A NEW PROOF OF FROBENIUS THEOREM AND APPLICATIONS

by

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ABSTRACT . A proof of Frobenius theorem on local integrability of a given distribution on a finite or infinite dimensional manifold under weak differentiability conditions is given.The Inverse Problem in the Calculus of Variations appears as a particular case. A local two-form which measures the non-integrability of a given distribution is also studied, with applications.

Key Words : integrability (of) distributions; Inverse Problem (in the) Calculus of Variations; holonomy.

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INTRODUCTION. Frobenius' Theorem establishes, roughly speaking, that a given sub-bundle of the tangent bundle to a given manifold is integrable if and only if for every pair of vector fields tangent to the sub-bundle, their Lie bracket is also tangent to the sub-bundle. This basic statement has been proved in different situations for finite or infinite dimensional manifolds.Each case assumes some kind of regularity condition on the sub-bundle (distribution) and the manifold (See [1],[5],[8],[9],[10],[12],[15])).There are various methods of proof (See [6] ,[13] and previous references) of the existence of a local integral manifold. Then existence of global integrals is usually established by some extension procedure.(See [20]). However this kind of global problem is beyond the scope of the present article.

In this paper we concentrate on a case of the local existence theory where (some of) the involved spaces are normed spaces, not necessarily complete and the hypothesis on regularity conditions on the distribution are expressed in terms of certain restrictions of the later to 2-dimensional subespaces. It is precisely because of this particular choice of the hypothesis, that our Theorem 1 generalizes known results on Potential Operators (See [16],[17]) as explained in § 2.We show that the usual integrability conditions in terms of Lie brackets of vector fields becomes,in the particular case of potential operators, the usual self-adjointness condition for existence of a potential. An interesting example of this is the inverse problem in the calculus of variations (See [14],[2],[18]) since the later can be aproached using potential operators theory, as shown in [14].

As a consequence of Theorem 2 , we can define (locally) a

2-form ω which vanishes if and only if the integrability condition holds. This is related to the curvature 2-form R as defined in [7]. We show an example in the context of nonholonomic constraints. See also [4] for related topics. Another example is provided by Caratheodory's inaccesibility theorem of thermodynamics. Those examples are of course not new in themselves, but the interpretations seems to be new.

Finally let us sketch how the basic idea of holonomy is used to prove Theorems 1 and 2, letting aside for the moment the required differentiability conditions. Let $E:H\times G \to L(H,G)$ be a map where H,G are linear spaces and $L(H,G)$ is the space of linear maps from H into G. For each $(x,y)\in H\times G$ $\pi(x,y)=(x,y)+\text{Graph } E(x,y)$ is a linear affine subespace of $H\times G$. Thus π is a distribution. Given a curve $q(t)$ on H a *lifting of q with origin* $y_0 \in G$ is a curve $y_q(t,y_0)=y(t)$ such that $\dfrac{\partial y(t)}{\partial t} = E(q(t),y(t))$ and $y(0)=y_0$. The distribution π is integrable provided that for any given curves q,q' s.t. $q(0)=q'(0)=x_0$ $q(1)=q'(1)$ and any $y_0 \in G$, we have $y_q(1,y_0)=y_{q'}(1,y_0)$. Then we can define, $S(x)=y_q(1,x_0)$, where $x=q(1)$. Then the graph of $y=S(x)$ becomes the integral manifold of π that contains (x_0,y_0). Our method of proof shows that independence of $S(x)=y_q(1,y_0)$ of the curve q s.t. $q(0)=x_0$, $q(1)=x$ is a consequence of the integrability condition

$$\omega(x,y)(h,k)=DE(x,y)(h,E(x,y).h).k - DE(x,y)(k,E(x,y).k).h = 0$$

Let γ be the boundary of the rectangle having vertices x_0, x_0+h, x_0+h+k , x_0+k. It is shown in Theorem 2 that $y_\gamma(1,y_0)-y_0$ and $\omega(x_0,y_0)(h,k)$ coincide up to order $|h||k|$ (where $|\ |$ is some norm on H) for $|h|,|k| \to 0$. Thus ω measures the "non-integrability" of the distribution.

§ 1.- Let H be a normed vector space and let G be a Banach space. Norms will be denoted "$| \ |$" in this paper. Let $U \subseteq H$ be an open ball centered at $x_0 \in H$. For each choice of a couple of unit linearly independent vectors $a, b \in H$, we define

$$H' = \{\lambda a + \mu b : \lambda, \mu \in \mathbf{R}\} \qquad , \qquad U' = (x_0 + H') \cap U$$

For given $\alpha, \beta > 0$ we write

$$R = \{ x_0 + \lambda a + \mu b : (\lambda, \mu) \in [0,\alpha] \times [0,\beta] \}$$

We always assume $R \subseteq U$. Thus shrinking U implies shrinking R. Let $E : U\times G\times H \to G$ be given. For each choice of U', x_0, a, b as before, we define Lipschitz (L) and integrability (I) conditions as follows

(L) $E(x,y).h$ continuous for $(x,y,h) \in U'\times G\times H'$ and linear in h. $E(x,y).h$ is k-Lipschitz in y in the following sense. There exists $k=k(U')$ such that

$$|E(x,y).h-E(x,y').h| \leq k|y-y'| \quad \text{for all } x\in U', |h|\leq 1;\ y,y' \in G$$

Now we define $DE: U\times G\times H\times G\times H \to G$ as follows

$$DE(x,y).(h,k).l = \frac{d}{ds}\Big|_{s=0} E(x+sh,y+sk).l$$

It is important to notice that this notion of derivative is weaker than that of *Frechet derivative*. It is usually called *Gateaux derivative*.

(I) $DE(x,y).(h,E(x,y).h).l$ is continuous for $(x,y,h,l)\in U'\times G\times H'\times H'$ and

$$DE(x,y)(h,E(x,y).h).l = DE(x,y)(l,E(x,y).l).h$$

for $(x,y,h,l) \in U'\times G\times H'\times H'$

Finally, fix $y_0 \in G$. Then, under the previous conditions we have

Theorem 1 There exists one and only one $S:U \to G$ such that

$$DS(x).h = E(x,S(x)).h \qquad \text{for all } (x,h) \in U\times H$$
$$S(x_0) = y_0$$

Here $DS(x).h = \frac{d}{d\lambda}S(x+\lambda h)\Big|_{\lambda=0}$,by definition

Remark: a) Using Zorn's lemma, we can always define an inner product norm on a given vector space H . In examples H is given to us and we have some freedom to choose the norm so as to satisfy (L),(I).

b) Replace the Lipschitz condition in (L) by the following: there exist balls $U\subseteq H$, $V\subseteq G$ about x_0,y_0 and a constant k such that $|E(x,y).h - E(x,y').h| < k|y - y'|$ for all $x\in U$, $y,y'\in V$, $h\in H$, $|h|\leq 1$. Then the conclusions of theorems *1* and *2* subsist, with a possibly smaller U.

c) Replace the Lipschitz condition in (L) by the following. Let

$x_0 \in H$, $y_0 \in G$ be given. Let $U \subseteq H$ be a ball centered at x_0. Let H', $U' = (x_0 + H') \cap U$ as before. Assume that for each ball $B \subseteq G$ centered at y_0, there is a constant $k = k(U', B)$ such that

$$|E(x,y).h - E(x,y').h| < k|y - y'|$$

for all $x \in U'$, $|h| \leq 1$, $y, y' \in B$

Then the conclusion of ***Theorem 1*** subsists in a weaker form, namely the domain of S is a star-shaped subset W of H about x_0. Moreover W has the following property: for each H' as before, $(x_0 + H') \cap W$ is a bidimensional ball centered at x_0.

The conclusion of ***Theorem 2*** subsists.

To prove this theorem we need some previous lemmas.

For each n = 1,2,3... we have an n-partition of R into n^2 smaller rectangles whose sides have lenght $\frac{\alpha}{n}$, $\frac{\beta}{n}$. We will work with piecewise-linear maps $q : [0, \alpha+\beta] \to R$ parametrized by arc-lenght, each linear piece being parallel to either a or b and having length $\frac{\alpha}{n}$ or $\frac{\beta}{n}$ respectively and coinciding with one side of some rectangle of the n-partition of R. We also assume that the length of q is $\alpha+\beta$ and $q(0) = x_0$, $q(\alpha+\beta) = x_0 + \alpha a + \beta b$. We denote by Q_n the set of such q's . Any $q \in Q_n$ (and more generally any piecewise linear curve) will be represented by the sequence of its vertices as follows

$$q = \left\langle q_0, q_1, \ldots q_{2n} \right\rangle , \quad \text{where } q_0 = x_0 , \ q_{2n} = x_0 + \alpha a + \beta b$$

We set $Q = \bigcup_{n=1}^{\infty} Q_n$. Given $q \in Q_n$, we can write the differential equation and initial condition problem

$$\dot{y}(t) = E(q(t),y(t)).\dot{q}(t)$$
$$y(0) = y_0 \qquad (1)$$

On each linear piece the differential equation has unique solution (as a consequence of the Lipschitz condition in y) and by glueing pieces together we get a unique continuous $y(t)$, $t\in[0,\alpha+\beta]$ called the "*lifting of the curve q with origin y_0*" denoted $y_q(t,y_0)$ or sometimes simply $y_q(t)$. Given two curves $C_1:[a_1, b_1] \to Z$, $C_2:[b_1, b_2] \to Z$ where Z is a given space, such that $C_1(b_1) = C_2(b_1)$ we can form the sum C_1+C_2: $[a_1, b_2] \to Z$ as follows : $(C_1+C_2)(t) = C_1(t)$ if $t\in [a_1, b_1]$ and $(C_1+C_2)(t) = C_2(t)$ if $t \in [b_1, b_2]$. We can also define $-C_1(t)=C_1(a_1+b_1-t)$. Thus a piecewise linear curve say $q = \{q_1,q_2,q_3,\ldots\}$ equals the sum of its linear pieces, namely $q = [q_1, q_2]+[q_2, q_3]+\ldots$ For liftings, we obviously have $y_{C_1+C_2}(.,y_0)=y_{C_1}(.,y_0)+y_{C_2}(.,y_{C_1}(b_1,y_0))$

By definition (See [5] pag 116) an ϵ-*approximate* solution to a differential equation $y'=f(t,y)$ where $f : U \to F$, U open in $\mathbf{R} \times F$ F a Banach space, is a differentiable map $\varphi : I \to F$, where $I \subset R$ is an open interval such that for $t\in I$ we have

(i) $(t,\varphi(t)) \in U$

(ii) $|\varphi'(t) - f(t,\varphi(t))| \leq \epsilon$

Lemma 1. Let $\varphi_i : I \to F$ be ϵ_i-approximate solutions of the equation

$y' = f(t,y)$ for $i=1,2$ .Let $x_i = \varphi_i(t_0)$ the initial values for φ_i, $i=1,2$. Then if f is k-Lipschitz in y and continuous in $(t,y)\in U$ we have for $t \in I$ the following

$$|\varphi_1(t) - \varphi_2(t)| \leq |x_1 - x_2| e^{k|t-t_0|} + (\epsilon_1+\epsilon_2) \frac{e^{k|t-t_0|} - 1}{k}$$

Proof: See [5] pag 116.

Let U' be as before and choose $y_0 \in G$. By shrinking U' if necessary and using continuity assumption (L) , for some $\epsilon>0$ we can find a ball $B_r(y_0)$ centered at $y_0 \in G$ and having radius $r>0$ such that $|E(C(t),y).\dot{C}(t)|<\epsilon$ for all piecewise linear curve $C:[t_1 , t_2] \to U'$ parametrized with arc lenght and all $y \in B_r(y_0)$. Then the constant function $\varphi(t)\equiv y'_0$ is an *ϵ-approximate solution* to (1) for each fixed $y'_0 \in B_r(y_0)$ (See ***Lemma 1***). There exists $L>0$ such that $\epsilon(\frac{e^{kt}-1}{k})< \frac{r}{2}$ for all $t \in [0 , L]$, where k is the Lipschitz constant appearing in (L). Choose $y'_0 \in B_{\frac{r}{2}}(y_0)$.

Since $y_C(t,y'_0)$ is the exact (0-approximate) solution to (1) with initial data y'_0, using ***Lemma 1*** we obtain $|y_C(t,y'_0) - y'_0|< \frac{r}{2}$ for all $t \in [0 , L]$. In particular $y_C(t,y'_0) \in B_r(y_0)$ for all $t\in [0 , L]$.Using this we can show that given $r>0$ we can shrink $R\subseteq U'$ so as to satisfy the following for some fixed $y_0 \in G$

$$|y_q(t,y_0) - y_0| < r \qquad \text{for all } t\in[0 , \alpha+\beta],\ q\in Q ,\ y_0 \in G$$

Let $V \subseteq G$ be an open ball centered at $y_0 \in G$.
In the following lemmas, x_0, y_0, H', will remain fixed while U, R, V, will be eventually shrinked so as to satisfy certain conditions. Using our continuity assumptions (I),(L) and shrinking U,V if necessary, we can assume that $E(x,y).h$ and $DE(x,y).(h,E(x,y).h).l$ are bounded for (x,y) $U' \times V$ and $h,l \in H'$ bounded. Besides we shall assume that the closure $\bar{K}$ of the set $K = \{ y_q(t,y_0) : q \in Q,\ t \in [0 , \alpha+\beta]\}$ satisfies $\bar{K} \subseteq V$. This can be achieved using the continuity and Lipschitz conditions (L), and the observation before ***Lemma 1***.
A number of bounds, either constants like C,D,etc.,or functions like ϵ,ϵ_1,etc., will appear along the line in the following lemmas. It is understood that they may depend on the choice of U,V,R,y_0, x_0, but apart of this, they are fixed.

Lemma 2 : Let $u,v \in R$, $v=u+h$, $h \neq 0$, $w \in V$. Set $\frac{h}{|h|} = \lambda$ and let $y(t,w)$ be the solution of

$$\dot{y} = E(u+t\lambda\ ,\ y(t)).\lambda$$

$$y(0) = w$$

Let

$$\varphi(t,w) = w + tE(u,w).\lambda + \frac{t^2}{2}DE(u,w)(\lambda,E(u,w).\lambda).\lambda$$

a) Then there is a continuous function $\epsilon(t,\lambda,u,w) > 0$ such that

$|y(t,w) - \varphi(t,w)| = \epsilon(t,\lambda,u,w)t^2$ for all $0 \leq t \leq |h|$, $u \in R, w \in V$.
In particular ϵ is bounded provided that U,V are small.

b) Moreover if w varies on a compact set $\bar{K} \subseteq V$, then there is a continuous function $\bar{\epsilon}(t)$ such that $\bar{\epsilon}(t) \to 0$ as $t \to 0$ and $\epsilon(t,\lambda,u,w) < \bar{\epsilon}(t)$ for all $t \in [0, |h|]$, $u \in R$, $w \in V$.

Proof: a) The Taylor's expansion of the solution $y(t) \equiv y(t,\lambda,u,v)$ is $y(t) = \varphi(t) + t^2\epsilon(t,\lambda,u,w)$. Let $A(u,v,\lambda) = DE(u,v)(\lambda, E(u,v).\lambda).\lambda$
Then the integral form of the remainder gives the following

$$\epsilon(t,\lambda,u,w) = \left| \int_0^1 (1-s)[A(u+ts\lambda, y(ts),\lambda) - A(u,w,\lambda)]ds \right|$$

Our continuity assumptions imply that ϵ is continuous, and therefore it is bounded provided that U,V are small.
b) It follows from the previous formula that $\epsilon(t,\lambda,u,w) \to 0$ as $t \to 0$ for each (λ,u,w). Since $(\lambda,u,w) \in S^1 \times R \times \bar{K}$ which is compact we can find $\bar{\epsilon}(t)$ having the required properties. ■

Let $u \in R$, $w \in V$ and for each positive integer n let $B = u + \frac{\alpha}{n}a$, $C = u + \frac{\beta}{n}b$, $D = u + \frac{\alpha}{n}a + \frac{\beta}{n}b$ such that u,B,C,D are vertices of a subrectangle contained in R. Set $\gamma = \{A,B,D\}$, $\gamma' = \{A,C,D\}$, piecewise linear maps parametrized by arc lenght.

Set $y_B = y_\gamma(\frac{\alpha}{n}, w)$, $y_D = y_\gamma(\frac{\alpha}{n}+\frac{\beta}{n}, w)$, $y_C = y_{\gamma'}(\frac{\beta}{n}, w)$, $y'_D = y_{\gamma'}(\frac{\alpha}{n}+\frac{\beta}{n}, w)$. We want to estimate the difference $|y_D - y'_D|$.

Lemma 3: a) There is a constant C such that $|y_D - y'_D| < C\frac{1}{n^2}$

b) Assume that w happens to vary on a compact set $\bar{K} \subseteq G$ and condition (I) is satisfied. Then for some $D(\frac{1}{n})$, $D(\frac{1}{n}) \to 0$ as $n \to \infty$, we have $|y_D - y'_D| < D(\frac{1}{n})\,\frac{1}{n^2}$.

Proof: a) Using the previous lemma with $v \equiv B$, $h \equiv \frac{\alpha}{n}a$, $t \equiv \frac{\alpha}{n}$, $\lambda \equiv a$ we obtain,

$$y_B - w = \frac{\alpha}{n}E(u,w).a + (\frac{\alpha}{n})^2\,\frac{1}{2}\,DE(u,w)(a,E(u,w).a).a + \epsilon(\frac{\alpha}{n},a,u,w)(\frac{\alpha}{n})^2 \qquad (1)$$

Using the previous lemma again with $u \equiv B$, $v \equiv D$, $w \equiv y_B$, $h \equiv \frac{\beta}{n}b$, $t \equiv \frac{\beta}{n}$, $\lambda \equiv b$ we obtain,

$$y_D - y_B = \frac{\beta}{n}E(B,y_B).b + (\frac{\beta}{n})^2\,\frac{1}{2}DE(B,y_B)(b,E(B,y_B).b).b + \epsilon(\frac{\beta}{n},b,B,y_B)(\frac{\beta}{n})^2 \qquad (2)$$

We replace B, y_B from (1) on the second side of (2) and using the continuity of E and DE we obtain, after some rearrangements,

$$y_D - y_B = \frac{\beta}{n} E(u,w).b + \left(\frac{\beta}{n}\right)^2 \frac{1}{2} DE(u,w)(b,E(u,w).b).b + \epsilon_1\left(\frac{1}{n},u,w\right) \frac{1}{n^2} +$$

$$+ \frac{\alpha\beta}{n^2} DE(u,w)(a,E(u,w).a).b \qquad (3)$$

Where ϵ_1 is continuous and therefore bounded if U,V are small. In a similar manner we can prove the following

$$y_C - w = \frac{\beta}{n} E(u,w).b + \left(\frac{\beta}{n}\right)^2 \frac{1}{2} DE(u,w)(b,E(u,w).b).b + \epsilon\left(\frac{\beta}{n},b,u,w\right) \left(\frac{\beta}{n}\right)^2 \qquad (4)$$

and

$$y'_D - y_C = \frac{\alpha}{n} E(u,w).a + \left(\frac{\alpha}{n}\right)^2 \frac{1}{2} DE(u,w)(a,E(u,w).a).a + \epsilon_2\left(\frac{1}{n},u,w\right) \frac{1}{n^2} +$$

$$+ \frac{\alpha\beta}{n^2} DE(u,w)(b,E(u,w).b).a \qquad (5)$$

Using (1),(3),(4) and (5) we can calculate $y_D - y'_D$ as follows

$$y'_D - y_D = \frac{\alpha\beta}{n^2} \left[DE(u,w)(b,E(u,w).b).a - DE(u,w)(a,E(u,w).a).b \right] +$$

$$+ \mu\left(\frac{1}{n},u,w\right) \frac{1}{n^2} \qquad (6)$$

where μ is continuous and therefore bounded if U,V are small. From this the proof of part a) follows easily

b) Note that, for each $(u,w) \in R \times V$ we have $\mu(\frac{1}{n},u,w) \to o$ as $n \to \infty$.

Since $(u,w) \in R\times\bar{K}$ compact, we can find $D(\frac{1}{n}) > \mu(\frac{1}{n},u,w)$ and

$D(\frac{1}{n}) \to 0$ as $n\to\infty$, satisfying the required condition.

■

Let $q,q' \in Q_n$. They are *contiguous* if and only if they differ by one vertex only namely $q=p+\gamma+s$, $q'=p+\gamma'+s$ where $p=\{q_1,\dots q_{k-1}\}$ $\gamma=\{q_{k-1},q_k,q_{k+1}\}$, $\gamma'=\{q_{k-1},q_k',q_{k+1}\}$, $s=\{q_{k+1},\dots q_{2n}\}$.

Let t_1=lenght p, $t_2-t_1 = \frac{\alpha}{n}+\frac{\beta}{n}$ =lenght γ = lenght γ'.Choose $t_3 \geq t_2$

$t_3-t_2 \leq$ lenght s.

Set $w_1 = y_p(t_1,y_0)$, $w_2 = y_\gamma(t_2-t_1,w_1)$, $w_2' = y_{\gamma'}(t_2-t_1,w_1)$

$w_3 = y_s(t_3-t_2,w_2) = y_q(t_3,y_0)$, $w_3' = y_s(t_3-t_2,w_2') = y_{q'}(t_3,y_0)$

Lemma 4: There is a constant C_1 such that $|w_3 - w_3'| < C_1\frac{1}{n^2}$

Proof : Using ***Lemma 3***,a) we have $|w_2 - w_2'| < C\frac{1}{n^2}$. Then applying ***Lemma 1*** to the *exact* solutions $y_s(t,w_2)$, $y_s(t,w_2')$ with *different* initial data w_2,w_2' we get

$$|w_3 - w_3'| \leq |w_2 - w_2'|\ e^{k(t_3-t_2)}$$

Since $t_3 - t_2 \leq \alpha+\beta$, we obtain

$$|w_3 - w_3'| \leqq C\, e^{k(\alpha+\beta)} \frac{1}{n^2} \equiv C_1 \frac{1}{n^2}$$

∎

Let $c_1, c_2 : [a, b] \to Z$, c_1, c_2 continuous, Z a metric space. We define as usual

$d(c_1,c_2) = \sup d(c_1(t), c_2(t))$

Lemma 5: There is a constant C_2 such that if $q, q' \in Q_n$ *satisfy* $d(q,q') < \delta$ and $q(t) = q'(t)$ for some $t = \bar{t}$ then

$$|y_q(\bar{t}, y_0) - y_{q'}(\bar{t}, y_0)| < C_2\, \delta$$

Proof: An elementary reasoning will show that there is a constant say F, such that the number of rectangles of the n-partition of R which lie "in between" the restrictions $q|[0, \bar{t}] \equiv \bar{q}$, $q'|[0, \bar{t}] \equiv \bar{q}'$ is less or equal than $F\delta n^2$. This means that there is a sequence of at most $N < F\delta n^2$ elements $q^1, q^2, \ldots q^N \in Q_n$ such that q^i, q^{i+1} are contiguous for $i=1,\ldots N-1$, $q^1 \equiv q$, $q^N \equiv q'$ and $q^i(\bar{t}) = q(\bar{t})$ for $i=1,\ldots N$. Using ***Lemma 4*** repeatedly with $\bar{t} \equiv t_3$ we can easily show that $C_2 = C_1 F$ satisfies the requirement of the lemma.

∎

Lemma 6: There is a constant C_3 such that for $q, q' \in Q$ we have

$$d(q,q') < \delta \quad \text{implies} \quad d(y_q, y_{q'}) < C_3 \delta$$

Proof: Let $t_0 \in [0 , \alpha+\beta]$. An elementary reasoning shows that we can find piecewise linear maps $\xi,\eta :[t_0 , t_0+\Delta] \to R$ satisfying the following conditions, where F_1 is an appropriate constant

i) $\xi(t_0+\Delta) = \eta(t_0+\Delta)$, $\xi(t_0) = q(t_0)$, $\eta(t_0) = q'(t_0)$
   Δ = *lenght* ξ = *lenght* $\eta < F_1\delta$

ii) Each linear piece of ξ and η is contained in one side of some rectangle of the n-partition of R.

Let $p=q|[0,t_0] + \xi$, $r = q'|[0,t_0] + \eta$ .Then $p(t_0+\Delta) = r(t_0+\Delta)$

Then we can apply the previous lemma with $\bar{t} = t_0+\Delta$ and we obtain

$$|y_p(\bar{t},y_0) - y_r(\bar{t},y_0)| < C_2\delta$$

Using ***Lemma 1*** (or more directly, the uniform Lipschitz condition on y) we can find a constant, say F_2, such that

$|y_p(t_0,y_0)-y_p(\bar{t},y_0)|$, $|y_r(t_0,y_0)-y_r(\bar{t},y_0)| < F_2\delta$. From this and the previous inequality we finally obtain

$$|y_q(t_0,y_0) - y_{q'}(t_0,y_0)| < (2F_2 + C_2)\delta = C_3\delta$$

■

Lemma 7: The set $K = \left\{ y_q(t,y_0) : q\in Q , t \in [0 , \alpha+\beta]\right\}$ is relatively compact as a subset of G

Proof: The family (of continuous functions from $[0 , \alpha+\beta]$ into R) Q is totally bounded as a consequence of Arzela's theorem. Then using the previous lemma, we can show that $\{y_q\}_{q \in Q}$ is totally bounded as well. Finally, the evaluation map

$$\{y_q\}_{q\in Q} \times [0 \ , \ \alpha+\beta] \to G$$
$$(y_q, t) \to y_q(t)$$

can be continuously extended to the closure of its domain which is compact. Since K is contained in the image of this extended map, it is relatively compact.

■

Lemma 8: Let the same situation as in ***Lemma 4***, and assume that condition (I) is satisfied. Then there is a function $\nu(\frac{1}{n})$ such that $\nu(\frac{1}{n}) \to 0$ as $n\to\infty$ and $|w_3 - w_3'| < \nu(\frac{1}{n}).\frac{1}{n^2}$

Proof: Using ***Lemma 7***, we can show that $w_1, w_2, w_2', w_3, w_3'$ all vary on a compact set $\bar{K}$. Therefore using ***Lemma 3 b)*** we see that $|w_2 - w_2'| < D(\frac{1}{n})\frac{1}{n^2}$ .Argueing like in the last part of the proof of ***Lemma 4*** we can show that

$$|w_3 - w_3'| \leq |w_2 - w_2'| \ e^{k(\alpha+\beta)} \leq e^{k(\alpha+\beta)} D(\tfrac{1}{n}) \frac{1}{n^2} \equiv \nu(\tfrac{1}{n}) \frac{1}{n^2}$$ ■

Proof of the theorem 1 : Let $q\equiv q^1, q^2 \ldots\ldots q^N \equiv q'$, where $N=n^2$ is a sequence of elements of Q_n such that q^i, q^{i+1} are contiguous for $i=1,\ldots N-1$ and q lies in the sum of the two edges $[x_0 \ , \ x_0+\alpha a]+ [x_0+\alpha a \ , \ x_0+\alpha a+\beta b]$ of R while q' lies in the sum of the remaining edges $[x_0 \ , \ x_0+\beta b] + [x_0+\beta b \ , \ x_0+\alpha a+\beta b]$. Using the previous lemma we have

$$|y_q(\alpha+\beta)-y_{q'}(\alpha+\beta)| \leqq |y_{q'}(\alpha+\beta) - y_{q^2}(\alpha+\beta)|+|y_{q^2}(\alpha+\beta) - y_{q^3}(\alpha+\beta)|+$$

$$\ldots+|y_{q^{N-1}}(\alpha+\beta) - y_{q'}(\alpha+\beta)| \leqq \nu(\tfrac{1}{n})$$

Since $\nu(\frac{1}{n}) \to 0$ as $n \to \infty$ we obtain $y_q(\alpha+\beta\ ,\ y_0) = y_{q'}(\alpha+\beta\ ,\ y_0)$

Now we define the map S appearing in the statement of the theorem. For this, given R as before, define $S_R : R \to G$ as follows For any $x_0+\lambda a+\mu b=x \in R$, let $q = \{x_0, x_0+\lambda a, x\}$, $q'= \{x_0, x_0+\mu b, x\}$ be piecewise linear maps (having two linear pieces each) contained in R. So far we have proved the following. Let $x_0 \in U', U'=(x_0+H')\cap U$ $y_0\in G$ and assume that (I),(L) are satisfied. Then there exist α,β 0 and $R\equiv R(x_0,a,b,\alpha,\beta) \subseteq U'$ as before such that, for each $x=x_0+\lambda a+\mu b$ $\in R$ we have

$$y_q(\lambda+\mu\ ,\ y_0) = y_{q'}(\lambda+\mu, y_0)$$

where $q=\{x_0,\ x_0+\lambda a\ , x\}$, $q'=\{x_0\ , x_0+\mu b,\ x\}$

Define $S_R : R \to G$ by $S_R(x) = y_q(\lambda+\mu\ ,\ y_0) = y_{q'}(\lambda+\mu, y_0)$. It follows that

$$\frac{\partial S_R(x)}{\partial\lambda} = E(x, S_R(x)).a \quad , \quad \frac{\partial S_R(x)}{\partial\mu} = E(x, S(x)).b$$

Consequently S_R is of differentiability class C^1 and therefore for every C^1 curve $x(s) = x_0 + \lambda(s)a + \mu(s)b \in R$ we have
$\frac{d}{ds}S_R(x(s)) = E(x(s)\ ,\ S_R(x(s))).\dot{x}(s)$. In particular for $x(s)=x+sh$
we obtain

$$\frac{d}{ds}\Big|_{s=0} S_R(x+sh) = E(x\ ,\ S_R(x)).h$$

This implies that for any piecewise-C^1 curve $\gamma(s)$ on R such that $\gamma(0) = x_0$ we have

$$y_\gamma(s\ ,\ y_0) = S_R(\gamma(s))$$

We can obviously find a finite family of rectangles $R_i \equiv R(x_0, a_i, b_i, \alpha_i, \beta_i)$ $a_i, b_i \in H'$ such that their union covers a ball centered at x_0 , say $D(x_0) \subseteq U'$. By glueing the S_{R_i} together we can find $S_{D(x_0)} : D(x_0) \to G$ satisfying the Frobenius differential equation and initial condition

$$DS_{D(x_0)}(x).h = E(x\ ,\ S_{D(x_0)}(x)).h$$
$$S_{D(x_0)}(x_0) = y_0$$

for all $x \in D(x_0)$, $h \in H'$

Next we shall extend $S_{D(x_0)}$ to a map $S_{U'} : U' \to G$ satisfying the same differential equation and initial condition for $x \in U'$. Let $U \equiv U_{r_0}$ be the ball of radius r_0 and center x_0. Let $U'_r = U_r \cap (H'+x_0)$ be the ball having the center x_0 and radius $r < r_0$ in the space $(H'+x_0)$, and let $\bar{U}'_r$ be its closure. Assume that $S_{D(x_0)}$ has an extension $S_{U'_r} : U'_r \to G$ satisfying the required conditions for some r. Using the Lipschitz condition we can show that $S_{U'_r}$ has a continuous extension $S_{\bar{U}'_r}$ to the closure $\bar{U}'_r$.

For each $x'_0 \in \partial\bar{U}'_r$ find $D(x'_0)$ and $S_{D(x'_0)}:D(x'_0) \to G$ satisfying the Frobenius differential equation and the initial condition: $S_{D(x'_0)}(x'_0) = S_{\bar{U}'_r}(x'_0)$. By compactness of $\partial\bar{U}'_r$ we can find a finite number of such $D(x'_0)$, covering $\partial\bar{U}'_r$ and glueing all the corresponding $S_{D(x'_0)}$ and $S_{U'_r}$ together, we find an extension $S_{U'_{r+\delta}}$ for some $\delta > 0$. From this we can easily conclude that there is an extension $S_{U'}$, where $U'=U\cap(x_0+H')$ with the originally given U.

Finally let $H',H''\subseteq H$ be given bidimensional subespaces and let $U'= (x_0+H')\cap U$, $U''= (x_0+H'')\cap U$. Let $x\in U'\cap U''$ and $C(s)=x_0+s(x-x_0)$. Then $y_C(s,y_0) = S_{U'}(C(s)) = S_{U''}(C(s))$. This shows that we can coherently define $S:U \to G$ using its restrictions to each $U' = (x_0+H')\cap U$ where H' is an arbitrary bidimensional subespace of H, namely $S|U' = S_{U'}$.

■

With the notation introduced in the previous theorem, we have as a corollary the following result:

Theorem 2. For each n, let $\gamma(t)$ be the boundary of the subrectangle R parametrized with arc length $t \in [0\ ,\ 2\frac{\alpha+\beta}{n}]$, such that

$$\gamma(0) = \gamma(2\frac{\alpha+\beta}{n}) = x_0\ ,\ \gamma(\frac{\alpha}{n}) = x_0+\frac{\alpha}{n}a\ ,\ \gamma(\frac{\alpha+\beta}{n}) = x_0+\frac{\alpha}{n}a+\frac{\beta}{n}b$$

$\gamma(2\frac{\alpha}{n}+\frac{\beta}{n}) = x_0+\frac{\beta}{n}b$

Define

$$\omega(x_0,y_0)(a,b) = \lim_{n\to\infty} \frac{y_\gamma(2(\frac{\alpha}{n}+\frac{\beta}{n}),y_0) - y_\gamma(0,y_0)}{\frac{\alpha\beta}{n^2}}$$

Set $y_0=y_\gamma(0)$. Then we have

$\omega(x_0,y_0)(a,b) =$

$$\left[DE(x_0,y_0)(a,E(x_0,y_0).a).b - DE(x_0,y_0)(b,E(x_0,y_0).b).a\right]$$

Proof: **The result follows directly from formula (6).** Notice that the denominator in the above limit is just the area of the subrectangle R.

■

§ 2.- APPLICATIONS

A) Potential operators

As stated in the Introduction, Theorem 1 generalizes in some sense, known results on potential operators (See [16],[17]). We shall briefly comment on those results, using our own notation, for convenience. Let H be a Banach space. H^* its dual and let $E:H \to H^*$ be an operator. Assume that : 1) E has a linear Gateaux differential $DE(x).h$ at every point x of a ball B centered at $0 \in H$. 2) The functional $DE(x).h.k$ is continuous in $x \in H' \cap B$ for each choice of a bidimensional subespace $H' \subseteq H$, $h,k \in H'$. Then in order for the operator E to be potential in the ball B (i.e. for existence of $S(x)$ s.t. $DS(x).h=E(x).h$, for $x \in B$, $h \in H$) it is necessary and sufficient that the bilinear functional $DE(x).h.k$ be symmetric i.e. $DE(x).h.k = DE(x).k.h$. (This is essentially the content of [16], Thm 5.1 and footnote). The later becomes a particular case of our Theorem 1 by taking the Banach space $G \equiv \mathbf{R}$ and also $E(x,y) \equiv E(x)$, independent of $y \in \mathbf{R}$ (this in turn implies that the Lipschitz condition is automatically satisfied). The nice fact that the Frobenius integrability condition $\omega=0$ in case E is independent of y, reduces to the typical symmetry (or self-adjointness) condition in potential theory is easily verified.
On the other hand, a result similar to Thm. 5.1 of [16] is proved in [17] (See Thm 6.3) using hypothesis involving the notion of *"hemicontinuity"* rather than *"continuity"* of the relevant operators. We could also provide a generalization of this result using a suitable generalization of the notion of

hemicontinuity. Thus let us say (as a provisional, ad-hoc notation only) that $DE(x,y).(a,E(x,y).b)$ is *hemicontinuous* on $(x,y,a,b) \in R\times G\times H\times H'$ if for each choice of $x_0, x_1 \in R$ the function $DE(x_0+t(x_1-x_0),y).(a,E(x_0+t(x_1-x_0),y).b$ is continuous for $(t,y,a,b) \in [0,1]\times G\times H\times H'$. Similary $E(x,y).h$ is hemicontinuous on $R\times G\times H'$ if $E(x_0+t(x_1-x_0),y).h$ is continuous on $(t,y,h) \in [0,1]\times G\times H'$. We can show that Theorem 1 remains valid after we restate it by replacing *continuous* by *hemicontinuous*.

It is also interesting that potential operator theory has been related to the inverse problem in the calculus of variations (see [14],[18], and references therein, specially, Tonti) This is done as follows. Let H be the vector space of C^∞ curves $q:[t_1,t_2] \to R^n$ such that $q(t_i)=q_i$, fixed for i=1,2. (To see the vector space structure just replace each q by $q-\phi$, where $\phi(t)$ is a suitable fixed linear function of t such that $(q-\phi)(t_i)=0$. Thus ϕ is the null vector of H,etc.). Besides we can choose the usual $\|\ \|_\infty$ norm in H). Let $L:TR^n \to R^n$ be a given Lagrangian of differentiability class C^∞. The Euler-Lagrange operator is by definition

$$E(L)(q).h=\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q}(q,\dot q) - \frac{d}{dt}\frac{\partial L}{\partial \dot q}(q,\dot q)\right].h\,dt \quad \text{where } q,h\in H.$$

This is an example of the situation described in§1, where $U\equiv H$, $G\equiv R$ $x\equiv q$ and $E(q,y).h = E(L)(q).h$ is independent of y. The inverse problem in the calculus of variations consists in finding necessary and sufficient conditions to ensure existence of L s.t. for a given $E(q).h=\int_{t_1}^{t_2} [A(q,\dot q)\ddot q + B(q,\dot q)].h\,dt$

$E(L)(q).h \equiv E(q).h$. The usual self-adjointness condition in the calculus of variations (usually written in terms of A,B and their partial derivatives) coincides, as observed by Tonti, with the symmetry condition of the operator E.

B) Caratheodory Inaccessibility Theorem

A nice application of the curvature ω is a geometrically inspired short proof of Caratheodory's inaccesibility theorem of Thermodynamics. Let us state the following version of this theorem (See [19]) for convenience.
Let M be a smooth n-manifold and $Q\in\Omega'(M)$ a nonwhere vanishing 1-form. Then the following are equivalent

- *(i) $Q=0$ is an integrable Pfaffian system, i.e. locally $Q=Tds$*
- *(ii) For each $x_0\in M$ there exist an open neighborhood V of x_0 on M such that each neighborhood W of x_0, $W\subseteq V$ contains a point $x\in W$ that cannot be connected to x_0 by a (piecewise smooth) quasi-static adiabatic path.*

Here Q is the heat delivered by the system and M is the phase space. For a given curve $\gamma(t)$ on M , the condition $Q(\dot{\gamma}(t)) = 0$ means that γ is a quasi-static adiabatic path.
We illustrate our method of proof by showing that $ii)\Rightarrow i)$
First chosse a local chart at $m_0\in M$ such that the neighborhood $V\subseteq M$ is of type $V\equiv\mathbf{R}^{n-1}\times\mathbf{R}$ and there is a function $f:\mathbf{R}^{n-1}\times\mathbf{R}\to\mathbf{R}$ such that $Q(x,y)(\dot{x},\dot{y})=0$ iff $\dot{y}=f(x,y).\dot{x}$.
In other words, the distribution defined by $Q=0$ is locally described by f. Now assume $\omega(x_0,y_0)(a,b)\neq 0$ for some $(x_0,y_0)\in\mathbf{R}^{n-1}\times\mathbf{R}$ and $a,b\in\mathbf{R}^{n-1}$. Let $\gamma(t)$ be the boundary of a rectangle R , as in theorem 2, and let n in that theorem take

real values. It can be shown by an elementary argument that as n varies, $y_\gamma(\frac{\alpha+\beta}{n})$ takes on every value on a certain interval ,say $(y_0-\delta\ ,\ y_0+\delta)$ about $y_0\in\mathbf{R}$. This is done by using the fact that $\omega(x_0,y_0)(\frac{\alpha a}{n}\ ,\ \frac{\beta b}{\pm n}) = \frac{\alpha\beta}{\pm n^2}\,\omega(x_0,y_0)(a,b)$ varies on $(\ -\delta\ ,\ \delta)$ and the fact that $\omega(x_0,y_0)(\frac{\alpha a}{n},\frac{\beta b}{\pm n})$ differ from $y_\gamma(\frac{\alpha+\beta}{n})-y_0$ by a higher order quantity. Thus every point in that interval is accessible. From this it follows also by an elementary argument that the union of all liftings $y_c(t,y_1)$ of linear curves $c(t)=x_0+tv$, $|v|=1$ with origin $y_c(0,y_1)\equiv y_1\ \epsilon\ (y_0-\delta\ ,\ y_0+\delta)$, fill a whole neighborhood of $(x_0,y_0)\epsilon\mathbf{R}^n\times\mathbf{R}$. Thus every point in that neighborhood is accesible. From this and $ii)$ it follows that $\omega(x,y)=0$ for all $(x,y)\epsilon\mathbf{R}^n$. Then f is integrable according to theorem 1. From this, existence of the integrating factor $\frac{1}{T}$ (where T is the absolute temperature) follows by standard arguments.

The proof $i)\Rightarrow ii)$ is easier and will be omited .

C) ω as a curvature form

The 2-form ω can be related to the curvature R defined in 4.41 of [7] as follows. Let H,G be Banach spaces and $U\subseteq H$, $V\subseteq G$ open sets. Let $\Gamma:U\times V\to L(H,G)$ be C^∞. Define the *horizontal* and *vertical* projectors as follows

$$h = \begin{bmatrix} I_H & 0 \\ -\Gamma & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 & 0 \\ \Gamma & I_G \end{bmatrix}$$

where I_H , I_G are the identity maps of H,G. Then $h^2=h$, $v^2=v$, $v+h=I_{H+G}$ $hov=0$, $voh=0$. A vector $x\in H\times G$ is vertical iff $vx=x$ and it is horizontal iff $hx=x$. Let $x,y\in H\times G$. Define

$$R(x,y)=-[hx , hy] + [hx , y] + h[x , hy] - h[x , y]$$

It can be easily shown that R is a tensor field i.e, it is R-linear in x,y and for any given $f\in C^{\infty}(U\times V)$ we have $R(fx,y)=fR(x,y)$, $R(x,fy)=fR(x,y)$. Moreover we can show that $R(x,y)=0$ if x or y is vertical. We can also show that for $x=a+a'$ $y=b+b'\in H\times G$ we have

$$\omega(a,b) = R(x,y)$$

This follows from the expression

$[U,V] =DV.U-DU.V$ and the formula for ω given in Theorem 2, by a straightfoward calculation.

D) Constrained Lagrangian Systems

Let us assume for simplicity $dim\ H = n$, $dim\ G = m$, E of class C^1. Let $L : T(U \times G) \to R$ be a given Lagrangian and let us interpret E as a (time independent) constraint (See [4]). A system is called *holonomic* or *non-holonomic* according to whether the imposed constraints are integrable or not.

A curve $P(t) = (q(t),y(t)) \in U \times G$ is *compatible with the constraint* E if $\dot{y}(t) = E(q(t),y(t)).\dot{q}(t)$ (i.e. if $y(t)$ is the lifting of $q(t)$ with $y(t_0) = y_0$, for some y_0). Choose variations

$P(t,\lambda) = (q(t,\lambda)\ ,\ y(t,\lambda))$ such that for each fixed t, $y(t,\lambda)$ is the lifting of $q(t,\lambda)$ with initial condition $y(t)$, and $q(t_i,\lambda) = q(t_i)\ ,\ i=0,1\ ,\ y(t_0,\lambda) = y_0\ ,\ y(t_1,\lambda) = y_1$ are fixed.

Remark. A different kind of variations is also interesting.

Namely $(q(t,\lambda),\tilde{y}(t,\lambda))$, where for each λ , $\tilde{y}(t,\lambda)$ is the lifting of $q(t,\lambda)$ with fixed origin y_0 . Thus in general

$\tilde{y}(t_1,\lambda) \neq \tilde{y}(t_1,0) = y(t,a)$.

Each vector $\dfrac{\partial P}{\partial \lambda} \equiv \delta P \in T_{P(t)}U\times G$ compatible with the constraint is called a *virtual velocity*, and D'Alambert-Lagrange Principle establishes that $P(t)$ is a motion if and only if it is a critical point of

$$\mathcal{L}(P) = \int_{t_0}^{t_1} L(P(t),\dot{P}(t))\ dt$$

with respect to variations δP of P compatible with the constraints (*virtual displacements*). This is equivalent to

$$\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right]\delta q + \left[\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}\right]E.\delta q = 0$$

$$\dot{y} = E(q\ ,\ y).\dot{q}$$

for all δq such that $\delta q(t_i) = 0, i=0,1$.

It is well known that for holonomic constraints we can simply restrict L to the integral manifold of E,obtaining an equivalent restricted unconstrained system . This is in turn equivalent to finding curves $(q(t),y(t))$ such that

$$0 = \frac{\partial}{\partial\lambda}\int_{t_0}^{t_1} L(q(t,\lambda),y(t,\lambda),\frac{\partial q}{\partial t}(t,\lambda),E(q(t,\lambda),y(t,\lambda)).\frac{\partial q}{\partial t}(t,\lambda))dt$$

The later is no longer true for non-holonomic constraints.However by using our two-form ω and expanding the previous equality we get the following formula (in case the distribution E is independent of y)

$$\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right]\delta q + \left[\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}\right] E.\delta q = - \frac{\partial L}{\partial \dot{y}}\,\omega(\dot{q},\delta q)$$

Thus restriction of the Lagrangian to a nonholonomic constraint is equivalent to adding an external force.If $n=2,m=1$ then $\omega(\dot{q},.)$ looks like a Coriolis force with *"angular velocity"* $\frac{\omega}{2}$.

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