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REMARKS ON A THEOREM OF Å. PLEIJEL AND RELATED TOPICS, II,

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RESUMEN. El problema de Neumann en una región de Jordan (acotada) plana D de contorno J suficientemente regular admite una sucesión de autovalores y autofunciones, $w_h \in C^2(D) \cap C(\overline{D})$, $\frac{\partial w_h}{\partial n_i} = 0$ en J, $-\Delta w_h = \lambda_h w_h$ en D, $h \geq 0$, tales que si J es C^2 vale en $\operatorname{Re} z > 1$

(*) $$\sum_1^{\infty} \frac{1}{\lambda_h^z} = \frac{áreaD}{4\pi} \frac{1}{z-1} + \frac{longJ}{8\pi} \frac{1}{z-1/2} + g(z)$$, $g(z)$ holomorfa en el semiplano derecho.

Pleijel demuestra esta fórmula para curvas C^{∞}. Sea $N(\lambda) = \#\{\lambda_j : \lambda_j \leq \lambda\}$. (*) puede demostrarse sin utilizar la siguiente fórmula asintótica de H. Weyl que es consecuencia de la precedente,

(**) $$N(\lambda) - \frac{áreaD}{4\pi}\lambda = o(\lambda).$$

Los planteos clásico y variacional dan lugar a los mismos autovalores con las mismas autofunciones. Estas son también autofunciones del operador de Green y tienen otras propiedades además de las indicadas. La exposición es independiente del vol. I y autocontenida.

Palabras clave: problema de Neumann, autofunciones, autovalores, serie de Dirichlet espectral.

ABSTRACT. We consider Neumann's problem for the Laplacian in a plane Jordan (bounded) region D with regular boundary J. If $\{\lambda_j : j = 0,1,2,\cdots\}$ is the set of eigenvalues of that problem, the counting function $N(\lambda) = \#\{\lambda_j : \lambda_j \leq \lambda\}$ satisfies H. Weyl asymptotic formula: $N(\lambda) = \frac{areaD}{4\pi}\lambda + o(\lambda)$.

Related to the monotonous function $N(\lambda)$ is the spectral Dirichlet series: $P(z) = \sum_1^{\infty} \lambda_n^{-z} = \int_{0+}^{\infty} x^{-z} dN(x)$, $\operatorname{Re} z > 1$. The behaviour of $P(z)$ follows from that of $N(\lambda)$ but results about $P(z)$ can be obtained without *a priori* knowledge of the counting function. A theorem of Å. Pleijel deals with this type of results. He proves that if D has a C^{∞}-boundary the following formula holds:

$$\sum_{1}^{\infty} \frac{1}{\lambda_n^z} = \frac{areaD}{4\pi} \frac{1}{z-1} + \frac{lengthJ}{8\pi} \frac{1}{z-1/2} + g(z),$$ $g(z)$ holomorphic at least in $\operatorname{Re} z > 0$.

In this paper we collect some results on eigenvalues, eigenfunctions and Green's kernel that hold for plane regular membranes. We show that the preceding equality holds for a Jordan region with a C^2-boundary and present a simplified proof for this case without using Weyl's asymptotic formula. In fact, it is a consequence of Pleijel's formula. Besides we show that the variational eigenvalues and eigenfunctions and the classical ones coincide for general C^2 Jordan regions proving then that in our framework we can use the results obtained by the powerful variational method.

This work is essentially of expository nature and mainly self-contained. No use is made of volume I. Its central core is in Chapters 5-9 that can be read almost independently of the first four chapters.

Key words: Neumann problem, eigenvalue, eigenfunction, spectral Dirichlet series.

CONTENTS

DEDICATORIA

Deseamos recordar aquí a ***Alberto González Domínguez*** *(1904-1982),* ***Mischa Cotlar*** *(1913-2007) y* ***Yanny Frenkel****. El primer autor recibió del Dr. González Domínguez un generoso apoyo cuando estudiante y en los inicios de su carrera. El segundo autor, quien fue discípulo del Dr. Cotlar, considera a la Dra. Frenkel su madrina científica.*

AGRADECIMIENTOS

A la Dra. María Inés Platzeck por la publicación del presente volumen. Al CONSEJO NACIONAL DE INVESTIGACIONES CIENTÍFICAS Y TÉCNICAS por la constante ayuda que desde 1960 ha brindado al segundo autor.

Bahía Blanca, Febrero de 2007.

CHAPTER 1

(1.1) INTRODUCTION. We study the so called Neumann problem for the Laplacian in bounded open plane regions. We assume that the problem is posed in one of two forms: the classical way and the variational one. The variational problem can be posed for more general regions and if nothing is said about a result, *it will mean that it has been obtained under the variational setting.*

One objective that is pursued in the study of Neumann problem is to know more and more precisely the set of eigenvalues. When this set is discrete an interesting problem is to determine its asymptotic behaviour. That is, to determine, if it exists, the asymptotic behaviour for $\lambda \to \infty$ of *the counting function* $N(\lambda) = \#\{0 \le \lambda_j : \lambda_j \le \lambda\}$=number of non negative eigenvalues not greater than λ counted according to their multiplicities. An actual problem is to know better the second term of the asymptotic expression of $N(\lambda)$ when the first one is determined or at least to estimate the difference between $N(\lambda)$ and the main term.

Sometimes to solve some problems related with the eigenvalues of a Neumann problem it is not necessary to know $N(\lambda)$, it is enough to know the behaviour of some related functions like $\int_0^\infty e^{-tx} dN(t)$ or $\int_0^\infty e^{-tx} dN(e^t) = \int_1^\infty v^{-x} dN(v)$.

(1.2) DEFINITION. a) A *plane region A* is a bounded, finitely connected, open set.

b) A is called ε*-semiregular*, $\varepsilon \in (0,1]$, if for $A(t) := \{z \in A : dist(z, \partial A) \le t\}$ it holds that $|A(t)| = O(t^\varepsilon)$. Here $|B| = m(B)$ =Lebesgue exterior plane measure of B. If A is ε-semiregular then $m(\partial A) = 0$. If $J = \partial A$ and A is a Jordan region we shall suppose J oriented in the positive sense and defined by continuous functions $x(s)$, $y(s)$, $s \in [0, S]$, $S > 0$. If J is rectifiable, $\langle J \rangle$ will denote the length of J, *lengthJ*, and s will represent the arc length parameter. In this case we have $S = \langle J \rangle$. Moreover, $A(t) \approx t$ and A is 1-semiregular. Note that there are Jordan regions that are semiregular for no $\varepsilon \in (0,1]$.

For a very general family of plane regions, H. Weyl's theorem holds, (regions with property C', cf. Note **1**. See also Note **6**). That is,

$$\textbf{(1.3)} \qquad \lambda_n \sim \frac{4\pi}{|A|} n, \quad \lambda \sim \frac{4\pi}{|A|} N(\lambda), \qquad n \to \infty,\ 0 < \lambda \to \infty.$$

If A satisfies an additional property, it is known that

$$\textbf{(1.4)} \qquad N(\lambda) = \alpha\lambda + O(\sqrt{\lambda}\log\lambda), \qquad \alpha = |A|(4\pi)^{-1}.$$

It is conjectured that for ∂A sufficiently regular one has

(1.5) $$N(\lambda)=\alpha\lambda+\beta\sqrt{\lambda}+o(\sqrt{\lambda}),\quad \beta=\langle J\rangle(4\pi)^{-1}.$$

Weyl's result **(1.3)** does not hold for Neumann problem on a general plane region A.

(1.6) THE CLASSICAL NEUMANN PROBLEM. In this boundary problem one is asked to find the eigenvalues and the corresponding eigenfunctions of the operator $-\Delta$ in a C^m plane region A, $m>0$, with a boundary $J=\partial A$, under the conditions:

(1.7) $-\Delta u(x)=\lambda u(x)$, $x\in A$; $\dfrac{\partial u}{\partial n_i}(y)=0$, n_i=interior normal at y, $y\in J$,

where it is required that the function $u\in C(\overline{A})\cap C^2(A)$.

If u satisfies the differential equation **(1.7)** in the sense of distributions then there is exactly one function in its equivalence class such that $u\in C^\infty$. It satisfies the equation in the usual sense. For regions with irregular boundary the classical problem has no sense and has to be posed in a weak form. However, for C^2 Jordan regions both provide the same sets of eigenvalues and eigenfunctions as we shall see in Ch. 8. What we understand by a C^2 Jordan region is the content of Note **3**. The definition there can be extended to C^m plane regions where $m=1,2,\cdots$.

(1.8) Any strongly Lipschitz region A (see definition in Note **2**) satisfies the following property **W)** that we call *Whitney condition*:

W) There is a constant c such that for any two points of A, x_1,x_2, there exists a rectifiable arc $p\subset A$ whose end points are x_1,x_2 and such that $\langle p\rangle\le c|x_1-x_2|$.

A theorem of H. Whitney asserts that if A satisfies **W)** and u, $v=\partial u/\partial x_1, w=\partial u/\partial x_2\in C(\overline{A})$ then they can be simultaneously extended to a region $A'\supset\overline{A}$ in such a way that their extensions U, V, W satisfy $V=\partial U/\partial x_1, W=\partial U/\partial x_2$.

In the case of **(1.7)**, if we had $\partial u/\partial x_k\in C(\overline{A})$ then we would have that U verifies $-\Delta U=\lambda U$ on A and, on J, $\dfrac{\partial U}{\partial n_e}=-\dfrac{\partial U}{\partial n_i}=-\dfrac{\partial u}{\partial n_i}=0$, n_e = exterior normal.

(1.9) The preceding argument suggests that it is convenient that the involved functions in Neumann problem can be extended outside A. Thus, naturally arises the idea of requiring to the region the extension property **S)** for Sobolev spaces, (cf. [La]). We introduce below this property **S)**.

A theorem due to P. Jones ([J]) asserts that a simply connected plane region A has property **S)** if and only if it is a uniform domain, (cf. Note **4**).

(1.10) DEFINITION. A plane region D is a quasidisc if it is simply connected and uniform, ([J]; [Le], Ch. 1; cf. also Note **8**).

Thus, a simply connected plane region D has property **S)** *if and only if* it is a quasidisc.

On the other hand, D is uniform if and only if it satisfies **W)** and some other additional property that we shall call **T)**, (cf. Note **4**).

For a quasidisc D the following property **E)** holds:

E) There exists a linear extension operator $E: W := W^{1,2}(D) \to \widetilde{W} := W^{1,2}(R^2)$ such that if $\theta \in W$ then $E(\theta)|_D = \theta$, $\|E(\theta)\|_{\widetilde{W}} \le K\|\theta\|_W$, K a constant, (cf. [J]).

We shall write $H^1(D)$ instead of $W^{1,2}(D)$, (cf. [A]). We have seen that on $H^1(D)$ it holds that $\|E(\theta)\|_{H^1(R^2)} =: \|\theta\| \approx \|\theta\|_{H^1(D)}$. Let us remember that the subspace of $H^1(D)$, $C_0^\infty(R^2)|_D (\supset C_0^\infty(D))$, is dense in $H^1(D)$.

(1.11) DEFINITION. The plane region D has the *extension property* **S)** if there exist a plane region $\widetilde{D} \supset \overline{D}$ and a continuous linear operator $\widetilde{E}: H^1(D) \to H^1(\widetilde{D})$ such that $\forall \theta \in H^1(D)$, $\widetilde{E}(\theta)|_D = \theta$ a.e.

Observe that a simply connected plane region D satisfies property **E)** if and only if it satisfies property **S)**. We are especially interested in these regions as domains for posing the Neumann problem.

(1.12) DEFINITION. By a *simple bounded open set* U we shall understand a finite union of plane regions $U = \bigcup_{h=1}^{q} U_h$ such that $h \neq k \Rightarrow \overline{U}_h \cap \overline{U}_k = \varnothing$ and $\forall h, U_h$ has property **E)**.

Thus, for this set U it holds the extension property for Sobolev spaces **S)**. Besides, $H^1(U)$ is compactly embedded ($\subset\subset$) in $L^2(U)$. In fact, we know that $H_0^1(B) \subset\subset L^2(B)$ on balls B. Since the domains U_j have property **E)**, it follows that $H^1(U_j) \subset\subset L^2(U_j)$. In consequence, on simple bounded open sets it holds *Rellich-Kondrachov theorem*: the embedding of $H^1(U)$ into $L^2(U)$ is completely continuous. Many results on Neumann problem hold even for simple bounded open sets.

(1.13) We supose *from now on* that *the Neumann problem is posed* on simple bounded open sets, in particular, on (finite) quasidiscs. A *variational* (or *weak*) *eigenfunction* is defined as follows:

DEFINITION. Let U be a simple bounded open set and $u \in H^1(U)$, real. u is an *eigenfunction* for the Laplace operator if there is a (real) number λ such that $\forall \phi \in H^1(U)$: $\int_U \nabla u \times \nabla \phi = \lambda \int_U u\phi$.

Thus, $u \in H^1(U)$ is a weak solution of the equation $-\Delta u = \lambda u$ according to the following definition.

(1.14) DEFINITION. Let U be a simple bounded open set and $f \in L^2(U)$. A function $u \in H^1(U)$ is called a *weak solution* of $(-\Delta+\lambda)u = f$ whenever the equality $\int_U \nabla u \times \nabla v\, dx + \lambda \int_U uv\, dx = \int_U fv\, dx$ holds for any $v \in H^1(U)$, (cf. [E]).

NB. This definition of weak solution is more restrictive than the definition of *weak solution in the distribution sense*, where $\int_U \nabla u \times \nabla v\, dx + \lambda \int_U uv\, dx = \int_U fv\, dx$ must hold only for $v \in C_0^\infty(U)$. Definition **(1.14)** implies that u satisfies an additional "boundary condition", (cf. **(8.8)** and **(8.9)**). This is why theorem **(1.16)** can assure uniquenes of the weak solution.

(1.15) DEFINITION. $B := B_\gamma(u,v) = I(u,v) + \gamma\langle u,v\rangle$ where $u,v \in H^1(U)$, $\gamma > 0$,

$I(u,v) := \int_U \nabla u \times \nabla v dx$ and $\langle u,v\rangle = \int_U uvdx$.

For positive constants $M = \sup(1,\gamma)$ and $\beta = \inf(1,\gamma)$ we have

$$M\|u\|_{H^1}^2 \geq B(u,u) = \int |\nabla u|^2 dx + \gamma\|u\|_2^2 \geq \beta\|u\|_{H^1}^2 .$$

Then, B is a bilinear functional, continuous and coercive on the Hilbert space $H^1(U)$.

(1.16) THEOREM (existence and uniqueness of a weak solution). Let $\gamma > 0$. For any $f \in L^2(U)$ there is one and only one weak solution of $(-\Delta+\gamma)u = f$.

PROOF. Because of Lax-Milgram theorem (cf. Note 5) there is a unique $u_f \in H^1(U)$ such that $\forall v \in H^1(U)$ one has $B(u_f,v) = I(u_f,v) + \gamma\langle u_f,v\rangle = \langle f,v\rangle$.

That is, $\int_U \nabla u_f \times \nabla v dx + \gamma \int_U u_f v dx = \int_U fv dx$, (cf. [E]), QED.

(1.17) COROLLARY. 1) $L_\gamma^{-1} : f \in L^2(U) \to u_f \in H^1(U)$ is a linear bounded operator,

2) $L_\gamma^{-1} : f \in L^2(U) \to u_f \in L^2(U)$ is a compact operator, i.e., L_γ^{-1}, as an operator from L^2 into L^2, is completely continuous, (cf. Note 7).

PROOF. 1) If $f_n \to f, u_{f_n} \to u$ then $B(u_{f_n},v) \to B(u,v)$ and $\langle f_n,v\rangle \to \langle f,v\rangle$. Thus, $u = u_f = L_\gamma^{-1} f$.

In consequence, $\|u_f\|_{H^1} = \|L_\gamma^{-1} f\|_{H^1} \leq K\|f\|_2$.

2) Because of $H^1(U) \subset\subset L^2(U)$ the operator L_γ^{-1} is completely continuous from $L^2(U)$ into $L^2(U)$, QED.

If in **(1.14)** $\gamma = 0$ and $f = 0$ then $\int_U \nabla u \times \nabla v = 0$, $\forall v \in H^1(U)$. This must hold in particular for any $v \in C_0^\infty(U)$. Then, $-\Delta u = 0$ in D'(*U*) and u is a harmonic function with $\|\nabla u\|_2 = 0$. Therefore, u is constant on each (connected) component of *U*.

(1.18) PROPOSITION (regularity). If u is a weak solution of $(-\Delta + \mu)u = f$, μ a constant and $f \in C^\infty(U)$ then $u \in C^\infty(U)$.

PROOF. It holds that $\int_U \nabla u \times \nabla v \, dx + \mu \int_U uv \, dx = \int_U f \, v dx$ for every $v \in H^1(U)$. Thus, in the sense of distributions, $\langle (-\Delta + \mu)u - f, \varphi \rangle = 0$ for every $\varphi \in C_0^\infty(U)$. Because of the hypoellipticity of the operator we obtain $u \in C^\infty(U)$, QED.

(1.19) PROPOSITION. Let $u \in H^1(U)$ and ρ be a real number such that $\rho u - \Delta u = f \in L^2(U)$. Then, $u \in H^2_{loc}(U)$.

PROOF. It is sufficient to prove the proposition for $\rho > 0$. Let $F = f$ on *U*, *F*=0 on $R^2 \setminus U$. The equation $\rho w - \Delta w = F \in L^2(R^2)$ has a solution whose Fourier transform is $\hat{w}(\xi) = \dfrac{\hat{F}}{\rho + |\xi|^2}$. Then, $w \in H^2(R^2)$ and $v = u - w$ on *U* is such that $\rho v - \Delta v = 0$.

Therefore, $v = u - w \in C^\infty(U)$. In consequence, $u \in H^2_{loc}(U)$, QED.

(1.20) THEOREM (the alternative). Either the equation $(-\Delta + \lambda)u = f$, $u \in H^1(U)$, $\lambda \le 0$, has a (unique) weak solution for any $f \in L^2$ or else there exists a non trivial weak solution of the homogeneous equation $(-\Delta + \lambda)u = 0$, $u \in H^1(U)$. The corresponding null space N_λ has finite dimension. The non homogeneous equation has a weak solution for $f \in L^2$ if and only if $f \perp N_\lambda$ in L^2.

PROOF. We know that $\forall g \in L^2(U)$ there exists $u = L_\gamma^{-1} g \in H^1(U)$ which is the only weak solution of $(-\Delta + \gamma)u = g$. Now, $u \in H^1(U)$ is a solution of the equation $(-\Delta + \gamma)u = (\gamma - \lambda)u + f$, if and only if

(1.21) $$u = L_\gamma^{-1}[(\gamma - \lambda)u + f] = (\gamma - \lambda)L_\gamma^{-1}u + L_\gamma^{-1}f = Ku + L_\gamma^{-1}f = Ku + h.$$

That is, if and only if $(I - K)u = h$ where $h := L_\gamma^{-1} f \in H^1$. Since in this last equation the operator $K = (\gamma - \lambda)L_\gamma^{-1}$ is completely continuous from $L^2(U)$ into $L^2(U)$, *Fredholm's alternative holds:* either there is a (unique) solution of $(I - K)u = h$ for any $h \in L^2$ or else there is a (maximal) subspace $N \subset L^2$ of positive finite dimension such that $(I-K)N=0$ and $(I-K)u=h$ will have a solution if and only if $h \perp N$. That is, if N is non trivial then $(-\Delta+\lambda)u=f, \quad u \in H^1(U)$, will have a solution if and only if $L_\gamma^{-1} f \perp N$ in L^2.

Let us see that $N=N_\lambda$. In fact, $v \in N$ if and only if $v=L_\gamma^{-1}(\gamma-\lambda)v$. Thus, if and only if $(-\Delta+\gamma)\dfrac{v}{\gamma-\lambda}=v$, which is equivalent to $(-\Delta+\lambda)v=0$, that is, it is equivalent to $v \in N_\lambda$.

Finally we show that $f \perp N_\lambda$ if and only if $L_\gamma^{-1} f \perp N$. We know that if $v \in N$ then $v \in N_\lambda$; this means that $\int \nabla v \times \nabla k dx + \lambda \int v k dx = 0$ for any $k \in H^1$.

Then, $\int \nabla v \times \nabla L_\gamma^{-1} f \, dx + \lambda \int v L_\gamma^{-1} f \, dx = 0$. But, $\int \nabla v \times \nabla L_\gamma^{-1} f \, dx + \gamma \int v L_\gamma^{-1} f \, dx = \int v f \, dx$. In consequence, $\forall v \in N_\lambda$,

(1.22) $$(\gamma-\lambda) \int v L_\gamma^{-1} f \, dx = \int v f \, dx .$$

Therefore, $L_\gamma^{-1} f \perp N$ if and only if $f \perp N_\lambda$, QED.

CHAPTER 2

(2.1) SIMPLE BOUNDED OPEN SETS. EIGENVALUES. We continue the study of the variational Neumann problem on *simple bounded open sets* in R^2, (cf. **(1.3)**).

THEOREM (characterization of the eigenvalues). A) There is a function $w_1 \in H^1(U)$, $\|w_1\|_2 = 1$, such that for

(2.2) $$\lambda_1 := \inf\{I(u,u) : u \in H^1, \|u\|_2 = 1\},$$

(2.3) $$I(w_1, v) = \lambda_1 \langle w_1, v \rangle \text{ for any } v \in H^1(U).$$

That is, w_1 is a nonzero weak solution of $(-\Delta - \lambda_1)w_1 = 0$ such that $I(w_1, w_1) = \lambda_1$.

B) There exist $w_2, w_3, \cdots \in H^1(U)$, $\|w_n\|_2 = 1$, such that for $n > 1$,

(2.4) $$\lambda_n := \inf\{I(v,v) : v \in H^1, \|v\|_2 = 1, \langle v, w_i \rangle = 0,\ i = 1,\ldots,n-1\}, \quad 0 \le \lambda_i \le \lambda_n,$$

(2.5) $$I(w_n, v) = \lambda_n \langle w_n, v \rangle \text{ for any } v \in H^1(U).$$

Thus, $w_n \in H^1(U)$ is a weak solution of $(-\Delta - \lambda_n)w_n = 0$ such that: $I(w_n, w_n) = \lambda_n$. Moreover, $w_n \perp [w_1, \cdots w_{n-1}]$.

PROOF. A) Obviously, $\lambda_1 = 0, w_1 =$ constant on open components. However, let us pay attention to the following argument that holds even for $\lambda_1 \ge 0$.

Assume that $\{v_n\} \subset H^1(U), \|v_n\|_2 = 1$, is a sequence such that $I(v_n, v_n) \to \lambda_1$. We have $\|v_n\|_{H^1}^2 = I(v_n, v_n) + \|v_n\|_2^2 \le I(v_n, v_n) + 1 \le K < \infty$, Because of the compact embedding of $H^1(U)$ into $L^2(U)$, it follows that there exists a convergent subsequence in L^2 to a function w_1, $\|w_1\|_2 = 1$. Without loss of generality we can assume that $\{v_n\}$ is this subsequence. We have,

(2.6) $$4\lambda_1 + \varepsilon \ge 2(I(v_n, v_n) + I(v_m, v_m)) = I(v_n + v_m, v_n + v_m) + I(v_n - v_m, v_n - v_m)) \ge$$
$$\ge \lambda_1 \|v_n + v_m\|^2 + \lambda_1 \|v_n - v_m\|^2 \ge 4\lambda_1 \|w_1\|^2 - \varepsilon \ge 4\lambda_1 - \varepsilon.$$

From **(2.6)** we get,

(2.7) $$I(v_n + v_m, v_n + v_m) \to 4\lambda_1, \qquad I(v_n - v_m, v_n - v_m) \to 0.$$

From the second limit we obtain: $v_n \to w_1$ in $H^1(U)$. Next, we show that w_1 is a weak solution. Let $F(t) := \dfrac{I(w_1 + t\varphi, w_1 + t\varphi)}{\|w_1 + t\varphi\|^2} \ge \lambda_1$, $\varphi \in H^1$. F has a minimum at $t = 0$.

Therefore, $F'(0)=0=2(I(w_1,\varphi)-\langle w_1,\varphi\rangle I(w_1,w_1))=2(I(w_1,\varphi)-\langle w_1,\varphi\rangle\lambda_1)$. Then, for every $\varphi\in H^1$, it holds that $I(w_1,\varphi)=\lambda_1\langle w_1,\varphi\rangle$.

We prove B) by induction. Let $n\geq 2$. Assume that we already have $w_1,\cdots,w_{n-1}$. Suppose that the family $\{v_h\}\subset H^1$ verifies $\|v_h\|_2=1$, $v_h\perp w_i$ for $i=1,\cdots,n-1$, ($\perp$ *in* L^2), $I(v_h,v_h)\underset{h\to\infty}{\to}\lambda_n$. The same argument as before shows the existence of a subsequence of $\{v_h\}$, that we denote in the same form, such that $\|v_h-w_n\|_2\to 0$ if $h\to\infty$ where $w_n\perp w_i$ in L^2 whenever $i=1,\cdots,n-1$. It verifies,

(2.8) $$I(v_h+v_m,v_h+v_m)\to 4\lambda_n,\quad I(v_h-v_m,v_h-v_m)\to 0.$$

Since $v_h\to w_n$ in L^2, from the second limit in **(2.8)** we get that $\|v_h-w_n\|_{H^1}\underset{h\to\infty}{\to}0$.

Let $\varphi\in H^1(U)$, $\psi:=\varphi-\langle\varphi,w_1\rangle w_1-\cdots-\langle\varphi,w_{n-1}\rangle w_{n-1}\in H^1$. ψ is orthogonal in L^2 to w_i, $i=1,\cdots,n-1$. Then, the function $F(t):=\dfrac{I(w_n+t\psi,w_n+t\psi)}{\|w_n+t\psi\|^2}\geq\lambda_n$ has a minimum at $t=0$ and therefore $F'(0)=0$. This is equivalent to

(2.9) $$I(w_n,\psi)-\lambda_n\langle w_n,\psi\rangle=0.$$

Because of $w_n\perp w_i$ and the inductive hypothesis, we obtain $I(w_i,w_n)=\lambda_i\langle w_i,w_n\rangle$=0, (cf. **(2.4)**). Applying this in **(2.9)** we arrive to $I(w_n,\varphi)-\lambda_n\langle w_n,\varphi\rangle=0$ for any $\varphi\in H^1(U)$, QED.

(2.10) THEOREM. For $u,w_j,v_k\in H^1\supset M$ and $\perp$ in L^2, it holds that

(2.11) $$\inf_{0\neq u\perp w_1,\dots,w_{n-1}}\frac{I(u,u)}{\|u\|_2^2}=\lambda_n,$$

(2.12) $$\inf_{\dim M=n}\ \sup_{0\neq u\in M.}\frac{I(u,u)}{\|u\|_2^2}=\lambda_n,$$

(2.13) $$\sup_{\{v_1,\dots,v_{n-1}\}lin.indep.}\ \inf_{0\neq u\perp v_1,\dots,v_{n-1}}\frac{I(u,u)}{\|u\|_2^2}=\lambda_n.$$

PROOF. **(2.11)** was already proved in **(2.1)**. Let us prove **(2.12)**. The case n=1 is trivial. Recall that $I(w_k,w_i)=0$ whenever $k\neq i$ and $I(w_k,w_k)=\lambda_k$. Assume that $M=\lfloor w_j:j=1,\dots n\rfloor$, $u=\sum_1^n c_iw_i\in M$. Then,

(2.14) $$\frac{I(u,u)}{\|u\|_2^2}=\frac{\sum c_i^2\lambda_i}{\sum c_i^2}\le\lambda_n$$

and its maximum λ_n is achieved at w_n. If M is an n-dimensional subspace then there exists a nonzero element $w\in M$ such that $w\perp[w_j : j=1,\dots,n-1]$. Threfore, from **(2.11)** it follows that $\infty>\frac{I(w,w)}{\|w\|_2^2}\ge\lambda_n$.

Next we prove **(2.13)**. There exists $u\in[w_1,\dots,w_n]\cap[v_1,\dots,v_{n-1}]^{\perp}$, $u\ne 0$, verifying $u=\sum_1^n c_iw_i$ and $\frac{I(u,u)}{\|u\|_2^2}=\frac{\sum c_i^2\lambda_i}{\sum c_i^2}\le\lambda_n$. In consequence,

$\sup_{\{v_1,\dots,v_{n-1}\}lin.indep.}\ \inf_{0\ne u\perp v_1,\dots,v_{n-1}}\frac{I(u,u)}{\|u\|_2^2}\le\lambda_n$. But, if $v_i=w_i$, $i=1,\dots,n-1$, then, because of **(2.11)** we obtain,

$$\inf_{0\ne u\perp v_1,\dots,v_{n-1}}\frac{I(u,u)}{\|u\|_2^2}=\lambda_n,\qquad \text{QED.}$$

(2.15) DEFINITION. Let *U* be a simple bounded open set, (cf. **(1.12)**). $\sigma=\sigma_U=\{\lambda_j : j=1,\cdots\}$ will denote the family (*spectrum*) of the variational Neumann eigenvalues of $-\Delta$ in *U*, $(-\Delta-\lambda_i)w_i=0$, counted according to their (finite) multiplicities.

$W_U=\{w_i : i=1,2,\cdots\}$ will denote the *normalized orthogonal* family (in L^2) of the corresponding weak eigenfunctions introduced in Theorem **(2.1)**.

(2.16) THEOREM. Let *D, E, U* be simple bounded open sets, $\overline{D}\cap\overline{U}=\varnothing, D\cup U=E$. If $\sigma_D=\{\lambda_j\}$, $\sigma_U=\{\lambda'_j\}$, $\sigma_E=\{\mu_j\}$ then

$$\mu_l\le\min_{j+k=l} max\{\lambda_j,\lambda'_k\}\qquad\text{and}\qquad N_D(\lambda)+N_U(\lambda)\le N_E(\lambda).$$

PROOF. It suffices to prove that $\mu_{j+k}\le\sup\{\lambda_j,\lambda'_k\}$. Let $M=M_1+M_2$, $M_1=[v_1,\dots,v_j]\subset H^1(D)$ and $M_2=[v'_1,\dots,v'_k]\subset H^1(U)$. Let $u=v+v', v\in M_1, v'\in M_2$. Then, $I(u,u)=I(v,v)+I(v',v')$. Because of **(2.12)**,

$$\mu_{j+k}\le\inf_{M=M_1+M_2}\ \sup_{0\ne u\in M}\frac{I(v,v)+I(v',v')}{\|v\|^2+\|v'\|^2}\le$$

$$\le\inf_{M=M_1+M_2}\ \sup_{v\in M_1,v'\in M_2}\frac{\sup_{t\in M_1}(I(t,t)/\|t\|^2)\|v\|^2+\sup_{t'\in M_2}(I(t',t')/\|t'\|^2)\|v'\|^2}{\|v\|^2+\|v'\|^2}.$$

If $M_1 = \lfloor w_1,\dots,w_j \rfloor$, $M_2 = [w_1',\dots,w_k']$ then, (cf. **(2.14)**,

$$\mu_{j+k} \leq \sup_{v\in M_1, v'\in M_2} \frac{\lambda_j \|v\|^2 + \lambda'_k \|v'\|^2}{\|v\|^2 + \|v'\|^2} \leq max\{\lambda_j, \lambda'_k\}.$$

Since $N_E(\lambda) = \#\{\mu_l : 0 \leq \mu_l \leq \lambda\}$ it follows that $N_D(\lambda) + N_U(\lambda) \leq N_E(\lambda)$, QED.

We leave to the reader the proof of the next result, (cf.[M], p. 141).

(2.17) THEOREM. Let *D*, *E*, *U* be simple bounded open sets, $\overline{D} \cap \overline{U} = \varnothing, D \cup U = E$. Let $\sigma_D = \{\lambda_j\}$, $\sigma_U = \{\lambda'_j\}$, $\sigma_E = \{\mu_j\}$. Then, $\mu_j \in \sigma_D$ or $\mu_j \in \sigma_U$ and its multiplicity is equal to the sum of the multiplicities it has as an eigenvalue in the spectra σ_D, σ_U.

If $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_l \leq \lambda < \mu_{l+1} \leq \cdots$ and λ_j is the greatest eigenvalue in σ_D not greater than λ, *i.e.* $\lambda_j \leq \lambda < \lambda_{j+1}$, and λ_k' is the greatest eigenvalue in σ_U not greater than λ, then $\mu_l = \mu_{j+k} = max(\lambda_j, \lambda'_k)$ and $N_D(\lambda) + N_U(\lambda) = N_E(\lambda)$.

(2.18) PLANE REGIONS. EIGENFUNCTIONS. *From now on* we restrict ourselves to plane regiones with the extension property **E)**. That is, we shall deal with connected simple bounded open sets satisfying **E)**. An instance is a quasidisc, (cf. Note **8**). For those regions the eigenspace corresponding to the first eigenvalue is one-dimensional and generated by $I_U(x)$, the characteristic function of *U*. σ is an infinite family of eigenvalues of *finite multiplicity* as it follows from **(1.20)** and next theorem.

(2.19) THEOREM. Let *U* be a plane region $\in$ **E)**. Then, it holds,

1) $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots;\ \lambda_n \to \infty$,

2) $W_U = \{w_i : i = 1,2,\cdots\}$ is a complete orthonormal system in $L^2(U)$.

PROOF. 1) The compact operator $S := L_\gamma^{-1} : L^2 \to L^2$, $\gamma > 0$, is such that $S(L^2) \subset H^1$,

$$B_\gamma(Sf, v) = \int \nabla Sf \times \nabla v dx + \gamma \int (Sf) v dx = \int f v dx.$$

For $v = Sg$ we have, $B_\gamma(Sf, Sg) = \int f Sg dx = B_\gamma(Sg, Sf) = \int g Sf dx$.

Thus, *S* is symmetric, (cf. note 7). On the other hand, w_n verifies $(-\Delta + \gamma) w_n = (\lambda_n + \gamma) w_n$. Therefore, $\frac{w_n}{\lambda_n + \gamma} = S w_n$. Then, $\mu_n = \frac{1}{\lambda_n + \gamma}$ is an eigenvalue of *S* corresponding to the eigenfunction w_n. Since the eigenvalues of *S* have 0 as its only point of accumulation, it follows that $\lambda_n \to \infty$.

2) Let $u \in H^1(U)$, $w := u - \sum_1^N c_n(u) w_n$ with $c_n(u) = \langle w_n, u \rangle$. Then $w \perp w_n$, $n = 1, \ldots, N$. Therefore,

$\frac{I(w,w)}{\|w\|^2} \geq \lambda_{N+1}$ or $\frac{I(w,w)}{\lambda_{N+1}} \geq \|w\|^2 = \left\| u - \sum_1^N c_n w_n \right\|^2$. But, (cf. **(2.4)**, **(2.5)**),

$0 \leq I(w,w) = I(u - \sum c_n w_n, u - \sum c_n w_n) = I(u,u) - \sum_1^N c_n^2 \lambda_n \leq I(u,u)$. In consequence,

$$\left\| u - \sum_1^N c_n w_n \right\|^2 \leq \frac{I(u,u)}{\lambda_{N+1}} \to 0, \qquad \text{QED.}$$

The following theorem is a complement of the preceding one and allows us to introduce a characterization of $H^1(U)$. Provisorily we denote with $W = W(U)$ the space $H^1(U)$ with the norm $|u| := \sqrt{(u,u)} := \sqrt{I(u,u) + \gamma \langle u,u \rangle}$.

(2.20) THEOREM. a) The family $\Gamma = \{\vartheta_n = w_n / \sqrt{\gamma + \lambda_n}\}$, $\gamma > 0$, is a complete orthonormal system in *W*.

b) If $w \in H^1(U)$ then its sequence of Fourier coefficients (in *W*), $\{h_n = (w, \vartheta_n) = I(w, \vartheta_n) + \gamma \langle w, \vartheta_n \rangle\}$, verifies $h_n / \sqrt{\gamma + \lambda_n} = c_n$ where $c_n = \langle w, w_n \rangle$ is the ordinary *n*th-Fourier coefficient of *w* (in $L^2(U)$). It holds that $w \in L^2$, $\sum_1^\infty c_n^2 \lambda_n < \infty$.

c) Conversely, if $u \in L^2(U)$ is such that $\sum_1^\infty c_n^2 \lambda_n < \infty$ then $u \in H^1(U)$.

d) If $n \geq 2$ then $\int_U w_n = 0$.

PROOF. a) Because of **(2.5)**, $(w, w_n) = (\lambda_n + \gamma) \langle w, w_n \rangle$. If $\vartheta_n = (\lambda_n + \gamma)^{-1/2} w_n$, $n = 1, 2, \cdots$ and $w \in W$ then we have $|\vartheta_n| = 1$ and

(2.21) $$(w, \vartheta_n) = (\lambda_n + \gamma)^{1/2} \langle w, w_n \rangle.$$

In consequence, $(\vartheta_m, \vartheta_n) = \left(\frac{\lambda_n + \gamma}{\lambda_m + \gamma} \right)^{1/2} \langle w_m, w_n \rangle$ and this implies that $\Gamma = \{\vartheta_n : n = 1, 2, \cdots\}$ is an orthonormal family in *W*. Besides, if $w \in W$ is orthogonal to Γ then because of **(2.21)** we obtain $\langle w, w_n \rangle = 0$ for every n. From the completeness of $\{w_n : n = 1, 2, \cdots\}$ in $L^2(U)$, we get $w = 0$. That is, $\Gamma = \{\vartheta_n\}$ is complete in *W*.

b) From Bessel's inequality we have $\sum h_n^2 = \sum c_n^2(\gamma + \lambda_n) \le \|w\|_W^2 < \infty$.

c) From the hypothesis it follows that $\sum c_n^2(u)(\gamma + \lambda_n) < \infty$. But then $\sum_1^N h_n \vartheta_n = \sum_1^N c_n(u)\sqrt{\gamma + \lambda_n}\,\vartheta_n$ converges in W.

d) $w_1 = 1/\sqrt{|U|}$, QED.

(2.22) QUASIDISCS. THE COUNTING FUNCTION. Let f be a K-quasiconformal mapping and Q $(\subset\subset R^2)$ the quasidisc $Q = f(B)$, B the open unit ball in R^2. Let $\Sigma := \{z = (x,y) : |z| = 1\}$ and $q = \partial Q = f(\Sigma)$, (cf. Note **8**).

NOTATION. $\dim_H A$ denotes the *Hausdorff dimension* of the set *A*, (cf. [F]). $D(q) = D_i(q)$ denotes the *interior upper Minkowski dimension* of the boundary of Q and M_μ represents the *interior upper Minkowski content* associated to μ, (cf. [L] or note **9**).

(2.23) THEOREM. I) $\dim_H q \in [1,2)$,

II) $D(q) \in [1,2]$,

III) There is a number $\mu = \mu(K) \in (1,2)$, ($\mu(K)$ is defined in the proof), such that $D = D(q) \le \mu$ and $M_\mu(q) = 0$.

PROOF. I) is proved in [GV], (cf. [Le]); a proof of II) for the boundary q of any bounded open set can be seen in [L]; III) is proved in note **10**, QED.

The same results hold if D denotes the *superior exterior Minkowski dimension*, D_e. From [L, Th. 2.1, p. 479] and Theorem **(2.23)**, we obtain,

(2.24) THEOREM. i) The Neumann counting function for a quasidisc Q verifies

(2.25) $$N(\lambda) = A\lambda + O(\lambda^{\mu/2}), \qquad A = |Q|/4\pi.$$

ii) If $D(q)=1$ and $M_1(q) < \infty$ then

(2.26) $$N(\lambda) = A\lambda + O(\lambda^{1/2}\log\lambda).$$

A strongly Lipschitz domain is a quasidisc with a rectifiable boundary that satisfies the hypothesis in ii) **(2.24)**, as it is easy to see. Thus, we have,

(2.27) COROLLARY. For a strongly Lipschitz domain Q it holds that $N(\lambda) = A\lambda + O(\lambda^{1/2}\log\lambda)$.

Thus we have Weyl's asymptotic approximation: $N(\lambda)/\lambda \sim |Q|/4\pi$, (cf. note **11**).

CHAPTER 3

(3.1) DEFINITION. Assume *H* is an infinite dimensional real Hilbert space with scalar product $\langle\cdot,\cdot\rangle$. We say that (V,H,a) is a *triplet* if *V* is a linear dense subset of *H* which is also a Hilbert space with a continuous embedding ($\forall v\in V, \|v\|_H \le C\|v\|_V$) and *a* is a real bilinear form defined on *V*, symmetric, such that *a* is bounded ($|a(x,y)| \le M\|x\|_V\|y\|_V$) with a non negative quadratic form ($\hat{a}(x) := a(x,x) \ge 0$) and coercive, as defined below.

We say that *a* is *strongly coercive* if the quadratic form $\hat{a}$ verifies, for a certain $m > 0$, that $\hat{a}(x) \ge m\|x\|_V^2$.

We say that *a* is c*oercive* if there exists a number $p \ge 0$ such that $(a+p)(x,y) := a(x,y) + p\langle x,y\rangle$ is strongly coercive on *V*, that is, there exists $m > 0$ such that $\widehat{(a+p)}(x) = \hat{a}(x) + p\|x\|_H^2 \ge m\|x\|_V^2$.

If *a* is strongly coercive on *V* we shall say that (V,H,a) is a *strong triplet.*

A triplet is called *variational* if $V \subset\subset H$, that is, if any bounded subset of *V* has a compact closure in *H, i.e.* the embedding is a completely continuous application.

EXAMPLES. For a Jordan region *D*, $(H_0^1(D), L^2(D), (\cdot,\cdot)_{H_0^1})$ is a strong variational triplet.

If *U* is a plane region with the extension property then $(H^1(U), L^2(U), (\cdot,\cdot)_{H^1})$ is also a strong variational triplet. However, in general, $(H^1(U), L^2(U), (\cdot,\cdot)_{H^1})$ is only a strong triplet, (cf. note **16**). $(H^1(U), L^2(U), \langle\cdot,\cdot\rangle)$ is not a triplet. However, if $I(f,g) := \int_U \nabla f . \nabla g$ then $(H^1(U), L^2(U), I(\cdot,\cdot))$ is a triplet.

(3.2) DEFINITION. Assume that (V,H,a) is a triplet and let

$$E_\lambda = E_\lambda(V,H,a) = \{subspaces\ E\ of\ V : a-\lambda\ is\ strongly\ coercive\ on\ E\}.$$

We define $N(\lambda,V,H,a) := \inf_{E\in E_\lambda} cod_V E$.

(3.3) Let us *denote* by $S_a := \{u\in V : \hat{a}(u) \le 1\}$. By the non negativity of *a*, the scalar product $a(x,y)$ induces a seminorm on *V* which satisfies Schwarz inequality, (cf. note **15**),

$|a(x,y)| \le \sqrt{a(x,x)}\sqrt{a(y,y)}$. Because of this inequality and the boundedness of *a* we conclude that the set $L := \{w : \hat{a}(w) = 0\}$, $L \subset S_a$, is a subspace of *V*. Let *l* be the dimension of *L*. Thus, $l = 0$ if *a* is strongly coercive, that is, for (V,H,a) a strong triplet.

Assume that (V,H,a) is a variational triplet. On L we have, $p\|u\|_H^2 \geq m\|u\|_V^2$. Therefore, $\{u : u \in L, \|u\|_H \leq 1\}$ is a precompact subset of *H*. Then, $\dim L < \infty$.

(3.4) LEMMA. Let (V,H,a) be a triplet. If $\lambda \geq 0$ and on the subspace $Y \subset V$ it holds that $(a-\lambda)(u,u) := \hat{a}(u) - \lambda\|u\|_H^2 \geq \varepsilon\|u\|_H^2$, $\varepsilon > 0$, then $(a-\lambda)(u,u) \geq \eta\|u\|_V^2$ for a certain $\eta = \eta(\varepsilon) > 0$. That is, $Y \in E_\lambda$.

PROOF. Because of the coercivity we have $(a-\lambda)(u,u) \geq m\|u\|_V^2 - (\lambda+p)\|u\|_H^2$ for each $u \in V$. From the hypothesis we get $\frac{\lambda+p}{\varepsilon}(a-\lambda)(u,u) \geq (\lambda+p)\|u\|_H^2$ whenever $u \in Y$. Then, on *Y*, $\left[1+\frac{\lambda+p}{\varepsilon}\right](a-\lambda)(u,u) \geq m\|u\|_V^2$, QED.

(3.5) In note **12** we define the *m*-diameter d_m and quote related results.

THEOREM. Assume that (V,H,a) is a triplet and *L* is of finite dimension *l*.

a) If $n = n(\lambda) := N(\lambda,V,H,a)$, $v = v(\lambda) := \#\{m \geq 0 : d_m(S_a,H) \geq 1/\sqrt{\lambda}\}$, then for any $\lambda > 0$, $n(\lambda) \leq v(\lambda)$.

b) If (V,H,a) is a strong triplet then $n(\lambda) = v(\lambda)$.

PROOF. Recall that $S_a := \{u \in V : \hat{a}(u) \leq 1\}$. Suppose that $\infty > n$. Observe that if we had $d_n(S_a,H) < 1/\sqrt{\lambda}$ then we would have $n := \inf_{E \in E_\lambda} cod_V E \geq v$. Let $E \subset V$ be of finite codimension *n* such that on *E*: $(a-\lambda)(u,u) \geq m\|u\|_V^2$, $(m>0)$. Because of $\lambda > 0$, $E \cap L = \{0\}$. Let $G = \{g \in V : \forall u \in E,\ a(g,u) = 0\}$. Because of Schwarz' inequality $G \supset L$, (cf. note **15**). Thus, $V \supset E+G$, $\{0\} = E \cap G$, $n \geq \dim G \geq l$.

b) Assuming a), it sufices to prove that $n \geq v$ for $\infty > n$. Since (V,H,a) is a strong triplet, *G* is an orthogonal complement of *E* with respect to the scalar product $a(\cdot,\cdot)$, (cf. note **15**). Therefore, $\dim G = n$ and $V = E+G$, (here $l=0$). If $u = e+g$ belongs to S_a then, for a certain $p > 0$, we have

$$1 \geq a(u,u) \geq a(e,e) \geq \lambda\|e\|_H^2 + m\|e\|_V^2 \geq (\lambda + m/C)\|e\|_H^2 = (\lambda+p)\|u-g\|_H^2 \geq (\lambda+p)dist^2(u,G).$$

Therefore, $1/\sqrt{\lambda} > \sup_{u\in S_a} dist(u,G) \geq \inf_{T\subset H, \dim T=n} (\sup_{u\in S_a} dist(u,T)) = d_n(S_a,H)$, and b) is proved.

a) Let us see that $n \leq \nu$. Assume that $\nu < \infty$. Observe that if $L/\{0\} \neq \varnothing$ from $S_a \supset L$ it follows that $d_0 = \infty$. Moreover, $d_j = \infty$ whenever $j = 0,\cdots,l-1$. Since $d_n\downarrow$ and $\nu(\lambda) < \infty$, then $\nu > l-1$ and there exists $\lambda' > \lambda$ such that $d_{\nu-1}(S_a,H) \geq 1/\sqrt{\lambda} > 1/\sqrt{\lambda'} > d_\nu(S_a,H)$. Let G be a subspace of H such that $\dim G = \nu$, $\sup_{x\in S_a} dist(x,G) < 1/\sqrt{\lambda'}$. Then $G \supset L$ and $\nu \geq l$. Let P be the projection of H onto G. Given $u \in H$ define $u' = u$ if $u \in L$ and $u' = u/\sqrt{\hat{a}(u)}$ if u does not belong to L (in which case $0 \neq u' \in S_a$). There exists $g \in G$ such that $\|u'-g\| < 1/\sqrt{\lambda'}$. In consequence, if $\hat{a}(u) > 0$ then $\|u - Pu\| < \sqrt{\hat{a}(u)}/\sqrt{\lambda'}$ and if $\hat{a}(u) = 0$ then $\|u - Pu\| = 0$.

Let $E = V \cap G^\perp$, ($G^\perp$ in H). This is a subspace of V since V is continuously embedded in H. Besides, the codimension of E in V is not greater than ν. Therefore, if $0 \neq u \in E$ then $u \notin G$ and $Pu=0$. Therefore, $\hat{a}(u) > 0$ and $\|u\|_H^2 = \|u - Pu\|_H^2 < \hat{a}(u)/\lambda'$.

Then we have, $(a-\lambda)(u,u) = \hat{a}(u) - \lambda\|u\|_H^2 \geq \varepsilon\|u\|_H^2$ for $u \in E$ and $\varepsilon = \lambda' - \lambda > 0$. But then by Lemma 2 there exists $\varepsilon_0 > 0$ such that $(a-\lambda)(u,u) \geq \varepsilon_0\|u\|_V^2$. Thus, $E \in E_\lambda$. Then, $\inf_{E\in E_\lambda} cod_V E \leq \nu$. That is, $n \leq \nu$, QED.

(3.6) When the bilinear functional $a(\cdot,\cdot)$ is not specified, it will be understood that $a(u,v) := (u,v)_V$. In the following lemma $V = H^1(\Omega)$, $H = L^2(\Omega)$, thus, according to our agreement $(H^1,L^2) = (H^1,L^2,(\cdot,\cdot)_{H^1})$ and $\hat{a}(u) = \|u\|_{H^1(\Omega)}^2$. In this case, (H^1,L^2) is a strong triplet. If Ω is a plane region with the extension property then it is a strong variational triplet.

LEMMA. Let Ω be a plane region with property **E**), (cf. note **14**).

a) If λ is great enough then $n_1 = n_1(\lambda) := N(\lambda,H^1(\Omega),L^2(\Omega)) \leq \left(\gamma|\Omega_{1/2\sqrt{\lambda}}|\right)\lambda$, where $\gamma = \gamma(\Omega) < \infty$.

Thus, $\overline{lim}_{0<\lambda\uparrow\infty} \dfrac{N(\lambda,H^1,L^2)}{\lambda} \leq \gamma|\Omega|$.

b) If Ω is a quasidisc then $1/4\pi \leq \gamma < \infty$ and $\overline{lim}_{0<\lambda\uparrow\infty} \dfrac{N(\lambda,H^1,L^2)}{\lambda} \leq \gamma|\Omega|$.

PROOF. a) Let $\nu \in Z^2$, $J_\nu = J_\nu(\delta) = \{x : |x_i - \delta \nu_i| < \delta/2, i = 1,2\}$. Define,

$$A = A(\delta) := \{\nu : \Omega \cap J_\nu \neq \varnothing\},\ \omega := (\bigcup_{\nu \in A} \bar{J}_\nu)^\circ,\ \Omega_{\sqrt{2}\delta} := \{x : dist(x,\Omega) < \sqrt{2}\,\delta\}.$$

Then, $\Omega = \Omega \cap (\bigcup_{\nu \in A} \bar{J}_\nu) = \Omega \cap (\bigcup_{\nu \in A} \bar{J}_\nu)^\circ$ or $\Omega = \Omega \cap \omega$. Thus,

$$\text{(3.7)} \qquad \Omega \subset \omega \subset \Omega_{\sqrt{2}\delta} \text{ and } |\Omega_{\sqrt{2}\delta}| \geq |\omega| \geq \sum_{\nu \in A} |J_\nu| = (\# A)\delta^2 .$$

From **(3.7)** we get, $\# A \leq |\Omega_{\sqrt{2}\delta}| / \delta^2$.

On the other hand assume that $f \in H^1(J_\nu)$ and that $f_\nu := \frac{1}{|J_\nu|}\int_{J_\nu} f(x)dx$. We have,

$\int_{J_\nu} |f - f_\nu|^2 dx = \int_{J_\nu} \left(\frac{1}{|J_\nu|} \int_{J_\nu} (f(x) - f(y))dy \right)^2 dx$. Because of Schwarz inequality and note **13**, the last integral is not greater than $\int_{J_\nu} \frac{dx}{|J_\nu|} \int_{J_\nu} |f(x) - f(y)|^2 dy \leq 8\delta^2 \int_{J_\nu} |\nabla f|^2 dx$. Thus, $\int_{J_\nu} |f - f_\nu|^2 dx \leq 8\delta^2 \|f\|^2_{H^1(J_\nu)}$. If $f \in H^1(\omega)$ and $g(x) = f_\nu$ whenever $x \in J_\nu$ then the following inequality holds,

$$\text{(3.8)} \qquad \int_\omega |f - g|^2 dx \leq 8\delta^2 \|f\|^2_{H^1(\omega)} .$$

Define G as the set of $L^2(\omega)$-functions that are constant on each $J_\nu, \nu \in A$. Thus, $G = \{g\} \subset L^2$ is a finite dimensional linear space such that $\dim G = \# A$.

Assume also that $f \perp G$ ($\perp$ in $L^2(\omega)$). Therefore, $f \in S := G^\perp \cap H^1(\omega)$. S is a subspace of $H^1(\omega)$ and $cod_{H^1} S \leq \# A$. In this case, for any $g \in G$, we obtain

$$\|f\|^2_{L^2(\omega)} \leq \|f - g\|^2_{L^2(\omega)} \leq 8\delta^2 \|f\|^2_{H^1(\omega)} .$$

Given $\lambda > 0$, if δ verifies $9\delta^2 \lambda = 1$ then, for ε such that $1/8\delta^2 = \lambda + \varepsilon$, it holds that,

$$\text{(3.9)} \qquad a(f,f) := \|f\|^2_{H^1(\omega)} \geq (\lambda + \varepsilon)\|f\|^2_{L^2(\omega)} .$$

Since $\varepsilon > 0$, $S \in E_\lambda(H^1(\omega), L^2(\omega), a(\cdot,\cdot))$. In fact, from **(3.9)** and **(3.4)** we obtain,

(3.10) $$(a-\lambda)(f,f)=\|f\|^2_{H^1(\omega)}-\lambda\|f\|^2_{L^2(\omega)}\geq\eta\|f\|^2_{H^1(\omega)},$$

that is, $a-\lambda$ is strongly coercive on S. In consequence,

(3.11) $$\mu:=N(\lambda,H^1(\omega),L^2(\omega))\leq cod_{H^1(\omega)}S\leq\# A\leq\frac{|\Omega_{\sqrt{2}\delta}|}{\delta^2}=9\lambda|\Omega_{\sqrt{2}\delta}|.$$

We have to replace ω by Ω. Assume M is a subspace of $H^1(\omega)$, $cod_{H^1(\omega)}M=\mu\,(<\infty)$, such that for any f in M, **(3.9)** holds. Assume that ω_0 is a bounded open set such that $\Omega\subset\omega\subset\omega_0\supset\overline{\Omega}$. Suppose that E_0 is a linear extension operator, $E_0:H^1(\Omega)\to H^1(\omega_0)$, of norm K_0 and $E:=I_\omega E_0$. If $F:=\{f\in H^1(\Omega):Ef\in M\}$ then $cod_{H^1(\Omega)}F\leq\mu$.

If K is the norm of E then $1\leq K\leq K_0<\infty$. Thus, for $\lambda>\lambda'>0$ and any $f\in F$, we get,

$$(\lambda+\varepsilon)\|f\|^2_{L^2(\Omega)}\leq(\lambda+\varepsilon)\|Ef\|^2_{L^2(\omega)}\leq\|Ef\|^2_{H^1(\omega)}\leq K\|f\|^2_{H^1(\Omega)},\qquad\left(\frac{\lambda}{K}+\varepsilon'\right)\|f\|^2_{L^2(\Omega)}\leq\|f\|^2_{H^1(\Omega)}.$$

Therefore, $n_1(\lambda/K)=N(\lambda/K,H^1(\Omega),L^2(\Omega))\leq cod_{H^1}F\leq\mu\leq 9\lambda|\Omega_{\sqrt{2}\delta}|=9\lambda|\Omega_{\sqrt{2}/3\sqrt{\lambda}}|$.

Then, we arrive to $n_1(\lambda)\leq(9K)\lambda|\Omega_{\sqrt{2}/3\sqrt{K\lambda}}|$. Therefore, there is $\gamma\leq 9K_0=9\|E_0\|$ such that $n_1(\lambda)\leq\gamma\lambda|\Omega_{1/2\sqrt{\lambda}}|$. In fact, $\sqrt{2}/3\sqrt{K\lambda}<1/2\sqrt{\lambda}$ and a) is proved.

b) Let Ω be a quasidisc. We shall estimate $\gamma=\gamma(\Omega)$. Define,

(3.12) $$n^D(\lambda)=N(\lambda,H^1_0(\Omega),L^2(\Omega)),$$

the counting function of the strong variational triplet corresponding to Dirichlet problem in Ω. Since H_0 is a subspace of H^1, the mth-diameter $d_m(S_{a_0},L^2)$ is not greater than the mth-diameter of $N(\lambda,H^1(\Omega),L^2(\Omega))$, $d_m(S_{a^1},L^2)$,. Because of b) theorem **(3.5)**, it follows that $n^D(\lambda)\leq n_1(\lambda)$. But, for a certain s, $0<s=\mu/2<1$, (cf. Th. **(2.23)**), we have

(3.13) $$n^D(\lambda)=\frac{|\Omega|}{4\pi}\lambda+O(\lambda^s)=\frac{|\Omega|}{4\pi}\lambda+o(\lambda),$$

(cf. note **17**, [La], Corollary 1, p. 479). In consequence, $\frac{|\Omega|}{4\pi}+o(1)\leq\gamma(\Omega)|\Omega_{1/2\sqrt{\lambda}}|$.

Letting $\lambda\to\infty$, one obtains $|\Omega|/4\pi\leq\gamma|\overline{\Omega}|=\gamma|\Omega|$. The last equality is due to the fact that $|\partial\Omega|=0$ since $\dim_H\partial\Omega<2$, (cf. Th. **(2.23)**), QED.

CHAPTER 4

(4.1) TRIPLETS AND THE NEUMANN PROBLEM. Let $0=\lambda_1<\lambda_2\leq\cdots\leq\lambda_j\leq\cdots$ be the eigenvalues corresponding to the variational normalized eigenfunctions, w_j, of Neumann problem in a plane region U with property **S)**, (*v.g.*, a quasidisc).

As usual we denote with l^2 the real Hilbert space given by the family $l^2=\left\{x=\sum_{i=1}^{\infty}x_ie_i:\sum_1^{\infty}x_j^2<\infty\right\}$ with scalar product $\langle x,y\rangle_{l^2}\equiv\langle x,y\rangle=\sum_1^{\infty}x_jy_j$, $\|x\|_{l^2}^2\equiv\|x\|^2=\sum_1^{\infty}x_j^2$. *Let us define,*

(4.2) $$W:=\left\{x\in l^2:\sum_1^{\infty}\lambda_ix_i^2<\infty\right\},\quad \langle x,y\rangle_W:=a(x,y)+\langle x,y\rangle,\ \text{where}$$

$$a(x,y):=\sum_1^{\infty}\lambda_ix_iy_i=\sum_2^{\infty}\lambda_ix_iy_i,\qquad \|x\|_W^2=\sum_1^{\infty}(1+\lambda_j)x_j^2.$$

$W(\subset l^2)$ is a Hilbert space dense in l^2.

Due to $\lambda_j\uparrow\infty$, the inclusion mapping of W into l^2 is completely continuous. In fact, if $x\in\left\{\|x\|_W\leq1\right\}$ and $y=\sum_{i=1}^{n}x_ie_i$ $(\in R^n)$ then $1\geq\sum_{i=n+1}^{\infty}\lambda_ix_i^2\geq\lambda_{n+1}\|x-y\|^2$ where $\|y\|\leq\|x\|_W\leq1$. That is, the unit ball of W is contained in a $\sqrt{1/\lambda_{n+1}}$-neighborhood of a compact set of l^2. Therefore, it is compact since it is a closed set of l^2.

(4.3) The bilinear form a is continuous in W, $|a(x,t)|\leq\|x\|_W\|t\|_W$ and verifies $\hat{a}(x)\geq0$ for any $x\in W$. For x such that $x_i=0$ whenever $i>1$ we have $\hat{a}(x)=0$. Thus, a is coercive but not strongly coercive. Then, $\tau:=(W,l^2,a)$ is a *variational* triplet, (cf. **(3.1)** and **(2.20)**).

The set $S_a:=\{x\in W:a(x,x)\leq1\}=\left\{x\in l^2:\sum_2^{\infty}\lambda_ix_i^2\leq1\right\}$ is not bounded in l^2 since it contains the non trivial subspace $L=\{te_1:t\in R\}$. Here, $l=\dim L=1$, (cf. **(3.3)**).

However, the set $S_a'=\{x:x_1=0,\ a(x,x)\leq1\}$ is contained in a bounded set of W and it is a closed set of l^2. Thus, it is also a compact set of l^2.

Let us recall that, $(\lambda\geq0)$,

$$E_\lambda=E_\lambda(W,l^2,a)=\{Y:Y\in\{\textit{subespaces of } W\}\ \textit{such that}\ a-\lambda\ \textit{is strongly coercive on}\ Y\}.$$

To verify that $Y \in E_\lambda$, it will be enough to check that for a certain $\varepsilon > 0$ and $\forall u \in Y$,

$(a-\lambda)(u,u) := a(u,u) - \lambda\|u\|^2 \geq \varepsilon\|u\|^2$, since this implies, as we have seen in Chapter 3, that for a certain $\eta > 0$, $(a-\lambda)(u,u) \geq \eta\|u\|_W^2$ holds.

(4.4) Assume that U is a plane region with property **S)**, $V = H^1(U)$, $H = L^2(U)$, $T = (H^1(U), L^2(U))$, $\|u\|_V = \sqrt{B_1(u,u)} = \sqrt{I(u,u) + \langle u,u \rangle_H}$. $\sqrt{B_1(\cdot,\cdot)}$ is a norm, $\|\cdot\|_{H^1}$, for the Sobolev space $H^1(U) = V$. Then, T is a *strong variational* triplet.

For $w \in L^2(U)$ define $x = x(w) \in l^2$ by $x_i = \int_U w(y) w_i(y) dy$, where the w_i are the weak eigenfunctions defined in **(2.15)**. This establishes an isomorphism between $L^2(U)$ and l^2 in such a way that the triplet $T = (H^1(U), L^2(U))$ becomes *equivalent to* the triplet $t = (W, l^2)$ where $W = \left\{ x \in l^2 : \sum_1^\infty \lambda_i x_i^2 < \infty \right\}$,

$\langle x,y \rangle_W = \sum_1^\infty (\lambda_i x_i y_i + x_i y_i) = a(x,y) + \langle x,y \rangle$. In fact, this is the content of Theorem **(2.20)** for $\gamma = 1$.

Thus, $t = (W, l^2)$ is a *strong variational* triplet. This fact can be proved directly.

The triplet $\tau = (W, l^2, a)$, $a(x,y) = \sum_1^\infty \lambda_i x_i y_i = \sum_2^\infty \lambda_i x_i y_i$ *is equivalent to*

$\mathrm{T} = (H^1(U), L^2(U), I)$ where $I(u,v) = \int_U \nabla u \times \nabla v$. In fact, this follows as before from $(\|u\|_{H^1})^2 = B_1(u,u) = I(u,u) + \langle u,u \rangle_{L^2}$ and Theorem **(2.20)**.

The existence of these equivalences allows us to work with the triplets $t = (W, l^2)$ and $\tau = (W, l^2, a)$ and then to translate the results to the triplets $T = (H^1(U), L^2(U))$ and $\mathrm{T} = (H^1(U), L^2(U), I(\cdot,\cdot))$, respectively.

(4.5) NOTATION. $n(\lambda) = N(\lambda; \tau) := N(\lambda; W, l^2, a) := \inf_{E \in E_\lambda} cod_W E$ and the counting function for $\lambda > 0$, $N(\lambda) := \#\{\lambda_j \leq \lambda\}$.

(4.6) LEMMA. If $\lambda > 0$ then $n(\lambda) = N(\lambda) \leq \nu(\lambda) = \#\{i \geq 0 : d_i(S_a, l^2) \geq 1/\sqrt{\lambda}\}$.

PROOF. Assume that $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \lambda < \lambda_{n+1} \leq \cdots$. Thus, $N(\lambda) = n$. If $u \in E^n$, $E^n = \{u \in W : u_i = 0,\ i = 1,\dots,n\}$, then $(a-\lambda)(u,u) = \sum_{n+1}^\infty (\lambda_i - \lambda) u_i^2 \geq (\lambda_{n+1} - \lambda)\|u\|^2$.

Therefore, $E^n \in E_\lambda$ and $n(\lambda) \le n$. If we had $n(\lambda) < n$ then there would exist $E \in E_\lambda$ such that $co\dim_W E < n$. Define $G = \{u \in W : u_j = 0, j = n+1, \cdots\}$. Then, $n = \dim G > co\dim E$ and there exists a $g \in W$ such that $0 \ne g \in G \cap E$. For such a g it holds that $0 < \varepsilon \|g\|_W^2 \le (a-\lambda)(g,g) = \sum_1^n (\lambda_i - \lambda) g_i^2 \le 0$, a contradiction. Hence, $n(\lambda) = n = N(\lambda)$.

From a) Th. **(3.5)** we obtain, $\lambda > 0 \Rightarrow N(\lambda) = n(\lambda) \le \nu(\lambda)$, QED.

(4.7) LEMMA. If $\lambda \ge \lambda_2$ then $N(\lambda) = \nu(\lambda)$.

PROOF. Define $\widetilde{l}^2 = \left\{ y = (y_2, y_3, \cdots) : \sum_2^\infty y_j^2 < \infty \right\}$, $\widetilde{W} = \left\{ x \in \widetilde{l}^2 : \sum_2^\infty \lambda_i x_i^2 < \infty \right\}$,

$$\widetilde{a}(x,y) := \sum_2^\infty \lambda_i x_i y_i,\ x, y \in \widetilde{l}^2,\ \langle x, y\rangle_{\widetilde{W}} := \sum_2^\infty (\lambda_i x_i y_i + x_i y_i) = \widetilde{a}(x,y) + \langle x, y\rangle_{\widetilde{l}^2},$$

$$S_{\widetilde{a}} := \{x \in \widetilde{W} : \widetilde{a}(x,x) \le 1\} = \left\{ x \in \widetilde{l}^2 : \sum_2^\infty \lambda_i x_i^2 \le 1 \right\}.$$

The triplet $\widetilde{\tau} := (\widetilde{W}, \widetilde{l}^2, \widetilde{a})$ is strong and variational, (cf. **(4.3)**). After applying b) of Th. **(3.5)** to the triplet $\widetilde{\tau}$, we obtain,

(4.8) $$\widetilde{n}(\lambda) = \widetilde{\nu}(\lambda).$$

Next we repeat the argument of Lemma **(4.5)** to get,

(4.9) $$\lambda \ge \lambda_2 \Rightarrow \inf\{cod_{\widetilde{W}} E : E \in E_\lambda(\widetilde{\tau})\} =: \widetilde{n}(\lambda) = \widetilde{N}(\lambda) := \#\{k : k \ge 2, \lambda \ge \lambda_k\}.$$

In fact, assume that

(4.10) $$\lambda_2 \le \cdots \le \lambda_n \le \lambda < \lambda_{n+1} \le \cdots \le \lambda_{n+m} < \lambda_{n+m+1} \cdots.$$

Thus, the counting function $\widetilde{N}(\lambda) = n-1$. If $u \in \widetilde{E}^{n-1} = \{u \in \widetilde{W} : u_i = 0,\ i = 2, \ldots, n\}$ then $(\widetilde{a} - \lambda)(u,u) = \sum_{n+1}^\infty (\lambda_i - \lambda) u_i^2 \ge (\lambda_{n+1} - \lambda)\|u\|^2$. Then, $\widetilde{E}^{n-1} \in E_\lambda(\widetilde{\tau})$ and $0 \le \widetilde{n}(\lambda) \le n-1$. If we had $\widetilde{n}(\lambda) < n-1$ then there would exist $\widetilde{E} \in E_\lambda(\widetilde{\tau})$ such that $co\dim_{\widetilde{W}} \widetilde{E} < n-1$. Define $\widetilde{G} = \{u \in \widetilde{W} : u_j = 0, j = n+1, \cdots\}$. Thus,

$n - 1 = \dim \widetilde{G} > co\dim \widetilde{E}$ and there would exist a $g \in \widetilde{W}$ such that $0 \ne g \in \widetilde{G} \cap \widetilde{E}$.

For this g there is an $\varepsilon > 0$ such that $0 < \varepsilon \|g\|_{\widetilde{W}}^2 \le (\widetilde{a} - \lambda)(g,g) = \sum_2^n (\lambda_i - \lambda) g_i^2 \le 0$, a contradiction. In consequence, $\widetilde{n}(\lambda) = n - 1 = \widetilde{N}(\lambda)$. It follows easily that,

(4.11) $\quad d_0 = \infty, d_1 = \widetilde{d}_0, d_2 = \widetilde{d}_1, \cdots,$

(4.12) $\quad \lambda \ge \lambda_2 \Rightarrow \nu(\lambda) = \widetilde{\nu}(\lambda) + 1$ and $N(\lambda) = \widetilde{N}(\lambda) + 1$.

Because of **(4.8)-(4.12)** we get,

$$\lambda \ge \lambda_2 \Rightarrow N(\lambda) = \widetilde{N}(\lambda) + 1 = \widetilde{n}(\lambda) + 1 = \widetilde{\nu}(\lambda) + 1 = \nu(\lambda), \qquad \text{QED.}$$

After collecting previous results we arrive to,

(4.13) THEOREM. Assume U is a plane region with property **S)**. $\{\lambda_j : j = 1.2,...\}$, $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \cdots$, is its family of Neumann eigenvalues; n, N, ν are the reckoning functions defined in **(4.5)** and **(4.6)** associated to the triplet $\mathrm{T} = (H^1(U), L^2(U), I(\cdot,\cdot))$ (or $\tau = (W, l^2, a)$); $\{d_i : i = 0,1,2,...\}$ is the family of diameters of the unit ball of H^1 evaluated in L^2. Then,

a) for $k \ge 0$, $d_k = 1/\sqrt{\lambda_{k+1}}$,

b) if $\lambda > 0$ then $n(\lambda) = N(\lambda) = \nu(\lambda)$.

PROOF. a) is a consequence of **(4.11)** and note **12**. b) We proved that if $\lambda \ge \lambda_2$ then $n(\lambda) = N(\lambda) = \nu(\lambda)$ and if $\lambda_2 > \lambda > 0$ then $n(\lambda) = N(\lambda) \le \nu(\lambda)$. But, because of a), for these values of λ we have $N(\lambda) = 1 = \nu(\lambda)$, QED.

(4.14) Next we repeat some known arguments on the triplet $t = (W, H), H = l^2$. (Recall that $t \approx T = (H^1, L^2)$). $\langle f, . \rangle_H, f \in H$, defines a continuos linear functional on W, $|\langle f, v \rangle_H| \le \|f\|_H \sqrt{B_1(v,v)}$. Then, there exists a continuous linear mapping, $R : H \to W$, such that $\langle f, . \rangle_H = (Rf, .)_W$ on W, $\|R\| \le 1$.

$Rf = 0$ implies $f \perp_H W$ and hence $f = 0$. Thus, R is *inyective*. From the definition we get, **(4.15)** $R(e_i) = e_i / (\lambda_i + 1)$.

Therefore, the range of R is *dense* in H. Since $W \subset\subset H$, R is a *completely continuous* linear application from H into H. Let us denote $A = R^{-1}$ where $dom(A) = R(H)$. If $u = Rf, v = Rg$ then,

(4.16) $\quad \langle Au, v \rangle_H = \langle f, Rg \rangle = (Rf, Rg)_W = (Rg, Rf)_W = \langle g, Rf \rangle = \langle Av, u \rangle = \langle u, Av \rangle_H$.

It follows that *R* and *A* are symmetric and *A* has a dense domain in *H*. The operator *A* is *selfadjoint*: in fact $A^{*-1} = A^{-1*} = R^* = R$ and because of this, $A^* = R^{-1} = A$. Recall that **(4.17)** $dom(A^*) = \{y \in H : \exists y^* \in H; \forall x \in dom(A), \langle Ax, y\rangle_H = \langle x, y^*\rangle_H\}$, $y^* = A^* y$.

If $g \in dom(A)$, $\langle Ag, v\rangle_H = \langle g, v\rangle_W$ and we have, $\langle Ag, g\rangle_H = \|g\|_W^2 \geq \|g\|_H^2$. *A* is then a *positive definite* operator on its domain.

Since $R = A^{-1}$ is a completely continuous selfadjoint operator from *H* into *H* there exists a *complete and orthonormal system* of eigenfunctions, φ_i, corresponding to *positive eigenvalues*: $R\varphi_i = \mu_i \varphi_i$, $0 < 1/\mu_i \uparrow \infty$, $\varphi_i \in dom(A) \subset W$.

From **(4.15)** we get, $\mu_k = (\lambda_k + 1)^{-1}$ and the spectrum of *R* is $\sigma_R = \{1/(\lambda_i + 1) : j = 0,1,...\}$. Besides we can assume that $\varphi_i = e_i$. *We denote* $\Lambda_i = \lambda_i + 1$. Then,

(4.18) $$Ae_i = \Lambda_i e_i .$$

If $y \in L = [(1.0,0,\cdots)] \subset W$ then $y^* = A^* y = y$, (cf. **(4.17)**, **(4.18)**). In the equivalence between the triplets $t = (W, l^2)$ and $T = (H^1, L^2)$, (cf. **(4.4)**), the subspace *L* is in correspondence with the family of constant functions of $H^1(U)$. Its orthogonal complement in *W* is $\left\{x \in l^2 : x_1 = 0, \sum_2^\infty x_i^2 \lambda_i < \infty\right\}$ which is in correspondence with the family of $H^1(U)$ of functions with zero mean.

(4.19) The domain of the operator *A* is $\Theta = \left\{x \in H = l^2 : \sum_1^\infty x_i^2 \lambda_i^2 < \infty\right\}$, (cf. note **18**). The action of *A* on *t* corresponds to the action of the operator $-\Delta + 1$ on *T*, (cf. **(1.16)**, **(1.20)**, for $\gamma = 1$).

(4.20) The indexes of the eigenvalues in Chapter 9 shall begin with 0 instead of 1. Then, we shall have $\lambda_0 = 0$ and a) of Th. **(4.13)** would read $d_k = 1/\sqrt{\lambda_k}$, $k \geq 0$. Obviously, we shall also have for $\lambda > 0$, $n(\lambda) = v(\lambda) = N(\lambda)$.

CHAPTER 5

(5.1) NORMAL DERIVATIVES. DEFINITION. Let u be a continuous function defined in a plane region D. To define the *interior normal derivative* of u, $\partial u / \partial n_x$, at a point x of the boundary J of the region D we assume that at x there is a tangent versor t_x, that n_x is the interior normal versor at x (that is: n_x orthogonal to t_x and a segment $I = (x,Y]$ of n_x is contained in the region) and that u on I has a continuous extension to $\bar{I} = [x,Y]$. The normal derivative is the limit $\frac{\partial u}{\partial n}(x) = \frac{\partial u}{\partial n_x} := \lim_{y \to x} \frac{u(y)-u(x)}{y-x}$, $y \in I$, whenever it exists and is finite. For $y \in I^{\circ} := (x,Y)$, $\frac{\partial u}{\partial n_x}(y)$ can be defined without the requirement that $u(y)$ has a limit at x. If $\frac{\partial u}{\partial n_x}(y)$ exists for all $y \in I^{\circ}$ and converges to a finite limit d for $y \to x$ then $u(x)$ can be defined in such a way that u becomes continuous on the segment $[x,Y]$ and it has a normal derivative at x equal to d. Therefore, if the continuous function u in D admits a continuous extension to $\overline{D}$ and has continuous derivatives in D, $u'_i = \partial u / \partial x_i$, $i = 1,2$, continuously extendable to $\overline{D}$ then u possesses a normal derivative at any point of the boundary where a tangent exists.

(5.2) THE MAXIMUM PRICIPLE FOR ELLIPTIC DIFFERENTIAL EQUATIONS. Assume that $a_{ik}(x) = a_{ki}(x)$, $a_i(x)$, $a(x)$ and $f(x)$ are continuous functions on $\overline{D}$, D a plane region. Let A be the *operator defined* by

(5.3) $$Au := \sum_{i,k=1}^{2} a_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i}, \; x = (x_1, x_2),$$

where $u \in C^2(D) \cap C(\overline{D})$. We assume that A is *uniformly elliptic*: for any $x \in \overline{D}$,

(5.4) $$\sum_{i,k=1}^{2} a_{ik}(x) h_i \, h_k > 0 \text{ whenever } h = (h_1, h_2) \neq (0,0).$$

For this operator, *Hopf's first lemma* holds, (cf. [H]),

(5.5) THEOREM. Assume that $Au \geq 0$ in D and $x_0 \in D$ is such that $u(x) \leq u(x_0)$ for any $x \in \overline{D}$. Then, $u(x) \equiv u(x_0)$ on $\overline{D}$.

Corollaries of Hopf's Lemma are the following theorems,

(5.6) THEOREM. (Th. of minimum/maximum.) Assume that $a \leq 0$.

If $f \le 0$ on $\overline{D}$ then any non constant solution $u(x)$ of $Au + au = f$ that has a negative minimum on $\overline{D}$ takes it on ∂D and not in *D*.

If $f \ge 0$ on $\overline{D}$ then any non constant solution $u(x)$ of $Au + au = f$ that has a positive maximum on $\overline{D}$ takes it on ∂D and not in *D*.

(5.7) THEOREM. The boundary problem $Au + au = f$ in *D*, $a \le 0$, $u = \phi$ on ∂D, $\phi \in C(\partial D)$, has at most one solution $u(x) \in C(\overline{D}) \cap C^2(D)$.

If the $u_i(x)$, $i = 1, 2$, are solutions such that $u_i = \phi_i$ on ∂D then $\|\phi_1 - \phi_2\|_\infty \ge \|u_1 - u_2\|_\infty$.

(5.8) THEOREM. (Maximum principle). Assume that the coefficients a_{ik} and a_i in *A* do not depend on *x* and that *D* is a plane region. Assume also that $c(x) \in L^1(D)$, $c(x) \le 0$ a.e. and that $f \ge 0$, $f \in L^1(D)$. Assume that $u \in C(\overline{D})$ verifies

$Au + c(x)\, u = f(x)$ in the sense of distributions in *D*.

Then, if $\max_{x \in \overline{D}} u(x) > 0$ then $\max_{x \in \overline{D}} u(x) = \max_{x \in \partial D} u(x)$, (cf. note **30**).

(5.9) DEFINITION. The plane region *D* satisfies the *ball property* if for any $x_0 \in \partial D$ there is an open ball $B(y_0) \subset D$ of radius $R = R(x_0) > 0$ such that $x_0 \in \partial B$.

Let us consider the following differential operator with $a(x) \le 0$, $\tilde{A} := A + a$. The next complement to the theorem **(5.6)** holds.

(5.10) THEOREM. Assume that *D* satisfies the ball property and $u \in C(\overline{D}) \cap C^2(D)$ is not a constant function. Assume that $\tilde{A}u \ge 0, a \le 0$ on *D* and that *u* has a strict positive maximum at $x_0 \in \partial D$: $0 < u(x_0) > u(x), x \in D$. If ν denotes the exterior normal to the ball *B* at x_0 then $\underline{lim} \dfrac{u(x_0) - u(x_0 - \delta\nu)}{\delta} > 0$ for $\delta \downarrow 0$.

(5.11) DEFINITION. Let us assume that $u \in C^1(D)$ and that *D* has the ball property. We shall say that *u* has a *normal derivative extendable to the boundary* if for any $x_0 \in \partial D$ there exists the (finite) limit $\lim_{x \to x_0} \nabla u \times n$ where *n* is the interior normal to the ball $B(y_0)$ at x_0, (cf. **(5.9)**) and $x \in D$ runs along *n* to x_0.

(5.12) Of course, this definition is really useful when for each $x_0 \in \partial D$ all the possible balls *B* have a common *n*, (cf. **(5.1)**). In this case, if $u \in C(\overline{D})$ has a normal derivative extendable to the boundary then

$\lim_{x \to x_0} \nabla u \times n = \frac{\partial u}{\partial n}(x_0)$ and $\nabla u \times n$ will become continuous on a segment $[x_0, Y]$ in the direction of *n*. So, if the function *u* of theorem **(5.10)** has a normal derivative extendable to the boundary then $0 > \frac{\partial u}{\partial n}(x_0)$.

(5.13) A FUNDAMENTAL SOLUTION FOR THE PLANE METAHARMONIC OPERATOR $\Delta_x + \lambda$. Assume that $\lambda = -\chi^2$, χ *positive.* We wish to find the radial solutions of $\Delta u + \lambda u = 0$, $u(p) = \phi(|p|)$. That is, the solutions of

(5.14) $$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\phi}{d\rho}\right) - \chi^2\phi = 0,\ \rho > 0.$$

Let $K(\chi\rho) := \phi(\rho)$. **(5.14)** is equivalent to

(5.15) $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dK}{dr}\right) - K = K'' + \frac{1}{r}K' - K = 0,\ r = \chi\rho > 0.$$

Kelvin's function of orden 0: $K_0(r) := \int_1^\infty \frac{e^{-rt}}{\sqrt{t^2-1}}dt$, $r > 0$, is a solution of **(5.15)**. In fact, $K_0''(r) - K_0(r) = \int_1^\infty \sqrt{t^2-1}\, e^{-rt}dt = \frac{1}{r}\int_1^\infty \frac{t\, e^{-rt}}{\sqrt{t^2-1}}dt = \frac{-K_0'(r)}{r}$. Another solution of **(5.15)** is the modified Bessel function of order 0, $I_0(r) := J_0(ir) = \sum_{n=0}^\infty \frac{r^{2n}}{(2^n.n!)^2}$. Because of $K_0(r) \to \infty$ as $r \to 0+$, K_0 and I_0 are linearly independent. Both solve the equation $(p\phi')' - q\phi = 0$ where $p(r) = q(r) = r$. Thus, their wronskian is

$W = W(I_0, K_0) = \frac{c}{p(r)} = \frac{c}{r} \neq 0$ and $\frac{d(K_0/I_0)}{dr} = \frac{W}{I_0^2(r)} = \frac{c}{rI_0^2(r)}$. Hence,

(5.16) $$\frac{d}{dr}\left(\frac{K_0}{I_0}\right) = \frac{c}{r} + h(r),\quad r \in U \cap R_+,$$

where *h* is a holomorphic function in a neighborhood *U* of the origin.
Denote $G := \mathbf{C}\backslash\{0\}$. After integrating **(5.16)** on *G* we get

(5.17) $$K_0(r) = c\, I_0(r)\log r + P(r),\quad r \in U/\{0\},$$

where *P*(*r*) is a holomorphic function in the neighborhood *U*. Since $K_0(r)$ is a solution of a differential equation with analytic coefficients in *G*, it is continuable along every continuous arc contained in *G*. The same happens to $I_0(r)\log r$. Therefore *P*(*r*) is arbitrarily continuable along any arc in **C**.

Because of the monodromy theorem we conclude that *P*(*r*) is an entire analytic function. Then, **(5.17)** holds in *G*. Next we determine *c*, proving thereby *i*) of the next theorem.

$$c = \lim_{r\to 0} r\,W = \lim_{r\downarrow 0} rI_0(r)K_0'(r) = \lim_{r\downarrow 0} -rI_0(r)\int_1^\infty \frac{t\,e^{-rt}}{\sqrt{t^2-1}}\,dt = \lim_{r\downarrow 0}\int_r^\infty \frac{-t\,e^{-t}}{\sqrt{t^2-r^2}}\,dt =$$

$$= \lim_{r\downarrow 0}\int_0^\infty \frac{-(t+r)e^{-(t+r)}}{\sqrt{t(t+2r)}}\,dt = \text{(using Lebesgue's theorem)} = -\int_0^\infty e^{-t}dt = -1.$$

(5.18) DEFINITION. $E^\chi(x) := -\dfrac{K_0(\chi|x|)}{2\pi}$.

(5.19) THEOREM. *i*) $W(I_0,K_0) = -1/r$ and $K_0(r) = -I_0(r)\log r + P(r),\ r>0$, where *P* is an entire function,

ii) $(\Delta - \chi^2)E^\chi = \delta$.

PROOF. *ii*) $T = (\Delta+\lambda)K_0(\chi|x|)$ is a distribution with support $\{0\}$. From *i*) we obtain $T = -2\pi\delta$ +function locally integrable. Therefore, $(\Delta+\lambda)K_0(\chi|x|) = -2\pi\delta$, QED.

Because of $K_0(\chi|x|) = \int_1^\infty e^{-\chi|x|t}(t^2-1)^{-1/2}dt > 0$, $E^\chi(x)$ is negative and real analytic in $R^2\backslash\{0\}$. Thus, $(\Delta - \chi^2)$ is an *analytic-hypoelliptic* operator. Therefore, for *D* a plane region, we have,

(5.20) COROLLARY. The solutions of $(\Delta-\chi^2)u = 0$ in *D* are real analytic.

Moreover, all the operators $\Delta+\mu$, $\mu\in$ **C**, are analytic-hypoelliptic (cf. [Hö], p.114).

(5.21) *Denote*, as usual, grad *w* = ∇w. One can estimate the quadratic mean of ∇u, for *u* the solution of $(\Delta+\lambda)u = f, -\chi^2 = \lambda < 0$, by the quadratic means of *u* and *f*:

We *denote* with $K \subset\subset K_1$ the fact that *K* is a compact set contained in the interior of the compact set K_1, *i.e.*, $K\subset K_1^\circ \subset K_1$.

PROPOSITION. Let *D* be a plane region, $f\in C(D)$, $u\in C^2(D)$ (real), $(\Delta-\chi^2)u = f$ in *D*. Let *K*, K_1 be compact subsets of *D* such that $K\subset\subset K_1$. Then,

$$\int_K |\nabla u|^2 dx \le C\int_{K_1} u^2 dx + \frac{1}{2\chi^2}\int_{K_1} f^2 dx,\quad C = C(K,K_1).$$

A proof of this proposition can be seen in note **19**.

(5.22) THE χ – HARMONIC (or METAHARMONIC) FUNCTIONS. The functions in the null space of the operator $\Delta-\chi^2$, $\chi>0$, defined on *D*, are the χ-harmonic functions (with domain *D*). As noted above, any distribution in the null space of the operator is a χ-harmonic function. They enjoy many properties in common with the ordinary harmonic functions ($\chi = 0$). Recall that

(5.22') $B_\rho(y)$ *denotes* the interior of the closed disk $S_\rho(y)$ and $\Sigma_\rho(y) = \partial B_\rho(y)$.

(5.23) THEOREM. (Phragmén-Lindelöf's maximum principle). Let w be a χ-harmonic function on D and $F := \{x_0,\ldots,x_n\} \subset J$. If w is bounded and continuous on $\overline{D}\backslash F$ and $w = 0$ on $J\backslash F$ then w=0.

PROOF. We shall give an argument in the particular case $F := \{x_0\}$. Asume that $D^\varepsilon = D \backslash S_\varepsilon(x_0)$ and that M=$\sup_{x\in D} |w(x)|$. The function

(5.24) $$W_\varepsilon(x) := MK_0(\chi|x - x_0|)/K_0(\chi\varepsilon),$$

is χ-harmonic on D. Besides $W_\varepsilon(x) \geq w(x)$ on ∂D^ε and therefore on D^ε, (cf. **(5.6)**). Let $x \in D$. Making $\varepsilon \to 0$ we obtain $0 \geq w(x)$. Applying the same reasoning to $-w$, it follows that $w(x) = 0$, QED.

(5.25) NOTATION. $A^\chi(D)$, $\chi \geq 0$, will *denote* the family of χ-harmonic functions in the region D. We shall simply write $A(D)$ whenever χ=0. If $u \in A^\chi(D)$ (for some $\chi > 0$) we shall also say that u *is metaharmonic.*

THEOREM. (Principle of the removable singularities). Assume that u is χ-harmonic and bounded in $D \backslash \{x_1\}$. Then, u can be extended as a continuous function $v(x)$ on D in such a way that $v \in A^\chi(D)$.

PROOF. Let $F = \{x_1\}$ and assume $S_\rho(x_1) \subset D$. As we shall see in theorem **(5.34')**, there exists a function $v \in A^\chi(B_\rho(x_1))$, continuous on $S_\rho(x_1)$, such that $u(y) = v(y)$ for $y \in \Sigma_\rho(x_1)$. Call $w(x) := u(x) - v(x)$. Then, $w = 0$ on Σ_ρ and is continuous and bounded on $S_\rho\backslash F$. From the theorem **(5.23)** we obtain that $w = 0$ on D, QED.

(5.26) THE KERNEL P_χ. Our aim is now to construct the analogous of Poisson's kernel on the disk of radius ρ and center 0 corresponding to the problem:

$$(\Delta - \chi^2)u = 0,\ u(\rho\, e^{i\varphi}) = f(\varphi),\ -\pi \leq \varphi \leq \pi.$$

After separation of variables in the differential equation, we have:

(5.27) $$u = R(r).\Phi(\varphi),\quad \frac{1}{r}\frac{(rR')'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} - \chi^2 = 0.$$

The periodicity of Φ implies that $\Phi(\varphi) = A\cos n\varphi + B\,\mathrm{sen}\; n\varphi$, $n = 0,1,2, \ldots$ and R is a solution of

(5.28) $$\frac{1}{r}(rR')' - \frac{n^2R}{r^2} - \chi^2 R = 0, \qquad 0 < r \leq \rho.$$

The bounded solutions of **(5.28)** are multiples of $R_n(r) = I_n(\chi r)$, where

(5.29) $$I_n(z) = e^{-in\pi/2} J_n(e^{i\pi/2} z), \quad -\pi \le \arg z \le \pi,$$

is the modified Bessel function $I_n(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{n+2k}}{k!(n+k)!}$.

Then $I_n(\chi r)(A\cos n\varphi + B\,\mathrm{sen}\, n\varphi)$, $n = 0,1,\cdots$, are bounded solutions of **(5.27)** on $B_\rho(0)$ and they are χ-harmonic functions on R^2.

We assume that the continuous periodic function f in **(5.26)** has the Fourier expansion $f(\varphi) \approx \sum_{n=0}^{\infty} (a_n \cos n\varphi + b_n\,\mathrm{sen}\, n\varphi)$. Then $\{a_n\}$ and $\{b_n\}$ are bounded sequences.

Define,

(5.30) $$u(re^{i\varphi}) = \sum_{n=0}^{\infty} \frac{I_n(\chi r)}{I_n(\chi\rho)} (a_n \cos n\varphi + b_n\,\mathrm{sen}\, n\varphi), \quad r \le \rho.$$

It is known that $\left|\frac{I_n(\chi r)}{I_n(\chi\rho)}\right| \le |r/\rho|^{n/2}$, (cf. [Wb], pgs.149-150). Moreover, (cfr. Note **31**),

(5.31) $$\left|\frac{I_n^{(j)}(\chi r)}{I_n(\chi\rho)}\right| \le |r_0/\rho|^n \frac{(n^j + C_j)}{(\chi r_0)^j} \qquad \text{for} \quad r \le r_0 < \rho.$$

Then, one obtains from **(5.30)**,

(5.32) $$u(re^{i\varphi}) = \frac{1}{2\pi} \frac{I_0(\chi r)}{I_0(\chi\rho)} \int_{-\pi}^{\pi} f(s)\, ds + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{I_n(\chi r)}{I_n(\chi\rho)} \int_{-\pi}^{\pi} \cos n(s-\varphi)\, f(s)\, ds =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \left[\frac{1}{2} \frac{I_0(\chi r)}{I_0(\chi\rho)} + \sum_{n=1}^{\infty} \frac{I_n(\chi r)}{I_n(\chi\rho)} \cos n(s-\varphi) \right] ds = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) P_\chi(r,\rho; s-\varphi)\, ds,$$

since, for $r \le r_0 < \rho$, both series converge uniformly.

By **(5.31)** the derivatives of $u(re^{i\varphi})$ can be evaluated term by term in $r \le r_0 < \rho$, so $u(re^{i\varphi})$ is a metaharmonic function in $r < \rho$.

Assume now that $\sum (|a_n| + |b_n|) < \infty$. This holds, for example, for *f* of bounded variation satisfying some Hölder condition (cf. [Z]); in particular, for *f* absolutely continuous with $f' \in L^2$. Then, **(5.30)** converges uniformly for $r \le \rho$ to a continuous function, equal to $\frac{1}{\pi} \int_{-\pi}^{\pi} f(s) P_\chi(r,\rho; s-\varphi)\, ds$ on $B_\rho(0)$ and equal to *f* on $\Sigma_\rho(0)$. Therefore, in this case,

(5.33) THEOREM. The function $u(re^{i\varphi})$ constructed in this way is the unique solution of the boundary problem **(5.26)**, (cf. **(5.2)-(5.8)**) and verifies, $\|u\|_\infty \le \|f\|_\infty$.

(5.34) DEFINITION. For $0 \le r < \rho$, $-\pi \le t \le \pi$,

$$P_\chi(r,\rho;t) := \frac{I_0(\chi r)}{2I_0(\chi\rho)} + \sum_{n=1}^{\infty} \frac{I_n(\chi r)}{I_n(\chi\rho)} \cos nt.$$

(5.34') THEOREM. α) P_χ is a *positive kernel*, that is,

i) $P_\chi \ge 0$; *ii)* $\frac{1}{\pi}\int_{-\pi}^{\pi} P_\chi(r,\rho;t)\,dt \underset{r\uparrow\rho}{\uparrow} 1$, *iii)* $\frac{1}{\pi}\int_{|t|>\varepsilon} P_\chi(r,\rho;t)dt \underset{r\uparrow\rho}{\to} 0$.

β) Assume that $\Phi(t)$ is a real, continuous, periodic function of period 2π. Then,

(5.35) $$u(re^{i\varphi}) = \frac{1}{\pi}\int_{-\pi}^{\pi} \Phi(s)\, P_\chi(r,\rho;\varphi - s)\, ds,\ 0 \le r < \rho,$$

is such that $(\Delta - \chi^2)u = 0$ on $B_\rho(0)$ and for any $\varphi \in [-\pi,\pi]$,

(5.36) $$\lim_{r\uparrow\rho} u(re^{i\varphi}) = \Phi(\varphi).$$

If $u(\rho e^{i\varphi}) := \Phi(\varphi)$ then $u \in C(S_\rho)$. Besides, $\Phi \ge 0 \Rightarrow u \ge 0$.

PROOF. α) In view of **(5.33)**, β) is true, for example, if the periodic function $\Phi \in C^\infty$. By the maximum principle, $\Phi \ge 0$ implies $u(re^{i\theta}) = \frac{1}{\pi}\int_{-\pi}^{\pi} \Phi(s)\, P_\chi(r,\rho;\theta - s)\, ds \ge 0$, (cf. **(5.8)**). If we had $P_\chi(r,\rho;\theta) < 0$ we would have $P_\chi(r,\rho;t) < 0$ in an ε-neighborhood of θ. Assume that $\Phi \ge 0$ with support contained in $(-\varepsilon,\varepsilon)$. Then,

$u(re^{i\theta}) = \frac{1}{\pi}\int_{-\pi}^{\pi} \Phi(s)\, P_\chi(r,\rho;\theta - s)\, ds < 0$, a contradiction and *i)* follows.

ii) On the other hand for $\Phi \equiv 1$, $\frac{1}{\pi}\int_{-\pi}^{\pi} P_\chi(r,\rho;t)\, dt = \frac{I_0(\chi r)}{I_0(\chi\rho)} \underset{r\to\rho}{\to} 1$.

iii) Assume now that $\Phi = |\Phi| \in C^\infty(-\pi,\pi)$, $\Phi(0) = 0$, $\Phi(t) = 1$ for $|t| > \varepsilon$. It holds that,

$\int_{|t|>\varepsilon} P_\chi(r,\rho;t)dt \le \int_{-\pi}^{\pi} P_\chi(r,\rho;t-0)\, \Phi(t)\, dt = \pi\, u(r) \underset{r\to\rho}{\to} \pi\,\Phi(0) = 0$. Therefore, α) is true.

β) A continuous periodic function Φ can be uniformly approximated by indefinitely differentiable functions $\{\Phi_n\}$. Calling u and u_n the functions associated by **(5.30)** to Φ and Φ_n respectively, we get from i), ii) and **(5.33)** that u_n converges uniformly on $S_\rho(0)$ and to u in $B_\rho(0)$ and to Φ on $\Sigma_\rho(0)$. Then β) follows, QED.

It is possible to extend β) to a related problem for circular sectors, (cf. note **20**). Note **21** deals with the behaviour of metaharmonic functions under a conformal change of variables. Note **22** is devoted to metasubharmonic functions. In note **23** a proof is given of theorem **(5.10)** in a conspicuous particular case. Note **24** deals with normal families of χ-harmonic functions. Next theorem is a more general formulation of Th. **(5.23)**.

(5.37) THEOREM (Phragmén-Lindelöf's maximum principle). Let $w(x)$ be a χ-harmonic function on the plane region D, continuous and bounded on $\overline{D}\backslash F$, where $F=\{x_0,\ldots,x_{n-1}\}\subset\partial D$. Call $N:=max\{0,\sup\{w(x):x\in\partial D\setminus F\}\}$.

Then, $N\geq w(x)$ on $\overline{D}\backslash F$.

PROOF. For the sake of simplicity we prove the theorem in the case $F=\{x_0\}$. Define for z fixed and M=$\sup\left\{|w(x)|:x\in\overline{D}\setminus F\right\}$,

$$W_\varepsilon(x):=M\frac{K_0(\chi|x-x_0|)}{K_0(\chi\varepsilon)}+N\,I_0(\chi|x-z|).$$

Then, on $\partial(D\setminus S_\varepsilon(x_0))$ we have $W_\varepsilon(x)\geq w(x)$. Therefore, $W_\varepsilon(x)\geq w(x)$ on the closure of $D\backslash S_\varepsilon(x_0)$, (cf. **(5.6)**, **(5.8)**). Assume $x'\in D$ fixed. Letting $\varepsilon\to 0$, we obtain $N\,I_0(\chi|x'-z|)\geq w(x')$ on D. Next, letting $z\to x'$, we arrive to $N\geq w(x')$. Thus, $N\geq w(x)$, for any $x\in\overline{D}\backslash F$, QED.

(5.38) THEOREM. If D is an infinite finitely connected region such that the proper boundary $J=\partial D$ is a bounded set then theorem **(5.37)** holds whenever $\lim\limits_{x\in D,x\to\infty} w(x)=0$.

PROOF. Let $D_R:=\{x\in D:|x|<R\}$. From the preceding theorem we get,

$$w(x)\leq max\left\{0,\ \sup\{w(x):x\in J\setminus F\},\ \sup_{|x|=R} w(x)\right\}\leq max\{N,\varepsilon\}.$$

If $R\to\infty$ then $\varepsilon\to 0$. In consequence, $w(x)\leq N$, QED.

Applying **(5.37)** and **(5.38)** to $-w$, we get with the *same hypotheses*,

(5.39) COROLLARY. $|w(x)| \le \sup\{|w(x)| : x \in \partial D \setminus F\}$ for all $x \in \overline{D} \setminus F$.

(5.40) THEOREM. Let $u \in C^2(D) \cap C(\overline{D})$ be a solution of the problem

(5.41) $\left(\Delta - \chi^2\right)u = h$ in the plane region D, $\qquad u = \phi$ on ∂D.

Assume that $|h| \le M$ and that $|\phi| \le m$, M and m constants. Then, if $x \in \overline{D}$,

(5.42)
$$|u(x)| \le \sup\{M / \chi^2, m\}.$$

PROOF. We can suppose that u is not a constant. Define $v_\pm = \pm u - M/\chi^2$. Then, $\left(\Delta - \chi^2\right)v_\pm = \pm h + M \ge 0$ on D and $v_\pm \le m - M/\chi^2$ on ∂D. Since v cannot reach a positive maximum on D (cf. **(5.5)**), we have $v(x) \le \sup\{0, m - M/\chi^2\}$, $x \in D$. In consequence, $\pm u \le \sup\{m, M/\chi^2\}$, QED.

(5.43) MEAN VALUE FORMULA; HARNACK's INEQUALITY. Let u be χ-harmonic in the plane region D and assume that $S_R(0) \subset D$. From **(5.35)**, we get,

(5.44)
$$u(0) = \frac{\int_0^{2\pi} u(re^{it})\, dt/2\pi}{I_0(\chi r)} = \frac{\int_{|x|=r} u(x)\, d\sigma^r(x)}{\int_{|x|=r} I_0(\chi|x|)\, d\sigma^r(x)} \qquad \text{for } r \le R.$$

That is, $u(0) \int_{|x|=r} I_0(\chi|x|)\, d\sigma^r(x) = \int_{|x|=r} u(x)\, d\sigma^r(x)$. Then,

(5.45)
$$u(0) = \frac{\iint_{S_R(0)} u(x)\, dx}{\iint_{S_R(0)} I_0(\chi|x|)\, dx}.$$

This is the *mean value formula*, which implies the inequality

(5.46)
$$|u(0)| \le \frac{1}{\pi R^2}\left|\iint_{S_R(0)} u(x)\, dx\right|.$$

Applying **(5.46)** to $\frac{\partial u}{\partial x_i}$, $\left|\frac{\partial u}{\partial x_i}(0)\right| \le \frac{1}{\pi R^2}\left|\iint_{S_R(0)} \frac{\partial u}{\partial x_i}\, dx\right| = \frac{1}{\pi R^2}\left|\int_{S_R(0)} u\, v_i\, ds\right| \le \frac{2\pi R}{\pi R^2} \cdot \max_{|x|=R} |u|$.

Thus, *Harnack's inequality* follows,

(5.47)
$$\left|\frac{\partial u}{\partial x_i}(0)\right| \le \frac{2}{R} \max_{|x|=R} |u(x)|.$$

(5.48) UNIQUENESS OF THE SOLUTION OF THE NEUMANN PROBLEM FOR THE METAHARMONIC EQUATION WITH $\lambda = -\chi^2$, $\chi > 0$.

Until now we have estimated derivatives of a solution u by means of the same solution . and data of the equation, (cf. **(5.21)**, also **(5.47)**). Next we shall see an inequality in the other direction, appropiated for Neumann problem. It is an interesting result on the *subordination* of the supremum of the modulus of the function to the supremum of the modulus of its normal derivative.

Let U be a C^2-*Jordan* region. That is, U is an open set with boundary a simple C^2-curve J, (cfr. note **3**). These regions have the (uniform) *ball property*. That is, there exists an $R > 0$ such that for every $x \in \partial U$ there is a disc $K_x \subset \overline{U}$, of radius R, verifying $K_x \cap \partial U = \{x\}$. This is achieved by taking, with the notation of note **3**, $R \le \delta/2$.

When dealing with such regions we shall *always suppose* that R satisfies this inequality. If $x \ne y \in K_x$ and y is on the *interior* normal n_x then $\hat{y}$ *denotes* its symmetric point with respect to the tangent to J (or K_x) at x.

NB. What we have said about a bounded U holds for $R^2 \setminus \overline{U}$.

(5.49) THEOREM. Let D be an open set with a C^2-Jordan boundary J and with the ball property satisfied with a radius R such that either

a) D is bounded or else *b)* $R^2 \setminus \overline{D}$ is bounded.

Assume that $u \in C^2(D) \cap C(\overline{D})$ is such that $\dfrac{\partial u}{\partial n_x}$ exists at every point $x \in \partial D$. If u verifies $\Delta u - \chi^2 u = 0$ in D, and in case *b)* also that $u(x) \underset{|x|\to\infty}{\to} 0$, then

(5.50) $$\max_{x\in\overline{D}} |u| \le \frac{I_0(\chi R)}{\chi I_0'(\chi R)} \max_{x\in\partial D} \left|\frac{\partial u}{\partial n_x}\right|.$$

PROOF. It is sufficient to consider the case $u \not\equiv 0$ with $\sup_{y\in D}|u(y)| = \sup_{y\in D} u(y)$. Then, from **(5.37)-(5.39)** we obtain, $\max_{y\in\overline{D}}|u(y)| = \max_{y\in\partial D} u(y) = M < \infty$. Also, $M = u(y_0)$ for a certain $y_0 \in \partial D$. Suppose that 0 is the centre of K_{y_0} and that x belongs to the segment $(0, y_0)$ of length R. Then, $u(x) \le \max_{y\in\partial D}|u(y)| = u(y_0)$.

Recall that $I_0(r) := \sum_{n=0}^{\infty} \dfrac{(r/2)^{2n}}{(n!)^2}$ and so $v(z) = M\dfrac{I_0(\chi|z-0|)}{I_0(\chi R)}$ satisfies also the equation $\Delta v - \chi^2 v = 0$.

Besides for $|z-0| = R$, $v(z) = M \ge u(z)$. By the maximum principle we have $v(z) \ge u(z)$ for $z \in K_{y_0} = \{z : |z-0| \le R\}$. Moreover, $v(y_0) = u(y_0)$.

In consequence, $(0 \leq) \dfrac{u(y_0) - u(x)}{|y_0 - x|} \geq \dfrac{v(y_0) - v(x)}{|y_0 - x|} \underset{x \to y_0}{\longrightarrow} M \dfrac{\chi I_0'(\chi R)}{I_0(\chi R)}$. Thus,

$-\dfrac{\partial u}{\partial n_{y_0}} \geq M \dfrac{\chi I_0'(\chi R)}{I_0(\chi R)}$. **(5.50)** follows from $\left| \dfrac{\partial u}{\partial n_x}\Big|_{x=y_0} \right| \geq \left(\max_{\partial D} |u| \right) \dfrac{\chi I_0'(\chi R)}{I_0(\chi R)}$, QED.

(5.51) COROLLARY. Let D be a C^2-Jordan region as in Theorem **(5.50)**. Assume that $u \in C^2(D) \cap C(\overline{D})$ is a χ − harmonic:function on D satisfying $\dfrac{\partial u}{\partial n_x}(x) = \varphi(x)$ for $x \in \partial D$. Then, u is the only function with these properties.

CHAPTER 6

(6.1) NEUMANN PROBLEM FOR THE METAHARMONIC OPERATOR.

We shall study the interior Neumann problem in a C^2-Jordan region *D* satisfying the ball property with radius *R*, (cf. **(5.11)**, **(5.48)**).

Let ρ be a *continuous* function defined on *J*, the boundary of *D* and $\chi > 0$.

The *simple layer potential of density* ρ is defined by, ($S = length(J) = \langle J \rangle$),

$$w(P) = w_\chi(P;\rho) := \int_0^S \rho(t)\, K_0(\chi|P - Q_t|)\, dt.$$

$w_\chi(P;\rho)$ is χ-harmonic on $D \cup D_e$ where $D_e := R^2 \setminus \overline{D}$.

(6.2) THEOREM. 1) The integral in **(6.1)** exists in the usual sense, *i.e.* absolutely for $P \in R^2$. *w(P)* is a continuous function on R^2. In particular, at $Q_s \in J$ takes the value $w(Q_s) = \int_0^S \rho(t)\, K_0(\chi|Q_s - Q_t|)\, dt$.

2) $w(P) \to 0$ whenever $P \to \infty$ and $W : \rho \to w$ belongs to $B(C([0,S]), C(R^2))$.

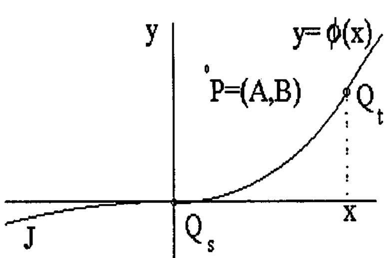


Fig. 1

PROOF. *s* is the arc length parameter that increases when the corresponding point Q_s moves along *J* in the positive sense. Taking into account that $\log^+ r^{-1} := \sup(0, -\log r) \in C(0,\infty)$, we have:

(6.3) $K_0(r) = \log^+(1/r) + F(r),\ F(r) \in C[0,\infty) \cap L^\infty[0,\infty)$.

Then, if $P \notin J$, we have,

(6.4)

$$w(P) = \int_0^S \rho(t) \log^+ \frac{1}{|P - Q_t|} dt + \int_0^S \rho(t) F(\chi|P - Q_t|) dt = L(P) + S(P),$$

where *S* is a well defined, bounded and continuous function on R^2. Let us consider *L(P)*, (see fig. 1). Define

$I_\varepsilon(P) = \int_{s-\varepsilon}^{s+\varepsilon} |\rho(t)|\, \log^+|P - Q_t|^{-1} dt$. Then,

$$I_\varepsilon \le \int_{s-\varepsilon}^{s+\varepsilon} |\rho(t) \log|P - Q_t||\, dt \le max|\rho| \int_{-h_1}^{h_2} \left|\log\sqrt{(x-A)^2 + (\phi(x) - B)^2}\right| (1 + \phi'(x)^2)^{1/2} dx,$$

where $h_i \to 0$ if $\varepsilon \to 0$. In consequence, for ε sufficiently small and *P* near enough to Q_s, there is a constant *M* such that,

$$I_\varepsilon \le M\|\rho\|_\infty \int_{-h_1}^{h_2} |\log|x-A||dx \le 2M\|\rho\|_\infty \int_0^{h_1+h_2} |\log x|dx .$$

The last integral is $o(1)$ for $\varepsilon \to 0$. Therefore, the integral in 1) exists, (cf. **(6.3)**), and also the integral in **(6.1)** exists. On the other hand, the following integral defines a continuous function in a plane neighborhood of Q_s, $\int_{s+\varepsilon}^{s-\varepsilon} \rho(t)\log^+ \frac{1}{|P-Q_t|}\,dt$, where Q_t runs from $Q_{s+\varepsilon}$ to $Q_{s-\varepsilon}$ in the positive sense of *J*. From this we obtain that $L(P)= \int_0^S \rho(t)\log^+ \frac{1}{|P-Q_t|}\,dt$ is a *continuous function* at each point of *J*, and therefore, at any point of the plane. Thus, 1) is proved. 2) is left to the reader, QED.

The *double layer potential* of density ρ is defined by

(6.5) $$u(P) = u_\chi(P;\rho) := \int_0^S \rho(t)\frac{\partial}{\partial n_t} K_0(\chi|P-Q_t|)\,dt,$$

$P \in R^2$, n_t is the interior normal at Q_t, where the derivative is calculated.

(6.6) *d=d(P)* will *denote* the function equal to 2π on *D*, equal to π on *J* and equal to 0 on D_e.

Let $w(P) = \int_0^S \rho(t)\,K_0(\chi|P-Q_t|)\,dt$ be the simple layer potential as before.

In $\left[\frac{\partial w}{\partial n_s}\right]_i (P), \left[\frac{\partial w}{\partial n_s}\right]_e (P)$ the suffixes *i* (interior), *e* (exterior), stress the fact that $P \in D$ is on n_s, $P \in D_e$ is on $-n_s$, respectively. In the case that $P = Q_s \in \partial D$, the same notation indicates that the derivatives in the direction of the interior normal are calculated from the interior or the exterior, respectively.

(6.7) THEOREM. 1) The function of $(s,t), \frac{\partial}{\partial n_s} K_0(\chi|Q_s - Q_t|)$, is continuous on $\{s \ne t\}$ and $\in L^\infty((0,S])$ for fixed *s*. Moreover, it is essentially bounded on $(0,S]\times(0,S]$.

2) $C(s) = \int_0^S \rho(t)\frac{\partial}{\partial n_s}K_0(\chi|Q_s - Q_t|)\,dt \in C([0,S])$, even for $\rho \in L^1([0,S])$. This holds also for

$$u_b(s) = \int_0^S \rho(t)\frac{\partial}{\partial n_t}K_0(\chi|Q_s - Q_t|)\,dt .$$

3) The following limits exist for $Q \in n_s$ and for $Q \in -n_s$ respectively and

$$\lim_{Q\to Q_s}\left(u+\left[\frac{\partial w}{\partial n_s}\right]_i\right)(Q) = u_b(s)+C(s) = \lim_{Q\to Q_s}\left(u+\left[\frac{\partial w}{\partial n_s}\right]_e\right)(Q).$$

4) Assume $\rho \equiv 1$. Then, $u(P) = \int_0^S \rho(t)\frac{\partial}{\partial n_t}K_0(\chi|P-Q_t|)\,dt$ is equal to $d(P)+V(P)$, where $V(P)$ is a continuous function on R^2 such that $V(\infty)=0$.

5) $$\left[\frac{\partial w}{\partial n_s}\right]_i(Q_s)+\pi\rho(s) = C(s) = \left[\frac{\partial w}{\partial n_s}\right]_e(Q_s)-\pi\rho(s).$$

6) $u_i(s) := \lim_{Q\to Q_s} u(Q)$ for $Q\in n_s$ and $u_e(s) := \lim_{Q\to Q_s} u(Q)$ for $Q\in -n_s$ exist and

$$u_i(s)-u_b(s)=\pi\rho(s)=u_b(s)-u_e(s), \quad (u_i-u_e)(s)=2(u_i-u_b)(s)=2\pi\rho(s).$$

7) If $u_i(P):=u(P),\ P\in D,\ u_i(P):=u_i(s),\ P=Q_s$ then $u_i(P)\in C(\overline{D})$.

Analogously, $u_e(P)\in C(\overline{D}_e)$.

PROOF. For fixed $Q\neq Q_t$ and for fixed $Q_t\neq Q$, respectively,we have, (cf. fig. 2),

(6.8) $$\frac{\partial}{\partial n_t}K_0(\chi|Q-Q_t|)=\chi K_0'(\chi|Q-Q_t|)\frac{\partial|Q-Q_t|}{\partial n_t}=$$
$$=\chi K_0'(\chi|Q-Q_t|).[-\cos(Q-Q_t,\eta_t)]=-\chi K_0'(\chi|Q-Q_t|).\cos\alpha.$$

(6.9) $$\frac{\partial}{\partial n_s}K_0(\chi|Q-Q_t|)=\chi K_0'(\chi|Q-Q_t|)\frac{\partial|Q-Q_t|}{\partial n_s}=\chi K_0'(\chi|Q-Q_t|).\cos(Q-Q_t,\eta_s)=$$
$$=\chi K_0'(\chi|Q-Q_t|).\cos\beta.$$

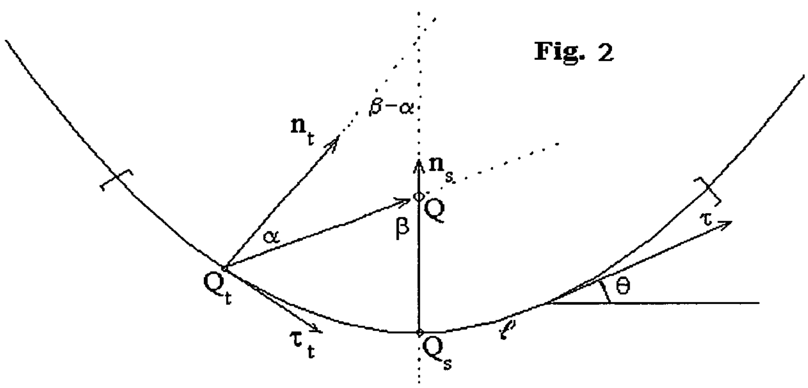


Fig. 2

(6.8) and **(6.9)** are continuous functions of (Q,Q_t) on the set $\{Q\neq Q_t\}$. In particular, $\frac{\partial}{\partial n_t}K_0(\chi|Q_s-Q_t|)$ is continuous on $\{s\neq t\}$.

For *s* near to *t* we have, (see figs. 2 and 3), (6.10) $$|\cos\alpha - \cos\beta| \le |\alpha - \beta| = |Ang(n_s, n_t)|$$

Also, $|\cos A| = |\operatorname{sen}\varepsilon| = \dfrac{|Q_s - Q'_s|}{|Q_s - Q_t|} = O(1)\dfrac{|Q_t - Q'_s|^2}{|Q_s - Q_t|}$. Thus,

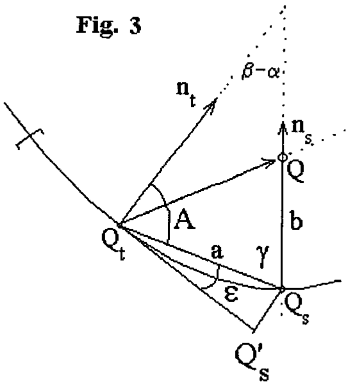
(6.11) $$|\cos A| = O(1)|Q_s - Q_t|.$$

For $Q = Q_s$ we have $A = \alpha$ and for a certain constant *M*, we obtain, (cf. **(6.10)**),

(6.12) $$|\cos(\pi - \gamma)| = |\cos\beta| \le |\cos A| + |Ang(n_s, n_t)| \le M|s - t|,$$

in fact, there is an $\eta > 0$ such that if the subinterval of *J*, $J(s)$, centered at Q_s verifies $\langle J(s)\rangle \le \eta$ then $\cos\gamma < 1/2$. Then, from

Fig. 3



$$a^2 + b^2 - 2ab\cos\gamma = |Q - Q_t|^2 \ge$$

$$\ge a^2 + b^2 - ab \ge (b - a/2)^2 + (3/4)a^2$$

$$\ge (3/4)a^2 = (3/4)|Q_s - Q_t|^2,$$

it follows that

(6.13) $$|Q - Q_t| \ge (\sqrt{3}/2)|Q_t - Q_s| \ge (\sqrt{3}/4)|s - t|.$$

Since $|s - t| \ge |Q_s - Q_t|$, we have $|s - t| \approx |Q_s - Q_t|$. On the other hand, we have

(6.14) $$|Ang(n_s, n_t)| = |Ang(\tau_s, \tau_t)| = \left|\int_s^t \dot{\theta}(\sigma)d\sigma\right| = \left|\int_s^t k(\sigma)d\sigma\right| \le (\sup k)|s - t|.$$

Thus, from **(6.13)-(6.14)** we obtain the last inequality in **(6.12)**. *Define,*

(6.15) $$h(Q,t) := \rho(t)\chi K_0'(\chi|Q - Q_t|).$$

Therefore, $C(s) = \int_0^S h(Q_s,t)\cos(Q_s - Q_t, n_s)dt = \int_0^S \rho(t)\dfrac{\partial}{\partial n_s}K_0(\chi|Q_s - Q_t|)\,dt.$

From **(6.15)** and note **32**, we get $K_0'(r) \underset{r\to\infty}{\to} 0$ and

(6.16) $$|K_0'(\chi|Q - Q_t|)| \le Ce^{-\chi|Q-Q_t|/2}\left(1 + \frac{1}{\chi|Q - Q_t|/2}\right),$$

$$|h(Q,t)| \le C\|\rho\|_\infty \chi e^{-\chi|Q-Q_t|/2}\left(1 + \frac{1}{\chi|Q - Q_t|/2}\right) = \frac{O(1)}{|Q - Q_t|}.$$

The first part of 1) follows from **(6.11)** and **(6.16)**. The second part is a consequence of the uniformity of the estimations.

2) follows from the continuity properties of $\frac{\partial}{\partial n_t} K_0(\chi|Q_s - Q_t|)$ and the Lebesgue dominated convergence theorem. Also in the same way for, (cf. **(6.8)**),

$$u_b(s) = -\int_0^s h(Q_s,t)\cos(Q_s - Q_t, n_t)dt = \int_0^s \rho(t)\frac{\partial}{\partial n_t} K_0(\chi|Q_s - Q_t|)\, dt$$

3) Let u be the double layer potential with density ρ. For $Q \notin J$, $Q \in n_s$, we have,

$$\textbf{(6.17)}\quad u(Q) + \frac{\partial w}{\partial n_s}(Q) = \int_0^s \rho(t)\frac{\partial}{\partial n_t} K_0(\chi|Q - Q_t|)\, dt + \int_0^s \rho(t)\frac{\partial}{\partial n_s} K_0(\chi|Q - Q_t|)\, dt =$$

$$=\text{(cf. \textbf{(6.8)}, \textbf{(6.9)})} = \int_0^s \rho(t)\chi K_0'(\chi|Q - Q_t|).(-\cos(Q - Q_t, n_t) + \cos(Q - Q_t, n_s))dt =$$

$$= -\int_0^s h(Q,t).(\cos(Q - Q_t, n_t) - \cos(Q - Q_t, n_s))dt.$$

If l is a small neighborhood of Q_s on J (fig. 2), because of **(6.10)** and **(6.16)**, the integral

$$\int_l |h(Q,t).(\cos(Q - Q_t, n_t) - \cos(Q - Q_t, n_s))|dt,$$

except for a constant factor, it is not greater than

$$\textbf{(6.17')}\qquad \int_l \frac{|\cos(Q - Q_t, n_t) - \cos(Q - Q_t, n_s)|}{|Q - Q_t|}dt \le 2\int_l \frac{|\text{sen}(1/2)(\alpha - \beta)|}{|Q - Q_t|}dt.$$

Therefore, if $Q \to Q_s$ along n_s, the last expression is equal to, (cf. **(6.12)**, **(6.13)**),

$$=2\int_l \frac{|\text{sen}\,(1/2)Ang(n_s,n_t)|}{|Q - Q_t|}dt \le \int_l \frac{|Ang(n_s,n_t)|}{|Q - Q_t|}dt \le \int_l \frac{M|s-t|}{(\sqrt{3}/4)|s-t|} \le 4M\langle l\rangle,$$

and the last integral in **(6.17)** tends to

$$\textbf{(6.18)}\quad \delta(s) = -\int_0^s h(Q_s,t)\,[\cos(Q_s - Q_t, n_t) - \cos(Q_s - Q_t, n_s)]dt = u_b(s) + C(s),$$

and the first equality in 3) is proved. *Idem* for the second equality.

NB. Observe that with the preceding arguments one can also prove that the function

$$f(t) = \cos(Q_s - Q_t, n_s)/|Q_s - Q_t| = ((x(t) - x(s))\dot{y}(t) + (y(s) - y(t))\dot{x}(s))/|Q_s - Q_t|^2$$

is continuous for $t \ne s$ and bounded. Since $\lim_{t\to s} f(t) = \dfrac{\dot{x}(s)\ddot{y}(s) - \ddot{x}(s)\dot{y}(s)}{2}$, it is also continuously extendable to $t = s$.

5) We have shown that for $Q \to Q_s$, $Q \neq Q_s$, $Q \in n_s$ or $Q \in -n_s$,

$$lim\left(u(Q)+\frac{\partial w}{\partial n_s}(Q)\right)-u_b(s)=C(s).$$

If $u_i(s):=\lim_{Q\to Q_s} u(Q)$ exists for $Q \in n_s$, $Q \neq Q_s$, as we shall show, then it exists.

$$\lim_{Q\to Q_s}\frac{\partial w}{\partial n_s}(Q)=\left[\frac{\partial w}{\partial n_s}\right]_i(Q_s).$$

Analogously, if $Q \in -n_s$ and $u_e(s):=\lim_{Q\to Q_s} u(Q)$ then $\lim_{Q\to Q_s}\frac{\partial w}{\partial n_s}(Q)=\left[\frac{\partial w}{\partial n_s}\right]_e(Q_s)$.

Then, putting s instead of Q_s, we would have,

(6.19) $$u_i(s)-u_b(s)+\left[\frac{\partial w}{\partial n_s}\right]_i(s)=C(s)=u_e(s)-u_b(s)+\left[\frac{\partial w}{\partial n_s}\right]_e(s).$$

Therefore, whenever 6) holds for some ρ, 5) also holds.

4) To prove 4) recall that if v is the conjugate function of the harmonic function u then from the Cauchy-Riemann equations we obtain,

$$\frac{\partial u}{\partial x}\cos\beta+\frac{\partial u}{\partial y}\operatorname{sen}\beta=\frac{\partial v}{\partial x}\cos(\beta+\pi/2)+\frac{\partial v}{\partial y}\operatorname{sen}(\beta+\pi/2).$$

Applying this to $Log(z)=\log|z|+iArg(z)$ we have,

$$-\frac{\partial}{\partial n_s}\log|P-Q_s|=\frac{\partial}{\partial(-n_s)}\log|P-Q_s|=\frac{d\alpha_P}{ds}(s)$$

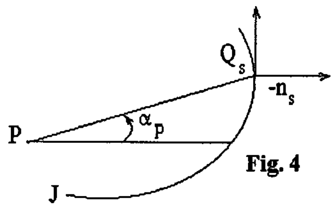


Fig. 4

and, (see fig. 4),

(6.20) $$\int_0^S\frac{\partial}{\partial n_s}\log|P-Q_s|^{-1}ds=\int_0^S d\alpha_P(s)=d(P).$$

Let $X(t,r):=e^{-rt}/(t^3+t^2\sqrt{t^2-1}-t)$, (cf. note **33**). Kelvin's function can be written for C a constant and $r>0$ as:

$$K_0(r)=\int_1^\infty(e^{-rt}/\sqrt{t^2-1})dt=-\log r+C+\int_r^1(e^{-t}-1)t^{-1}dt+\int_1^\infty Xdt.$$

After replacing r by $\chi|P-Q_s|$ in

(6.21) $$(\partial/\partial n_s)\left[\log 1/r+C+\int_r^1(e^{-t}-1)t^{-1}dt+\int_1^\infty(e^{-rt}/(t^3+t^2\sqrt{t^2-1}-t))dt\right],$$

we obtain, $\int_0^S \rho(s) \frac{\partial}{\partial n_s} K_0(\chi|P-Q_s|)\, ds =$

$$= d(P) + \int_0^S \left[\frac{\partial}{\partial n_s} \int_{\chi|P-Q_s|}^{1} \frac{e^{-t}-1}{t} dt + \frac{\partial}{\partial n_s} \int_1^{\infty} \frac{e^{-\chi|P-Q_s|t}}{t^3 + t^2\sqrt{t^2-1} - t} dt \right] ds = d(P) + V(P),$$

(6.22) $$V(P) := \int_0^S v(|P-Q_s|) ds =$$

$$= \int_0^S \left[\frac{1-e^{-\chi|P-Q_s|}}{\chi|P-Q_s|} - \int_1^{\infty} \frac{e^{-\chi|P-Q_s|t}}{\sqrt{t^2-1}\,(\sqrt{t^2-1}+t)} dt \right] \chi \frac{\partial}{\partial n_s}(|P-Q_s|) ds$$

$$= -\chi \int_0^S \frac{1-e^{-\chi|P-Q_s|}}{\chi|P-Q_s|} \cos(P-Q_s, n_s) ds + \chi \int_0^S \left[\int_1^{\infty} \frac{e^{-\chi|P-Q_s|t}}{\sqrt{t^2-1}\,(\sqrt{t^2-1}+t)} dt \right] \cos(P-Q_s, n_s) ds$$

$$= V_1(P) - V_2(P),$$

since $(\partial/\partial n_s)(|P-Q_s|) = -\cos(P-Q_s, n_s)$. Hence. the functions $V_i, i=1,2$, are defined and continuous on R^2 and they tend to 0 as $P \to \infty$.

6) and 7). Let $u(Z) := \int_0^S \rho(s) \frac{\partial}{\partial n_s} K_0(\chi|Z-Q_s|) ds$. Assume that $\rho \in C^1$. Then, from **(6.8), (6.16)** and 4) we obtain,

(6.23) $$u(Z) = \int_0^S (\rho(s)-\rho(t)) \frac{\partial}{\partial n_s} K_0(\chi|Z-Q_s|) ds + \rho(t)(d(Z)+V(Z)) =$$

$$= \int_0^S \left\{ |Z-Q_s| \chi K_0'(\chi|Z-Q_s|) \frac{-\cos(Z-Q_s, n_s)}{|Z-Q_s|} \right\} (\rho(s)-\rho(t)) ds + \rho(t)(d(Z)+V(Z)) =$$

$$= \int_0^S O(1) \frac{\rho(s)-\rho(t)}{|Q_s - Z|} ds + \rho(t)(d(Z)+V(Z)).$$

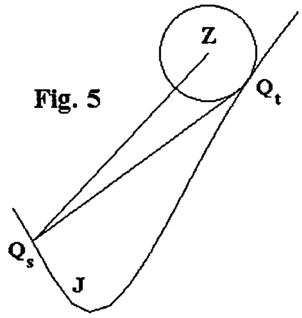


Fig. 5

Suppose $Z \in J_\theta = \theta -$ neighborhood of *J*. If θ is small enough, because of the ball property, there is a unique Q_t on *J* such that $|Z-Q_t| = \textit{distance from } Z \textit{ to } J$, (see fig. 5).

Thus, $t = t(Z)$ and $\frac{|Q_s - Q_t|}{|Q_s - Z|} \le \frac{|Z-Q_s| + |Z-Q_t|}{|Z-Q_s|} \le 2$.

For σ near to *s* we obtain $|\sigma - s|/2 \le |Q_\sigma - Q_s|$ and therefore,

$$O(1)\frac{\rho(s)-\rho(t)}{|Q_s - Z|} = O(1)\frac{|s-t|}{|Q_s - Z|} = O(1)\frac{|Q_s - Q_t|}{|Q_s - Z|} = O(1).$$

Also, $Z \to Q_s \Rightarrow Q_t \to Q_s \Rightarrow t \to s$. Then, $Z \to Q_h$ implies $\rho(t(Z)) \to \rho(h)$.

Therefore, the second integral in **(6.23)** tends to a number $L(h)$ *independent* of the way *Z* approaches Q_h. Thus,

(6.24) $$L(h) = \int_0^S \{-\chi K_0'(\chi|Q_h - Q_s|)\cos(Q_h - Q_s, n_s)\}(\rho(s) - \rho(h))ds + .$$

$$+\rho(h)(d(Q_h) + V(Q_h))$$

Besides, $L(h) \in C([0,S])$. In consequence, $u(Z) = \int_0^S \rho(s)\frac{\partial}{\partial n_s}K_0(\chi|Z - Q_s|)ds$ has a limit for $Z \to Q_h$ that *depends only* on the set $D_i = D, J, D_e$ containing *Z* and we have,

(6.25) $L(h) = u_i(h) - (2\pi + V(h))\rho(h) \qquad Z \in D,$

(6.26) $L(h) = u_b(h) - (\pi + V(h))\rho(h) \qquad Z \in J,$

(6.27) $L(h) = u_e(h) - V(h)\rho(h) \qquad Z \in D_e.$

With the same abuse of notation as before, we write $V(h) := V(Q_h)$, $L(Q_h) := L(h)$.

(6.28) It was proved that if $\rho \in C^1$, $u_i(h)$ and $u_e(h)$ exist as limits independent of the way *Z* approaches Q_h as far as *Z* remains in *D* or in D_e respectively. A consequence of **(6.25)-(6.27)** is that **(6.19)** is a valid formula. And 6) and 5) hold for $\rho \in C^1$.

Let us return to $u(Z) = \int_0^S [\rho(s) - \rho(t(Z))]\frac{\partial}{\partial n_s}K_0(\chi|Z - Q_s|)ds + \rho(t(Z))(d(Z) + V(Z))$. The integral is equal to $L(Z) := \int_0^S \{-\chi K_0'(\chi|Z - Q_t|)\cos(Z - Q_t, n_t)\}(\rho(t) - \rho(t(Z)))dt$. Since *J* is a compact set, if $Z, Z' \in J_0$ and $Z \to Z' \notin J$ then $t(Z) \to t(Z')$. Then, $L(Z) \to L(Z')$. Thus, *L is a continuous function on* J_0.

Therefore, from $u(Z) = L(Z) + \rho(t(Z))(d(Z) + V(Z))$ we conclude that the double layer potential $u(P)$ is continuously extendable up to the boundary as well for $P \in D$ or for $P \in D_e$. So, 7) is also proved if $\rho \in C^1$.

(6.29) Assume that $\{\rho_j\} \subset C^1$ is a sequence that converges uniformly to $\rho \in C$ and call $u_{i,j}, u_{b,j}, u_{e,j}$ the functions associated to the double layer potential u_j of density ρ_j. Let us prove **(6.26)** for $\rho \in C$. Because of $u_{b,j}(s) \xrightarrow{\bullet} u_b(s)$, it follows that $L_j(s) \xrightarrow{\bullet} L(s) = u_b(s) - (\pi + V(s))\rho(s)$. In consequence, $u_{i,j}(s) \xrightarrow{\bullet} U(s)$, $u_{e,j}(s) \xrightarrow{\bullet} W(s)$. The sequence of χ-harmonic functions $\{u_j(P) : P \in D\}$ which are continuously extendable up to the boundary, converges uniformly to $A(P)$, (cf. **(5.37)-(5.39)**), a χ-harmonic function on D, continuous on $\overline{D}$ that verifies on the boundary: $A(s)=U(s)$. However, $u_j(P) \to u(P) = \int_0^S \rho(s) \frac{\partial}{\partial n_s} K_0(\chi|P - Q_s|) ds$, $P \in D$. That is, $A(P)=u(P)$, $U(s) = u_i(s)$ and **(6.25)** holds for $\rho \in C$.

Moreover, we get $u_i(s) \in C([0,S])$. Also, $u_i(s) = \lim_{Q \to Q_s} u(Q)$, $Q \in D$, and not only for Q on n_s as was originally defined.

Let us prove **(6.27)** for $\rho \in C$. It holds that $u_{e,j}(\infty) = 0$, (cf. **(6.16)**). Also, given $\varepsilon > 0$ there exists m such that for $|P| \geq m$ and any j we have $|u_{e,j}(P)| \leq \varepsilon$. Thus, we can repeat the preceding argument but using this time the theorem of maximum of Corollary **(5.39)**, proving so **(6.27)** for ρ continuous. On the other hand, $u(P)$, $P \notin \overline{D}$, is continuously extendable up to ∂D; therefore, we can arrive to similar conclusions as in the case $P \in D$.

In particular, $u(P) \in C(\overline{D})$, $u(P) \in C(\overline{D}_e)$ for $\rho \in C$ and 7) holds. Then, 6), **(6.19)** and 5) also hold for ρ continuous. Thus, Theorem **(6.7)** is proved, QED

(6.30) Next we prove the *fundamental* result for Neumann's problem of existence and uniqueness of the solution of metaharmonic equations. Recall that if $\rho(t)$ is continuous then the simple layer potential

$$w(P) = \int_0^S K_0(\chi|P - Q_t|)\rho(t)dt \in C(R^2) \cap A^{\chi}(R^2 \setminus \partial D).$$

The derivative of $w(P)$ at Q_s in the direction of the interior normal n_s is, (cf. **(6.7)**),

$$\left[\frac{\partial w}{\partial n_s}\right]_i (s) = -\pi\rho(s) + C(s) = -\pi\rho(s) + \int_0^S \frac{\partial}{\partial n_s} K_0(\chi|Q_s - Q_t|)\rho(t)dt,$$

and from the exterior is,

$$\left[\frac{\partial w}{\partial n_s}\right]_e (s) = \pi\rho(s) + C(s) = \pi\rho(s) + \int_0^S \frac{\partial}{\partial n_s} K_0(\chi|Q_s - Q_t|)\rho(t)dt.$$

Then, because of **(5.51)** we have i) of the following theorem.

(6.31) THEOREM. i) If $\rho \in C$ then $w(P) = \int_0^S K_0(\chi|P-Q_t|)\rho(t)dt$ is the solution of the following problem: $\Delta w - \chi^2 w = 0$ on D, $w \in C(\overline{D})$, $\left[\frac{\partial w}{\partial n_s}\right]_i (s) = -\pi\rho(s) + C(s)$.

ii) If $\rho(t)$ verifies $-\pi\rho(s) + C(s) \equiv 0$ then $\rho(t) \equiv 0$.

PROOF of ii). The hypothesis implies that for any s, $\left[\frac{\partial w}{\partial n_s}\right]_i (s) = 0 = -\pi\rho(s) + C(s)$. Because of theorem **(5.49)**, we have $w(P) = 0$ for $P \in \overline{D}$. Since $w(P) \underset{P\to\infty}{\to} 0$, again by **(5.49)**, we obtain $w(P) = 0$ also for $P \in R^2 \setminus \overline{D}$. Then, $0 = \left[\frac{\partial w}{\partial n_s}\right]_e (s) = \pi\rho(s) + C(s)$. Therefore, $\rho(s) = 0$, QED.

(6.32) DEFINITION. $K(s,t) := \frac{\partial}{\partial n_s}\frac{K_0(\chi|Q_s - Q_t|)}{\pi}$, $K^*(s,t) := \frac{\partial}{\partial n_t}\frac{K_0(\chi|Q_s - Q_t|)}{\pi}$.

Thus, $K^*(s,t) = K(t,s)$. Because of 1), 2) of **(6.7)**, both are bounded kernels on $[0,S]\times[0,S]$ that define completely continuous operators, one adjoint of the other, that we call K and K*. Precisely, they have finite Hilbert-Schmidt norms and

$$\mathrm{K}(\rho) = C(s)/\pi = \int_0^S K(s,t)\rho(t)\,dt, \qquad \mathrm{K}^*(\rho) = u_b(s)/\pi = \int_0^S K^*(s,t)\rho(t)\,dt.$$

Moreover, K, K*$\in B(L^1([0,S],C([0,S])$. From ii) of theorem **(6.31)** we know that $\lambda = 1$ is not an eigenvalue of K. But then, from the theory of completely continuous operators it follows that K−I is a surjective mapping from L^2 onto L^2. Then, for any $h \in C(J)$, the equation,

$$\text{(6.33)} \qquad \frac{h(s)}{\pi} = -\rho(s) + \int_0^S K(s,t)\rho(t)dt = (-\mathrm{I}+\mathrm{K})(\rho).$$

has a solution $\rho \in L^2$. Necessarily, $\rho \in C(J)$.

Because of i) of theorem **(6.31)** we have $\left[\frac{\partial w}{\partial n_s}\right]_i (s) = \pi(-\rho + \mathrm{K}\rho)(s)$. Then, the next *existence* theorem holds.

(6.34) THEOREM. For any $h \in C(J)$ there exists a $\rho \in C(J)$ such that the simple layer potential

$w(P) = \int_0^S K_0(\chi|P - Q_t|)\rho(t)dt$ is a solution of the following Neumann's problem:

$$\Delta w - \chi^2 w = 0 \text{ on } D,\ w \in C(\overline{D}),\ \left[\frac{\partial w}{\partial n_s}\right]_i (s) = h(s).$$

(6.35) THEOREM. i) The Neumann problem for $h \in C(J)$, (n = interior normal),

$$(\Delta - \chi^2)w = 0,\ w \in C(\overline{D}),\ \frac{\partial w}{\partial n}(x) = h(x),\ x \in J,$$

has a unique solution w.

ii) It is of the form $w(P) = w_h(P) = \int_0^S K_0(\chi|P - Q_t|)\rho_h(t)dt$ with $\rho_h \in C(J)$.

iii) $T = (1/\pi)(-I+K)^{-1}$ is a surjective continuous mapping from $L^2([0,S])$ onto $L^2([0,S])$.

iv) $T : h \to \rho_h$ is a suryective continuous mapping from $C([0,S])$ onto $C([0,S])$.

v): The mapping $T : h \to \rho_h$ is continuous from $L^1([0,S])$ into $L^1([0,S])$.

PROOF. i), ii) The existence of the solution is asserted in **(6.34)**. The unicity of this solution is a consequence of the subordination principle **(5.49)** of chapter 5.

iv) From **(6.33)** we get $(-I+K)^{-1}(h/\pi) = \rho$. Thus, if $h \to 0\,(C([0,S]))$ then $\rho \to 0\,(L^2(0,S))$ and therefore $\rho \to 0\,(L^1(0,S))$. Then, $K(\rho) \to 0\,(L^\infty(0,S))$. Since $K(\rho) \in C([0,S])$, we obtain $K(\rho) \to 0\,(C([0,S]))$. From **(6.33)** we know that $\rho \in C([0,S])$ and now that $\rho \to 0\,(C([0,S]))$.

v) Let us see that $T \in B(L^1([0,S]), L^1([0,S]))$. $K: L^1 \to L^1$ is a completely continuous operator that has not the eigenvalue 1. From Fredholm theory of Banach spaces we know that $\forall h \in L^1\ \exists_1 \rho \in L^1$ such that $(-I+K)(\rho) = h/\pi$. Thus, $(-I+K)^{-1} : L^1 \to L^1$ is a bounded operator that defines a bijection on L^1. In particular, since $(-I+K)^{-1}(h/\pi) = \rho$, we get for $h \in C$ that $h \to 0$ in L^1 implies $\rho = \rho_h \to 0$ in L^1, QED.

(6.36) Recall that if $u \in C^1(D)$ and D has the ball property we say that u has a normal derivative extendable to the boundary if for any $x_0 \in \partial D$ there exists the (finite) limit $\lim_{x \to x_0} \nabla u \times n$, n is the interior normal to the ball at x_0 and $x \in D$ runs along n, (**(5.11)**).

DEFINITION. We denote *N(D)* the familiy of these functions that belong to $C^2(D) \cap C(\overline{D})$, (see note **25)**.

NOTATION. We denote $J^\delta = J_\delta \cap D = \{P \in D : dist(P,J) < \delta\}$ and write sometimes $Q = Q^s$ if $Q \in D, Q \in n_s$, $Q \neq Q_s$.

(6.37) THEOREM. Assume D and w are as in **(6.1)**. Then, i) $w \in N(D)$,

ii) $\nabla w \in L^1(D)$ and $\|\nabla w\|_1 \leq M\|\rho\|_\infty$ where $M = M(D,\chi)$ is a constant,

iii) $\frac{\partial w}{\partial n_s} \in C\left(\overline{J^\delta}\right)$.

PROOF. i) follows from **(6.1)**, 2) **(6.2)** and 5) **(6.7)**.

ii) $\nabla \int_0^S K_0(\chi|P - Q_t|)\rho(t)dt = \int_0^S \chi K_0'(\chi|P - Q_t|)\nabla|P - Q_t|\rho(t)dt$ implies

$$\left|\nabla \int_0^S K_0(\chi|P - Q_t|)\rho(t)dt\right| \leq \int_0^S \chi|K_0'(\chi|P - Q_t|)||\rho(t)|dt \leq \chi\|\rho\|_\infty \int_0^S \frac{C}{|P - Q_t|}dt,$$

(cf. **(6.16)**), where C is a constant. Then,

$$\|\nabla w\|_1 \leq C\chi\|\rho\|_\infty \int_0^S dt \int_D \frac{dP}{|P - Q_t|} \leq C\chi S\|\rho\|_\infty \int_{|P| \leq diamD} \frac{dP}{|P|} = M\|\rho\|_\infty .$$

iii).We know from **(6.7)** that the double layer potential u with density ρ verifies $u \in C(\overline{D})$ and takes the value $u_i(s) \in C([0,S])$ in the boundary. Therefore, we have for $h(Q^s,t) = \rho(t)\chi K_0'(\chi|Q^s - Q_t|)$, that (cf. **(6.17)** and 3), 5) of **(6.7)**),

$$\frac{\partial w}{\partial n_s}(Q^s) = \int_0^S h(Q^s,t)\left(\cos(Q^s - Q_t,n_s) - \cos(Q^s - Q_t,n_t)\right)dt - u(Q^s) \to u_b(s) + C(s) - u_i(s)$$ =the sum of three continuous functions. We shall prove, for $Q^s \to Q_s$, that

(6.38) $$\int_0^S \Theta(Q^s,t)dt, \qquad \Theta(Q^s,t) := h(Q^s,t)\left(\cos(Q^s - Q_t,n_s) - \cos(Q^s - Q_t,n_t)\right),$$

converges *uniformly*. Since $u(Q^s) \xrightarrow[Q^s \to Q_s]{\bullet} u_i(s)$, it will follow that

(6.39) $$\frac{\partial w}{\partial n_s}(Q^s) \textit{ converges uniformly to } \frac{\partial w}{\partial n_s}(Q_s).$$

After fixing conveniently the origin of the parameter t, (see Fig. 2), we have for $s \in (-\varepsilon/2, \varepsilon/2)$,

(6.40) $$I_\varepsilon(Q^s) := \int_{t \notin (-\varepsilon,\varepsilon)} \Theta(Q^s,t)dt \xrightarrow{\bullet} \int_{t \notin (-\varepsilon,\varepsilon)} \Theta(Q_s,t)dt = I_\varepsilon(Q_s).$$

On the other hand, (cf. **(6,16)** and **(6.17')**), we obtain,

(6.41) $$\left|\int_{-\varepsilon}^{\varepsilon}\Theta(Q^s,t)dt\right| \le M_0 \int_{-\varepsilon}^{\varepsilon}|\rho(t)|\frac{|Ang(n_s,n_t)|}{|Q^s-Q_t|}dt \le M_1\|\rho\|_\infty \int_{-\varepsilon}^{\varepsilon}\frac{|s-t|}{|Q_s-Q_t|}dt \le$$

$$\le 2M_1\|\rho\|_\infty \int_{-\varepsilon}^{\varepsilon}\frac{|s-t|}{|s-t|}dt = 4M_1\|\rho\|_\infty\varepsilon = M\varepsilon,$$

with *M independent* of *s*. Then, from

$$I(Q^s) = I_\varepsilon(Q^s) + O(\varepsilon) = I_\varepsilon(Q^s) \pm I_\varepsilon(Q_s) \pm I(Q_s) + O(\varepsilon) = I(Q_s) + O(\varepsilon),$$

we get the local uniform convergence and from this the uniform convergence on all *J*, proving that **(6.38)** converges uniformly and then **(6.39)**.

A consequence of this fact for our regions is that

(6.42) $$\frac{\partial w}{\partial n_s}(Q_s) \in C(J).$$

(6.39) and **(6.42)** imply that $\frac{\partial w}{\partial n_s}(Q)$ is well defined and continuous on $\overline{J^\delta}$, QED.

Since $K_0'(r) = \frac{I_0'(r)K_0(r)}{I_0(r)} - \frac{1}{rI_0(r)}$, (cf. i) **(5.19)**), we have, for *t* outside a neighborhood of *t=s*, that $h(Q^s,t) = \rho(t)\chi K_0'(\chi|Q^s - Q_t|) \xrightarrow{\bullet} h(Q_s,t)$, uniformly in a small neighborhood of *s* and uniformly in ρ, whenever $\|\rho\|_\infty$ remains bounded.

(6.43) THEOREM. i) Assume we have a family $\Pi = \{\rho\} \subset C([0,S])$ for which there is a constant *C* such that for any $\rho \in \Pi$ it holds that $\|\rho\|_\infty \le C < \infty$. Then given ε there exists a δ such that for any $\rho \in \Pi$ and *s*, $|Q^s - Q_s| \le \delta$ implies $\left|\frac{\partial w_\rho}{\partial n_s}(Q^s) - \frac{\partial w_\rho}{\partial n_s}(Q_s)\right| \le \varepsilon$.

ii) The same holds if for any $\rho \in \Pi$, $\|\rho\|_2 \le C < \infty$.

PROOF. i) The same proof as that given for iii) of theorem **(6.37)** but taking into account the previous observation and using more precisely **(6.40)** and **(6.41)**.

ii) In this case instead of **(6.41)** we use

(6.44) $$\left|\int_{-\varepsilon}^{\varepsilon}\Theta\right| \le M_0 \int_{-\varepsilon}^{\varepsilon}|\rho(t)|\frac{|Ang(n_s,n_t)|}{|Q^s-Q_t|}dt \le M_1\|\rho\|_2\sqrt{\int_{-\varepsilon}^{\varepsilon}\left(\frac{|s-t|}{|Q_s-Q_t|}\right)^2 dt} \le M'\|\rho\|_2\sqrt{\varepsilon}$$

obtained by means of the Cauchy-Bunjakowsky-Schwarz inequality. With a similar approach we arrive to an analogous of **(6.40)**, and ii) follows, QED.

CHAPTER 7

(7.1) GREEN's KERNEL FOR THE METAHARMONIC OPERATOR $\Delta-\chi^2$.

Green's kernel $G(p,q;\lambda)$, $\lambda=-\chi^2$, $\chi>0$, for Neumann problem is defined as follows,

$$\begin{cases} G(p,q;\lambda):=\dfrac{1}{2\pi}K_0(\chi|p-q|)-H(p,q;\lambda), & p,q\in D \\ (\Delta_q+\lambda)H(p,q;\lambda)=0, \quad H(p,.;\lambda)\in A^{\chi}(D)\cap C(\overline{D}) & \\ \dfrac{\partial}{\partial n_q}H(p,q;\lambda)=\dfrac{\partial}{\partial n_q}\dfrac{1}{2\pi}K_0(\chi|p-q|), & q\in\partial D, p\in D \end{cases}$$

Because of theorems **(6.35)**, **(6.37)**, of chapter 6 we know that $H(p,.;\lambda)$ is uniquely determined and belongs to $N(D)\subset C(\overline{D})\cap C^2(D)$ and also to $A^{\chi}(D)\cap H^1(D)$. Fixing $p\in D$, the function $G(p,.;\lambda)$ is positive near p. If $G(p,q;\lambda)<0$ for a $q\in D$ then it would have a negative minimum on at a point $q_0\in\partial D$ and (cf. **(5.10)**) its exterior normal derivative would be negative there, a contradiction. Thus, the next result holds.

THEOREM. $G(p,q;\lambda)=\dfrac{1}{2\pi}K_0(\chi|p-q|)-H(p,q;\lambda)\geq 0,\ p\in D,\ q\in\overline{D}$.

(7.2) Recall that ∂D is, by hypothesis, a curve C^2 and that

$J=\partial D, J^\delta=\{P\in D: dist(P,J)<\delta\}$ and as usual, $J_\delta=\{P\in R^2: dist(P,J)\}<\delta$.

We can introduce in J_δ, δ *sufficiently* small, say $0<\delta<\delta_0<1$, coordinates $(s,r)\in(0,S)\times(-\delta,\delta)$ where $P=P(s,r)\in n_s$ and $r=\begin{cases} dist(P,\partial D), & P\in D \\ -dist(P,\partial D), & P\notin D \end{cases}$.

The mapping $P=(x,y)\to(s,r)$ and its inverse are C^2-mappings with (non null) uniformly bounded jacobians, (cf. note **3**).

If $P\in n_s$ and $P\in J$ we write P_s instead of *P*, in accordance with our notation in Chapter 6. If $P\in n_s$ and $P\in J^\delta\setminus J$, we write sometimes P^s instead of *P*, as we already did. The following four lemmas collect very useful auxiliary results.

(7.3) LEMMA (data lemma). There is a constant *M* independent of $P\in D$ such that,

i) $h_P(s):=\dfrac{1}{2\pi}\dfrac{\partial}{\partial n_s}K_0(\chi|P-Q_s|)$ verifies $\int_0^S |h_P(s)|ds=\|h_P\|_1\leq M<\infty$.

ii) In general, for $\varepsilon>0$, $\|h_P\|_{1+\varepsilon}$ is not uniformly bounded as a function of *P*.

PROOF. i) $\frac{\partial}{\partial n_s} K_0(\chi|P-Q_s|) = \frac{\partial}{\partial n_s}\log\frac{1}{|P-Q_s|} + v(|P-Q_s|)$ where, (cf. **(6.21)**, **(6.22)** and note **33**),

$$v(|P-Q_s|) = \left[\frac{1-e^{-\chi|P-Q_s|}}{\chi|P-Q_s|} - \int_1^\infty \frac{e^{-\chi|P-Q_s|t}}{\sqrt{t^2-1}(\sqrt{t^2-1}+t)}dt\right]\chi\frac{\partial|P-Q_s|}{\partial n_s},$$

is a bounded function of (P,s). To prove that $\|h_P\|_1 \le M < \infty$, it is sufficient to show that if $P \in n_0$, n_0 is the interior normal at $Q_0 \in J$, $\operatorname{dist}(P,\partial D) < \delta$, δ small enough, then,

$$\int_0^S \left|\frac{\partial}{\partial n_s}\log|P-Q_s|\right| ds \le C(\delta) < \infty.$$

But this is the content of the basic lemma **(7.5)** for $Q = Q_s$.

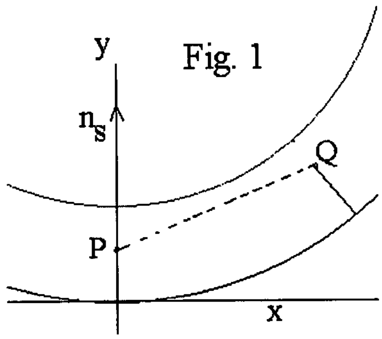
ii) see note **34**, QED.

(7.4) LEMMA (area lemma). There exists a constant C such that for any $P \in n_s \cap J^\delta$,

$$A(P) = \int_{J^\delta}\left|\frac{\partial}{\partial n_s}\log|P-Q|\right| dQ \le C\delta\log\frac{1}{\delta} = O(area\, J^\delta\, |\log area\, J^\delta|).$$

PROOF. Without loss of generality we can assume that $P \in y-axis$, that the $x-axis$ is tangent to ∂D at the origin and that $J = \{(x(s),y(s)) : s \in (-S/2, S/2]\}$, (see Fig. 1).

Fig. 1



Thus, $\dot{x}^2(s)+\dot{y}^2(s)=1$. Since $|\nabla_Q \log|P-Q|| = \frac{1}{|P-Q|}$ is bounded whenever $|P-Q| > a > 0$, we shall have

$$\int_{J^\delta\setminus\{|P-Q|>a\}}\left|\frac{\partial}{\partial y}\log|P-Q|\right| dQ = \int_{J^\delta} O(1)dQ = O(\delta).$$

Therefore, it will suffice to prove that for $P=(0,y)$, $Q = (x(s)-r\dot{y}(s), y(s)+r\dot{x}(s))$, the integral

$$\int_{-\varepsilon}^{\varepsilon} ds \int_0^\delta \left|\frac{\partial}{\partial y}\log|P-Q|\right| dr = O(\delta\log\delta).$$

Taking into account that $|P-Q|^2 = (x(s)-r\dot{y}(s))^2 + (y-y(s)-r\dot{x}(s))^2$ we have

$$F(r,s) := \left|\frac{\partial}{\partial y}\log|P-Q|\right| = \frac{1}{2}\left|\frac{\partial}{\partial y}\log|P-Q|^2\right| = \left|\frac{y-y(s)-r\dot{x}(s)}{(x(s)-r\dot{y}(s))^2+(y-y(s)-r\dot{x}(s))^2}\right|.$$

For ε and δ sufficiently small we get $\dot{x}(s) > 1/2$ and $|(x(s) - r\dot{y}(s))/\dot{x}(s)| > |s|/2$ whenever $|s| < \varepsilon$ and $r < \delta$, (cf. note **35**).

If $A(s) := \dfrac{y - y(s)}{\dot{x}(s)}$ then,

$$F(r,s) \le \frac{1}{\dot{x}(s)} \left| \frac{((y - y(s))/\dot{x}(s) - r)}{s^2/4 + ((y - y(s))/\dot{x}(s) - r)^2} \right| \le 2 \left| \frac{4(A(s) - r)}{s^2 + 4(A(s) - r)^2} \right| \le 8 \left| \frac{A(s) - r}{s^2 + (A(s) - r)^2} \right|.$$

Hence $\frac{1}{8}\int_0^\delta F(r,s)dr \le \int_0^\delta \left| \frac{A(s) - r}{s^2 + (A(s) - r)^2} \right| dr = \int_{A(s)}^{\delta + A(s)} \frac{|x|}{s^2 + x^2} dx$. Thus, due to the auxiliary result below,

$$\frac{1}{4}\int_0^\delta F(r,s)dr \le 4\log\left(1 + \frac{\delta}{|s|}\right).$$

(7.4') PROPOSITION. Let us *define* for $a, s \in R^1$ and $\delta > 0$, $G(a) := \int_a^{a+\delta} \frac{|2x|}{s^2 + x^2} dx$. Then,

$G(a) \le 4\log\left(1 + \frac{\delta}{|s|}\right)$, (cf. note **36**).

Integrating $\int_0^\delta F(r,s)dr \le 16\log\left(1 + \frac{\delta}{|s|}\right)$ with respect to s between $-\varepsilon, \varepsilon$, we arrive to,

$$\int_{-\varepsilon}^{\varepsilon} ds \int_0^\delta F(r,s)dr \le 16 \int_{-\varepsilon}^{\varepsilon} \log\left(1 + \frac{\delta}{|s|}\right) ds = 32\delta \int_0^{\varepsilon/\delta} \log\left(\frac{u+1}{u}\right) du.$$

From $\int \log\left(\frac{u+1}{u}\right) du = u\log\left(\frac{u+1}{u}\right) + \log(1+u)$, we obtain,

$$\int_{-\varepsilon}^{\varepsilon} ds \int_0^\delta F(r,s)dr \le 32\delta \left(u\log\left(\frac{u+1}{u}\right) + \log(u+1) \right)\Bigg|_0^{\varepsilon/\delta}.$$

The first summand in the parenthesis of the preceding formula is a bounded function on the positive axis. In consequence,

$$\int_{-\varepsilon}^{\varepsilon} ds \int_0^\delta F(r,s)dr \le 32\delta \left(C + \log\frac{\varepsilon + \delta}{\delta} \right).$$

Thus, for fixed ε, δ small, it holds that,

$$\int_{-\varepsilon}^{\varepsilon} ds \int_0^\delta F(r,s)dr = O(\delta)(O(1) + \log\delta^{-1}) = O(1)\delta\log\frac{1}{\delta},$$ QED.

(7.5) LEMMA (basic lemma). Let $Q \in J_\delta$ and $p_s = p_s(y) = P_s + yn_s$, where $P_s \in \partial D$, $-\delta < y < \delta$. Then, the integral $I = I(y,Q) = \int_0^S \left|\frac{\partial}{\partial n_s}\log|p_s - Q|\right| ds$ is bounded, as a function of y and Q, for δ sufficiently small.

PROOF. The definition of p_s implies $\operatorname{dist}(p_s, \partial D) = |y|$. Therefore $p_s \in J_\delta$ and could be equal to P_s or not. We assume that we have a coordinate system as in the previous lemma and also that $Q = (0, v)$. However, unlike lemma **(7.4)**), here the integration is with respect to the length parameter s. Recall that $1 - \dot{x}^2 = \dot{y}^2 = O(s^2)$, (cf. note **35**).

Write $I = (\int_{-\varepsilon}^{\varepsilon} + \int_{|s|>\varepsilon})\left|\frac{\partial}{\partial n_s}\log|p_s - Q|\right| ds = I_1 + I_2$. Because of $|\nabla_X \log|X - Y|| = \frac{1}{|X-Y|}$, it follows that $I_2 \le M < \infty$, $M = M(\varepsilon)$. To estimate I_1, observe that,

$$p_s = (x(s) - y\dot{y}(s), y(s) + y\dot{x}(s)), \quad |p_s - Q|^2 = (x(s) - y\dot{y}(s))^2 + (y(s) + y\dot{x}(s) - v)^2,$$

$$\frac{\partial}{\partial n_s}\log|p_s - Q| = \frac{\partial((x(s) - y\dot{y}(s))^2 + (y(s) + y\dot{x}(s) - v)^2)/\partial y}{2((x(s) - y\dot{y}(s))^2 + (y(s) + y\dot{x}(s) - v)^2)} =$$

$$= \frac{y + (y(s) - v)\dot{x}(s) - x(s)\dot{y}(s)}{(x(s) - y\dot{y}(s))^2 + (y(s) + y\dot{x}(s) - v)^2}.$$

For $|s| < \varepsilon$ and ε sufficiently small, we have: $|x(s) - y\dot{y}(s)|^2 > |s|^2/2$, $\dot{x}(s) > 1/2$. After multiplying the numerator and the denominator of the last quotient by $\dot{x}(s)$, we obtain,

$$\left|\frac{\partial}{\partial n_s}\log|p_s - Q|\right| \le \frac{|(y(s) + y\dot{x}(s) - v) + (v - y(s))(1 - \dot{x}^2(s)) - x(s)\dot{y}(s)\dot{x}(s)|}{(s^2/2 + (y(s) + y\dot{x}(s) - v)^2)\dot{x}(s)}.$$

Therefore, $\frac{1}{2}\left|\frac{\partial}{\partial n_s}\log|p_s - Q|\right| \le \frac{|A(s)| + Cs^2}{s^2/2 + A(s)^2}$, where $A(s) := y(s) + y\dot{x}(s) - v$..

But $A(s) = y - v + y(s) - y(1 - \dot{x}(s)) = y - v + O(s^2)$. So, the last denominator verifies $s^2/2 + A(s)^2 \ge s^2(1/2 - c|y - v|) + (y - v)^2 \ge s^2(1/2 - c\delta) + (y - v)^2$ for some $c > 0$.

Choosing δ such that $c\delta < 1/4$, we have $s^2/2 + A(s)^2 \ge s^2/4 + (y - v)^2$. In consequence, $\frac{1}{2}\left|\frac{\partial}{\partial n_s}\log|p_s - Q|\right| \le \frac{|A(s)| + Cs^2}{s^2/2 + A(s)^2} \le \frac{|y - v| + \tilde{C}s^2}{s^2/4 + (y - v)^2}$.

Then, $I_1/2 \le \int_{-\varepsilon}^{\varepsilon} \frac{|y-v|}{s^2/4+(y-v)^2} ds + 8\widetilde{C}\varepsilon$. If $|y-v|>0$, by the change of variables $s=|y-v|t$, we obtain $I_1/2 \le \int_{-\infty}^{\infty} \frac{ds}{s^2/4+1}+8\widetilde{C}\varepsilon < \infty$, QED.

(7.6) LEMMA (boundary lemma). Assume that $\rho(Q,t)$ is a continuous function on $D\times(-S/2,S/2]$. Let $P=P_s+zn_s$, $z\in[0,\delta)$. Then,

i) $\frac{\partial}{\partial n_s}\int_{-S/2}^{S/2}(\log|P_s-Q_t|)\rho(Q,t)dt = \int_{-S/2}^{S/2}\frac{\partial}{\partial n_s}(\log|P_s-Q_t|)\rho(Q,t)dt + \pi\rho(Q,s)$,

ii) $\Sigma=\Sigma(z,s)=\int_{-S/2}^{S/2}\left|\frac{\partial}{\partial n_s}(\log|P-Q_t|)\right|dt \le R<\infty$, R independent of z and s.

PROOF. i) Without loss of generality we can assume that we have the same framework of the preceding lemma and that $s=0$, $P_s=(0,0)$.

For z positive, we have,

$$I(z)=\frac{\partial}{\partial z}\int_{-S/2}^{S/2}\rho(Q,t)\log|(0,z)-Q_t|dt = \int_{-S/2}^{S/2}\rho(Q,t)\frac{\partial}{\partial z}\log|(0,z)-Q_t|dt=$$

$$=\int_{-S/2}^{S/2}\rho(Q,t)\frac{z-y(t)}{x^2(t)+(y(t)-z)^2}dt=$$

$$=\int_{-S/2}^{S/2}\frac{z}{x^2(t)+(y(t)-z)^2}\rho(Q,t)dt+\int_{-S/2}^{S/2}\frac{-y(t)}{x^2(t)+(y(t)-z)^2}\rho(Q,t)dt=I_1(z)+I_2(z).$$

If $I(z)$ has a limit L for $z\downarrow 0$ then $\int_{-S/2}^{S/2}\rho(Q,t)\log|(0,z)-Q_t|dt$ has L as its derivative at $z=0$. Then, to prove i) it will be sufficient to prove the following equalities,

(7.7) $\lim_{z\to 0} I_2 = \int_{-S/2}^{S/2}\rho(Q,t)\frac{-y(t)}{x^2(t)+y^2(t)}dt$,

(7.8) $\lim_{z\to 0} I_1 = \pi\rho(Q,0)$.

Using the fact that $\frac{|y(t)|}{x^2(t)}\le M$ for small t, one sees that the integrand of I_2 is uniformly bounded. So, **(7.7)** follows from the Lebesgue dominated convergence theorem.

Also, the integrand of I_1 is uniformly bounded in z for $|t| > \varepsilon$. Then,

$$\lim_{z\to 0} I_1 = 0 + \lim_{z\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{z}{x^2(t) + (y(t)-z)^2} \rho(Q,t)dt.$$

If ε and z are sufficiently small, the denominator in the last integral verifies,

$x^2(t) + (y(t)-z)^2 = (t + O(t^2))^2 + (O(t^2)-z)^2 = t^2(1 + O(z) + O(t)) + z^2 > (t^2 + z^2)/2$. We have, after the change of variable $t = zu$ in the last integral, that

$$\lim I_1 = \lim \int_{|u|\le \varepsilon/z} \frac{\rho(Q,zu)}{u^2(1 + O(z) + O((uz)^2)) + 1} du.$$

The integrand is bounded by $\frac{C}{u^2+1}$ whenever ε and z are small enough. Thus,

$$\lim_{z\to 0} I_1 = \int_{-\infty}^{\infty} \frac{\rho(Q,0)}{u^2+1} du = \pi\rho(Q,0).$$

ii) Using the same estimations one can show that

$\int_{-S/2}^{S/2} \left| \frac{z - y(t)}{x^2(t) + (y(t)-z)^2} \right| dt$ is uniformly bounded in z, QED.

(7.9) Because of **(6.30)** and theorems **(6.7)**, **(6.31)-(6.35)**, we obtain from **(7.1)**,

(7.10) $$H(p,q;\lambda) = \frac{1}{2\pi}\int_0^S \rho_p(s).\, K_0(\chi|q - Q_s|)ds =$$

$$= \int_0^S \left(\frac{C_p(s)}{\pi} - \frac{\partial}{2\pi^2 \partial n_s} K_0(\chi|p - Q_s|) \right) . \frac{K_0(\chi|q - Q_s|)}{2\pi} ds = \gamma(p,q;\lambda) - \eta(p,q;\lambda),$$

where, (cf. **(6.7)**),

(7.11) $$\eta(p,q;\lambda) = \frac{1}{4\pi^3}\int_0^S \frac{\partial}{\partial n_s} K_0(\chi|p - Q_s|).K_0(\chi|q - Q_s|)ds,$$

(7.12) $$\gamma(p,q;\lambda) = \frac{1}{2\pi^2}\int_0^S \left(\int_0^S \rho_p(t) \frac{\partial}{\partial \eta_s} K_0(\chi|Q_s - Q_t|)dt \right) K_0(\chi|q - Q_s|)ds.$$

Recall that $K_0(r) = \log(1/r) + O(1)$, $K_0'(r) = e^{-r}(-1/r + O(1))$.

If U and V are the χ-harmonic functions $U(\cdot) = K_0(\chi|P - \cdot|)$, $V(\cdot) = K_0(\chi|Q - \cdot|)$, $P \neq Q$, on the bounded region $A = D \setminus \{S(P,\varepsilon) \cup S(Q,\varepsilon)\}$ then we have by Green's formula,

$$0 = \int_A (U(\Delta - \chi^2)V - V(\Delta - \chi^2)U)dx = \int_{\partial A} (UV_n - VU_n)d\sigma.$$

Then, $0 = \int\limits_{\partial D \cup \partial S(P,\varepsilon) \cup \partial S(Q,\varepsilon)} (UV_n - VU_n) d\sigma = \int\limits_{\partial D} (UV_n - VU_n) d\sigma$. In fact, using the radial symmetry of

$K_0(\chi|\cdot|)$ we obtain, $\int\limits_{\partial S(P,\varepsilon)} U_n V = \int\limits_{\partial S(Q,\varepsilon)} UV_n$, $\int\limits_{\partial S(Q,\varepsilon)} U_n V = \int\limits_{\partial S(P,\varepsilon)} UV_n$. Therefore,

$\eta(P,Q;\lambda) - \eta(Q,P;\lambda) = 0$ and a) of the following proposition holds.

(7.13) PROPOSITION. a) $\eta(p,q;\lambda) = \eta(q,p;\lambda), \quad p,q \in D$.

b) There is a constant $c = c(\lambda) > 0$ such that $(\int_D \eta^2(p,q;\lambda)\, dq)^{\frac{1}{2}} \le c(\lambda) < \infty$.

c) $\eta(p,q;\lambda) \in L^2(D \times D)$ and $\|\eta\|_2 \le c(\lambda)\sqrt{|D|}$.

PROOF. c) follows from b). To prove b) define

(7.14) $$T(\lambda) := (\int_{\{|q| \le diam D\}} K_0^2(\chi|q|)\, dq)^{1/2} < \infty.$$

Because of Minkowski's integral inequality and the data lemma **(7.3)**, we have

$$\left(\int\limits_D \eta^2(P,Q;\lambda) dQ\right)^{1/2} \le \frac{T(\lambda)}{4\pi^3} \int\limits_0^S \left|\frac{\partial}{\partial n_s} K_0(\chi|P - Q_s|)\right| ds \le \frac{M\,T(\lambda)}{4\pi^3}, \qquad \text{QED.}$$

(7.15) THEOREM (properties of Green's kernel).

i) $G(p,q;\lambda) = G(q,p;\lambda)$ if $p \ne q,\ p,q \in D$.

ii) $\gamma(p,q;\lambda)$, $H(p,q;\lambda)$ and $\eta(p,q;\lambda)$ are symmetric functions.

iii) $\int\limits_D G(p,q;\lambda) dq = \int\limits_D |G(p,q;\lambda)| dq = 1/\chi^2$.

iv) $H(p,q;\lambda) \in C(D \times \overline{D})$; $H(\cdot,q;\lambda) \in A^{\chi}(D)$ for any $q \in \overline{D}$.

v) Given $r \in [1,\infty)$, there is a positive constant $C_0 = C_0(\lambda,r)$, independent of $p \in D$, such that $(\int_D |H(p,q;\lambda)|^r\, dq)^{\frac{1}{r}} \le C_0$. Besides, $\int\limits_{J^\delta} |H(p,q;\lambda)|^r dq = O(\delta)$.

vi) $H(p,q;\lambda) \in L^r(D \times D)$, for any $r \in [1,\infty)$, and $\|H\|_r \le C_0 \sqrt[r]{|D|} < \infty$.

vii) $G(p,q;\lambda)$ verifies v) and vi) with a constant $C = C(\lambda,r)$ instead of C_0.

viii) Let $\Phi(p) = \int_D H(p,q;\lambda)\, \phi(q)\, dq$, $\phi \in L^\infty(D)$. Then $(\Delta - \chi^2)\Phi = 0$.

ix) If $\operatorname{supp}\phi := \overline{\{x : \phi(x) \ne 0\}} \subset D$ then $\Phi \in N(D)$, (see definition **(6.36)**).

PROOF. i)-iii). Assume that $t \in D, \eta > 0$ and recall that $B_\eta(t) := \{x : |x-t| < \eta\}$, $\Sigma_\eta(t) := \{x : |x-t| = \eta\}$ and $S_\eta(t) = B_\eta(t) \cup \Sigma_\eta(t) \subset D$. Suppose that $u \in N(D)$.

(7.15') $D_\eta(t) := D \setminus S_\eta(t)$; $d\sigma$ or $d\sigma_q$ will *denote* de differential of the arclength.

(7.16)
$$\int_{D_\eta(t)} G(t,q;\lambda)\,(\Delta - \chi^2)u(q)\,dq =$$
$$= \int_{D_\eta} \big(G(t,q;\lambda)(\Delta + \lambda)u(q) - u(q)(\Delta_q + \lambda)G(t,q;\lambda)\big)\,dq =$$
$$= \int_{D_\eta} \big(G(t,q;\lambda)\,\Delta u(q) - u(q)\,\Delta_q G(t,q;\lambda)\big)\,dq .$$

Since D is a C^2-region and $\partial H(p,.;\lambda)/\partial n \in C(\overline{J^\delta})$, (cf. **(7.10)**, **(6.37)**), we can use Green's theorem to obtain,

(7.17)
$$\int_{D_\eta} G(t,q;\lambda)\,(\Delta - \chi^2)u(q)\,dq = \int_{\partial D_\eta} \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} \right) d\sigma =$$
$$= \int_J \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} \right) d\sigma_q - \int_{\Sigma_\eta(t)} \left(G\frac{\partial u}{\partial r} - u\frac{\partial G}{\partial r} \right) d\sigma_q =$$
$$= \int_J G\frac{\partial u}{\partial n}\,d\sigma_q - \int_{|t-q|=\eta} \left(G\frac{\partial u}{\partial r} - u\frac{\partial G}{\partial r} \right) d\sigma_q .$$

In particular, if $u \equiv 1$, we have,

(7.18)
$$-\int_{D_\eta} G(t,q;\lambda)\,\chi^2 dq = \int_{|t-q|=\eta} \frac{\partial G}{\partial r}(t,q;\lambda)\,d\sigma_q .$$

From the definition of G and for v a continuous function in a neighborhood of $t \in D$, it holds that,

(7.19)
$$\int_{|t-q|=r} \frac{\partial G}{\partial r}(t,q;\lambda)\,v(q)\,d\sigma_q \xrightarrow[r\to 0]{} -v(t) .$$

Thus, from **(7.18)** and **(7.19)** we get iii):

$$1 - \chi^2 \int_D G(t,q;\lambda)\,dq = 0 .$$

Assume that $p \neq q$, $S = S_\eta(p) \cup S_\eta(q)$, $S_\eta(p) \cap S_\eta(q) = \varnothing$, $S \subset D$. If $u(s) = G(p,s;\lambda)$, $v(s) = G(q,s;\lambda)$ then it follows as before that,

$$0 = \int_{D\setminus S} (u\Delta v - v\Delta u)\,ds = \left(\int_J - \int_{\partial S}\right)\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma = -\int_{\partial S} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma .$$

Thus, for $\eta \downarrow 0$,

$$0 = -\int_{|s-q|=\eta} u(s)\frac{\partial G}{\partial n}(q,s;\lambda)\,d\sigma + \int_{|s-p|=\eta} v(s)\frac{\partial G}{\partial n}(p,s;\lambda)\,d\sigma + f(\eta),$$

and $f(\eta)=o(1)$, (cf. **(7.1)**). From **(7.19)** we get, $0=u(q)-v(p)=G(p,q;\lambda)-G(q,p;\lambda)$, and i) is proved. From i) and the definition of G we obtain the symmetry of H. ii) is a consequence of this fact, **(7.10)** and **(7.13)**.

iv). Assume that $\{p_n\}\subset D, p_n \to p\in D$. Then, $\{H(p_n,.;\lambda)\}\subset C(\overline{D})$ is a sequence of χ-harmonic functions on D uniformly bounded (cf. **(6.2)**, **(6.35)** and **(7.10)**).

From $\frac{\partial}{\partial n_q}H(p,q;\lambda)=\frac{\partial}{\partial n_q}\frac{1}{2\pi}K_0(\chi|p-q|),\ q\in\partial D, p\in D$, one obtains

$$\sup\left\{\left|\frac{\partial}{\partial n}H(p_n,q;\lambda)-\frac{\partial}{\partial n}H(p_m,q;\lambda)\right| : q\in\partial D\right\}\to 0 \text{ if } m,n\to\infty.$$

Thus, from **(5.49)**, it follows now that $\{H(p_n,q;\lambda)\}$ converges uniformly in $C(\overline{D})$. But, since for x,y in D, $H(x,y;\lambda)=H(y,x:\lambda)$ we have $H(p_n,q;\lambda)\to H(p,q;\lambda),\ q\in D$.

Then, $H(p_n,q;\lambda)\xrightarrow{\bullet}H(p,q;\lambda),\ q\in\overline{D}$. This implies that $H(p,q;\lambda)$ is continuous on $D\times\overline{D}$. The proof of the second part of iv) is included in the proof of viii)

v) We already know that there exists a constant M such that if $P\in D$ then $\|h_P\|_1\le M<\infty$. Then, from v) **(6.35)**, there is a constant $C=C(M)$ such that $\|\rho_P\|_1\le C$.

From **(7.10)** and the fact that $K_0(\chi|q-Q_t|$ is uniformly bounded if $q\in D\setminus J^\delta$, we get $H(p,q;\lambda)\in L^\infty(D\times(D\setminus J^\delta))$. For $q=Q^s$, $|Q^s-Q_s|=y\le\delta$, we have, (cf. **(6.13)** and note 3),

$$|H(P,Q^s;\lambda)|=\left|\int_0^S\rho_P(t)K_0(\chi|Q^s-Q_t|)dt\right|=O(1)+O(\int_0^S|\rho_P(t)|.\left|\log|s-t|\right|dt).$$

Hence, $\int_0^S|H(P,Q^s;\lambda|^r ds\le A$, with $A=A(r)$ independent of P and y.

Then, $\int_0^\delta dy\int_0^S|H(P,Q^s;\lambda)|^r ds\le A\delta$. Therefore, v) follows.

vi) is an immediate consequence of v).

vii) follows from the definition of G, v) and vi).

viii) From $H(p,q;\lambda)\in C(D\times\overline{D})$ we deduce that if $\{q_n\}\subset D$, $q_n\to q\in\overline{D}$ and $p\in K$, K a compact subset of D, then $H(p,q_n;\lambda)\xrightarrow{\bullet}H(p,q;\lambda)$. In consequence, $(\Delta_p+\lambda)H(p,q;\lambda)=0$ and $H(.,q;\lambda)$ is

χ – harmonic for any $q \in \overline{D}$. From Harnack's inequality **(5.47)** we conclude that if $p \in K_1 \subset \operatorname{int} K' \subset K' \subset D$, $q \in \overline{D}$, K_1, K' compact subsets of *D*, then,

(7.20) $$\left| \frac{\partial H(p,q;\lambda)}{\partial p_i} \right| \leq 2 \sup_{(p,q) \in K' \times \overline{D}} \frac{|H(p,q;\lambda)|}{dist(K_1, \partial K')},$$

and if $p \in K \subset \operatorname{int} K_1$, $q \in \overline{D}$, K a compact subset of *D*, then,

$$\left| \frac{\partial^2 H(p,q;\lambda)}{\partial p_i \partial p_j} \right| \leq 4 \sup_{(p,q) \in K' \times \overline{D}} \frac{|H(p,q;\lambda)|}{dist(K, \partial K_1).dist(K_1, \partial K')}.$$

That is, the derivatives of any order of $H(p,q;\lambda)$ with respect to the components of *p* are uniformly bounded for $p \in K \subset\subset D$ and $q \in \overline{D}$. Then, (cf. notes **27, 28**),

(7.21) $$\frac{\partial^{i+j} \Phi}{\partial p_1^i \partial p_2^j}(p_1, p_2) = \iint_D \frac{\partial H^{i+j}}{\partial p_1^i \partial p_2^j}((p_1,p_2),(q_1,q_2);\lambda)\, \phi(q_1,q_2)\, dq_1 dq_2 .$$

Therefore, $\left(\Delta - \chi^2\right) \Phi = \int_D \left(\Delta_p - \chi^2\right) H(p,q;\lambda)\, \phi(q)\, dq = 0$.

ix) Since $\phi(q) = 0$ in J^δ for some $\delta > 0$, the continuity of Φ up to the border follows from iv). Also, for $p \in J^\delta$, $\frac{\partial \Phi}{\partial n}(p) = \iint_D \frac{\partial H}{\partial n}(p,q;\lambda)\, \phi(q)\, dq$. But, because of **(6.43)**, $\frac{\partial H}{\partial n}(p,q)$ converges uniformly for $q \in \operatorname{supp} \phi$, $p = P^s \to P_s \in J$. Therefore,

(7.21') $$\frac{\partial \Phi}{\partial n}(p) \overset{\bullet}{\to} \frac{\partial \Phi}{\partial n}(P_s),$$

QED.

(7.22) DEFINITION. We shall say that v belongs to $N_0(D)$ if $v \in C(\overline{D}) \cap C^2(D)$ verifies, for $Q^s \in J^\delta$, $dist(Q^s, J) = y$, that $\frac{\partial v}{\partial n_s}(Q^s)$ converges *uniformly* to 0 for $y \to 0$.

Then, $N_0(D) \subset N(D)$ and $\frac{\partial v}{\partial n_s}(Q_s) = 0$.

(7.23) DEFINITION. $\mathrm{G}[\phi](p) = \int_D G(p,q;\lambda)\, \phi(q)\, dq$ is the Green operator applied to ϕ, evaluated at *p*.

(7.24) THEOREM. If $u \in N_0(D)$, $\phi \in L^\infty(D)$, $(-\Delta - \lambda)u = (-\Delta + \chi^2)u = \phi$ then, for any $p \in D$, $u(p) = \int_D G(p,q;\lambda)\, \phi(q)\, dq = \mathrm{G}[\phi](p)$.

If $\phi \in L^{\infty}(D)$ then $\sigma\phi \in C^{1}(D)$, (cf. fundamental theorem, note **26**).

If $\phi \in L^{\infty}(D) \cap C^{1}(D)$ then $\sigma\phi \in C^{2}(D)$. Accordingly, we have $u \in C^{1}(D)$ or $u \in C^{2}(D)$, respectively.

We get, because of iii) of **(7.15)** and the auxiliary proposition of note **26**, that

(7.27)
$$\frac{\partial u}{\partial p_i} = \int_D \frac{\partial G}{\partial p_i}(p,q;\lambda)\phi(q)dq\,,$$

and i) follows. Let us prove iii).

iii) If $\phi \in C^{1}(D) \cap L^{\infty}(D)$, $v(p) = \frac{1}{2\pi}\int_D K_0(\chi|p-q|)\,\phi(q)\,dq \in C^2$, as we have just shown. But by ii) of **(5.19)**, we have $(\Delta - \chi^2)(-v) = \phi$ in the sense of distributions and therefore also in the ordinary sense. Since $L \in A^{\chi}(D)$, $(-\Delta + \chi^2)u = (-\Delta + \chi^2)(v - L) = \phi$ and iii) is proved.

ii) Assume $\{\phi_n(q)\}$ is a sequence of functions of $L^{\infty}(D)$ such that $\phi_n(q) \to \phi(q)$ almost everywhere and $\|\phi_n\|_{\infty} \le A\|\phi\|_{\infty}$, with A a finite positive constant. Then, for any $p \in D$, $u_m(p) := \int_D G(p,q;\lambda)\,\phi_m(q)\,dq \to u(p)$. From **(7.15)** and Hölder's inequality, we obtain,

(7.28)
$$|u_n(p) - u_m(p)| \le \int_D |G(p,q;\lambda)|\,|\phi_n(q) - \phi_m(q)|\,dq \le C\|\phi_n(q) - \phi_m(q)\|_2 = o(1)$$

whenever $m,n \to \infty$. That is, $\{u_m\}$ converges uniformly in D and converges to u. Assume now that I_n is the characteristic function of the closed set $\{x \in D : dist(x,J) \ge 1/n\}$. Call $\phi_n = \phi I_n$. From the definition of G and ix) of **(7.15)** we have, $u_n \in C(\overline{D})$. Therefore, the sequence of extended functions $\{u_n\}$ converges uniformly to an extension of u. In particular we obtain, $u \in C(\overline{D})$.

What remains to prove of ii) is mainly a consequence of ix) of **(7.15)**; $\frac{\partial u^{\delta}}{\partial n_s}(Q^{s})$ converges uniformly because of **(7.21')** and that $\frac{\partial u^{\delta}}{\partial n_s}(Q_s) = 0$ follows from the definition of G. In consequence, $u^{\delta} \in N_0(D)$, (cf. **(7.22)**).

iv) If $\phi \in C^{1}(D) \cap L^{\infty}(D)$, the sequence $\{\phi_n(q)\}$, $\|\phi_n\|_{\infty} \le A\|\phi\|_{\infty}$, can be chosen in $C_0^{1}(D)$.

Thus, $u_n \in N_0(D)$, QED.

PROOF. Using the same notation of the previous theorem and **(7.16)**, we obtain, $-\int_{D_\eta(t)} G(t,q;\lambda)\,\phi(q)\,dq = \int_{D_\eta(t)} G(t,q;\lambda)\,((\Delta-\chi^2)u)(q)\,dq = \int_{|t-q|=\eta}\left(u\frac{\partial G}{\partial r} - G\frac{\partial u}{\partial r}\right) d\sigma_q$.

For $\eta \to 0$, we get, (cf. **(7.19)**),

$$-\int_D G(t,q;\lambda)\,\phi(q)\,dq = -u(p) - \lim_{\eta\to 0}\int_{|t-q|=\eta} G(t,q;-\chi^2)\frac{\partial u}{\partial r}(q)\,d\sigma_q = -u(p), \qquad \text{QED.}$$

(7.25) The relevant fact in the preceding proof is that for any $t \in D$,

$$\lim_{\eta\to 0}\int_{|t-q|=\eta} G(t,q;-\chi^2)(\partial u/\partial r)(q)\,d\sigma_q = 0.$$

We proved that G defines the left inverse $(-\Delta+\chi^2)^{-1}$ on the space $(-\Delta+\chi^2)(N_0(D))\cap L^\infty(D)$, (see **(8.3)**). In iii) **(7.26)** is shown that it is also a right inverse on $C^1(D)\cap L^\infty(D)$.

(7.26) THEOREM. i) Let $\phi \in L^\infty(D)$, $p \in D$. Then, $\mathrm{G}[\phi] \in (C^1 \cap L^\infty)(D)$ and for i=1,2, it holds that $(\partial/\partial p_i)(\mathrm{G}[\phi])(p) = \int_D \frac{\partial G}{\partial p_i}(p,q;\lambda)\phi(q)dq$.

ii) Let us call $u(p) = \int_D G(p,q;\lambda)\,\phi(q)\,dq = \mathrm{G}[\phi](p)$ and $u^\delta(p) = \int_{D^\delta} G(p,q;\lambda)\,\phi(q)\,dq$, where $D^\delta := D\setminus J^\delta$. Then, $u^\delta \in C^1(D)\cap C(\overline{D})$ and $\|u^\delta - u\|_\infty \to 0$ for $\delta \downarrow 0$. Moreover, $\frac{\partial u^\delta}{\partial n_s}(Q^s)$ is well defined on J^δ, is uniformly extendable to J and $\frac{\partial u^\delta}{\partial n_s}(Q_s) = 0$ for any s. Also $u = \mathrm{G}[\phi] \in C(\overline{D})$.

iii) Assume that $\phi \in C^1(D)\cap L^\infty(D)$. Then, $u \in C^2(D)$ and $(-\Delta+\chi^2)u = \phi$ in D.

iv) If $\phi \in C^1(D)\cap L^\infty(D)$ then $u(p) = \mathrm{G}[\phi](p)$ is the uniform limit of functions $u_n = \mathrm{G}[\phi_n] \in N_0(D)$, $\phi_n \in C_0^1(D)$.

PROOF. i) $u(p) = \frac{1}{2\pi}\int_D K_0(\chi|p-q|)\,\phi(q)\,dq - \int_D H(p,q;\lambda)\,\phi(q)\,dq =: v(p) - L(p)$. From viii) **(7.15)** we know that $L(p) \in A^\chi(D)$. Therefore $L(p) \in C^\infty(D)$. $v(p)$ can be written in the following form, (cf. note **26**),

$$v(p) = \sigma\phi(p) + \int_D T(p,q)\,\phi(q)\,dq = \sigma\phi(p) + \widetilde{v}(p),$$

here $T(p,q) = (|p-q|^2\log|p-q|)R(|p-q|^2) + P(|p-q|)$, R and P being entire functions. Then, $\widetilde{v}(p) \in C^2(R^2)$, (cf. note **28**). That is, $u(p) - \sigma\phi(p) = -L(p) + \widetilde{v}(p) \in C^2(D)$.

(7.29) THEOREM. Assume that $\phi \in L^{\infty}(D)$, $u(p) = \int_D G(p,q;\lambda)\,\phi(q)\,dq = \mathrm{G}(\phi)(p)$. For any $\vartheta \in C(\overline{D}) \cap C^2(D)$, it holds that $\lim_{\delta \downarrow 0} \int_{\partial D^\delta} \vartheta(p) \frac{\partial u}{\partial n_s}(p)\,d\sigma = 0$.

PROOF. Let $p = P^s \in J^\delta$ and δ small enough. Then,

$$\frac{\partial u}{\partial n_s}(P^s) = \int_D \frac{\partial}{\partial n_s} G(P^s,q;\lambda)\phi(q)\,dq = \int_{D^\delta} \frac{\partial}{\partial n_s} G(P^s,q;\lambda)\phi(q)\,dq + \int_{J^\delta} \frac{\partial}{\partial n_s} G(P^s,q;\lambda)\phi(q)\,dq$$

$$= \int_{D^\delta} \left\{ \frac{1}{2\pi} \frac{\partial}{\partial n_s} K_0\left(\chi\left|P^s - q\right|\right) - \frac{\partial}{\partial n_s} H(P^s q;\lambda) \right\} \phi(q)\,dq + \int_{J^\delta} \frac{\partial}{\partial n_s} G(P^s,q;\lambda)\phi(q)\,dq =$$

$$= I(P^s) + B(P^s).$$

From the definition of G and **(6.43)**, we have $\{\cdots\}(P^s,q) \overset{\bullet}{\to} \{\cdots\}(P_s,q) = 0$, uniformly on $(s,q) \in [0,S] \times D^\delta$. Then, $\left|I(P^s)\right| \le \varepsilon$ whenever $\left|P^s - P_s\right| \le \eta = \eta(\varepsilon)$. Therefore,

$I(P^s) = \frac{\partial}{\partial n_s} \int_{D^\delta} G(P^s,q;\lambda)\phi(q)\,dq \overset{\bullet}{\to} 0$ and

(7.30)
$$\lim_{\delta \downarrow 0} \int_{\partial D^\delta} \vartheta(p) I(p)\,d\sigma = 0.$$

On the other hand, it holds that, (cf. **(7.10)**),

(7.31)
$$B(P^s) = \int_{J^\delta} \frac{\partial}{2\pi \partial n_s} K_0\left(\chi\left|P^s - q\right|\right)\phi(q)\,dq - \int_{J^\delta} \frac{\partial}{\partial n_s} H(P^s,q;\lambda)\phi(q)\,dq =$$

$$= B_K(J^\delta)(P^s) - B_H(J^\delta)(P^s) = B_K(J^\delta)(P^s) - \int_{J^\delta} \phi(q)\,dq \int_0^S \rho_q(t) \frac{\partial}{2\pi \partial n_s} K_0\left(\chi\left|P^s - Q_t\right|\right) dt.$$

Let $P^s, q \in J^\delta$. Then we have $\frac{\partial}{\partial n_s} K_0(\chi\left|P^s - q\right|) = \frac{\partial}{\partial n_s} \log\left|P^s - q\right|^{-1} + v(\left|P^s - q\right|)$, with $v(\left|x - y\right|)$ a bounded function, continuous on $x \ne y$, (cf. proof of i) **(7.3)**).

Call $b(P^s) := \int_{J^\delta} \phi(Q) \frac{\partial}{\partial n_s} \log\left|P^s - Q\right|^{-1} dQ$. From the area lemma **(7.4)** we obtain,

$$\left|b(P^s)\right| \le \left\|\phi\right\|_\infty \int_{J^\delta} \left|\frac{\partial}{\partial n_s} \log\left|P^s - Q\right|\right| dQ = O(\delta \log\frac{1}{\delta}),$$ uniformly on $P^s \in J^\delta$.

Therefore, $\lim_{\delta\downarrow 0} \int_{\partial D^\delta} \vartheta(p)b(p)d\sigma = 0$ and because of **(7.31)**, we get,

(7.32) $$\lim_{\delta\downarrow 0} \int_{\partial D^\delta} \vartheta(p)B_K(J^\delta)(p)d\sigma = 0.$$

We want to prove that

(7.33) $$\lim_{\delta\downarrow 0} \int_{\partial D^\delta} \vartheta(p)B_H(J^\delta)(p)d\sigma = 0.$$

Define $h(P^s) := \int_{J^\delta} \phi(Q)dQ \int_0^S \rho_Q(t)\frac{\partial}{\partial n_s}\log\left|P^s - Q_t\right|^{-1} dt$. Note that due to the data lemma **(7.3)** and v) **(6.35)**, $\int_0^S \left|\rho_Q(t)\right|dt$ is uniformly bounded on $Q \in D$. Thus, because of **(7.31)**, it will be sufficient to prove that $\lim_{\delta\downarrow 0} \int_{\partial D^\delta} \vartheta(p)h(p)d\sigma = 0$.

The basic lemma **(7.5)** implies that $\int_{\partial D^\delta} \left|\frac{\partial}{\partial n_s}\log\left|P^s - Q_t\right|\right| d\sigma$ is uniformly bounded on $Q_t \in J$ and δ small.

In consequence, we obtain,

(7.34) $$\left|\int_{\partial D^\delta} \vartheta(p)h(p)d\sigma\right| \le \int_{J^\delta} \left|\phi(Q)\right|dQ \int_0^S \left|\rho_Q(t)\right|dt \int_{\partial D^\delta} \left|\frac{\partial}{\partial n_s}\log\left|P^s - Q_t\right|\right| d\sigma \le$$

$$\le O(1) \int_{J^\delta} \left|\phi(Q)\right|dQ \int_0^S \left|\rho_Q(t)\right|dt \le O(1) \int_{J^\delta} \left|\phi(Q)\right|dQ = O(\delta).$$

From **(7.30)**, **(7.32)** and **(7.33)** the theorem follows, QED.

(7.35) One could describe the content of Theorem **(7.29)** saying that the normal derivatives of the functions in $G\left[L^\infty\right]$ have a null *weak uniform limit* at the boundary.

CHAPTER 8

(8.1) CLASSICAL EIGENFUNCTIONS OF THE NEUMANN PROBLEM.

The Green operator G=G$_{\lambda}$=G$_{-\chi^2}$ defined by $\mathrm{G}[f](p) = \int_D G(p,q;\lambda)\, f(q)\, dq$, is a Hilbert-Schmidt operator. We show that the family of eigenfunctions of this operator is complete in $L^2(D)$ and characterize them.

(8.2) DEFINITION. The function $f \in C(\overline{D}) \cap C^2(D)$ will be called *admissible* if for any $\vartheta \in C_0^\infty(R^2)\big|_D$ it holds that $\lim_{\delta \downarrow 0} \int_{\partial D^\delta} \vartheta(p) \frac{\partial f}{\partial n_s}(p) d\sigma = 0$. We shall denote **A**=**A**(D) the family of admissible functions.

Observe that always $N_0(D) \subset$ **A**(D) but it could be that $N_0(D) \neq$ **A**(D), (cf. **(8.6)**).

To characterize the eigenfunctions we need the following extension of theorem **(7.24)**.

(8.3) THEOREM. If u is admissible, $\phi \in L^\infty(D)$, $-(\Delta + \lambda)u = \phi$ then, for any $p \in D$, $u(p) = \int_D G(p,q;\lambda)\, \phi(q)\, dq$=G$[\phi](p)$.

PROOF. Let us repeat well known arguments. Assume that $\eta > 0$, $S_\eta(t) \subset D^\delta$ and call $D_\eta^\delta(t) = D^\delta \setminus S_\eta(t)$. By hypothesis $u \in C^2(D)$. Then,

$$\int_{D_\eta^\delta(t)} G(t,q;\lambda)\, (\Delta + \lambda) u(q)\, dq = \int_{D_\eta^\delta} \big(G(t,q;\lambda)(\Delta + \lambda)u(q) - u(q)(\Delta_q + \lambda) G(t,q;\lambda)\big) dq =$$

$$= \int_{D_\eta^\delta} \big(G(t,q;\lambda)\, \Delta u(q) - u(q)\, \Delta_q G(t,q;\lambda)\big) dq .$$

Since D is a C^2-region and $\partial H(t,\cdot;\lambda)/\partial n \in C(J^\delta)$, we obtain using Green's theorem, (cf. **(6.43)** and note **39**),

$$\int_{D_\eta^\delta} G(t,q;\lambda)\, (\Delta - \chi^2) u(q)\, dq = \int_{\partial D_\eta^\delta} \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} \right) d\sigma =$$

$$= \int_{\partial D^\delta} \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} \right) d\sigma_q - \int_{\Sigma_\eta(t)} \left(G\frac{\partial u}{\partial r} - u\frac{\partial G}{\partial r} \right) d\sigma_q =$$

$$= \int_{\partial D^\delta} G\frac{\partial u}{\partial n}\, d\sigma_q + o(1) - \int_{|t-q|=\eta} \left(G\frac{\partial u}{\partial r} - u\frac{\partial G}{\partial r} \right) d\sigma_q ,$$

where $o(1)$ is for $\delta \to 0$.

Letting $\delta \to 0, \eta \to 0$, we get, because of the hypothesis and **(7.19)**,

$$-\int_D G(t,q;\lambda)\,\phi(q)\,dq = 0-\lim_{\eta\to 0}\int_{|t-q|=\eta} G(t,q;-\chi^2)\frac{\partial u}{\partial r}(q)\,d\sigma_q - u(p)=0-u(p), \qquad \text{QED.}$$

(8.4) THEOREM. Assume that $\mu \neq 0$, *f* non null and $-\lambda = \chi^2$, $\chi > 0$. Then, the following propositions i) and ii) are equivalent and imply iii), (cf. also **(8.20)**),

i) $f \in L^2(D)$, $f(p) = \mu\int_D G(p,q;\lambda)\, f(q)\, dq$.

ii) f is admissible, $-(\Delta+\lambda)f = \mu f$ on *D*.

iii) *f* is the uniform limit of a sequence of functions of $N_0(D)$.

PROOF. Suppose *f* satisfies i). From vii) **(7.15)** and the Cauchy-Schwarz inequality we obtain $f \in L^\infty(D)$. From i) **(7.26)** we get that $f \in C^1(D)$. From iii) **(7.26)** it follows that $f \in C^2(D)$ and $-(\Delta+\lambda)f = \mu f$. Now, theorem **(7.29)** implies that $\lim_{\delta\downarrow 0}\int_{\partial D^\delta} \vartheta(p)\frac{\partial f}{\partial n_s}(p)\,d\sigma = 0$ and i)$\Rightarrow$ii) is proved.

From iv) **(7.26)** we know that *f* is the uniform limit of a sequence of functions of $N_0(D)$, (cf. **(7.22)**), and i)$\Rightarrow$iii).

Moreover, from ii) **(7.26)** it follows that it is also the uniform limit of the sequence of functions $f^\delta = \mu\int_{D^\delta} Gf\,dq$, $\delta = 1/n$, with null normal derivative on the boundary of *D*.

ii)$\Rightarrow$i). $\phi := \mu f \in L^\infty$. Because of **(8.3)** we have $f(p) = \mathrm{G}[\phi](p) = \mathrm{G}[\mu f](p)$, QED.

(8.5) Since *f* is admissible, $f \in L^\infty(D)$. We also have $\nabla f \in L^1$, (cf. **(8.8)**). In fact, it follows from i) **(7.26)** and the next theorem where we prove that $\dfrac{\partial G}{\partial p_i} \in L^1(D\times D)$.

THEOREM. $\int_{D\times D}|\nabla_p G(P,Q)|\,dPdQ \le K < \infty$.

PROOF. Because of the Fundamental Theorem of note **26**, it is sufficient to prove that $\dfrac{\partial H}{\partial p_i} \in L^1(D\times D)$.

We have, (cf. **(6.37)**),

$$\nabla w = \nabla\int_0^S K_0(\chi|P-Q_t|)\rho(t)dt = \int_0^S \chi K_0'(\chi|P-Q_t|)\nabla|P-Q_t|\rho(t)dt.$$

This implies that $\left|\nabla\int_0^S K_0(\chi|P-Q_t|)\rho(t)dt\right| \le \int_0^S \chi|K_0'(\chi|P-Q_t|)||\rho(t)|dt \le \chi\int_0^S \frac{C|\rho(t)|}{|P-Q_t|}dt$,

where *C* is a constant. Then, if $\rho = \rho_Q$,

$$|\nabla_P H(P,Q)| = \left|\nabla_P \int_0^S K_0(\chi|P-Q_t|)\rho_Q(t)dt\right| \le \chi C \int_0^S \frac{|\rho_Q(t)|}{|P-Q_t|}dt \Rightarrow \int_D |\nabla_P H| dP \le C' \int_0^S |\rho_Q(t)|dt.$$

Using the data lemma **(7.3)** and **(6.35)**, we obtain,

$$\|\nabla_P H\|_{L^1(D\times D)} \le C' \int_D \|\rho_Q\|_1 dQ \le C'' M|D| = K, \qquad \text{QED.}$$

(8.6) EXAMPLE. There is a function F=$F(p)$, $p=(x,y)$, admissible, such that $\partial F/\partial n = 0$ does not hold at each point of the boundary. We suppose that D is any C^2-Jordan region such that: $V := [-1,1]\times(0,1] \subset D \subset \{(x,y): y>0\}$, $[-1,1]\times\{0\} \subset \partial D$.

Call $\varphi(x) \in C_0^\infty(-1,1)$, $\varphi(x)=1$ for $-1/2<x<1/2$ and for $y\ge 0$, $\psi(y)=\varphi(y)y^a$ $0<a<1$, $\tau(y)=y^{a/2}$. Define, $F(x,y)=\psi(y)\varphi(x/\tau(y))$ for $(x,y)\in V$, $F(x,0)=0$ and $F(x,y)=0$ for $(x,y)\in \overline{D}\setminus V$. Then, $F\in C(\overline{D})\cap C^2(D)$. Besides, $F(x,y)=0$ for $|x|\ge\tau(y)$, or equivalently, $F(x,y)=0$ if $0\le y\le |x|^{2/a}$. Therefore, if $x\ne 0$, then $F_y'(x,y)\to 0$ for $y\downarrow 0$. On the contrary, for $x=0$, $F_y'(0,y)=\psi'(y)\to\infty$. Thus $F\notin N_0(D)$.

Finally, on D, $F_y'(x,y)=\psi'(y)\varphi(x/\tau(y))-\psi(y)\varphi'(x/\tau(y))x\tau'(y)\tau^{-2}(y)$.

Hence, for $y<1/2$, $F_y'(x,y)=ay^{a-1}\varphi(x/\tau(y))-(a/2)xy^{a/2-1}\varphi'(x/\tau(y))$. Therefore,

$$\int_{-1}^{1} |F_y'(x,y)|dx \le C \int_{-\tau(y)}^{\tau(y)} \left(y^{a-1}+|x|y^{a/2-1}\right)dx = C\left(2\tau(y)y^{a-1}+\tau^2(y)y^{a/2-1}\right) \le C' y^{3a/2-1}.$$

Choosing a such that $3a/2>1$ we get $\int_{-1}^{1}|F_y'(x,y)|dx \to 0$ for $y\downarrow 0$. This proves that F is admissible. We conclude then that $F\in \mathbf{A}(D)\setminus N_0(D)\ne\varnothing$.

NB. F satisfies iii), **(8.4).** Assume that the sequence of functions $\chi_n\in C^\infty(0,\infty)$ verifies $0\le\chi_n(y)\le 1$ and also that $\chi_n=0$ on $(0,1/n]$, $\chi_n=1$ on $[2/n,\infty)$. If we define $F_n(x,y)=\psi(y)\chi_n(y)\varphi(x/\tau(y))$, we have $F_n\in C(\overline{D})\cap C^2(D)$. Its normal derivative on ∂D is null. Moreover, $|F-F_n|\le 2\sup_{0<y<2/n}|\psi(y)|$.

That is, $\|F-F_n\|_\infty\to 0$.

(8.7) DEFINITION. We call a function f a *classical eigenfunction of the Neumann problem for the metaharmonic operator* $-\Delta+\chi^2$, *corresponding to the eigenvalue* μ, if f is admisible and $(-\Delta+\chi^2)f=\mu f$. We can also say in this case that f is a *classical eigenfunction of the Neumann problem* *for the operator* $-\Delta$, *corresponding to the eigenvalue* $\mu - \chi^2 = \mu + \lambda$. *Variational eigenfunctions* for the operator $-\Delta$ were introduced in **(1.13)**.

(8.8) THEOREM. Any classical eigenfunction of the Neumann problem for the metaharmonic operator corresponding to the eigenvalue μ is a variational one for $-\Delta$ and the eigenvalue $\mu + \lambda$. (Therefore, $\mu \geq -\lambda = \chi^2 > 0$).

PROOF. Let *f* be admisible and $(-\Delta + \chi^2)f = \mu f$ in *D*. Assume that $\vartheta \in C_0^\infty(R^2)\big|_D$ Applying Gauss' theorem to $\nabla f \times \nabla \vartheta = div(\vartheta \nabla f) - \vartheta \Delta f$ we obtain $\int\limits_{\partial D^\delta} \vartheta \frac{\partial f}{\partial n_e} - \int\limits_{D^\delta} \vartheta \Delta f = \int\limits_{D^\delta} \nabla f \times \nabla \vartheta$. Then,

$$-\int\limits_{\partial D^\delta} \vartheta \frac{\partial f}{\partial n_i} + \int\limits_{D^\delta} \vartheta(\mu + \lambda) f = \int\limits_{D^\delta} \nabla f \times \nabla \vartheta. \tag{8.9}$$

Observe that for δ small enough the interior normal n_s at $Q_s \in J$ coincides with the interior normal n_i to ∂D^δ at the intersection of n_s with ∂D^δ, $n_s = n_i$, (cf. note **39**). Therefore, since $f \in$ **A**(*D*), we have $\lim\limits_{\delta \downarrow 0} \int\limits_{\partial D^\delta} \vartheta \frac{\partial f}{\partial n_i} = 0$. Letting $\delta \to 0$ in **(8.9)**, we obtain $-(\mu + \lambda) \int\limits_D \vartheta f + \int\limits_D \nabla f \times \nabla \vartheta = 0$, $\forall \vartheta \in C_0^\infty(R^2)\big|_D$.

But $C_0^\infty(R^2)\big|_D$ is dense in $H^1(D)$. Thus, *f* is a non trivial weak solution of $(-\Delta + (-\mu - \lambda))f = 0$. That is, $f \in H^1(D)$ is a weak eigenfunction of $-\Delta$ corresponding to the eigenvalue $\mu + \lambda$:

$$-\Delta f = (\mu + \lambda) f, \qquad \text{QED.}$$

(8.10) THEOREM. Any variational Neumann eigenfunction for a bounded C^2-Jordan region is also a classical eigenfunction.

PROOF. Let *f* be admisible and $(-\Delta + \chi^2)f = \mu f$ in *D*. This is, by theorem **(8.4)**, equivalent to say that *f* is an *eigenfunction of the metaharmonic Green operator* G=G$_\lambda$=G$_{-\chi^2}$ corresponding to the eigenvalue $1/\mu$. That is, $f = \mu\, \mathsf{G}(f)$.

A function $u \in C_0^\infty(D)$ verifies $\partial u / \partial n = 0$ on J and also $-(\Delta + \lambda)u = \phi \in L^\infty$. Then, $u = \int\limits_D G\phi\, dq = \mathsf{G}[\phi]$. In consequence, G has a range dense in L^2. Since G is symmetric, 0 is not an eigenvalue. Let E=$\{e_k\}$ be the orthonormal family of classical eigenfunctions such that $\mathsf{G}(e_k) = \mu_k^{-1} e_k$.

Any function in the range of G admits an expansion in the eigenfunctions of the operator. Therefore, the family E=$\{e_k\}$ is complete.

Because of Theorem **(8.8)**, E is also a complete normalized family of variational eigenfunctions., QED.

Taking into account **(8.7)-(8.10)** we get i) and ii) of next theorem.

(8.11) THEOREM. Assume that the Neumann eigenvalue problem is posed on a bounded C^2-Jordan region D. Then,

i) Classical and variational eigenfunctions coincide.

ii) The eigenvalues of Neumann problem for the metaharmonic differential operator are positive and of finite multiplicity.

iii) $\mathrm{G}_{-\chi^2} = \mathrm{G} = L_{\chi^2}^{-1}$.

PROOF. iii) Observe that for $e_k \in \mathrm{E}$ we have $(-\Delta + \chi^2)e_k = \mu_k e_k$. Theorems **(8.3)** and **(8.8)** yield respectively $\mathrm{G}(\mu_k e_k) = e_k$, and $\int_D \nabla e_k \times \nabla v\, dx + \chi^2 \int_D e_k v dx = \int_D \mu_k e_k v dx \quad \forall v \in H^1(D)$. Therefore,

$L_{\chi^2}^{-1}(\mu_k e_k) - \mathrm{G}(\mu_k e_k) = e_k - e_k = 0$, (cf. **(1.17)**). Hence, iii) follows, QED.

(8.12) COROLLARY. i) $0 < \mu_j \uparrow \infty,\ j = 0,1,2,\cdots;\ \chi^2 = \mu_0 < \mu_1 \le \mu_2 \le \cdots$.

ii) $\sigma(\mathrm{G}) = \{1/\mu_j : j = 0,1,2,\cdots\}$.

iii) $f \ne 0 \Rightarrow (\mathrm{G}f, f) > 0$.

iv) $1/\mu_0 = \sup_{\|f\|_2 = 1} \iint G(p,q,\lambda) f(p) f(q) dpdq$ =((G))=operator norm of G.

v) χ^{-2} = ((G)) $\le \|G\|_2 = \sqrt{\iint G^2 dpdq}$ =Hilbert-Schmidt norm of G.

PROOF. i) In fact, $-\Delta f_i = (-\chi^2 + \mu_i) f_i$, $-\chi^2 + \mu_i \ge 0$, (cf. Introduction).

ii)-v) follow from i), properties of Hilbert-Schmidt operators, (cf. [RN]), iii) **(7:15)** and the fact that the eigenfunction for μ_0 is a constant function, QED.

(8.13) COROLLARY. i) $G(p,q,\lambda) = \sum_0^\infty \mu_j^{-1} e_i(p) e_i(q),\ L^2(D \times D)$.

ii) If $f \in L^2$ then $\mathrm{G}(f)(p) = \sum \mu_j^{-1}(f, e_i) e_i(p)$, where the expansion converges uniformly in $p \in D$ and absolutely for each p.

PROOF. In fact, $e_j = \overline{e_j}$ and $\int G^2 dq \le C^2 < \infty$ with C independent of p, (cf. vii) **(7.15)** and [RN]), QED.

(8.14) THEOREM. Let D be a bounded C^2-Jordan region.

i) The integrand in $H(P,Q;\lambda) = (1/2\pi)\int_0^S K_0(\chi|P-Q_s|)\rho_Q(s)ds,\ P,Q \in D$, (cf. **(7.10)**), is such that the function $\rho(Q,s) := \rho_Q(s) \in C(D\times[0,S])$.

ii) $\rho(Q,s)$ belongs to $L^1([0,S])$ uniformly in Q and to $L^1(D)$ uniformly in s.

iii) $k(P) := \int_0^S |K_0(\chi|P-Q_s|)|ds \in C(\overline{D})$.

iv) $\varphi(s) = \int_D f(Q)\rho(Q,s)dQ \in C([0,S])$ if $f \in L^\infty$.

v) $F(P) := \mathrm{G}[f](P) = \int_D G(P,Q;\lambda)\, f(Q)\, dQ$, $f \in L^\infty$, verifies $\dfrac{\partial F}{\partial n_s}(P_s) = 0$ at any $P = P_s \in J$.

PROOF. i) $\rho_Q(s)$ satisfies the equation

(8.15) $$-\rho_Q(s) + \int_0^S K(s,t)\rho_Q(t)dt = \frac{h_Q(s)}{\pi} = \frac{1}{2\pi^2}\frac{\partial}{\partial n_s}K_0(\chi|P_s - Q|),$$

where K is a bounded kernel defined on $[0,S]\times[0,S]$, (cf. **(6.33)**). Because of this and iv) **(6.35)**, it follows that

(8.16) $\rho_Q(s) = \rho(Q,s)$ is a continuous function of $(Q,s) \in D\times[0,S]$.

ii) Because of the data lemma of Chapter 7 and **(6.35)**, we have $\int_0^S |\rho_Q(s)|ds \le M < \infty$ where M is independent of Q.

Since K is a bounded kernel on $[0,S]\times[0,S]$, the integral in **(8.15)** is bounded by $\|K\|_\infty M$. Thus,

$$\int_D |\rho(Q,s)|dQ \le M'\left(|D| + \int_{|Q|\le diamD} |Q|^{-1}dQ\right) = M'' < \infty .$$

iii) From **(6.2)** we obtain,

$k(P) = \int_0^S |K_0(\chi|P-Q_s|)|ds = \int_0^S K_0(\chi|P-Q_s|).1\, ds \le N < \infty$ with N independent of $P \in \overline{D}$. Moreover, $k \in C(\overline{D})$.

iv) Let us write $\varphi(s) = \int_{D\setminus J^\delta} f(Q)\rho(Q,s)dQ + \int_{J^\delta} f(Q)\rho(Q,s)dQ = I(s) + R(s)$.

Since $D\setminus J^{\delta}$ is a compact set included in *D*, from **(8.16)** it follows that $I(s)\in C([0,S])$. It remains to prove that $R(s)=\int_{J^{\delta}} f(Q)\rho(Q,s)dQ\in C([0,S])$. But,

$$R(s)=-\int_{J^{\delta}} f(Q)dQ\int_0^S K(s,t)\rho_Q(t)dt-\int_{J^{\delta}} f(Q)\frac{1}{2\pi^2}\frac{\partial}{\partial n_s}K_0(\chi|P_s-Q|)dQ=L(s)+M(s).$$

Since $K(.,.)\in C([0,S]\times[0,S])$ and $\|\rho_Q\|_1$ is uniformly bounded, $\int_0^S K(s,t)\rho(Q,t)dt$ is bounded on (s,Q) and continuous as a function of *s* for fixed *Q*. Therefore, $L(s)$ is a continuous function.

$\frac{\partial}{\partial n_s}K_0(\chi|P_s-Q|)$ is of the form $\cos(n_s,Q-P_s)\left(c(|P_s-Q|)+\frac{b(|P_s-Q|)}{|P_s-Q|}\right)$ with $c(x)$, $b(x)$ continuous and locally bounded functions of $x\in[0,\infty)$, (cf. **(6.16)**). Thus, *M* is also continuous.

v) We have, $G(P,Q;\lambda)=K_0(\chi|P-Q|)/2\pi-H(P,Q;\lambda)$. Then

$$\int_D G(P,Q;\lambda)f(Q)dQ=(1/2\pi)\int_D K_0(\chi|P-Q|)f(Q)dQ-\int_D H(P,Q;\lambda)f(Q)dQ.$$

Because of ii) **(7.26)**, the first integral defines a function in $C(\overline{D})$. The second integral defines a continuous function of $P\in\overline{D}$, as it is easy to see. Therefore, the third integral defines also a function belonging to $C(\overline{D})$. Thus, to prove v) it suffices to show that,

(8.17) $$\frac{\partial}{\partial n_s}\int_D G(P,Q,\lambda)f(Q)dQ=\int_D\frac{\partial}{\partial n_s}G(P,Q,\lambda)f(Q)dQ,$$

since the last integral is equal to $\int_D 0.fdQ$.

From the fundamental theorem of note **26**, it follows that,

$$\frac{\partial}{\partial n_s}\int_D\frac{K_0(\chi|P-Q|)}{2\pi}f(Q)dQ=\int_D\frac{\partial}{\partial n_s}\frac{K_0(\chi|P-Q|)}{2\pi}f(Q)dQ,$$

then, to prove **(8.17)** we need to show that next equality holds,

(8.18) $$\frac{\partial}{\partial n_s}\int_D H(P,Q;\lambda)f(Q)dQ=\int_D\frac{\partial}{\partial n_s}H(P,Q;\lambda)f(Q)dQ.$$

But, $\frac{\partial}{\partial n_s}\int_D H(P,Q,\lambda)f(Q)dQ = \frac{\partial}{\partial n_s}\int_D f(Q)dQ\int_0^S K_0(\chi|P-Q_t|)\rho_Q(t)dt =$

$$= \frac{\partial}{\partial n_s}\int_0^S K_0(\chi|P-Q_t|)dt\int_D f(Q)\rho(Q,t)dQ = \frac{\partial}{\partial n_s}\int_0^S K_0(\chi|P-Q_t|)\varphi(t)\,dt.$$

Because of iv), since f is a bounded function, $\varphi(s) = \int_D f(Q)\rho(Q,s)dQ$ is a continuous function on $[0,S]$.

Then, φ is a bounded function. Because of **(6.34)** we have,

(8.19) $$\frac{\partial}{\partial n_s}\int_0^S K_0(\chi|P-Q_t|)\varphi(t)dt = \int_0^S \frac{\partial}{\partial n_s}K_0(\chi|P-Q_t|)\varphi(t)dt - \pi\varphi(s) =$$

$$= \int_0^S \pi K(s,t)\left(\int f(Q)\rho(Q,t)dQ\right)dt - \pi\int_D f(Q)\rho(Q,s)dQ.$$

We have seen that $\rho(Q,s) \in L^1(D\times[0,S])$. Therefore, the last member is equal to,

$$= \pi\int_D f(Q)\left(\int_0^S K(s,t)\rho_Q(t)dt - \rho(Q,s)\right)dQ.$$

The last expression is equal to, (cf. **(8.15)**),

$$= \int_D f(Q)\frac{1}{2\pi}\frac{\partial}{\partial n_s}K_0(\chi|P_s-Q|)dQ = \int_D f(Q)\frac{\partial}{\partial n_s}H(P_s,Q,\lambda)dQ,$$

and v) is proved, QED.

(8.20) THEOREM. Assume D is a bounded C^2-Jordan region.

i) Let $\mu>0$, $-\lambda=\chi^2, \chi>0$ and $0\neq f\in L^2(D)$. If $f(p)=\mu\int_D G(p,q;\lambda)\,f(q)\,dq$ then $f\in C^2(D)\cap C(\overline{D})$, $-(\Delta+\lambda)f=\mu f$ on D, $f\in$ **A**(D) and $\frac{\partial f}{\partial n_s}(Q_s)=0$, $Q_s\in J$.

ii) $\mu_n \approx n,\ n>0$.

PROOF. i) is a consequence of theorem **(8.14)** and is a complement to theorem **(8.4)**.

ii) follows from **(2.26),** QED.

CHAPTER 9

(9.1) In this chapter D will denote a bounded C^2-Jordan region as in chapters 6-8. Recall that these regions have the uniform ball property: there exists a positive number R such that for any point x of the boundary of D there is a disk of radius R, K_x, contained in $\overline{D}$ such that $K_x \cap \partial D = \{x\}$. If $y \neq x$ belongs to K_x and is on the interior normal n_x then $\hat{y}$ will *denote* the symmetric point of y with respect to the tangent at x.

If R is sufficiently small then $\hat{y} \notin \overline{D}$ and $\hat{y} = \hat{z}$ implies $y = z$, (cfr. Note **3**). We shall *assume* that R is so chosen. We shall eventually add extra conditions, making R smaller.

(9.2) Recall that for such a region, the *metaharmonic Green's kernel for the Neumann problem* and $\chi > 0$ verifies:

1) $G(x,y;-\chi^2) = (2\pi)^{-1} K_0(\chi|x-y|) - H(x,y;-\chi^2) \geq 0$, $x,y \in D$,

2) $-\Delta_x H(x,y;-\chi^2) + \chi^2 H(x,y;-\chi^2) = 0$, $(x,y) \in D \times D$,

3) $H(x,\cdot;-\chi^2) \in A^{\chi}(D) \cap C(\overline{D})$ for $x \in D$,

4) $H(x,y;-\chi^2) \in C(D \times \overline{D}) \cap L^r(D \times D)$, $r \in [1,\infty)$,

5) $H(y,x;-\chi^2) = H(x,y;-\chi^2)$ on $D \times D$,

6) $\dfrac{\partial}{\partial n_x} H(x,y;-\chi^2) = \dfrac{1}{2\pi} \dfrac{\partial}{\partial n_x} K_0(\chi|x-y|)$ for $x \in \partial D$, $y \in D$,

7) $\left(\int_D G^2(x,y;\lambda)\, dy\right)^{\frac{1}{2}} \leq M = M(\lambda) < \infty$, $\lambda = -\chi^2$,

8) $(-\Delta_y + \chi^2) G(x,y;-\chi^2) = \delta(x-y)$, $x,y \in D$, (cf. **(5.19)**).

The function $\mathrm{H}_\lambda(x) := H(x,x;\lambda) \in C(D)$. We shall prove that $\mathrm{H}_\lambda \in L^1$ and also estimate its integral,

(9.3)
$$I = I(-\chi^2) = \int_D \mathrm{H}_\lambda(x)dx = \int_D H(x,x;-\chi^2)dx.$$

For this we shall use the approximation ,

(9.4)
$$H(x,y;\lambda) = -K_0(\chi|x-\hat{y}|)/2\pi + \Gamma(x,y;\lambda),$$

and find a bound for the integral of the error Γ.

(9.5) *Define* $Q(\chi,R) := \dfrac{I_0(\chi R)}{\chi I_0'(\chi R)}$, $Q(\chi) := Q(\chi,1)$. The behaviour of Q for $\chi \geq \chi_0 > 1$ is $Q \approx 1/\chi$, *i.e.*, there are constants A and B such that $A/\chi \leq Q \leq B/\chi$. We restrict ourselves to prove only the following lemma.

(9.6) LEMMA. i) There is a constant B such that for $x \geq \alpha > 0$ it holds that

$$Q(x) := Q(x,1) \leq \frac{B}{x}.$$

ii) Also $Q(\chi, R) \leq B/\chi$, $\chi R \geq \alpha$.

PROOF. i) For $x > 0$ we have, (cf. [MO], p.32),

$$I_0(x)K_0(x) = \int_0^\infty \frac{J_0^2(t)t}{x^2+t^2}dt, \tag{9.7}$$

where $tJ_0^2(t) \leq M$, M a constant.

On the other hand the wronskian, (cfr. **(5.19)**),

$$I_0'(x)K_0(x) - I_0(x)K_0'(x) = 1/x. \tag{9.8}$$

From **(9.7)** we obtain, $(I_0'K_0 + I_0K_0')(x) = -2x\int_0^\infty \frac{tJ_0^2(t)}{(x^2+t^2)^2}dt$.

Then, $(I_0'K_0)(x) = \dfrac{1}{2x} - x\displaystyle\int_0^\infty \frac{tJ_0^2(t)}{(x^2+t^2)^2}dt$. Therefore,

$$Q(x) = \frac{I_0(x)K_0(x)}{x\,I_0'(x)K_0(x)} = \frac{\displaystyle\int_0^\infty \frac{tJ_0^2(t)}{x^2+t^2}dt}{\dfrac{1}{2} - x^2\displaystyle\int_0^\infty \frac{tJ_0^2(t)}{(x^2+t^2)^2}dt}. \tag{9.9}$$

The numerator is not greater than $\displaystyle\int_0^\infty \frac{M}{x^2+t^2}dt = \frac{M'}{x}$. The integral in the denominator is not greater than $\displaystyle\int_0^\infty \frac{M}{\left(x^2+t^2\right)^2}dt = \frac{M''}{x^3}$. Thus, if $x \geq \alpha := 4M''$ one gets from **(9.9)**:

$$0 < Q(x) \leq \frac{M'/x}{1/2 - M''/x} \leq \frac{M'/x}{1/2 - 1/4} = B/x, \text{ with } B = 4M'.$$

ii) Observe that $Q(\chi, R) = RQ(\chi R, 1)$. Using i), one gets $Q(\chi, R) \leq RB/(R\chi) = B/\chi$ for $\chi R > \alpha$.

QED.

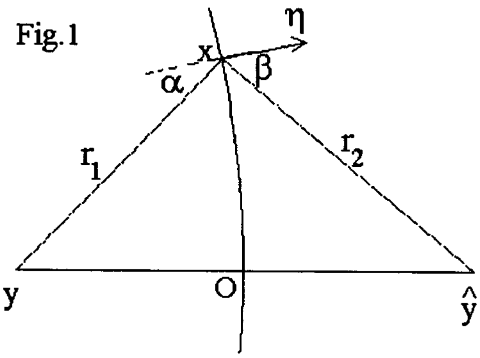
(9.10) If $(-\Delta + \chi^2)u = 0$ on D, $u \in C(\overline{D})$ and $\partial u/\partial n_x$ exists at each point $x \in \partial D$, the subordination principle **(5.50)** can be written as

$$\sup_{x\in D}|u| \leq \frac{B}{\chi}\sup_{x\in\partial D}\left|\frac{\partial u}{\partial n_x}\right| \text{ for } \chi > \alpha/R.$$

From **(9.4)**, $\Gamma(x,y;-\chi^2) = H(x,y;-\chi^2) + K_0(\chi|x-\hat{y}|)/2\pi$. *Define,*

(9.11) $\Lambda = \Lambda(x,y;\lambda) := \dfrac{\partial \Gamma}{\partial n_x}(x,y;\lambda) = \dfrac{\partial}{\partial n_x} K_0(\chi|x-y|)/2\pi + \dfrac{\partial}{\partial n_x} K_0(\chi|x-\hat{y}|)/2\pi$,

Fig.1



see Fig. 1 where $\eta = -n_x$.

Our objective is now to find a bound for Λ.

Recall that, (cfr. also notes **32** and **33**),

$K_0''(r) + K_0'(r)/r = K_0(r)$.

$$K_0(r) = \int_1^\infty \frac{e^{-tr}}{\sqrt{t^2-1}} dt = \int_r^\infty \frac{e^{-t}}{t} dt + F(r),$$

$$F(r) = \int_1^\infty \frac{e^{-tr}}{t(t+\sqrt{t^2-1})\sqrt{t^2-1}} dt.$$

Then, for $r > 1$, $K_0(r) = O(e^{-r})$.

But, if $r \in (0,1]$, $\displaystyle\int_r^\infty \frac{e^{-t}}{t} dt \le \int_r^1 \frac{1}{t} dt + \int_1^\infty e^{-t} dt = \log(1/r) + e$.

Therefore, if $r \in (0,\infty)$ then

(9.12) $$K_0(r) = \log^+(1/r) + O(e^{-r}).$$

On the other hand, we have,

(9.13) $$K_0'(r) = \frac{-e^{-r}}{r} - \int_1^\infty \frac{e^{-tr}}{(t+\sqrt{t^2-1})\sqrt{t^2-1}} dt = \frac{-e^{-r}}{r} + O(e^{-r}) = e^{-r}\left(\frac{-1}{r} + O(1)\right),$$

$$|K_0'(r)| \le e^{-r} C(1 + 1/r).$$

From the differential equation satisfied by K_0 we get,

(9.14) $K_0''(r) = e^{-r}/r^2 + O(1)e^{-r}/r + O(1)e^{-r} + \log^+(1/r) = e^{-r}\left(1/r^2 + O(1)/r + O(1)\right)$.

That is, $K_0''(r) = e^{-r}\left(O(1)/r^2 + O(1)\right)$. In consequence, there is a constant C such that,

$$|K_0''(r)| \le e^{-r} C(1 + 1/r^2).$$

(9.15) Assume that $y \in D$, $d = dist(y, \partial D) \le R$. Take coordinates in such a way that $y = (y_1, 0)$, $\hat{y} = (-y_1, 0)$ (hence $O = (0,0) \in \partial D$). Suppose $x \in \partial D$, $|x_2| \le R$. Call $\eta = -n_x$ and $\tilde{x}$ the symmetric point of x with respect to the tangent at O, (see Figs. 1 and 2). Call $r_1 := |x-y|$, $r_2 := |x-\hat{y}| = |\tilde{x}-y|$. Then,

$|r_1 - r_2| = \big||x-y| - |\tilde{x}-y|\big| \le |x-\tilde{x}|$.

Besides, $r_1^2 = (x_1 - y_1)^2 + x_2^2 \geq x_2^2$, $r_2^2 = (x_1 + y_1)^2 + x_2^2 \geq x_2^2$, $\inf(r_1, r_2) \geq |x_2|$.

Observing Fig. 1, we see that

$$\text{(9.16)} \quad \frac{\partial|x-y|}{\partial\eta} = lim\frac{|\Delta r_1|}{|\Delta\eta|} = \cos\alpha\,, \qquad \frac{\partial|x-\hat{y}|}{\partial\eta} = lim\frac{-|\Delta r_2|}{|\Delta\eta|} = -\cos\beta\,.$$

Since,

$$\text{(9.17)} \quad \frac{\partial K_0}{\partial\eta}(\chi r_1) = \chi K_0'(\chi r_1)\frac{\partial r_1}{\partial\eta} = \chi K_0'(\chi r_1)\cos\alpha\,, \quad \frac{\partial K_0}{\partial\eta}(\chi r_2) = -\chi K_0'(\chi r_2)\cos\beta\,,$$

from **(9.11)**, for $dist(y, \partial D) \leq R$, we obtain,

$$\text{(9.18)} \quad 2\pi|\Lambda| \leq |(\cos\alpha - \cos\beta)\chi K_0'(r_1\chi)| + |(\chi\cos\beta)(K_0'(r_1\chi) - K_0'(r_2\chi))| \leq$$
$$\leq \chi|\alpha-\beta|.|K_0'(r_1\chi)| + \chi^2|r_1 - r_2|.|K_0''(\tilde{r}\chi)|,$$

where $\tilde{r}$ is a number in the interval determined by r_1 and r_2. Then, $\tilde{r}^2 \geq x_2^2$.

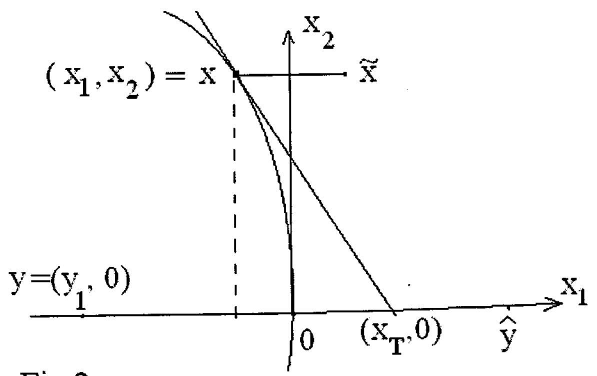


Fig.2

On the other hand,

$|r_1 - r_2| \leq |x - \tilde{x}| = 2|x_1| = O(x_2^2)$,

in fact, (see Fig. 2),

$x_1 = f(x_2)$, $f \in C^2$, $f(0) = 0 = f'(0)$,

implies that

(9.19) $f(x_2) = O(x_2^2)$, $f'(x_2) = O(x_2)$.

Thus, (C denotes constants),

(9.20) $|r_1 - r_2| \leq Cx_2^2 \leq C\inf(r_1^2, r_2^2)$.

We also have,

(9.21). $x_T = f(x_2) - x_2 f'(x_2) = O(x_2^2)$

Therefore, $|x_1| + |x_T| = O(x_2^2)$.

From Fig.3 we see that,

$$\tilde{\alpha} = \alpha + \gamma\,,\ \tilde{\beta} = \beta - \gamma\,,\ |\alpha - \beta| \leq |\tilde{\alpha} - \tilde{\beta}| + 2|\gamma|.$$

If R is small enough then for $|x_2| \leq R$, it holds that,

$$|\tilde{\alpha} - \tilde{\beta}| \leq \frac{\pi}{2}|\text{sen}(\tilde{\alpha} - \tilde{\beta})| \leq \frac{\pi}{2}\frac{|x - \tilde{x}|}{\inf(r_1, r_2) - |x - \tilde{x}|} \leq \frac{\pi}{2}\frac{\tilde{C}\,x_2^2}{|x_2| - 2|x_1|} \leq \pi\tilde{C}\frac{x_2^2}{|x_2|} \leq C|x_2|,$$

$$|\gamma| \leq (\pi/2)|\text{sen}\,\gamma| \leq \frac{\pi}{2}\frac{|x_1| + |x_T|}{|x_2|} \leq C|x_2|.$$

(Notice that C may have different values in different formulae!). In consequence,

(9.22) $$|\alpha-\beta|\le|\tilde{\alpha}-\tilde{\beta}|+2|\gamma|\le C|x_2|.$$

From **(9.18)** and the preceding estimations, (cf. **(9.13)**, **(9.14)**, **(9.20)** and **(9.22)**), for $x\in\partial D, y\in D$, $d=dist(y,\partial D)\le R$, $\varepsilon>0$ and small, we obtain,

$$2\pi|\Lambda(x,y;\lambda)|\le \chi|\alpha-\beta|.|K_0'(r_1\chi)|+\chi^2|r_1-r_2|.|K_0''(\tilde{r}\chi)|\le$$
$$\le C_1e^{-\chi r_1}(\chi\inf r_j+1)+C_2(\chi\inf r_j)^2K_0''(\tilde{r}\chi)\le$$
$$\le C_1e^{-\chi r_1}(\chi\inf r_j+1)+C_2'e^{-\chi\tilde{r}}((\chi\inf r_j)^2+1)\le C_0e^{-\chi\inf r_j}(1+\chi\inf r_j+(\chi\inf r_j)^2)\le$$
$$\le Ce^{-\chi(1-\varepsilon)\inf(r_1,r_2)}.$$

Then, for $M=M(\varepsilon)$ and $\chi>0$, it holds that,

(9.23) $$|\Lambda(x,y;\lambda)|\le Me^{-\chi(1-\varepsilon)d}.$$

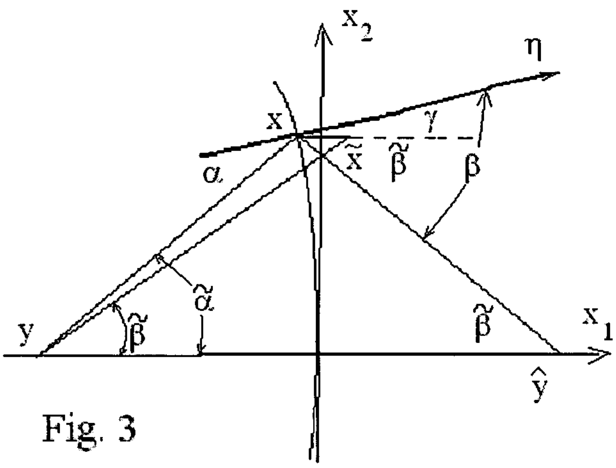


Fig. 3

(9.19), and also **(9.20)** and **(9.22)**. are valid whenever $|x_2|\le R$, independent of the position of O on the boundary. Therefore, the constant M of **(9.23)** can be chosen to hold for any x and y as far as they remain in the same relative position.

Thus, under these conditions $M=M(\varepsilon)$ depends only on ε.

Let Q be the square of side $2R$ with center O and a side parallel to the tangent at O.

Assume that $x\notin Q$, $x\in\partial D$. Thus, $|x-y|\ge R\ge d$ where $dist(y,\partial D)=d$. Then,

(9.23') $$|\Lambda(x,y;\lambda)|\le Ce^{-\chi(1-\varepsilon)d},\qquad C=C(\varepsilon)\text{ independent of } x \text{ and } O.$$

In fact, by **(9.13)**, we get,

$$\left|\frac{\partial}{\partial\eta}K_0(\chi|x-y|)\right|\le|\chi K_0'(\chi|x-y|)|\le\chi Ce^{-\chi|x-y|}(1+\frac{1}{\chi|x-y|})\le\frac{Ce^{-\chi|x-y|}}{|x-y|}(\chi|x-y|+1).$$

Thus, because of $|x-y|\ge R\ge d$, we deduce that, (cf. **(9.11)**,

(9.24) $$|\Lambda|\le Ce^{-\chi(1-\varepsilon)|x-y|}\le Ce^{-\chi(1-\varepsilon)d},\quad C=C(\varepsilon).$$

From **(9.23)** and **(9.24)**, it follows that there exist C and R such that for any $x\in\partial D$ and y verifying $d=dist(y,\partial D)\le R$, next inequality holds,

(9.25) $$|\Lambda(x,y;\lambda)| \le Ce^{-\chi d(1-\varepsilon)}.$$

Therefore, $\Gamma(x,y;-\chi^2) = H(x,y;-\chi^2) + K_0(\chi|x-\hat{y}|)/2\pi$ verifies $\left|\dfrac{\partial\Gamma}{\partial\eta}\right| \le Ce^{-\chi d(1-\varepsilon)}$.

(9.26) From **(9.10)** we obtain, for any $x \in D$, $d = dist(y,\partial D) \le R$, the estimation,

$|\Gamma(x,y;-\chi^2)| = O(\chi^{-1}e^{-\chi(1-\varepsilon)d})$. As a consequence of this we have, for $x \in D$, $d = dist(y,\partial D) \le R$,

(9.27) $$G(x,y;-\chi^2) = \frac{K_0(\chi|x-y|)}{2\pi} - H(x,y;-\chi^2) =$$

$$= \frac{K_0(\chi|x-y|)}{2\pi} + \frac{K_0(\chi|x-\hat{y}|)}{2\pi} - \Gamma(x,y;-\chi^2) =$$

$$= \frac{K_0(\chi|x-y|)}{2\pi} + \frac{K_0(\chi|x-\hat{y}|)}{2\pi} + O(\frac{e^{-\chi(1-\varepsilon)d}}{\chi}).$$

(9.28) From **(9.27)** and for $x \in D$, $dist(x,\partial D) \le R$, we obtain, (cf. **(9.3)**),

$$\mathrm{H}_\lambda(x) = -K_0(\chi 2\,dist(x,\partial D)) + O(1/\chi) = O(|\log dist(x,\partial D)|) + O(|\log\chi|) + O(1/\chi).$$

Taking into account that H_λ is continuous, we conclude that

$$\int_D |\mathrm{H}_\lambda| dx < \infty \text{ for any } \chi = \sqrt{|\lambda|} > 0.$$

(9.29) The normalized eigenfunctions of the Neumann problem for the metaharmonic differential operator verify for $\lambda_h := \lambda + \mu_h$, (cf. **(8.11)-(8.13)**, **(4.20)** and note **38**),

$$(-\Delta_x + \chi^2)\varphi_h(x) = (\lambda_h + \chi^2)\varphi_h(x) = \mu_h\varphi_h(x),\ \ h = 0,1,2,\cdots;\mu_h > 0,$$

$$\partial\varphi_h/\partial n_x = 0,\ x \in \partial D;\ \ \|\varphi_h\|_2 = 1,\ \int_D G(x,y;-\chi^2)\varphi_h(y)dy = \varphi_h(x)/(\lambda_h + \chi^2).$$

From **(9.2)**, **(9.29)** and the Cauchy-Schwarz inequality, we get for any $x \in D$,

(9.30) $$|\varphi_h(x)/(\lambda_h + \chi^2)| \le M, \qquad G(x,\cdot;-\chi^2) = \sum_{h=0}^{\infty}\frac{\varphi_h(x)\varphi_h(\cdot)}{\lambda_h + \chi^2} \quad (L^2(D)),$$

(9.31) $$\int_D |G(x,y;\lambda)|^2 dy = \sum_{h=0}^{\infty}\frac{\varphi_h^{\,2}(x)}{(\lambda_h + \chi^2)^2} \le M^2,$$

(9.32) $$\iint_{D\times D} |G|^2 dxdy = \sum_{h=0}^{\infty}\frac{1}{(\lambda_h + \chi^2)^2} \le M^2|D|.$$

Therefore, $G(x,y;-\chi^2) = \sum_{h=0}^{\infty}\dfrac{\varphi_h(x)\varphi_h(y)}{\lambda_h + \chi^2}$ $(L^2(D\times D))$ holds, (cf. also i) **(8.13)**).

Then,

(9.33) $G(x,y;-\chi^2)-G(x,y;-\chi_0{}^2)=(\chi_0{}^2-\chi^2)\sum_{h=0}^{\infty}\frac{\varphi_h(x)\varphi_h(y)}{(\lambda_h+\chi^2)(\lambda_h+\chi_0{}^2)}$, $(L^2(D\times D))$.

The series in **(9.31)** converges *uniformly* on *D*, (cf. note **37**). This implies that the series in **(9.33)** converges *uniformly* on $D\times D$.

(9.34) Recall that there are positive constants, *A*, *B*, such that (cf. [M] or Chapter 2),

$$n>0\Rightarrow An\le\lambda_n+\chi^2\le Bn,\qquad n\to\infty\Rightarrow\frac{\lambda_n+\chi^2}{n}\to\frac{4\pi}{|D|}.$$

However we shall not use the last result, on the contrary, we shall prove it at the end of this chapter.
Our next step is to study the behaviour of **(9.33)** for *x* tending to *y*.

(9.35) LEMMA. $\lim_{t\downarrow 0}\left(K_0(\chi t)-K_0(\chi_0 t)\right)=\log(\chi_0/\chi)$.

PROOF. We have,

$$K_0(\chi t)-K_0(\chi_0 t)=\int_1^{\infty}\frac{e^{-ut\chi}-e^{-ut\chi_0}}{\sqrt{u^2-1}}du=\int_1^{\infty}\left[\frac{1}{\sqrt{u^2-1}}-\frac{1}{u}\right](e^{-ut\chi}-e^{-ut\chi_0})du+\int_1^{\infty}\frac{e^{-ut\chi}-e^{-ut\chi_0}}{u}du$$

The function inside the square brackets belongs to $L^1(1,\infty)$. Hence,

$$\lim_{t\downarrow 0}(K_0(\chi t)-K_0(\chi_0 t))=\lim_{t\downarrow 0}\int_1^{\infty}\frac{e^{-ut\chi}-e^{-ut\chi_0}}{u}du=\lim_{t\downarrow 0}\int_t^{\infty}\frac{e^{-u\chi}-e^{-u\chi_0}}{u}du=\int_0^{\infty}\frac{e^{-u\chi}-e^{-u\chi_0}}{u}du.$$

The last integral is equal to

$$\int_0^{\infty}\frac{dx}{x}\int_{x\chi}^{x\chi_0}e^{-t}dt=\int_0^{\infty}dx\int_{\chi}^{\chi_0}e^{-ux}du=\int_{\chi}^{\chi_0}du\int_0^{\infty}e^{-ux}dx=\log\chi_0-\log\chi,\qquad\text{QED.}$$

(9.36) Letting $x\to y$ in formula **(9.33)**, taking into account that $\lambda_0=0$, $\varphi_0=1/\sqrt{|D|}$, the first formula in **(9.27)** and Lemma **(9.35)**, we arrive to

$$\frac{1}{2\pi}(\log\chi_0-\log\chi)-H(y,y;-\chi^2)+H(y,y;-\chi_0{}^2)=\left(\chi_0{}^2-\chi^2\right)\sum_{h=0}^{\infty}\frac{\varphi_h^2(y)}{(\lambda_h+\chi^2)(\lambda_h+\chi_0{}^2)}=$$

$$=\frac{\chi_0{}^2-\chi^2}{|D|\chi_0{}^2\chi^2}+\left(\chi_0{}^2-\chi^2\right)\sum_{h=1}^{\infty}\frac{\varphi_h^2(y)}{(\lambda_h+\chi^2)(\lambda_h+\chi_0{}^2)}.$$

After integrating on *D*, we obtain, (cf. **(9.28)**),

(9.37) $$\frac{|D|}{2\pi}(\log\chi_0-\log\chi)-\int_D H(y,y;-\chi^2)dy+\int_D H(y,y;-\chi_0{}^2)dy=$$

$$=\frac{1}{\chi^2}-\frac{1}{\chi_0{}^2}+\left(\chi_0{}^2-\chi^2\right)\sum_{h=1}^{\infty}\frac{1}{(\lambda_h+\chi^2)(\lambda_h+\chi_0{}^2)}.$$

(9.38) DEFINITION. $F(\chi_0) := \frac{|D|}{2\pi}\log\chi_0 + \int_D H(y,y;-\chi_0{}^2)dy + \frac{1}{\chi_0{}^2}$.

Thus, from **(9.37)** we get, $F(\chi_0) = L(\chi,\chi_0)$, where

$$L(\chi,\chi_0) = \frac{|D|}{2\pi}\log\chi + \int_D H(y,y;-\chi^2)dy + \frac{1}{\chi^2} + \left(\chi_0^2 - \chi^2\right)\sum_{h=1}^{\infty}\frac{1}{(\lambda_h + \chi^2)(\lambda_h + \chi_0^2)}$$

Letting $\chi_0 \downarrow 0$ in the preceding formula we arrive to

$$L(\chi) := \lim_{\chi_0 \downarrow 0} L(\chi,\chi_0) = \frac{|D|}{2\pi}\log\chi + \int_D \mathrm{H}_\lambda(y)dy + \frac{1}{\chi^2} - \chi^2\sum_{h=1}^{\infty}\frac{1}{(\lambda_h + \chi^2)\lambda_h}.$$

Thus, the function $L(\chi) = \lim F(\chi_0)$, $\chi_0 \downarrow 0$, is independent of χ; that is, $L(\chi) = A$ and *A*, given *D*, is a real constant. Now, the preceding equality looks like,

(9.39) $$A = A(D) = \frac{1}{\chi^2} + \int_D \mathrm{H}_\lambda(y)dy - \chi^2\sum_{h=1}^{\infty}\frac{1}{(\lambda_h + \chi^2)\lambda_h} + \frac{|D|}{2\pi}\log\chi.$$

(9.40) THEOREM (Å. Pleijel). If D is a C^2-Jordan region then on $\mathrm{Re}\, z > 1$ it holds that $\int_{0+}^{\infty}\frac{dN(\lambda)}{\lambda^z} = \sum_1^{\infty}\frac{1}{\lambda_n^z} = \frac{|D|}{4\pi}\frac{1}{z-1} + \frac{\langle J\rangle}{8\pi}\frac{1}{z-1/2} + G(z)$, where G(z) is a holomorphic function on $\mathrm{Re}\, z > 0$, (cf. **(9.29)**).

(9.41) For a proof of the theorem we need some results that are interesting in themselves.

THEOREM. There exists $u > 0$ such that for $\chi \geq u$ it holds that,

$$\int_D \mathrm{H}_{-\chi^2}(p)dp = -\frac{\langle J\rangle}{8\chi} + O\left(\frac{\log\chi}{\chi^2}\right).$$

PROOF. Recall that $J^h = D \cap J_h$. Then, if $h \leq R$,

$$I := \int_D \mathrm{H}_\lambda dp = \int_{J^h}\mathrm{H}_\lambda(x)\, dx + \int_{D\setminus J^h}\mathrm{H}_\lambda(x)\, dx = \int_{J^h}\mathrm{H}_\lambda(x)dx + O(e^{-\chi h}).$$

In fact, to prove that the last terms are equal, it is sufficient to show that for great values of χ and $dist(q,J) \geq h > 0$, one has,

(9.42) $$H(p,q;\lambda) = O(e^{-h\chi}).$$

If $dist(Q,J) \geq h$ then, $\left|\frac{\partial}{\partial n_t}H(Q,Q_t;\lambda)\right| = \left|\frac{\partial}{2\pi\,\partial n_t}K_0(\chi|Q - Q_t|)\right| \leq C\chi|K_0{}'(\chi h)| = O(\chi e^{-h\chi})$.

Because of the subordination theorem **(9.10)** we get, $H(Q,P;\lambda) = O(e^{-h\chi})$, for $P \in \overline{D}$, $\chi > \alpha / R$. Thus, **(9.42)** is proved. Then, $\int_{D \setminus J^h} \mathrm{H}_\lambda(x)dx = O(e^{-\chi h})$.

In **(9.26)** we have proved that the function $\Gamma(x,p;\lambda) = H(x,p;-\chi^2) + \frac{K_0(\chi|x-\hat{p}|)}{2\pi}$ verifies $|\Gamma(x,p;-\chi^2)| = O(\chi^{-1}e^{-\chi d/4})$ for $d := dist(p,\partial D) \le R$, $x \in \overline{D}$ and $\chi \ge 1$. Hence,

(9.43) $$|\Gamma(p,p;\lambda)| \le M \inf(1/\chi, e^{-\chi\, dist(p,J)/4}).$$

where M is independent of p. Now we write, with $0 < h \le R$

$$\int_{J^h} \mathrm{H}_\lambda(p)dp = \int_{J^h} \Gamma(p,p;\lambda)dp - \int_{J^h} K_0(\chi|p-\hat{p}|)dp/2\pi .$$

Then, for $\varepsilon \le h \le \delta$,

$$\left|\int_{J^h} \Gamma(p,p;\lambda)dp\right| \le \int_{J^h} |\Gamma| dp \le \int_{J^h} |\Gamma| I_{\{dist(p,J) \ge \varepsilon\}} dp + \int_{J^h} |\Gamma| I_{\{dist(p,J) < \varepsilon\}} dp = \mathbf{I} + \mathbf{II}.$$

But, $\mathbf{I} \le M|D|e^{-\chi\varepsilon/4}$, $\mathbf{II} \le M'\varepsilon/\chi$. Choosing χ great enough and $\varepsilon = \frac{4}{\chi}\log\chi^2$ we arrive to $\int_{J^h} |\Gamma(p,p;\lambda)|dp \le C'/\chi^2 + C''(\log\chi)/\chi^2$. Therefore, for great values of χ,

(9.44) $$\int_{J^h} |\Gamma(p,p;\lambda)|dp = O(1)\frac{\log\chi}{\chi^2}.$$

Thus, $I := \int_D \mathrm{H}_\lambda dp = \int_{J^h} \mathrm{H}_\lambda(p)dp + O(e^{-h\chi}) = -\int_{J^h} K_0(\chi|p-\hat{p}|)dp/2\pi + O\left(\frac{\log\chi}{\chi^2}\right)$.

Define $\Theta := -\int_{J^h} \frac{K_0(\chi|p-\hat{p}|)}{2\pi} dp$. Θ could be called the first approximation of $\int_D \mathrm{H}_\lambda(p)dp$. Recall that $K_0(r) = O(e^{-r/2})$ and $\int_0^\infty K_0(r)dr = \frac{\pi}{2}$. Using the coordinate system on J_h that we introduced in note **3**, we obtain,

(9.45) $$\Theta = \frac{-1}{2\pi}\int_0^{\langle J \rangle} d\xi_1 \int_0^h K_0(\chi 2\xi_2)[1 - c(\xi_1)\xi_2]d\xi_2 =$$

$$= \frac{-1}{2\pi}\int_0^{\langle J \rangle} \{\frac{1}{2\chi}\int_0^{2h\chi} K_0(t)dt - \frac{c(\xi_1)}{(2\chi)^2}\int_0^{2h\chi} K_0(t)\,t\,dt\} d\xi_1 =$$

$$=\frac{-1}{2\pi}\int_0^{\langle J\rangle}\{\frac{1}{2\chi}\int_0^{\infty}K_0(t)\,dt-\frac{1}{2\chi}\int_{2h\chi}^{\infty}K_0(t)dt\}\,d\xi_1+O(1/\chi^2)=$$

$$=\frac{-1}{2\pi}\int_0^{\langle J\rangle}(\frac{1}{2\chi}\int_0^{\infty}K_0(t)\,dt)\,d\xi_1+O(e^{-\chi h}/\chi)+O(1/\chi^2)=(\chi>>1)=-\int_0^{\langle J\rangle}\frac{1}{8\chi}d\xi_1+O\left(\frac{1}{\chi^2}\right)=$$

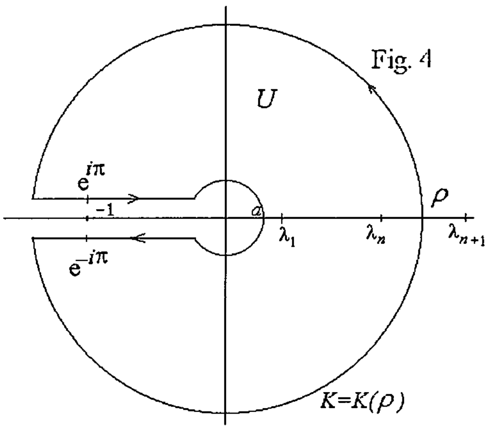
$$=-\frac{\langle J\rangle}{8\chi}+O\left(\frac{1}{\chi^2}\right).$$

Hence, for great values of χ, we have, $I=-\frac{\langle J\rangle}{8\chi}+O\left(\frac{1}{\chi^2}\right)+O\left(\frac{\log\chi}{\chi^2}\right)+O(e^{-\chi h/2})$.

The theorem follows, QED.

(9.46) *Define* $k(w):=w\sum_{h=1}^{\infty}\frac{1}{(\lambda_h-w)\lambda_h}$.

$k(w)$ is a meromorphic function on the complex plane w with poles on the positive real axis. Because of **(9.32)** we have $\sum_1^{\infty}\frac{1}{(\lambda_n+1)^2}<\infty$. Then, $\mathrm{K}:=\sum_1^{\infty}\frac{1}{\lambda_n{}^2}<\infty$.



Thus, if $w=-\chi^2$, from **(9.39)** we have,

(9.47) $k(-\chi^2)=A-\frac{|D|}{4\pi}\log\chi^2-\frac{1}{\chi^2}-\int_D \mathrm{H}_\lambda dx$.

(9.48) LEMMA. Assume that $0<\lambda_n\le\lambda_{n+1}$, $\mathrm{K}=\sum_1^{\infty}\frac{1}{\lambda_n^2}<\infty$, $k(w)=w\sum_{h=1}^{\infty}\frac{1}{(\lambda_h-w)\lambda_h}$ and the parameter $s>5$. Then, there exists a sequence of positive numbers $R_j\uparrow\infty$ such that $\int_{|w|=R_j}w^{-s}k(w)dw\to 0$ for $j\to\infty$.

PROOF. Call $\Delta_N:=\sup\{\lambda_{n+1}^2-\lambda_n^2:n\ge N\}$. Then $\Delta_N=\infty$. In fact, if it were not so it would exist $C>0$ such that $\lambda_{n+1}^2-\lambda_n^2\le C$. Then, for any positive integer p we would have, $\lambda_{n+p}^2\le\lambda_n^2+pC$. Therefore,

$$\infty=\sum_{p=1}^{\infty}\frac{1}{\lambda_n^2+pC}\le\sum_{p=1}^{\infty}\frac{1}{\lambda_{n+p}^2}<\infty.$$

Observe now that if $N(\rho) := \#\{\lambda_h \le \rho\}$ then

(9.49) $$\frac{N(\rho)}{\rho^2} \le \sum_1^\infty \frac{1}{\lambda_h^2} = K < \infty$$

Choose $R_j = (\lambda_n + \lambda_{n+1})/2$ for *n* such that $\lambda_{n+1}^2 - \lambda_n^2 \ge j$. In consequence, for any *h*,

(9.50) $$2|R_j - \lambda_h| \ge \lambda_{n+1} - \lambda_n > \frac{j}{\lambda_{n+1} + \lambda_n} = \frac{j}{2R_j}.$$

Let $\rho = R_j$. Thus,

(9.51) $$\frac{1}{2\pi} \int_{|w|=R_j} |w^{-s} k(w)||dw| \le \sum_{h=1}^\infty \frac{\rho^{2-s}}{|\lambda_h - \rho|\lambda_h} =$$

$$= \sum_{\lambda_h < 2\rho} \cdots + \sum_{\lambda_h \ge 2\rho} \cdots \le \frac{\rho^{2-s} N(2\rho)}{\min\limits_{\lambda_h < 2\rho}(|\lambda_h - \rho|\lambda_h)} + 2\rho^{2-s} \sum_1^\infty \frac{1}{\lambda_h^2}.$$

Taking into account **(9.49)-(9.50)**, it follows that $\frac{1}{2\pi} \int_{|w|=\rho} |w^{-s} k(w)||dw| = O(\rho^{5-s})$, QED.

(9.52) Assume that $\rho = R_j$ as in Lemma **(9.48)** and let *U* be the region shown in Fig 4. Let γ be an arc as in Fig. 5. *Define*, $\gamma_\infty = \gamma \cap \{\operatorname{Re} z \le -1\}$, $\gamma_0 = \gamma \setminus \gamma_\infty$.

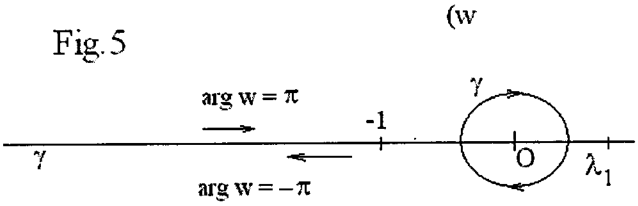
LEMMA. Suppose $s > 5$. Then, there is an entire function $g_0(w)$ such that

$$\frac{1}{2\pi i} \int_{\gamma_\infty} \frac{k(w)}{w^s} dw + g_0(s) = -\sum_1^\infty \frac{1}{\lambda_n^s} = -\int_{0+}^\infty \lambda^{-s} dN(\lambda).$$

PROOF. From the formula $(1/2\pi i) \int_{\partial U} k(w) w^{-s} dw = -\sum_{h=1}^{N(R)} 1/\lambda_h^s$, using Lemma **(9.48)**, one obtains, $(1/2\pi i) \int_\gamma k(w) w^{-s} dw = -\sum_{h=1}^\infty 1/\lambda_h^s$. But, $g_0(z) := (1/2\pi i) \int_{\gamma_0} k(w) w^{-z} dw$ is an entire function of *z*, QED.

(9.53) LEMMA. Assume $s > 5$. Then, $\frac{1}{2\pi i} \int_{\gamma_\infty} \frac{k(w)}{w^s} dw = -\frac{|D|/4\pi}{s-1} - \frac{\langle J \rangle / 8\pi}{s - 1/2} + h_0(s)$, where $h_0(w)$ is an holomorphic function on $\operatorname{Re} w > 0$.

Fig. 5



PROOF. In what follows we shall denote with the letter g entire functions and with the letter h holomorphic functions defined on the half plane $\mathrm{Re}(z) > 0$.

Because of formula **(9.47)** and theorem **(9.41)**, for $\{t \geq u,\ t \geq 1\}$, we have,

(9.54) $$k(-t) = A - \frac{|D|}{4\pi}\log t + \frac{\langle J\rangle}{8\sqrt{t}} + \mathrm{O}\left(\frac{1+\log t}{t}\right).$$

It follows then that the same formula holds for $t \geq 1$.

On the other hand, we have,

$$(2\pi i)^{-1}\int_{\gamma_\infty} k(w)w^{-s}dw = \frac{1}{2\pi i}\int_{-1}^{-\infty}|w|^{-s}e^{i\pi s}k(-|w|)dw + \frac{1}{2\pi i}\int_{-\infty}^{-1}|w|^{-s}e^{-i\pi s}k(-|w|)dw =$$

$$= \frac{1}{2\pi i}\int_1^\infty t^{-s}k(-t)\left(-e^{i\pi s} + e^{-i\pi s}\right)dt = \frac{1}{\pi}\int_1^\infty t^{-s}k(-t)(-\operatorname{sen}\pi s)\,dt =$$

$$= \frac{-\operatorname{sen}\pi s}{\pi}\int_1^\infty\left(-\frac{|D|}{4\pi}\log t + A + \frac{\langle J\rangle}{8\sqrt{t}} + \mathrm{O}\left(\frac{1+\log t}{t}\right)\right)\frac{dt}{t^s} =$$

$$= \frac{-\operatorname{sen}\pi s}{\pi}\left(-\frac{|D|}{4\pi(s-1)^2} + \frac{A}{s-1} + \frac{\langle J\rangle}{8(s-1/2)}\right) + h_1(s).$$

Taking into account that $\frac{\operatorname{sen}\pi s}{s-1} = -\pi + g_1(s),\ g_1(1) = 0$ and $\operatorname{sen}\pi s = 1 + (s-1/2)g_2(s),\ g_2(1/2) = 0$, we get,

$$(2\pi i)^{-1}\int_{\gamma_\infty} k(w)w^{-s}dw = \left(-\frac{|D|}{4\pi(s-1)} + g_3(s)\right) + \left(-\frac{\langle J\rangle}{8\pi(s-1/2)} + g_4(s)\right) + h_1(s), \quad \text{QED.}$$

(9.55) PROOF OF THEOREM **(9.39)**. The following equality, whenever $s > 5$, follows from Lemmas **(9.52)** and **(9.53)**,

$$\sum_1^\infty \frac{1}{\lambda_n^s} = \frac{|D|/4\pi}{s-1} + \frac{\langle J\rangle/8\pi}{s-1/2} + h(s), \quad h(z) \text{ holomorphic on } \mathrm{Re}\, z > 0.$$

Then, the Dirichlet series has 1 as an abscissa of convergence, that is, $\sum \lambda_n^{-z}$ converges on $\mathrm{Re}\, z > 1$, (cf. [Wd]). Therefore,

$$\sum_1^\infty \frac{1}{\lambda_n^z} = \frac{|D|/4\pi}{z-1} + \frac{\langle J\rangle/8\pi}{z-1/2} + h(z) \text{ holds for } \mathrm{Re}\, z > 1, \qquad \text{QED.}$$

Observe that 1 is also the abscissa of absolute convergence since $\lambda_n > 0$ if $n \geq 1$.

(9.56) THEOREM (H. Weyl). For $n \to \infty$, $\dfrac{n}{\lambda_n} \to \dfrac{|D|}{4\pi}$.

PROOF. From **(9.40)** we have,

(9.57) $$\int_{1+}^{\infty} \frac{dN(\lambda)}{\lambda^z} = \frac{|D|}{4\pi}\frac{1}{z-1} + F(z), \quad F(z) \text{ holomorphic on } \operatorname{Re} z \geq 1.$$

From the following theorem, (cf. [C]), we obtain, $\dfrac{N(\lambda)}{\lambda} \underset{\lambda \leftarrow \infty}{\to} \dfrac{|D|}{4\pi}$. For $\lambda = \lambda_n$ we have,

$\dfrac{n+1}{\lambda_n} \underset{n \to \infty}{\to} \dfrac{|D|}{4\pi}$, and the theorem is proved, QED.

(9.58) THEOREM (S. Ikehara). Assume that $\alpha(x)$ is a nonnegative, nondecreasing function, defined on $(1, \infty)$ and

(9.59) $$f(s) = \int_{1+}^{\infty} x^{-s} d\alpha(x) = \int_{0+}^{\infty} e^{-ts} d\alpha(e^t).$$

If for $\operatorname{Re}(s) > 1$ the integral converges and if there is a constant A such that the function,

(9.60) $$g(s) := f(s) - \frac{A}{s-1},$$

is holomorphic on $\operatorname{Re}(s) > 1$ and has a continuous limit on $\operatorname{Re} s = 1$, then

(9.61) $$\lim_{x \to \infty} \frac{\alpha(x)}{x} = A.$$

CHAPTER 10. NOTES.

1. A bounded open set A has *property* C if there exist constants δ and K such that

i) if $x \in \partial A$ there exists a versor $\vec{h}$ such that for $y \in B(x,\delta) \cap A$ and t such that $0 < t < K$, it holds that $y + t\vec{h} \in A$,

ii) the convex hull of $K\vec{h} + B(x,\delta) \cap A$ is included in A.

A bounded open set A has *property* C' if there are a finite number of open sets $A_i \subset A$, $i = 0,1,\ldots,N$, such that

1) $\overline{A_0} \subset A$, $\left| A \setminus \bigcup_{i=0}^{N} A_i \right| = 0$,

2) for $i = 1,\ldots,N$ there exists a C^1 diffeomorfism, θ_i, from a neighborhood of $\overline{A_i}$ onto an open set of R^2 such that $\theta_i(A_i) \in C$.

THEOREM (T. 5.12, case 3, [M], p. 188) Weyl's theorem holds for Neumann problem in an open bounded set A whenever $A \in C'$ and $|\partial A| = 0$.

2. A bounded region A will be called *strongly Lipschitz* if each point x_0 of its boundary can be covered by a neighborhhod U with an adequate coordinate system that verifies:

$U = \{x : |x_i| < d_i, i = 1,2\}$,

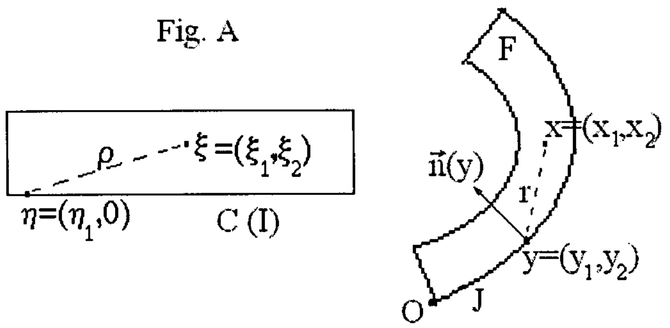
$U \cap \partial A = \{x : |x_1| < d_1, x_2 = F(x_1)\}$,

$U \cap A = \{x : |x_1| < d_1, -d_2 < F(x_1) < x_2 < d_2\}$,

where F satisfies a Lipschitz condition: $|F(x) - F(y)| \le K|x - y|$. Because of the boundedness of the region a finite family of U's cover the boundary for which the same K can be chosen. It follows that ∂A has finitely many components and therefore A is a plane region. Moreover, A has property C'.

3. To deal with a regular region A it is convenient to introduce local coordinates around the boundary in the

Fig. A



following way. To avoid unnecesary complications we assume next that A is a Jordan region with a rectifiable boundary. Let s be the parameter arc legth on J starting at the origin O, (see Figs. A, B).

The points in J will be denoted by $y = y(s) = (y_1(s), y_2(s))$, $0 \le s < \langle J \rangle$. We *assume* that $y_i(s) \in C^2([0,\langle J \rangle]), y_i(0) = y_i(\langle J \rangle)$. Let $n_i = \vec{n} = \vec{n}(s) = (\vec{n}_1, \vec{n}_2)$ be the interior normal versor at y. Suppose $\delta > 0$ sufficiently small and let I be an interval such that $\langle I \rangle \le \langle J \rangle$. *Define* the map T: $(s,t) \to x := y(s) + t\,\vec{n}$ on the rectangle $C(I) := I \times (-\delta, \delta)$ to the strip $J_\delta := \{x : dist(x,J) < \delta\}$.

We denote with $\xi := (\xi_1,\xi_2)$ a point of the rectangle $C(I)$ and with $x=(x_1,x_2)$ its image in the strip J_δ. Then, T is written as $T: \xi=(\xi_1,\xi_2) \to x=(x_1,x_2)=(y_1(\xi_1),y_2(\xi_1))+\xi_2(\vec{n}_1(\xi_1),\vec{n}_2(\xi_1))$, $0 \le \xi_1 < \langle I \rangle$, $|\xi_2| < \delta$. Thus, if $\xi_2 = 0$ then $x \in J$. Given $\eta = (\eta_1, 0)$ *its image will be represented by* $y = y(\eta_1)$ to underline that it is in J. One can get, taking δ sufficiently small, that the map T of the rectangle $\{\xi=(\xi_1,\xi_2): 0 \le \xi_1 \le \langle J \rangle, |\xi_2| < \delta\}$ with vertical sides identified, be a homeomorphism onto J_δ. In fact, since by hypothesis J is C^2, T is a C^1 map and can be written as (note that $n_i = (-\dot{y}_2(\xi_1), \dot{y}_1(\xi_1))$, $|n_i| = 1$ because of $\xi_1 = s$):

$$T(\xi)=\begin{cases} x_1(\xi) = y_1(\xi_1) - \xi_2 \dot{y}_2(\xi_1) \\ x_2(\xi) = y_2(\xi_1) + \xi_2 \dot{y}_1(\xi_1) \end{cases}.$$

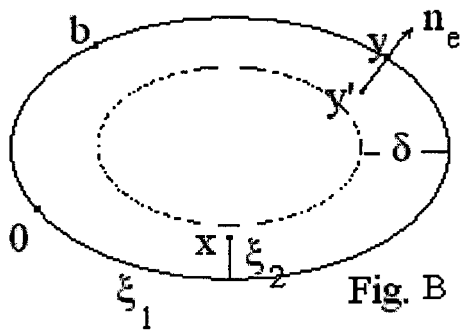


Fig. B

Its jacobian B is the modulus of the determinant of the following matrix,

$$\frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)} = \begin{vmatrix} \dot{y}_1(\xi_1) - \xi_2 \ddot{y}_2(\xi_1) & -\dot{y}_2(\xi_1) \\ \dot{y}_2(\xi_1) + \xi_2 \ddot{y}_1(\xi_1) & \dot{y}_1(\xi_1) \end{vmatrix} = 1 - \xi_2 c(\xi_1).$$

$c(\xi_1)$ is the curvature of J at the point $T(\xi_1, 0) \in J$. For δ sufficiently small $1 - \xi_2 c(\xi_1) > 0$ whenever $|\xi_2| < \delta$ because of the continuity of c. In this case $0 < B = 1 - \xi_2 c(\xi_1)$ and T is *locally* a homeomorphism. Let us see, perhaps by using a smaller δ, that T is *globally* a homeomorphism. Let $y(s) \in J$ and ε be such that T is a homeomorphism from $(s-2\varepsilon, s+2\varepsilon) \times (-\delta, \delta)$ onto $U := T((s-2\varepsilon, s+2\varepsilon) \times (-\delta,\delta))$, a neighborhood of $y(s)$. Let $V := \{x \in U : dist(x, J \setminus U) > 2dist(x,J)\}$. Then, V is also a neighborhood of $y(s)$ and there exists $0 < \tilde{\delta} \le \delta$ such that $\tilde{U} := T((s-\varepsilon, s+\varepsilon) \times (-\tilde{\delta}, \tilde{\delta})) \subset V$. Since J is compact it is possible to cover it with a finite number of such neighborhoods, $\{\tilde{U}_h : h = 1, \cdots, N\}$. Let $\delta_1 = \min_h \tilde{\delta}_h$. We claim that T is a homeomorphism from $\{\xi = (\xi_1,\xi_2): 0 \le \xi_1 \le \langle J \rangle, |\xi_2| < \delta_1\}$ onto J_{δ_1}. In fact, by the construction, for every $x \in \tilde{U}$, if $y \in J$ verifies $dist(x,J) = |x-y|$ then $y \in U$ and x is on the normal to J at y. Assume that $dist(x,J) = |x - y_1| = |x - y_2|$. Writing $y_h = y(s_h)$ and defining $\xi_2 := \pm dist(x,J)$, (with the sign + whenever $x \in A$), we deduce that $x = T(s_1, \xi_2) = T(s_2, \xi_2)$. Since T is a homeomorphism on U, we get $s_1 = s_2$, *i.e.* $y_1 = y_2$. Thus we have proved that T is one to one and onto J_{δ_1}.

Moreover, $x \in$ normal at $y_1 = y(s_1)$ and $|x - y_1| <$ radius of curvature at y_1. Let $0 < \rho <$ radius of curvature at y_1 for any $y_1 \in J$. Assume that $\delta = \inf(\delta_1, \rho)$. Then, if $R < \delta/2$, there is a circle contained in A tangent to J at y_1.

Because of the definition of T we also proved that in $J_{\delta_1} := \{x : dist(x,J) < \delta_1\}$ two different normals at J have no points in common and therefore the same happens in J_δ.

DEFINITION. Given $x = y(\xi_1) + \xi_2 n_i(\xi_1) \in F_\delta = A \cap J_\delta$, $\hat{x}$ denotes the *symmetric point with respect to J*: $\hat{x} := y(\xi_1) - \xi_2 n_i(\xi_1)$.

4. The simply connected plane region D is said to be *uniform* if there exists a number $\varepsilon \in (0,1]$ such that the following conditions W) and T) are satisfied:

W) For any two points of D, x_1, x_2, there exists a rectifiable arc $p \subset A$ whose end points are x_1, x_2 such that $\langle p \rangle \leq |x_1 - x_2| / \varepsilon$,

T) For any point $x \in p$, $dist(x, \partial D) \geq \varepsilon \dfrac{|x_1 - x||x_2 - x|}{|x_1 - x_2|}$.

5. **THEOREM** (Lax-Milgram). Let H be a Hilbert space and B a continuous bilinear functional $B : H \times H \to R$ ($|B(u,v)| \leq \alpha \|u\|_H \|v\|_H$), strongly coercive ($\exists \beta > 0$ such that $\beta \|u\|_H^2 \leq |B(u,u)|$). Given a continuous linear fuctional F on H there exists a unique $u \in H$ such that $B(u,v) = F(v)\ \forall v \in H$, (cf. [E]).

6. For the set of eigenvalues of Dirichlet problem in a Jordan region A, H. Weyl's theorem holds:(cf. [L], [M] or [BP]): $N(\lambda)/\lambda \to \alpha = |A|/4\pi$. There is a Jordan region D that is not a quasidisc such that for the Neumann problem one has $N(\lambda)/\lambda \to (|D| + 1/2)/4\pi$. However, there are Jordan regions that are not quasidiscs for which the behaviour of $N(\lambda)$ is as in Weyl's theorem. It is interesting the fact that there is a Jordan region not a quasidisc such that $N(\lambda) \sim c\lambda^{1+p}$, c, p positive, (cf. [M]).

7. The operator $S = L_\gamma^{-1}$ is symmetric and it has domain $L^2(U)$. Because of this, as an operator in L^2 it verifies $S = S^*$. S^{-1} exists and has a domain dense in $L^2(U)$. Then, $(S^{-1})^* = S^{-1}$, (cf. [RN]). Therefore, $S^{-1} = L_\gamma \ (= -\Delta + \gamma)$ is a selfadjoint operator such that $dom(L_\gamma) = range(S) \subset H^1(U) \cap H^2_{loc}(U)$, (cf. (1.19)). Besides,

$$dom\, L_\gamma = dom\, L_\gamma^* = \{ y : \exists y^*; \forall x \in dom\, L_\gamma, \langle L_\gamma x, y \rangle = \langle x, y^* \rangle \}.$$

8. A *quasidisc* is the image Q of the unit (open) disc B by a quasiconformal mapping f. Q is contained in the Riemann sphere. We shall be interested only in quasidiscs such that $\overline{Q} \subset R^2$. The boundary of a quasidisc is a *quasicircle*: $\partial Q = f(\{z : |z| = 1\}) = f(\Sigma)$. By definition, a *quasiconformal mapping* f is a one-to-one, sense-preserving transformation of the Riemann sphere such that a parameter called *maximal dilatation* associated with f is finite. If K is a finite number greater than or equal to the maximal dilatation then f is called K-quasiconformal. Always $K \geq 1$.

There are several useful characterizations of quasidiscs, for instance, a quasidisc is a Jordan domain whose boundary satisfies the arc condition, (cf. [Le], Ch. 1).

A disc with a Lebesgue spine removed is a Jordan domain but it is not a quasidisc.

9. Let $q_\varepsilon = \{z \in U : dist(z,q) < \varepsilon\}$ be the open set $(\varepsilon - neighborhood\, of\, U) \cap U$, $\varepsilon > 0$. $M_\mu := \overline{\lim_{\varepsilon \downarrow 0}} \left(\varepsilon^\mu \frac{|q_\varepsilon|}{\varepsilon^2} \right)$ is the so called μ – dimensional upper Minkowski content.

The Minkowski dimension of q is defined by $D = D(q) = \inf\{\mu \geq 0 : M_\mu(q) = 0\}$. Or, by $D = \sup\{d \geq 0 : M_d(q) = \infty\}$. More precisely, they should be called *interior* μ – *dimensional upper Minkowski content* and *interior Minkowski dimension*. One can define analogously the exterior content and dimension.

10. PROOF OF III) THEOREM (2.22): Let Q be a quasidisc defined by a K-quasiconformal mapping. Then, there is a number μ defined below, $\mu = \mu(K) \in (1,2)$, such that $D = D(q) \leq \mu$ and $M_\mu(q) = 0$.

Let $h = \{Y \in Q : dist(Y,q) < t\}$, $H = f^{-1}(h)$. If J is the jacobian of f and $p \in (1,\infty)$ then it holds that $|h| = \int_h dudv = \int_H Jdxdy \leq |H|^{1/p'} \left(\int_H J^p dxdy \right)^{1/p}$, $\lim_{t \downarrow 0} |H| = 0$. A theorem due to Bojarski asserts that J is locally p-integrable for a certain $p = p(K)$ such that

$$\infty > K/(K-1) \geq p(K) > 1, \quad K > 1.$$

Thus, $|h| = o(|H|^{1/p'})$. Observe that this holds for any $p \in (1, p(K)]$. If $K = 1$ the mapping is conformal and $|h| \approx |H|$ and $|h| = o(|H|^{1/p'})$ for any $p \in (1,\infty)$.

For $x, y \in \overline{B}$ we have $|f(x) - f(y)| \geq c|x-y|^{\sqrt{K}}$, (cf. [GV]). Then, if $x \in H$ and $y \in \Sigma$ we obtain $dist(x,\Sigma) \leq |x-y| \leq (|f(x)-f(y)|/c)^{1/\sqrt{K}}$. If $y \in \Sigma$ is such that $|f(x)-f(y)| < t$ then it follows that for every $x \in H$, $dist(x,\Sigma) \leq (t/c)^{1/\sqrt{K}}$. In consequence, $|H| = O(t^{1/\sqrt{K}})$ and

(1) $$|h| = |\{Y \in Q : dist(Y,q) < t\}| = o(t^{1/p'.\sqrt{K}}), \quad t \downarrow 0.$$

But $D = 2 - \underline{\text{lím}} \frac{\log|h|}{\log t}$, (cf. [L], Corollary 3.1). Then, $\frac{\log 1/|h|}{\log 1/t} \leq 2 - D + \varepsilon$, $\varepsilon > 0$, for a sequence $t = t_k \downarrow 0$. But, in this case, $t_k^{2-D+\varepsilon} \leq |h| = o(t_k^{1/p'.\sqrt{K}})$. Thus, $2 - D + \varepsilon - 1/p'\sqrt{K} > 0$ whenever $\varepsilon > 0$. Therefore,

(2) $$D \leq 2 - 1/p'\sqrt{K} =: \mu(K) < 2.$$

Since $K \geq 1$ we get $\mu(K) > 1$. On the other hand, $M_\mu(q) := \overline{\text{lím}} \frac{|h(t)|}{t^{2-\mu}}$, $t \downarrow 0$. From (1) and (2) we have $M_\mu(q) := \overline{\text{lím}} \frac{o(1)t^{2-\mu}}{t^{2-\mu}}$, *i.e.*, $M_\mu(q) = 0$, QED.

11. If we restrict ourselves to the first term of the asymptotic aproximation of $N(\lambda)$ for a second order elliptic differential operator then Lemma 4.4 of Lapidus [L] says that it is sufficient to consider only the principal parte of the operator.

Moreover, Corollary 5.11 of Métivier [M] asserts that what is known about the first term of the asymptotic aproximation of $N(\lambda)$ for the operator $-\Delta$ will be enough to know what happens with the first approximation of $N(\lambda)$ for most of the elliptic second order differential operators.

Thus, in a sense, the operator $-\Delta$ is a paradigm with respect to the asymptotic behaviour of the eigenvalues, at least for the main term of the asymptotic expansion of $N(\lambda)$

12. Assume that *X* is a Banach space of infinite dimension; G^n will denote the famuly of subspaces of *X* of dimension *n*.

DEFINITION. Let $A, A' \subset X$. The deviation of *A* measured from *A'* is $E_{A'}(A) = \sup\{dist(x, A') : x \in A\}$. The *n*th-diameter of *A* is defined by (Kolmogoroff),

$$d_n(A) = d_n(A, X) := \inf\{E_Y(A) : Y \in G^n\}, \quad n=0,1,...$$

Obviously d_n is non increasing. If $Z \in G^n$ and $d_n(A) = E_Z(A)$ then *Z* is an optimal approximation to *A* in G^n and is called an extremal subspace. It can be proved that if *A* is a compact set then $d_n(A) \to 0$. Also that $d_n(B_1(0)) = 1$. Moreover, (cf. [Lo]),

THEOREM (Gohberg and Krein). If $V_{n+1}(0)$ is the unit ball of the subspace *Y* of dimension *n*+1 then $d_n(V_{n+1}(0)) = 1$.

Let $X = l^2$ real, $\infty \geq \delta_1 \geq \delta_2 \geq \cdots > 0$ and *D* be the ellipsoid $\left\{x \in l^2 : \sum_1^\infty \left(\frac{x_k}{\delta_k}\right)^2 \leq 1\right\}$. Then,

THEOREM. $d_n(D) = \delta_{n+1}, \quad n = 0,1,2,\cdots$.

13. Assume that $J \subset R^2$ is a convex open set and that $A, B \subset J$ are measurable sets.
The following result holds, (cf. [L], [M]),

THEOREM. If $f \in H^1(J)$ then $I = \int_B dy \int_A |f(x) - f(y)|^2 dx \leq 2(|A| + |B|)(diamJ)^2 \|\nabla f\|_2^2$.

14. In Métivier [M], chapter VII, Corollary 7.2, an example is shown where $n_1(\lambda) \approx \lambda^{1+\xi}$, $\xi > 0$. For plane regions $\in$ **E**) we have $n_1(\lambda) = O(\lambda)$.

15. Asuume that *a(x,y)* is a real symmetric bounded bilinear form on the real Hilbert space *V* such that $\hat{a}(x) = a(x,x) \geq 0$. Let us define $\|x\| = \sqrt{\hat{a}(x)}$. Then, the parallelogram law holds,

(1) $$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

If λ is a non null real number then

(2) $$0 \leq \|x - \lambda y\|^2 = \|x\|^2 - 2\lambda a(x,y) + \lambda^2 \|y\|^2.$$

We wish to prove that

(3) $$|a(x,y)| \leq \|x\| \|y\|.$$

If $\|x\| = 0$ then $2|a(x,y)| \leq |\lambda|^2 \|y\|^2$; thus, $|a(x,y)| = 0$. Assume that $0 < |a(x,y)|$. Then $\|x\| > 0$ and from (2) for $\lambda = \|x\|^2 / |a(x,y)|$ we get $0 \leq -1 + \|x\|^2 \|y\|^2 / |a(x,y|^2$ and Schwarz inequality (3) follows. From this we obtain, Minkowski's inequality,

(4) $$\|x + y\| \leq \|x\| + \|y\|.$$

Assume that *a* also satisfies

(5) $$\hat{a}(x) = 0 \Rightarrow x = 0.$$

Then, $\|\cdot\|$ is a norm. (5) holds if a is strongly coercive. In this case $\|\cdot\| \approx \|\cdot\|_V$. In consequence, $\{V, \|\cdot\|\}$ is a Banach space. Because of (1), $\{V, \|\cdot\|\}$ is a Hilbert space with scalar product $a(x,y)$. In fact, from the polarization process we know that the scalar product has to be equal to

(6) $$\left\|\frac{x+y}{2}\right\|^2 - \left\|\frac{x-y}{2}\right\|^2 .$$

But (6)$= a(x,y)$.

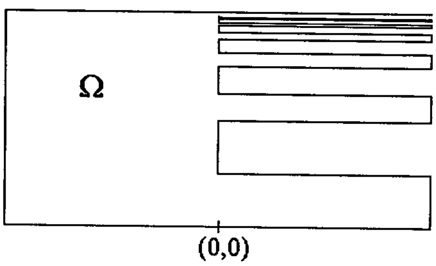
16. We give next an example of a plane region Ω such that the inclusion mapping of $H^1(\Omega)$ into $L^2(\Omega)$ is not completely continuous. Let $I_i \subset (0,1)$ $i = 1,2,...$ be a sequence of disjoint open finite intervals of length $|I_j| = \varepsilon_j^2 \downarrow, \sum \varepsilon_j^2 < 1$. Let us define, (see Fig.),

$$\Omega = \left\{(x,y) : x \in (-1,1) \; if \; y \in I_i \, , x \in (-1,0) \; if \; y \in (0,1) \setminus \bigcup_i^{\infty} I_i \right\},$$

$$g_j = \begin{cases} 0 & si \; x < 0 \\ 0 & si \; y \notin I_j \\ x & si \quad y \in I_j, x > 0 \end{cases}$$

Then $g_j \in H^1(\Omega)$ and verifies,

$$\|g_j\|_{H^1} = \varepsilon_j (1 + 1/3)^{1/2}, \; \|g_j\|_{L^2} = \varepsilon_j (1/3)^{1/2}.$$



Thus, the set $\Sigma = \left\{ f_j := \frac{\sqrt{3}}{\varepsilon_j} g_j : j = 1, \cdots \right\}$ is bounded in $H^1(\Omega)$. However, $\forall i \neq j$ $\|f_i - f_j\|_{L^2}^2 = \|f_i\|_{L^2}^2 + \|f_j\|_{L^2}^2 = 2$. Therefore, Σ is not precompact in $L^2(\Omega)$, QED.

17. As a matter of fact, in the cited Corollary, Lapidus shows instead of **(3.13)** that

$$N(\lambda) = \frac{|\Omega|}{4\pi}\lambda + O(\lambda^{\mu/2}),$$

where $N(\lambda)$ is the counting function of the eigenvalues of Dirichlet problem. He also proves that $n^D(\lambda) \approx N(\lambda)$. The main objective of our Chapter 4.will be to prove that $n(\lambda) = N(\lambda)$ for the Neumann problem, (cf. Th. **(4.13)**).

18. PROPOSITION. $dom(A) = \Theta := \left\{ x \in H = l^2 : \sum_1^{\infty} x_i^2 \lambda_i^2 < \infty \right\}$,

$$\Theta \subset W = \left\{ x \in H : \sum x_i^2 \lambda_i < \infty \right\}.$$

PROOF. Assume that $x \in \Theta$ and $x^N = \sum_1^N x_i e_i$. Then, $x^N \underset{H}{\to} x$ and $\|Ax^N\|_H^2 = \sum_1^N x_i^2 \Lambda_i^2$. Thus, $Ax^N \underset{H}{\to} y = \sum_1^\infty x_i \Lambda_i e_i$. $Ax = y$ since A is selfadjoint and therefore a closed operator, (cf. **(4.13)-(4.17)**). In consequence, $\Theta \subset dom(A)$.

Let us see that $\Theta \supset dom(A)$. If $g \in dom(A) = range(R)$ then $g = Rf$, $f \in H$. That is, $g = \sum_{i=1}^\infty \frac{f_i}{\Lambda_i} e_i = \sum_{i=1}^\infty g_i e_i$. Then, $\sum_1^N f_i^2 = \sum_1^N g_i^2 \Lambda_i^2 \le \|f\|_2^2 < \infty$. In particular, $\sum_1^\infty g_i^2 \lambda_i^2 < \infty$ and $g \in \Theta$, QED.

19. PROOF OF PROPOSITION **(5.21)**. Assume that $\phi \in C_0^\infty(K_1)$, $0 \le \phi \le 1$, $\phi = 1$, in a neighborhhod of K. From $\phi^2 |\nabla u|^2 = div(\phi^2 u \nabla u) - 2\phi\, u \nabla u \times \nabla\phi - \phi^2(\chi^2 u^2 + uf)$, we obtain

$$\int_D \phi^2 |\nabla u|^2 dq \le 0 + 2\int_D |\phi \nabla u| . |u \nabla \phi|\, dq - \int_D \phi^2 (\chi u + f/2\chi)^2 dq + \int_D \phi^2 (f^2/4\chi^2) dq \le$$

$$\le 2\int_D |\phi \nabla u||u \nabla \phi| dq + \frac{1}{4\chi^2} \int_D \phi^2 f^2 dq.$$

Using the inequality $2ab \le \frac{a^2}{2} + 2b^2$ and the Cauchy-Bunjakowski-Schwarz inequality we see that the last expression is not greater than

$$\frac{1}{2}\int_D \phi^2 |\nabla u|^2 dq + 2\int_D u^2 |\nabla \phi|^2 dq + \frac{1}{4\chi^2}\int_{K_1} f^2 dq.$$

Therefore, $\int_D \phi^2 |\nabla u|^2 dq \le 4\int_D u^2 |\nabla\phi|^2 dq + \frac{1}{2\chi^2}\int_{K_1} f^2 dq$. Then,

$$\int_K |\nabla u|^2 dq \le C(K, K_1) \int_{K_1 \setminus K} u^2 dq + \frac{1}{2\chi^2}\int_{K_1} f^2 dq, \qquad \text{QED.}$$

20. We state without proof the following result,

THEOREM. Let $f(t) \in C_0^\infty(\alpha, 2\pi - \alpha)$, $\alpha \in (0, \pi)$. There exists a function u, which is a solution of $(\Delta - \chi^2)u = 0$ in the sector $Q = \{re^{i\varphi} : 0 < r < \rho, \alpha < \varphi < 2\pi - \alpha\}$, u is continuous on $\overline{Q}$ and it verifies for $x = re^{it} \in \overline{Q}$ the relations,

$$u(x) = 0 \text{ if } t = \alpha \text{ or } t = 2\pi - \alpha;\ u(x) = f \text{ if } r = \rho.$$

21. Let $\xi = X + iY = \xi(z) = \xi(x + iy)$ be a conformal mapping in the plane. Assume that $\Delta_{x,y} w - \chi^2 w = 0$. Then, $u(\xi) := w(z)$ is not necessarily a χ-harmonic function since $\Delta_{X,Y} u = \left|\frac{dz}{d\xi}\right|^2 \Delta_{x,y} w$. If $\xi = X + iY = \xi(z) = cz + d, |c| = 1$ and as before $\Delta_{x,y} w - \chi^2 w = 0$ then $\Delta_{X,Y} u - \chi^2 u = 0$.

If $\xi(z) = X + iY = 1/z$ then $\Delta_{X,Y}u - \chi^2|\xi|^{-4}u = 0$, but we have, $\sup(\left|\frac{\partial u}{\partial X}\right|, \left|\frac{\partial u}{\partial Y}\right|) \le \frac{|\nabla w|}{|\xi|^2}$.

22. χ **SUBHARMONIC FUNCTIONS.** Assume that $u \in C(D)$ and $S = S_\rho(x) \subset D$, D a plane egion. *Define* for $y = x + re^{i\varphi}$,

$$u_S(y) := \begin{cases} u(y), & y \in D \setminus B_\rho(x) \\ U(y) = \frac{1}{\pi}\int_0^{2\pi} P_\chi(r,\rho;t-\varphi)\, u(x+\rho e^{it})\, dt, & y \in B_\rho(x) \end{cases}$$

u_S is continuous in *D* and

$$u_S(x) = U(x) = \frac{1}{I_0(\chi\rho)}\frac{1}{2\pi}\int_0^{2\pi} u(x+\rho e^{it})\, dt = \frac{1}{I_0(\chi\rho)}\int_{|z|=\rho} u_S(x+z)\, d\sigma^\rho(z),$$

σ_x^ρ is a unit mass uniformly distributed on $\Sigma_\rho(x)$. If $x = 0$, we simply write σ^ρ.

We call χ*-subharmonic* a function $u \in C(D)$ such that for every $x \in D$ and $S = S_\rho(x) \subset D$ verifies $u \le u_S$ on *D*. Whenever $u \equiv u_S$ for every *x* and any $S = S_\rho(x) \subset D$, we shall say that *u* is χ*-harmonic.* That is, *u* is χ-harmonic if *u* and $-u$ are χ-subharmonic. This definition of χ-harmonicity coincides with that already introduced. From this definition we have again that if $u \ge 0$ is χ-harmonic in *D* and $u(x) = 0$ for $x \in D$ then $u \equiv 0$.

For $\chi \ge 0$, $SUB^\chi(D)$, will denote the family of χ-subharmonic functions in the region *D*. If χ=0, we simply write *SUB(D)*. If for some $\chi > 0$, $u \in SUB^\chi(D)$), we shall say that *u* is *metasubharmonic*.

THEOREM (maximum principle for χ-subharmonic functions). Let $u \in C(\overline{D}) \cap SUB^\chi(D)$, *D* a plane region. Then, *u* does not take its positive maximum at a point $p \in D$.

The following propositions hold,

i) $\max_{\partial D} u \ge 0 \Rightarrow \max_{\partial D} u = \max_{\overline{D}} u$,

ii) $\max_{\overline{D}} u \ge 0 \Rightarrow \max_{\partial D} u = \max_{\overline{D}} u$,

iii) $u \ne 0$, $\max_{\partial D} u \le 0 \Rightarrow u < 0$ on *D*.

PROOF. Assume that *u* takes its maximum value at $p \in D$ and $u(p) > 0$. Then, for certain $\varepsilon > 0$ we would have: $u(p) \le \frac{1}{I_0(\chi\varepsilon)}\int_{|p-q|=\varepsilon} u(q)\, d\sigma_p^\varepsilon(q) < u(p)$, because of $I_0(\chi r) > 1$ for $r > 0$. Thus, *u* does not take its positive maximum at a point $p \in D$. From this *i)* follows.

i) implies $\max_{\overline{D}} u > \max_{\partial D} u \Rightarrow \max_{\overline{D}} u < 0$. In fact, if $\max_{\overline{D}} u = u(x_0)$, $u(x_0) \ge 0, x_0 \in D$ then from the preceding argument we get $\max_{\overline{D}} u = u(x_0) = 0$. This implies $u \equiv 0$. And this is in contradiction with $\max_{\overline{D}} u > \max_{\partial D} u$. The proved proposition is equivalent to *ii)*.

iii) If $\max_{D} u \geq 0$ then from *ii*) we obtain $\max_{D} u \leq 0$. Thus, $\max_{D} u = 0$. If $\max_{D} u \leq 0$ and u is not identicaly zero then $u(x) \neq 0$ for $x \in D$. Then, $u(x) < 0$ for $x \in D$, QED.

23. We prove next a result which is precisely theorem **(5.10)** in the particular case where $A = \Delta,\ a = -\chi^2$.

THEOREM. Let D be a plane region with the ball property. Assume that $u \in C^2(D) \cap C(\overline{D})$, verifies $\Delta u - \chi^2 u = 0$ and m is its positive maximum on $\overline{D}$. Suppose that y_0 is the centre of a ball $B = \{x : |x - y_0| < R\}$ of radius R, contained in D, whose circumference intersects ∂D at x_0 where $u(x_0) = m$. Let $\eta = \frac{x_0 - y_0}{|x_0 - y_0|}$. Then, if $c(\chi, R) := \frac{\chi I_0'(\chi R)}{I_0(\chi R)}$ we have,

$$\frac{\partial_{\inf} u}{\partial \eta}(x_0) \geq c(\chi, R) u(x_0).$$

DEMOSTRACION. Recall that $x_0 \in \partial D$. Define $w(x) := u(x_0)\frac{I_0(\chi|x - y_0|)}{I_0(\chi R)}$. Then, $w(x)$ is a χ-harmonic function on $\{x \neq y_0\}$ and because of **(5.23)** on the whole plane. $w(x)$ is such that for $x \in \partial B$, $w(x) = u(x_0) \geq u(x)$ holds. Thus, if $x \in B$ we have $w(x) \geq u(x)$, (cf. Th. **(5.6), (5.8)**). Therefore, it follows for $x \in (x_0, y_0)$ the inequality $\frac{u(x_0) - u(x)}{|x - x_0|} \geq \frac{w(x_0) - w(x)}{|x - x_0|}$. In consequence,

$$\lim_{x \to x_0} \frac{u(x_0) - u(x)}{|x - x_0|} \geq \frac{\partial w}{\partial \eta}(x_0) = c(\chi, R) u(x_0), \qquad \text{QED.}$$

24. NORMAL FAMILIES OF χ-HARMONIC FUNCTIONS. We say that the sequence of functions $\{u_n(x) : x \in D\}$, D a plane region, converges *almost uniformly* to the function u, $u_n \underset{a.u.}{\to} u$, if the sequence converges uniformly, $u_n \underset{\bullet}{\to} u$, on compact sets of D. A family $F = \{u_l : l \in \Lambda\}$ will be called *normal* if any sequence contained in F has a subsequence almost uniformly convergent. Next we prove theorems A and B analogous to theorems of Harnack and Montel, respectively.

THEOREM A. Let $\{u_n : n = 1, 2, \ldots\} \subset A^\chi(D)$. Assume that for any n and $x \in D$, $u_n(x) \geq u_{n-1}(x)$. If the numerical sequence $\{u_n(x_0)\}$, $x_0 \in D$, converges then the sequence of functions converges almost uniformly to a function $u(x) \in A^\chi(D)$.

PROOF. Without loss of generality assume that $x_0 = 0$ and $S = S_\rho(0) \subset D$. Let $C_\varepsilon := \sup\{P_\chi(r, \rho; t) : r \leq \rho - \varepsilon, t \in [0, 2\pi]\}$. Then, $0 < C_\varepsilon < \infty$ since $P_\chi(r, \rho; t)$ is a continuous function on $\{(r, t) \in [0, \rho - \varepsilon] \times [-\pi, \pi]\}$. Therefore,

$$0 \leq (u_n - u_{n-1})(re^{i\varphi}) = \frac{1}{\pi}\int_0^{2\pi} P_\chi(r, \rho; \varphi - t)(u_n - u_{n-1})(\rho e^{it})\, dt \leq \frac{C_\varepsilon}{\pi}\int_0^{2\pi} (u_n - u_{n-1})(\rho e^{it})\, dt =$$

$= 2C_\varepsilon I_0(\chi\rho).(u_n - u_{n-1})(0)$. The last equality is a consequence of the formula of the mean value for metaharmonic functions, (cf. **(5.44)**). Thus, it follows the existence of $u \in C(D)$ such that $u_n \underset{a.u.}{\to} u$. From

$u_n(re^{i\varphi}) = \frac{1}{\pi}\int_0^{2\pi} P_\chi(r,\rho;\varphi - t)\, u_n(\rho e^{it})\, dt$, $r < \rho$, we obtain $u(re^{i\varphi}) = \frac{1}{\pi}\int_0^{2\pi} P_\chi(r,\rho;\varphi - t)\, u(\rho e^{it})\, dt$.

That is, $u = u_S$ for any $S \subset D$, QED.

THEOREM B. Let $\{u_n : n = 1,2,...\} \subset A^\chi(D)$ be a sequence uniformly bounded by a constant *M*. Then, there exists a subsequence $\{u_{n_j}(x)\}$ and a function $u \in A^\chi(D)$ such that $u_{n_j} \underset{a.u.}{\to} u$ for $j \to \infty$.

PROOF. Let $r', r'' \le r < \rho$. Assume that $0 \in D$. Using **(5.31)** to obtain a bound for the remainder of the series that defines the kernel P_χ, we arrive to,

$$|P_\chi(r',\rho; s - \varphi') - P_\chi(r'',\rho; s - \varphi'')| \le \varepsilon \quad \text{for} \quad |r'e^{i\varphi'} - r''e^{i\varphi''}| \le \delta = \delta(\varepsilon, r, \rho).$$

In this case, $|u_n(r'e^{i\varphi'}) - u_n(r''e^{i\varphi''})| \le 2M\varepsilon$ and $\{u_n\}$ is relatively compact on $S_r(0)$ because of the theorem of Arzelá-Ascoli. Therefore, $\{u_n\}$ contains a subsequence almost uniformly convergent whose limit is necessarily a χ-harmonic function, QED.

25. Observe that if $u \in N(D)$ then $\lim_{x \to x_0} \nabla u \times n = \frac{\partial u}{\partial n}(x_0)$ and $\nabla u \times n$ is continuous on $(y_0, x_0]$. Moreover, if *u* satisfies the hypothesis of theorem **(5.10)** and has a positive strict maximum at x_0 then $0 > \frac{\partial u}{\partial n}(x_0)$.

26. We cite in this section and the next one some auxiliary results that we use in the main text. In particular, about the *fundamental solution of the Laplacian.* Let $s(a,x) = \frac{1}{2\pi}\log\frac{1}{|x-a|}$, $a, x \in R^2, a \ne x$. Then, $\Delta_x s(a,x) = 0$ in $R^2 \setminus \{a\}$.

Assume that the function $f(x) \in L^\infty(R^2)$ is of compact support that is contained in the compact set *K*, $K^o \ne \varnothing$. Under this hypothesis the following theorem holds. Its content is known and we shall not prove it here.

$C_0\{K\}$ denotes the functions of $C(R^2)$ with compact support contained in *K*.

$D'(R^2)$ denotes the space of distributions on R^2.

FUNDAMENTAL THEOREM. I) If $x = (x_1, x_2)$, $y = (y_1, y_2)$ and

$$u(x) := (\sigma f)(x) := -\int_{R^2} s(x,y)\, f(y)\, dy = \frac{1}{2\pi}\int_{R^2} f(y) \log|x - y|\, dy,$$

then $u(x) \in C^1(R^2)$. For any $x \in R^2$, one has,

$$: \frac{\partial u}{\partial x_i}(x) = -\int_{R^2} \frac{\partial s}{\partial x_i}(x,y)\, f(y)\, dy = \frac{1}{2\pi}\int_{R^2} \frac{x_i - y_i}{|x-y|^2} f(y)\, dy.$$

II) Assume that *D* is a plane region, $D \supset K$, $f_1 \in L^\infty(R^2)$, $f_1 = 0$ on $R^2 \setminus D$, $f_1 \in C^1(D)$. Then, $u_1(x) = (\sigma f_1)(x) \in C^2(D)$ and $\Delta u_1 = f_1$ on *D*.

III) $C(D) \cap L^\infty(D) \not\subset \Delta C^2(D)$.

IV) $\sigma(C_0\{K\}) \not\subset C^2(D)$.

V) It holds that, $\dfrac{\partial^2}{\partial x_1{}^2}\left(\dfrac{\log|x|}{2\pi}\right) = \dfrac{1}{2\pi} vp\dfrac{x_2^2 - x_1^2}{|x|^4} + \dfrac{\delta}{2}$ ($D'(R^2)$),

$$\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}\right)\left(\frac{\log|x|}{2\pi}\right) = \delta\,, (D'(R^2)).$$

VI) Assume f is null on $R^2 \backslash D$ and Hölder continuous of order α on D. That is,

$$f \in C^{0,\alpha}(D)\,,\ \alpha \in (0,1]\,,\ f = f.I_D\,.$$

Then, σf is a function with continuous second derivatives on D.

27. AUXILIARY THEOREM. Let $a \le t \le b$, $0 \in (a,b)$, $f(t,y)$ absolutely continuous with respect to t for almost every y and $\int |f(0,y)|\, dy < \infty$.

If $\int_a^b dt \int \left|\dfrac{\partial}{\partial t} f(t,y)\right| dy < \infty$ then $f(t,\cdot)$ is (absolutely) integrable for any $t \in [a,b]$, $F(t) := \int f(t,y)\, dy$ is absolutely continuous for $t \in [a,b]$,

and $F'(t) = \int \dfrac{\partial f}{\partial t}(t,y)\, dy$, almost everywhere on $[a \le t \le b]$.

Moreover, if $\int \dfrac{\partial f}{\partial t}(t,y)\, dy$ is continuous with respect to t then everywhere on $[a \le t \le b]$, it holds that, $\dfrac{d}{dt}\int f(t,y)\, dy = F'(t) = \int \dfrac{\partial f}{\partial t}(t,y)\, dy$.

PROOF. $f(x,y) - f(0,y) = \int_0^x \dfrac{\partial f}{\partial u}(u,y)du$ holds for almost every y. Then,

$$|f(x,y)| \le |f(0,y)| + \int_a^b \left|\frac{\partial f}{\partial u}\right|(u,y)du\,,$$

$$\int |f(x,y)|dy \le \int |f(0,y)|dy + \int_a^b \int \left|\frac{\partial f}{\partial u}\right|(u,y)dydu < \infty\,.$$

Therefore $f(x,\cdot) \in L^1$ for each $x \in [a,b]$. Moreover, for $x \in [a,b]$,

$$F(x) = \int f(x,y)dy = \int f(0,y)dy + \int_0^x \left(\int \frac{\partial f}{\partial u}(u,y)dy \right) du\,.$$

F is an absolutely continuous function since the function inside the braces,

$$I(u) = \int \frac{\partial f}{\partial u}(u,y)dy\,,$$

is (absolutely) integrable. Thus, for almost every x,

$$F'(x) = \int \frac{\partial f}{\partial x}(x,y)dy\,.$$

If $I(\cdot) \in C([a,b])$ then the last equality is verified at every point $x \in [a,b]$, QED.

28. The preceding auxiliary theorem can be applied to obtain I) of the fundamental theorem of note **26**. The following lemma is also a consequence of it. Applications of this lemma can be seen in the proofs of **(7.6)** and **(7.26)**.

AUXILIARY LEMMA. Assume that $f(t,y)$ is an absolutely continuous function with respect to the variable t, $a \le t \le b$, for any fixed y, and

$$\int_a^b dt \int |f(t,y)|\, dy < \infty, \quad \int_a^b dt \int \left|\frac{\partial}{\partial t} f(t,y)\right| dy < \infty, \quad \int \frac{\partial f}{\partial t}(t,y)\, dy \text{ continuous.}$$

Call $F(t) := \int f(t,y)\, dy$. Then, $F'(t) := \int f'(t,y)\, dy$. Precisely,

$$\frac{d}{dt}\int f(t,y)\, dy = \int \frac{\partial f}{\partial t}(t,y)\, dy \qquad \text{for any } t \in (a,b).$$

29. EXAMPLE. The function $(x, j(x))$ defined in a neighborhood of the origin by $j(x) = x^5 \operatorname{sen}\frac{1}{x} \in C^2$, describes a rectifiable arc Γ. It can be completed in such a way as to be a part of a C^2-Jordan curve J, like the boundaries we consider here. For its interior domain D, it holds the theorem **(8.14)**. However, Γ is not convex at either side of the origin, (cf. [P]).

30. PROOF OF **(5.8)**. *i)* Case $c \equiv 0$. Let $0 \le \varphi \in C_0^\infty(B)$, $\int_B \varphi(y) dy = 1$, where B is the unit ball and define

$$u_\varepsilon(x) := \varepsilon^{-2}\int_D u(y)\varphi\left(\frac{x-y}{\varepsilon}\right) dy \text{ on } D.$$

Then $u_\varepsilon \in C^\infty(\overline{D(\varepsilon)})$, where $D(\varepsilon) := \{x \in D : dist(x, \partial D) > \varepsilon\}$.

It verifies, $Au_\varepsilon = f_\varepsilon \ge 0$. The hypothesis implies, for ε small enough, that

$\max_{x \in \overline{D}(\varepsilon)} u_\varepsilon(x) > 0$. Therefore, by **(5.6)**, it follows that,

$$\max_{x \in \overline{D}(\varepsilon)} u_\varepsilon(x) = \max_{x \in \partial D(\varepsilon)} u_\varepsilon(x) \le \max\{u(x) : x \in D, dist(x, \partial D) < 2\varepsilon\}.$$

Letting ε tend to zero one gets for $x \in D$, $u(x) \le \max_{y \in \partial D} u(y)$.

ii) Case $c(x) \in L^1(D)$, $c(x) \le 0$ a.e. Define $\widetilde{D} := \{x \in D : u(x) > 0\}$ and $\widetilde{f} := f - c(x)u$. On any connected component of $\widetilde{D}$, $Au = \widetilde{f} \ge 0$ a.e.. Thus, by *i)* and the hypothesis, $0 < \max_{x \in \overline{D}} u(x) = \max_{x \in \widetilde{D}} u(x) = \max_{x \in \partial\widetilde{D}} u(x)$. Noting that for $x \in D \cap \partial\widetilde{D}$ necessarily $u(x) = 0$, one obtains $\max_{x \in \partial\widetilde{D}} u(x) = \max_{x \in \partial D} u(x)$, QED.

31. LEMMA. The derivatives of the modified Bessel function $I_n(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{n+2k}}{k!(n+k)!}$ verify the inequality

$$\left|\frac{I_n^{(j)}(r)}{I_n(\rho)}\right| \le |r_0/\rho|^n \frac{(n^j + C_j)}{r_0^j}, \text{ for } r < r_0 < \rho, \text{ where } C_j = C_j(r_0/\rho).$$

PROOF. Let $0 < r < r_0 < \rho$. $I_n^{(j)}(r) = r^{-j} \sum_{k=0}^{\infty} \frac{(r/2)^{n+2k}}{k!(n+k)!} \frac{(n+2k)!}{(n+2k-j)!} \le I_n^{(j)}(r_0) =$

$$= r_0^{-j} \left(\frac{r_0}{\rho} \right)^n \sum_{k=0}^{\infty} \frac{(\rho/2)^{n+2k}}{k!(n+k)!} \left[\frac{(r_0/\rho)^{2k}(n+2k)!}{(n+2k-j)!} \right].$$

Observe that the square bracket is zero for $n + 2k < j$. It is bounded by:

$$\left[\frac{(r_0/\rho)^{2k}(n+2k)!}{(n+2k-j)!} \right] \le (n+2k)^j (r_0/\rho)^{2k} \le \max_{0<x} (n+x)^j (r_0/\rho)^x .$$

If $F(x) := (n+x)^j (r_0/\rho)^x$, $F'(x) = F(x)(j + (n+x)\log(r_0/\rho))/(n+x)$.

If $j + n\log(r_0/\rho) < 0$, then $F'(x) < 0$ for all $0 < x$ and

(1) $F(x) \le F(0) = n^j$.

Else $F'(x_0) = 0$ for $x_0 = -n - j/\log(r_0/\rho)$, and

(2) $F(x) \le F(x_0) = \left(\frac{j}{-\log(r_0/\rho)} \right)^j (r_0/\rho)^{x_0} < \left(\frac{j}{\log(\rho/r_0)} \right)^j =: C_j$.

So, in any case $F(x) < C_j + n^j$, QED.

32. From **(5.13)-(5.16)** we obtain $-K_0'(r) = \int_1^{\infty} \frac{te^{-rt}}{\sqrt{t^2-1}} dt$. Using the identity $\frac{t}{\sqrt{t^2-1}} = 1 + \frac{1}{(t+\sqrt{t^2-1})\sqrt{t^2-1}}$, we have $-K_0'(r) = \frac{e^{-r}}{r} + \int_1^{\infty} \frac{e^{-rt}}{(t+\sqrt{t^2-1})\sqrt{t^2-1}} dt$. So $|K_0'(r)| \le e^{-r} \left(\frac{1}{r} + \int_1^{\infty} \frac{1}{(t+\sqrt{t^2-1})\sqrt{t^2-1}} dt \right) \le Ce^{-r}\left(1 + \frac{1}{r}\right)$. In particular $K_0'(r) \underset{r\to\infty}{\to} 0$.

33. Using $\frac{1}{\sqrt{t^2-1}} = \frac{1}{t} + \frac{1}{t(t+\sqrt{t^2-1})\sqrt{t^2-1}}$, one obtains

$K_0(r) = \int_1^{\infty} \frac{e^{-rt}}{\sqrt{t^2-1}} dt = \int_r^{\infty} \frac{e^{-t}}{t} dt + \int_1^{\infty} X(t,r)\,dt$, where $X(t,r) := \frac{e^{-rt}}{t^3 + t^2\sqrt{(t^2-1)} - t}$.

Therefore, $K_0(r) \le \int_r^{\infty} \frac{e^{-t}}{t} dt + e^{-r} \int_r^{1} \frac{1}{t^3 + t^2\sqrt{(t^2-1)} - t} dt \le \int_{r\wedge 1}^{1} \frac{1}{t} dt + \int_{r\vee 1}^{\infty} e^{-t} dt + Ce^{-r} =$

$= \log^+(1/r) + O(e^{-r})$.

34. Suppose that the boundary of D contains the segment $\{(x,0):-1\le x\le 1\}$ and that $P=(0,y)\in D$ for $0<y<1$. Then

$$\int_0^S\left|\frac{\partial}{\partial n_s}\log|P-Q_s|\right|^{1+\varepsilon}ds\ge\frac{1}{2}\int_{-1}^{1}\left|\frac{\partial}{\partial z}\log(x^2+(y-z)^2)\Big|_{z=0}\right|dx\ge\int_{-1}^{1}\left|\frac{y}{y^2+x^2}\right|^{1+\varepsilon}dx=$$

$$=y^{-\varepsilon}\int_{-1/y}^{1/y}\frac{1}{\left(1+t^2\right)^{1+\varepsilon}}dt\ge y^{-\varepsilon}C\xrightarrow[y\to 0]{}\infty.$$

35. Let s be the length parameter of a C^2 curve $\{(x(s),y(s)):s\in(-S/2,S/2)\}$ such that $(x(0),y(0))=(0,0)$ and suppose that $\dot y(0)=0$, (see Fig. 1, Ch. 7).

Then $\dot x(0)=1$ and $\ddot x(0)=0$. In fact, the first equality follows from $\dot x^2(s)+\dot y^2(s)=1$, and the second one from its derivative: $2\dot x(s)\ddot x(s)+2\dot y(s)\ddot y(s)=0$. Therefore, for $s\to 0$, we have $x(s)=s+o(s^2)$, $\dot x(s)=1+o(s)$ and $y(s)=O(s^2)$, $\dot y(s)=O(s)$.

36. PROOF OF PROPOSITION (7.4'): for $-\infty<a,\ s<\infty$ and $\delta>0$,

$$G(a):=\int_a^{a+\delta}\frac{|2x|}{s^2+x^2}dx\le 4\log\left(1+\frac{\delta}{|s|}\right).$$

In fact, $G(a)\to 0$ for $a\to\infty$ and $G(a)$ is increasing for small positive *a*. So, on the half-line $0\le a<\infty$, *G*(*a*) has a maximum at $a\ (=a_{max})$ such that

$$\frac{G'(a)}{2}=\frac{(a+\delta)(s^2+a^2)-a(s^2+(a+\delta)^2)}{(s^2+a^2)(s^2+(a+\delta)^2)}=\delta\frac{s^2-a\delta-a^2}{(s^2+a^2)(s^2+(a+\delta)^2)}=0.$$

This occurs at $a_{max}=\dfrac{\sqrt{\delta^2+4s^2}-\delta}{2}=\dfrac{2s^2}{\sqrt{\delta^2+4s^2}+\delta}$. Therefore, $G(a)\le G(a_{máx})$ for $a\ge 0$. From the identity $G(a)=G(-a-\delta)$, valid for every *a*, we get $G(a)\le G(a_{máx})$ also for $a\le -\delta$.

On the other hand, if $-\delta<a<0$, $G(a)\le\int_0^{a+\delta}+\int_a^0\le 2G(0)$. Therefore, $G(a)\le 2G(a_{max})$ for any $a\in R^1$. But

$$G(a_{max})=\log\frac{s^2+(a_{máx}+\delta)^2}{s^2+a_{max}^2}=\log(1+\frac{\delta^2+2\delta a_{max}}{s^2+a_{max}^2}).$$

Calling $u:=\dfrac{\delta}{|s|}$ one gets $\dfrac{a_{max}}{|s|}=\dfrac{\sqrt{u^2+4}-u}{2}=\dfrac{2}{\sqrt{u^2+4}+u}$ and

$$1+\frac{\delta^2+2\delta a_{max}}{s^2+a_{max}^2}=1+\frac{\delta^2+2\delta a_{max}}{2s^2-\delta a_{max}}=1+\frac{u^2+u\sqrt{u^2+4}-u^2}{2-2u/(\sqrt{u^2+4}+u)}=$$

$$=1+\frac{u^2\sqrt{u^2+4}+u^3+4u}{2\sqrt{u^2+4}}=1+\frac{u^2}{2}+u\frac{\sqrt{u^2+4}}{2}=1+u^2+u\frac{\sqrt{u^2+4}-u}{2}\le(1+u)^2.$$

Thus, $G(a)\le 2G(a_{max})=2\log\left(1+\dfrac{\delta^2+2\delta a_{max}}{s^2+a_{max}^2}\right)\le 2\log\left(1+\dfrac{\delta}{|s|}\right)^2$, QED.

37. A PROOF OF THE UNIFORM CONVERGENCE OF THE SERIES IN **(9.31)**. $\int_D G(X,y;-\chi^2)\varphi_h(y)dy=\varphi_h(X)/(\lambda_h+\chi^2)$ holds even for $X\in\partial D$, (see **(9.29)**). In fact, since φ_h is continuous on $\overline{D}$, it is enough to prove, for $x\to X$, that we have $\int_D G(x,y;-\chi^2)\varphi_h(y)dy\to\int_D G(X,y;-\chi^2)\varphi_h(y)dy$. This is true if we put K_0 in the place of G. If we had H instead of G, we arrive at the same conclusion making use of iv) and the second part of v) of **(7.15)**. (Observe that $\int_{J^\delta}|H(p,q;\lambda)|^r dq\le C\delta$ holds for $p\in\overline{D}$ as it follows from Fatou's Lemma.) Because of this, **(9.31)** holds for $x\in\overline{D}$.

Using the same argument, it is possible to prove that for $X,Z\in\overline{D}$, $\int_D|G(X,y,:\lambda)-G(Z,y,:\lambda)|^2dy\to 0$ whenever $X\to Z$. Thus, $\int_D|G(x,y;\lambda)|^2dy$ is a continuouus function on $\overline{D}$. Because of this, from a result due to Dini, it follows that the convergence in **(9.31)** is uniform.

38. We have, $(-\Delta_x+\chi^2)\varphi_h=(\lambda_h+\chi^2)\varphi_h=\mu_h(\chi^2)\varphi_h, \mu_h>0$, $\{\varphi_h: h=0,1,2,\cdots\}$ a complete orthonormal system in $L^2(D)$. Then, $(0\le)\,\lambda_h$ does not depend of χ^2. Besides, $-\Delta_x\varphi_h=\lambda_h\varphi_h$, $\partial\varphi_h/\partial n_x=0$ for $x\in\partial D$. Because of this, $\lambda_0=0$ and $\varphi_0=|D|^{-1/2}$. Then, $0<\lambda_1\le\lambda_2\le\cdots$.

39. Let s be the arclength parameter on $J=\partial D$ and $\delta>0$, small. Call $D^\delta:=D\setminus J_\delta$. Then, ∂D^δ can be parametrized in the following way, (see note **3**),
$\partial D^\delta=\{(\tilde{x}(s),\tilde{y}(s)):-S/2\le s<S/2\}$ where $\tilde{x}(s)=x(s)-\delta\,\dot{y}(s)$, $\tilde{y}(s)=y(s)+\delta\,\dot{x}(s)$. The parameter s is possibly not the arclength σ of ∂D^δ, however,
$d\sigma^2=\left((\dot{x}-\delta\,\ddot{y})^2+(\dot{y}+\delta\,\ddot{x})^2\right)ds^2=\left(1-2\delta(\dot{x}\ddot{y}-\ddot{x}\dot{y})+\delta^2(\ddot{x}^2+\ddot{y}^2)\right)ds^2\approx ds^2$ for small δ. Besides, the interior normal $n_s=(-\dot{y}(s),\dot{x}(s))$ to ∂D at $(x(s),y(s))$ is also the interior normal to ∂D^δ at $(\tilde{x}(s),\tilde{y}(s))$ since $(-\dot{y}(s),\dot{x}(s))\times(\dot{x}(s)-\delta\,\ddot{y}(s),\dot{y}(s)+\delta\,\ddot{x}(s))=0$.

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ERRATA

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BEHAVIOUR OF THE EIGENVALUES OF CLASSICAL BOUNDARY PROBLEMS IN THE PLANE

page	**line**	**written**	**should be**
4	13	a the	a
5	1	L $\int_r^1$	$=\int_r^1$
7	15	*Therefore*	*Moreover*
12	10	If $y_1 \in \tilde{U} \cap J$ then, by construction, $y_2 \in U \cap J$.	If $x \in \tilde{U}$ then, by construction, $y_1, y_2 \in U \cap J$.
13	1	Assumed	assume
15	12	(19)	(19) and (15)
18	2	$\geq\geq$	$\geq$
21	17	$O(\log\lvert x-\xi\rvert)$	$O(\log\frac{2}{\lvert x-\xi\rvert})$
22	16	the map	a map
22	18	Theorem and y	Theorem 1 and
24	6	$H_0^1(D_n) \square H_0^1(D_{n-1})$	$H_0^1(D_n) \ominus H_0^1(D_{n-1})$
25	15	$\{n,m : n^2+m^2<\lambda/4\}$	$\{0<n,m : n^2+m^2<\lambda/4\}$

ERRATA

REMARKS ON A THEOREM OF Å. PLEIJEL AND RELATED TOPICS, II,

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