(S_n * S_n)^{(\gamma)}(0) = \beta_n^{\gamma+1}.(\rho * \rho)^{(\gamma)}(0) \tag{4}$$

for $\gamma = 0, 1, 2, \ldots$.

Let $r + m$ be odd, $1 - p - q = 0$, then by formula d1) of theorem 2, formula a) of lemma 3 and (4) we obtain:

$$\lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{-r-p}.x_-^{-m-q}] * S_n)(y)\ dy}{\beta_n^{m+r}} = -\frac{\Gamma^2(p).\Gamma^2(q)}{\Gamma(p+r).\Gamma(q+m)}\ .(\rho * \rho)^{(r+m-1)}(0). \tag{5}$$

Let $r + m$ be even, $r + m \geq 2$, $1 - p - q = 0$, then by formula d1) of theorem 2, formula b) of lemma 3 and (4) we get:

$$\lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{-r-p}.x_-^{-m-q}] * S_n)(y).y\ \ dy}{\beta_n^{m+r-1}} = \tag{6}$$

$$= \frac{\Gamma^2(p).\Gamma^2(q).(m - r + q - p)}{2.\Gamma(p+r).\Gamma(q+m)}\ .(\rho * \rho)^{(r+m-2)}(0).$$

From formulae (5) and (6), a1) follows.

We consider now case a2). The following is an identity if $0 \leq \text{Re } p < 0, p \neq 0,$ $0 \leq \text{Re } q < 0, q \neq 0, r,m = 0, 1, 2, \ldots$:

(7) $$\frac{(p-r)}{(m+q)} \cdot \int ([x_+^{r-1-p}.x_-^{-m-q}] * s_n)(y)\, dy = \int ([x_+^{r-p}.x_-^{-m-1-q}] * s_n)(y)\, dy$$

which is obtained integrating the derivative of $([x_+^{r-p}.x_-^{-m-q}] * s_n)(y)$. We want to prove that the following formula holds if $r + m$ is odd, $m - r \geq 1$, $1 - p - q = 0$, $0 \leq \mathrm{Re}\, p < 1$, $p \neq 0$, $0 \leq \mathrm{Re}\, q < 1$, $q \neq 0$, $m,r = 0, 1, 2, \ldots$:

(8) $$\lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{r-p}.x_-^{-m-q}] * s_n)(y)\, dy}{\beta_n^{m-r}} =$$

$$= - \frac{\Gamma^2(q).\Gamma^2(p)}{\Gamma(q+m).\Gamma(p-r)} .(\rho * \rho)^{(m-r-1)}(0).$$

If $r = 0$ then this formula is a particular case of formula (5). Using (7) repeatedly we see that it is valid in any case.
Finally, if $0 \leq \mathrm{Re}\, p < 1$, $p \neq 0$, $0 \leq \mathrm{Re}\, q < 1$, $q \neq 0$, $r,m = 0, 1, 2, \ldots$ then from the following equality:

$$\int ([x_+^{r-p}.x_-^{-m+1-q}] * s_n)^{(1)}(y).y\, dy = -\int ([x_+^{r-p}.x_-^{-m+1-q}] * s_n)(y)\, dy$$

we deduce

(9) $$\int ([x_+^{r-p}.x_-^{-m-q}] * s_n)(y).y\, dy =$$

$$= \frac{(-1)}{(q+m-1)} .\left[\int ([x_+^{r-p}.x_-^{-m+1-q}] * s_n)(y)\, dy + (r-p).\int ([x_+^{r-1-p}.x_-^{-m+1-q}] * s_n)(y).y\, dy\right].$$

Next we prove that the following formula (10) is valid if $r + m$ is even, $1 - p - q = 0$, $m - r \geq 2$, $0 \leq \mathrm{Re}\, p < 1$, $p \neq 0$, $0 \leq \mathrm{Re}\, q < 1$, $q \neq 0$, $r,m = 0,1,2,\ldots$:

(10) $$\lim_{n \to \infty} \frac{\int_{-\infty}^{+\infty} ([x_+^{r-p}.x_-^{-m-q}] * s_n)(y).y\, dy}{\beta_n^{m-r-1}} =$$

$$= \frac{(m + r + q - p).\Gamma^2(p).\Gamma^2(q)}{2.\Gamma(p-r).\Gamma(q+m)} .(\rho * \rho)^{(m-r-2)}(0).$$

The formula holds for $r = 0$ since it is a particular case of (6). The general case follows by induction using (9) and (8), and a2) is proved.
Let us consider case b).
By formula d1) of theorem 2 we have for $-2 < \mathrm{Re}\,(-p-q-1) \leq -1$, $p + q \neq 0$ that

$$\int_{-\infty}^{+\infty} ([x_+^{-p}.x_-^{-1-q}] * s_n)(y)\, dy =$$

$$= \frac{1}{q}\cdot\int_{-1}^{1}\left(\int_{t}^{1}|s - t|^{1-p-q}.K^{0}_{0,p,q}(s,t).S_n^{(1)}(s)\ ds\right).S_n(t)\ dt =$$

$$= \frac{(p + q - 1).B(1-p,1-q)}{q}\cdot\int_{-1}^{1}\left(\int_{t}^{1}|s - t|^{-p-q}.S_n(s)\ ds\right).S_n(t)\ dt =$$

$$= \frac{(p + q - 1).B(1-p,1-q)}{2.q}\cdot\iint |s - t|^{-p-q}.S_n(s).S_n(t)\ ds\ dt.$$

Now by theorem 1, a), we get:

$$(11)\qquad \lim_{n\to\infty}\frac{\int_{-\infty}^{+\infty}([x_+^{-p}.x_-^{-1-q}] * S_n)(y)\ dy}{\beta_n^{p+q}} =$$

$$= \frac{(p + q - 1).B(1-p,1-q)}{2.q}\cdot\int |u|^{-p-q}.(\rho * \rho)(u)\ du.$$

By lemma 1 we can choose ρ such that

$$\int |u|^{-p-q}.(\rho * \rho)(u)\ du \neq 0$$

and therefore

$$x_+^{-p}.x_-^{-1-q}$$

does not exist if $-2 < \mathrm{Re}\ (-1-p-q) \leq -1$, $p + q \neq 0$, $0 \leq \mathrm{Re}\ p < 1$, $p \neq 0$, $0 \leq \mathrm{Re}\ q < 1$, $q \neq 0$.

Assume that r is an integer. For p,q as above we get

$$x_+^{r-p}.x_-^{-r-q} = 0.$$

Differentiating we obtain

$$(r - p).x_+^{r-1-p}.x_-^{-r-p} + (r + p).x_+^{r-p}.x_-^{-r-1-q} = 0.$$

Therefore if one product does not exist then the other one does not exist either.

Using the fact that

$$x_+^{-p}.x_-^{-1-q}$$

does not exist we conclude that

$$x_+^{r-p}.x_-^{-r-1-q}$$

cannot exist.

To prove c) we need some auxiliary formulae.

Let $\mathrm{Re}\ \lambda > -1$, $k = 2, 4, 6, \ldots$. Then

$$\int_{-\infty}^{+\infty}([x_+^{\lambda}.S^{(k-1)}] * S_n)(y)\ dy = \int_{-\infty}^{+\infty}\left(\int_{0}^{+\infty} x^{\lambda}.S_n(t-x)\ dx\right).S_n^{(k-1)}(t)\ dt =$$

$$= -\int_{-\infty}^{+\infty}\left(\int_{0}^{+\infty} x^{\lambda}.s_n^{(k-1)}(t-x)\ dx\right).s_n(t)\ dt = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{t} |t - x|^{\lambda}.s_n(x)\ dx\right).s_n^{(k-1)}(t)\ dt =$$

$$= -\int_{-\infty}^{+\infty}\left(\int_{t}^{+\infty} |t - x|^{\lambda}.s_n(x)\ dx\right).s_n^{(k-1)}(t)\ dt$$

and

(12) $$\int_{-\infty}^{+\infty}([x_+^{\lambda}.s^{(k-1)}] * s_n)(y)\ dy =$$

$$= \frac{1}{2}.\iint |t - x|^{\lambda-1}.(t - x).s_n(x).s_n^{(k-1)}(t)\ dx\ dt.$$

Also from formula (9) of §4 if $k = 2, 4, 6, \ldots$, $\mathrm{Re}\ \lambda > -1$, $-2 < \mathrm{Re}\ (\lambda-k) \leq -1$ and $\{s_n\}_{n=1,2,\ldots}$ is a sequence as in theorem 1 we get

(13) $$\lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty}([x_+^{\lambda-1}.x_-^{-(k-1)} + (-1)^{(k-1)+1}.x_+^{-(k-1)}.x_-^{\lambda-1}] * s_n)(y)\ dy}{\beta_n^{k-1-\lambda}.\log \beta_n} =$$

$$= \lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty}([x_+^{\lambda-1}.x_-^{-(k-1)}] * s_n)(y)\ dy}{\beta_n^{k-1-\lambda}.\log \beta_n}.2 =$$

$$= \frac{1}{\lambda.(k-2)!}.\int |u|^{\lambda-1}.u.(\rho * \rho)^{(k-1)}(u)\ du.$$

Now for the values of λ, k pointed out above it follows from theorem 4 a) that

$$x_+^{\lambda}.x_-^{-k+1} = 0$$

and differentiating

$$\lambda.x_+^{\lambda-1}.x_-^{-k+1} + (k-1).x_+^{\lambda}.x_-^{-k} - \frac{1}{(k-1)!}.x_+^{\lambda}.s^{(k-1)} = 0.$$

Therefore, using (12) and (13) and theorem 1 a) we get

$$\lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty}([x_+^{\lambda}.x_-^{-k}] * s_n)(y)\ dy}{\beta_n^{k-1-\lambda}.\log \beta_n} = -\frac{1}{(k-1)!.2}.\int |u|^{\lambda-1}.u.(\rho * \rho)^{(k-1)}(u)\ du$$

and by lemma 1 case c) follows.

Case d). From theorem 3 b) we know that

$$x_+^{-k}.x_-^{k-1}$$

does not exist if k is odd $k \geq 1$.

We want to show that the same product does not exist if k is even, $k \geq 2$. (If $k = 0$ it follows from theorem 3 b) for $\lambda = 0$, $k = 1$).

From theorem 4 a) we know that

$$x_+^{-k}.x_-^k = 0$$

if k = 1,2, ... Differentiating this it follows that

$$(-k).x_+^{-k-1}.x_-^k + \frac{(-1)^k}{k!}.S^{(k)}.x_-^k + (-k),x_+^{-k}.x_-^{k-1} = 0$$

Let us prove that $S^{(k)}.x_-^k$ exists for k = 1,2 , ... It will follow then that $x_+^{-k}.x_-^{k-1}$ does not exist for k even, k ≧ 2. Now

$$([S.x_-^0] * S_n)(y) = S_n(y).\int_y^{+\infty} S_n(u)\, du = -\frac{1}{2}.[(\int_{+\infty}^y S_n(u)\, du)^2]^{(1)}$$

Hence $S.x_-^0$ exists. By Fisher's theorem 0 it follows that

$$S^{(k-1)}.x_-^k = 0$$

for k = 1, 2, ... Differentiating this we get

$$S^{(k)}.x_-^k - k.S^{(k-1)}.x_-^{k-1} = 0$$

and applying this formula we obtain the desired result. QED.

7. COMPILATION OF SOME OF THE RESULTS OBTAINED IN THEOREMS 3, 4 AND 5.

THEOREM 6.

a)
$$x_+^{-r-1/2}.x_-^{-r-1/2} = \frac{(-1)^r.\pi}{(2r)!.2}.S^{(2r)}$$

if r = 1, 2, ...

b)
$$x_+^{r-p}.x_-^{p-r-1} = \frac{(-1)^r.\pi}{\text{sen } \pi p.2}.S$$

if r *is an integer*, 0 ≦ Re p < 1, p ≠ 0.

c) $x_+^\lambda.x_-^\mu = 0$ *if* Re (λ+μ) > -1, *with* λ,μ ε C.

d) The product $x_+^\lambda.x_-^\mu$ *does not exist for other values of* λ,μ *as those appearing in a), b) or c).*

However it holds that

e)
$$x_+^{-r-p}.x_-^{-m-q} + (-1)^{m+r}.x_+^{-m-q}.x_-^{-r-p} = \frac{(-1)^r.\pi}{\text{sen } \pi p.(m+r)!}.S^{(m+r)}$$

if m,r *are integers*, 0 ≦ Re p < 1, p ≠ 0, 0 ≦ Re q < 1, q ≠ 0, 1-p-q = 0, r+m ≧ 0.

f)
$$x_+^\lambda.x_-^{-k} + (-1)^{k+1}.x_+^{-k}.x_-^\lambda = 0$$

if k = 2, 4, 6, ..., Re (λ-k) > -2, Re λ > -1, λ ε C.

g) $$x_+^{r-p}.x_-^{-q-r-1} - x_+^{-q-r-1}.x_-^{r-p} = 0$$

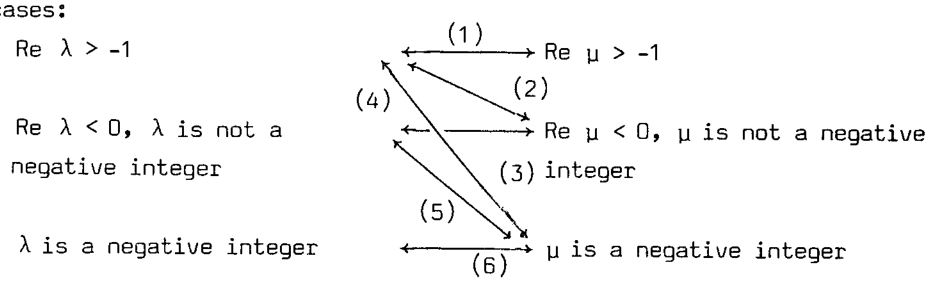
if r *is an integer,* $0 \leqq \text{Re } p < 1$, $p \neq 0$, $0 \leqq \text{Re } q < 1$, $q \neq 0$, $-1 < \text{Re } (-p-q)$.

PROOF. Cases b) and c) follow from cased b2) and a) of theorem 4.

Case a) follows from case b1) of theorem 4.

From Th. 3 and 5 follows that the product $x_+^{\lambda}.x_-^{\mu}$ does not exist if λ, μ are not as in a), b) or c).

In fact, to study the product $x_+^{\lambda}.x_-^{\mu}$, $\lambda, \mu \in C$, $\text{Re } (\lambda+\mu) < -1$ we consider different subcases:



Notice that by symmetry we need to consider only these six subcases.

SUBCASE 1. From the fact that $\text{Re } (\lambda+\mu) < -1$ we deduce that

$$-1 < \text{Re } \lambda < 0,$$
$$-1 < \text{Re } \mu < 0,$$

and therefore we can set $\lambda = -p$, $\mu = -q$ and use theorem 3 d1).

SUBCASE 2. Since $\text{Re } \mu < 0$ and μ is not a negative integer we set $\mu = -m - q$ with $m = 0, 1, 2, \ldots$, $0 \leqq \text{Re } q < 1$, $q \neq 0$.

Re $\lambda > -1$ →

- $\lambda = 0, 1, 2, \ldots$. Set $\lambda = r$. The condition $\text{Re } (\lambda+\mu) < -1$ is equivalent to $\text{Re } (r-m-q) < -1$. Therefore hte product does not exist by theorem 3 c).
- $\lambda \neq 0, 1, 2, \ldots$. Set $\lambda = r - p$ ($r = 0, 1, 2, \ldots$, $0 \leqq \text{Re } p < 1$, $p \neq 0$); then the condition $\text{Re } (\lambda+\mu) < -1$ is equivalent to $\text{Re } (r-p-m-q) < -1$.
  - $1 - p - q \neq 0$
    - $m + r$ is even. The product does not exist by theorem 3 d2).
    - $m + r$ is odd.
      - $\text{Re } (r-p-m-q) \leqq -2$. The nonexistence follows from theorem 3 d2).
      - $-2 < \text{Re } (r-p-m-q) < -1$. Then r-m must be equal to -1. Then the product is of the form $x_+^{r-p}.x_-^{-r-1-q}$ with $-2 < \text{Re } (-p-q-1) < -1$ which does not exist by theorem 5 b).
  - $1 - p - q = 0$. Then $m - r \geqq 1$ and we use theorem 5 a2) to prove the nonexistence of the product.

SUBCASE 3. Set $\mu = -k$. Then the condition Re $(\lambda+\mu) < -1$ reduces to

Re $(\lambda-k) < -1$

- k is odd. The product does not exist by Th. 3 b).
- k is even.
  - Re $(\lambda-k) \leqq -2$. Use theorem 3 b) to prove the nonexistence.
  - $-2 <$ Re $(\lambda-k) < -1$. Use theorem 5 c) to prove the nonexistence.

SUBCASE 4. Let us write $\mu = -m-q$ ($m = 0, 1, 2, \ldots$, $0 \leqq$ Re $q < 1$, $q \neq 0$), $\lambda = -r-p$ ($r = 0, 1, 2, \ldots$, $0 \leqq$ Re $p < 1$, $p \neq 0$). Then Re$(\lambda+\mu) < -1$ reduces to

Re $(-m-q-r-p) < -1$.

- $1 - p - q \neq 0$
  - $r + m$ is even. Use theorem 3 d1) to prove the nonexistence.
  - $r + m$ is odd
    - Re $(-m-q-r-p) \leqq -2$. The product does not exist by theorem 3 d1)
    - $-2 <$ Re $(-m-q-r-p) < -1$. Then $-r-m = -1$. Use theorem 5 b) for the nonexistence.
- $1 - p - q = 0$ (see below)

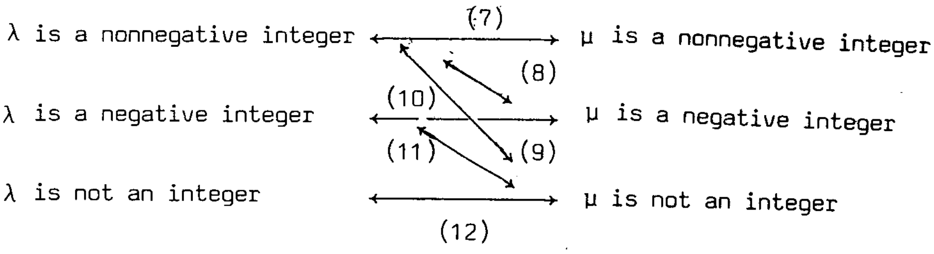
$1 - p - q = 0$

- $r \neq m$. it does not exist. Use theorem 5 a1).
- $r = m$. (notice that $r \geqq 1$).
  - $p \neq q$. Use theorem 5 a1) to prove the nonexistence.
  - $p = q$. The product exists and is the case a) of this theorem.

SUBCASE 5. Set $\mu = -k$, $\lambda = -r - p$ ($r = 0, 1, 2, \ldots$, $0 \leqq$ Re $p < 1$, $p \neq 0$). Then the product does not exist by theorem 3 c).

SUBCASE 6. Use theorem 3 a1), a2) (nonexistence).

Next we study $x_+^\lambda . x_-^\mu$ with Re $(\lambda+\mu) = -1$. Here again we divide in subcases the problem.



SUBCASES 7 & 10. These cases are excluded since Re $(\lambda+\mu) \neq -1$ for both.
SUBCASE 8. The product does not exist by theorem 5 d).
SUBCASE 9. Set $\lambda = r$ ($r = 0, 1, 2, \ldots$), $\mu = -m - q$ (m an integer, $0 \leqq \text{Re } q < 1$, $q \neq 0$). Then Re $(\lambda+\mu) = -1$ can be written as Re $(r-m-q) = -1$ and then Re $q = 0$ (ie. q is purely imaginary), $r - m = -1$. We can rewrite the product as

$$x_+^r . x_-^{-r-1-q}$$

and then we can apply theorem 3 e) to prove the nonexistence.
SUBCASE 12. Set $\lambda = r - p$ (r integer, $0 \leqq \text{Re } p < 1$, $p \neq 0$), $\mu = m - q$ (m an integer, $0 \leqq \text{Re } q < 1$, $q \neq 0$). Then Re $(\lambda+\mu) = -1$ can be written as Re $(m-q+r-p) = -1$. Observe that Re $(q+p)$ must be an integer. We have

Re $(m-q+r-p) = -1$

→ Re $(p+q) = 0$. Then $m + r = -1$ and we can write

$$x_+^{r-p} . x_-^{-r-1-q}$$

- $p + q \neq 0$. The product does not exist by theorem 5 b).
- $p + q = 0$. The product exists by theorem 4 b2). See b) in the statement of the present theorem.

→ Re $(p+q) = 1$. (*) Then the product can be written as

$$x_+^{r-p} . x_-^{-r-q}$$

(*)

- $p + q = 1$. The product exists by theorem 4 b2). See b) of the present theorem.
- $p + q \neq 1$. Then the product does not exist by theorem 3 d2).

SUBCASE 11. Set $\lambda = -k$ ($k = 1, 2, \ldots$); Then the condition Re $(\lambda+\mu) = -1$ can be written as Re $(\mu-k) = -1$. Therefore Re $\mu > -1$.
The product is $x_+^{-k} . x_-^{\mu}$.

$$\mathrm{Re}\ (\mu - k) = -1$$

k an odd integer.
The product does not exist by theorem 3 b).

k an even integer.
Use theorem 5 c) to prove the nonexistence.

In case e) of the present theorem 6 we put together case b1) and d1) of theorem 4.
Case f) coincides with case c1) of theorem 4.
Finally, case g) is the case d2) of theorem 4 for r an integer. QED.

APPENDIX.

DEFINITION OF x_+^λ, x_-^μ. If λ is a complex number such that Re $\lambda > -1$ then the following function defined in R:

$$x_+^\lambda = \begin{cases} x^\lambda = e^{\lambda \log x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is locally integrable and defines a distribution. Here log x is real for $x > 0$. One obtains easily the following relations:

i) $$x . x_+^\lambda = x_+^{\lambda+1} \quad \text{if Re } \lambda > -1.$$

ii) $$\frac{d}{dx} x_+^\lambda = \lambda\, x_+^{\lambda-1} \quad \text{if Re } \lambda > 0.$$

We wish to extend the definitions of x_+^λ to all $\lambda \varepsilon C$ preserving properties i), ii).

Some exceptions must occur because if one takes $\lambda = 0$ in ii) then

$$\frac{d}{dx} x_+^0 = S \neq 0.$$

Let $\phi \varepsilon C_0^\infty(R)$. We define the function

$$\lambda \to I_\lambda(\phi) = \langle x_+^\lambda, \phi \rangle = \int_0^{+\infty} x^\lambda . \phi(x)\, dx,$$

which is holomorphic in Re $\lambda > -1$.

Property ii) means that

$$I_\lambda(\phi') = -\lambda\, I_{\lambda-1}(\phi) \quad \text{if Re } \lambda > 0, \text{ for all } \phi \varepsilon C_0^\infty(R).$$

Then one can write for Re $\lambda > -1$, k a positive integer,

(1-Ap) $$I_\lambda(\phi) = \frac{(-1)^k\, I_{\lambda+k}(\phi^{(k)})}{(\lambda + 1). \ldots .(\lambda + k)} .$$

The right hand side of this equality is holomorphic in Re $\lambda > -k-1$ except for the simple poles at -1, -2, ..., -k. In this way we can define $I_\lambda(\phi)$ by analytic continuation if λ is not a negative integer or, what is the same, by formula (1-Ap).

We define then for $\lambda \neq$ negative integer

$$\langle x_+^\lambda, \phi \rangle := I_\lambda(\phi).$$

In $\lambda = -k$ the residue of the function $\lambda \to I_\lambda(\phi)$ is

$$\lim_{\lambda \to -k} (\lambda + k)\, I_\lambda(\phi) = \frac{(-1)^k\, I_0(\phi^{(k)})}{(1 - k). \ldots .(-1)} = \frac{\phi^{(k-1)}(0)}{(k - 1)!} .$$

Therefore

$$(\lambda + k)\, x_+^{\lambda} \to \frac{(-1)^{k-1}\, \delta^{(k-1)}}{(k-1)!} \quad \text{if } \lambda \to -k.$$

Subtracting the singular part we obtain, if $\lambda + k = \varepsilon \to 0$,

$$I_\lambda(\phi) - \frac{\phi^{(k-1)}(0)}{(k-1)!.\varepsilon} =$$

$$= (-1)^k.\int_0^{+\infty} \frac{(x^{\varepsilon}-1).\phi^{(k)}(x)}{(\varepsilon+1-k).\ \dots\ .\varepsilon}\, dx + \frac{\phi^{(k-1)}(0)}{\varepsilon}.\left(\frac{1}{(k-1-\varepsilon).\ \dots\ .(1-\varepsilon)} - \frac{1}{(k-1)!}\right) \to$$

$$\to -\frac{1}{(k-1)!}.\int_0^{+\infty} \log x.\phi^{(k)}(x)\, dx + \frac{\phi^{(k-1)}(0)}{(k-1)!}.\left(\sum_{j=1}^{k-1} 1/j\right).$$

Then we can define x_+^{-k} by

$$<x_+^{-k},\phi> = -\frac{1}{(k-1)!}.\int_0^{+\infty} \log x.\phi^{(k)}(x)\, dx + \frac{\phi^{(k-1)}(0)}{(k-1)!}.\left(\sum_{j=1}^{k-1} 1/j\right).$$

x_+^{λ} has the following properties:

i') $$x.x_+^{\lambda} = x_+^{\lambda+1} \quad \text{if } \lambda \neq -1, -2, \dots ,$$

ii') $$\frac{d}{dx}\, x_+^{\lambda} = \lambda\, x_+^{\lambda-1} \quad \text{if } \lambda \neq 0, -1. -2, \dots ,$$

and by a straightforward computation we obtain

iii) $$\frac{d}{dx}\, x_+^{-k} = -k\, x_+^{-k-1} + (-1)^k\, \frac{\delta^{(k)}}{k!} \quad \text{if } k = 0, 1, 2, \dots .$$

We define, as usual x_-^{λ} by

$$< x_-^{\lambda},\phi> = <x_+^{\lambda},\check{\phi}>$$

for all $\lambda \in C$, all $\phi \in C_0^{\infty}(R)$, where $\check{\phi}(x) := \phi(-x)$.

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