DISTRIBUTIVE LATTICE CONGRUENCES AND PRIESTLEY SPACES

by

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One of the nicest features of Priestley duality for distributive lattices is the existence of an isomorphism from the congruence lattice of a bounded distributive lattice onto the the lattice of the open subsets of its Priestley space. My aim in this brief note is to show how this property can be proved by adding some simple topological considerations to a result given by Monteiro in a course on De Morgan algebras which he taught in the early sixties. The main advantage of this approach is a very clear description of the relation existing between congruences and sets of prime filters in a bounded distributive lattice.

A *totally order-disconnected topological space* is a triple $(X,\leq,\tau)$ such that $(X,\leq)$ is a poset, (X,τ) is a topological space and given x, y in X such that $x \not\leq y$, there is a clopen (= closed and open) increasing set U such that $x \in U$ and $y \notin U$. A *Priestley space* is a compact totally order-disconnected topological space. Given a Priestley space X, D(X) will denote the lattice of increasing clopen subsets of X.

Given a bounded distributive lattice L, X(L) will denote the Priestley space of L, i.e. X(L) is the set of prime filters of L, ordered by inclusion and with the topology having as a sub-basis the sets of the form $\sigma_L(a) = \{P \in X(L) | a \in P\}$ and $X(L)\setminus\sigma_L(a)$ for each $a \in L$.

It was shown by H.A. Priestley [2],[3] (see also the survey article [5]) that $\sigma_L : L \to D(X(L))$ is a lattice isomorphism and that the mapping $\varepsilon_X : X \to X(D(X))$ defined by the prescription $\varepsilon_X(x) = \{U \in D(X) | x \in U\}$ is both a homeomorphism and an order isomorphism.

Finally, recall that given bounded distributive lattices L and M, and a 0-1-preserving homomorphism $h: L \to M$, the mapping $X(h): X(M) \to X(L)$ defined by the prescription $X(h)(P) = h^{-1}(P)$ for each $P \in X(M)$ is continuous and monotonic.

In what follows L will always denote a bounded distributive lattice, and for any $Y \subseteq X(L)$, Cl(Y) will denote the closure of Y in **X**(L). Moreover **Con**(L) will denote the congruence lattice of L, and for each topological space X, **Op**(X) will denote the lattice formed by the open subsets of X ordered by inclusion.

1 Lemma (A. Monteiro [1]): For each set Y of prime filters of L, the relation:

$\Theta(Y) = \{(a,b) \in L \times L \mid \text{For each } P \in Y,\ a \in P \text{ if and only if } b \in P\}$ is a congruence on L. Moreover, given any congruence Θ on L, if X denotes the set of prime filters of the quotient lattice L/Θ, $h: L \longrightarrow L/\Theta$ denotes the natural projection and $Z = \{h^{-1}(Q) \mid Q \in X\}$, then $\Theta = \Theta(Z)$.∎

In general, the correspondence $Y \mapsto \Theta(Y)$ is not one-to-one, as the following example shows: Let L be the unit segment of the reals numbers with its natural lattice structure, $Y = \{(x,1] \mid x \in [0,1)\}$ and $Z = \{[x,1] \mid x \in (0,1]\}$. Then Y and Z are disjoint sets of prime filters of L, but $\Theta(Y) = \Theta(Z) = \{(x,x) \mid x \in [0,1]\}$.

Note that each set Y of prime filters of L is a subset of **X**(L), and that $\Theta(Y) = \{(a,b) \in L \times L \mid \sigma_L(a) \cap Y = \sigma_L(b) \cap Y\}$.

2 Theorem: The following properties hold for any subsets Y, Z of **X**(L):

(i) $\Theta(Z) \subseteq \Theta(Y)$ if and only if $Y \subseteq Cl(Z)$.

(ii) $\Theta(Y) = \Theta(Cl(Y))$.

(iii) $\Theta(Y) = \Theta(Z)$ if and only $Cl(Y) = Cl(Z)$.

Proof: Suppose that $Y \not\subseteq CL(Z)$, and take $P \in Y \backslash Cl(Z)$. Then there are a, b in L such that

$$P \in \sigma_L(a) \cap X \backslash \sigma_L(b) \tag{1}$$

and

$$(\sigma_L(a) \cap X \backslash \sigma_L(b)) \cap Z = \emptyset \tag{2}$$

Let $Q \in Z$. If $a \in Q$, then by (2) $b \in Q$ and $a \wedge b \in Q$. Since $a \wedge b \in Q$ implies $a \in Q$, we have that $(a, a \wedge b) \in \Theta(Z)$. On the other hand, by (1) $a \in P$ and $a \wedge b \notin P$, and then $(a, a \wedge b) \notin \Theta(Y)$. Therefore $\Theta(Z) \not\subseteq \Theta(Y)$. Suppose now that $Y \subseteq CL(Z)$, and let $(a,b) \notin \Theta(Y)$. Then there is $P \in Y$ such that $a \in P$ and $b \notin P$ or there is $Q \in Y$ such that $a \notin Q$ and $b \in Q$. Without loss of generality we can consider just the first situation, i.e. $P \in \sigma_L(a) \cap X \backslash \sigma_L(b)$. Since $P \in Y \subseteq CL(Z)$, there is $Q \in Z$ such that $Q \in \sigma_L(a) \cap X \backslash \sigma_L(b)$, and this implies that $(a,b) \notin \Theta(Z)$. Therefore

$\Theta(Z) \subseteq \Theta(Y)$. This completes the proof of (i). Properties (ii) and (iii) are obvious consequences of (i).■

3 Corollary (Priestley [4],[5]): Let L be a bounded distributive lattice and $X = X(L)$. Then the correspondence $V \mapsto \Theta(X\setminus V)$ establishes an isomorphism from **Con**(L) onto **Op**(X).

Proof: By Lemma 1, the correspondence $V \mapsto \Theta(X\setminus V)$ defines a function ϕ:**Op**(X) $\longrightarrow$ **Con**(X), and given $\Theta \in$ **Con**(L), there is $Y \subseteq X$ such that $\Theta = \Theta(Y)$. Let $V = X\setminus Cl(Y)$. Then by Theorem 2 (ii), $\Theta = \Theta(X\setminus V)$, and ϕ is an onto mapping. Finally, by Theorem 2 (i), for each V, W in **Op**(L), $V \subseteq W$ if and only if $\Theta(X\setminus V) \subseteq \Theta(X\setminus W)$.■

4 Corollary: Let L be a bounded distributive lattice, $Y \subseteq X(L)$ and $f:L \longrightarrow L/\Theta(Y)$ be the natural projection. Then the mapping $Q \mapsto f^{-1}(Q)$ is an order isomorphism and a homeomorphism from $X(L/\Theta(Y)$ onto $Cl(Y)$.

Proof: Let $Z = \{f^{-1}(Q) | Q \in X(L/\Theta(Y))\} = X(f)^{-1}(X(L/\Theta(Y))$. Since Priestley spaces are compact and Hausdorff and $X(f)$ is continuous, Z is a closed subset of $X(L)$. By Lemma 1, $\Theta(Y) = \Theta(Z)$, and then by Theorem 2 (iii), $Cl(Y) = Cl(Z) = Z$.■

5 Corollary (Priestley [2], [3]): Let L, M be bounded distributive lattices, and $h:L \longrightarrow M$ be a 0-1-preserving homomorphism. Then h is surjective if and only if $X(h):X(M) \longrightarrow X(L)$ is an order embedding.

Proof: Let P, Q be prime filters of M. If h is surjective, then $h^{-1}(P) \subseteq h^{-1}(Q)$ implies $P = h(h^{-1}(P)) \subseteq h(h^{-1}(Q)) = Q$, and hence $X(h)$ is an order embedding from $X(M)$ into $X(L)$. Conversely, suppose now that $X(h)$ is an order embedding. Then since Priestley spaces are compact and Hausdorff and $X(h)$ is continuous, it follows that $Y = \{h^{-1}(Q) | Q \in X(M)\}$ is homeomorphic and order isomorphic to $X(M)$. On the other hand, it is easy to check that $\Theta(Y) = Ker(h) = \{(a,b) \in L \times L | h(a) = h(b)\}$. Hence by Corollary 4, if $f:L \longrightarrow L/Ker(h)$ denotes the natural projection, then $X(f):X(L/Ker(h)) \longrightarrow Y$ is a homeomorphism and an order isomorphism. If $j:L/Ker(h) \longrightarrow M$ denotes the natural monomorphism such that $h = jf$, we have that $X(h) = X(f)X(j)$, and since $X(h):X(M) \longrightarrow Y$ and $X(f):X(L/Ker(h)) \longrightarrow Y$ are both isomorphisms in the category of Priestley spaces, so is $X(j):X(M) \longrightarrow X(L/Ker(h))$. Therefore $j:L/Ker(h) \longrightarrow M$ is an isomorphism in the category of bounded distributive lattices, and, in particular, must be surjective.

Consequently h, being the composition of surjective mappings, is surjective.∎

References:

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