I_3-ALGEBRAS WITH AN ADITIONAL OPERATION

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ABSTRACT. In this paper we consider the algebraic counterpart of the fragment of the three-valued Lukasiewicz propositional calculus where the connectives are $\rightarrowtail$ (Lukasiewicz implication) and $\wedge$ (conjunction). Then, we define a new equational class of algebras of type (2,2,0), called I_3-algebras with infimun (or SI_3-algebras). They are an extention of the implicative three-valued Lukasiewicz algebras (or I_3-algebras).

1. INTRODUCTION.

A.Rose [13] (see also [7]) gave a formalization of the implicative three-valued Lukasiewicz propositional calculus (or three-ILPC) by means of the following axiom schemes

(C1) $x \rightarrowtail (y \rightarrowtail x)$,

(C2) $(x \rightarrowtail y) \rightarrowtail ((y \rightarrowtail z) \rightarrowtail (x \rightarrowtail z))$,

(C3) $((x \rightarrowtail y) \rightarrowtail y) \rightarrowtail ((y \rightarrowtail x) \rightarrowtail x)$,

(C4) $((x \rightarrowtail y) \rightarrowtail (y \rightarrowtail x)) \rightarrowtail (y \rightarrowtail x)$,

(C5) $((x \rightarrowtail (x \rightarrowtail y) \rightarrowtail x) \rightarrowtail x$,

and the rules of procedure

(R0) substitution rule ,

(R1) "Modus ponens" rule $\dfrac{x,\ x \rightarrowtail y}{y}$.

Adopting the notations

(N1) $x \vee y = (x \rightarrowtail y) \rightarrowtail y$,

(N2) $x \rightarrow y = x \rightarrowtail (x \rightarrowtail y)$.

we can write C3, C4 and C5 as follows

(C3) $(x \vee y) \rightarrowtail (y \vee x)$,

(C4) $(x \rightarrowtail y) \vee (y \rightarrowtail x)$,

(C5) $(x \rightarrow y) \vee x$.

The followings formulas and rules are consequence of R0, R1 , C1,...,C5 (See [10]) .

(R2) $\dfrac{x}{x \rightarrowtail t}$,

(R3) $\dfrac{x \rightarrowtail y}{(y \rightarrowtail z) \rightarrowtail (x \rightarrowtail z)}$,

(C6) $x \rightarrowtail x$,

(C7) $(x \rightarrowtail y) \rightarrowtail ((z \rightarrowtail x) \rightarrowtail (z \rightarrowtail y))$,

(R4) $\dfrac{(x \rightarrowtail y)}{(z \rightarrowtail x) \rightarrowtail (z \rightarrowtail y)}$,

(C8) $(x \rightarrowtail (x \rightarrow y)) \rightarrowtail (x \rightarrow y)$,

(C9) $(x \rightarrowtail (y \rightarrowtail z)) \rightarrowtail (y \rightarrowtail (x \rightarrowtail z))$,

(R5) $\dfrac{(x \rightarrowtail (y \rightarrowtail z)}{(y \rightarrowtail (x \rightarrowtail z)}$,

(C10) $(x \rightarrow (y \rightarrowtail z) \rightarrowtail ((x \rightarrow y) \rightarrowtail (x \rightarrow z))$,

(C11) $x \rightarrow (y \rightarrow x)$,

(C12) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))$,

(C13) $((x \rightarrow y) \rightarrow x) \rightarrow x$, (R6) $\dfrac{x,\ x \rightarrow y}{y}$ (weak modus ponens).

We denote by F_c and T_c the sets of all the formulas and all the formulas which can be obtain from R0,R1,C1,...,C5, respectively. Let α,β be the formulas of F_c , we write $\alpha \equiv \beta$ if and only if $\alpha \rightarrowtail \beta$, $\beta \rightarrowtail \alpha \in T_c$.

Then, it holds ([10]) .

(R7) $x \equiv x$,

(R8) $\dfrac{x \equiv y}{y \equiv x}$,

(R9) $\dfrac{x \equiv y,\ y \equiv z}{x \equiv z}$,

(R10) $\dfrac{x \equiv y}{(x \rightarrowtail z) \equiv (y \rightarrowtail z)}$,

(R11) $\dfrac{x \equiv y}{(z \rightarrowtail x) \equiv (z \rightarrowtail y)}$,

In order to study the 3-ILPC with algebraic techniques, in 1968 A. Monteiro [10] introduced the notion of implicative three-valued Lukasiewicz algebra (or I_3-algebra) as algebras $(A,\rightarrowtail,1)$ of type (2,0) satisfying :

(I1) $1 \rightarrowtail x = x$,

(I2) $x \rightarrowtail (y \rightarrowtail x) = 1$,

(I3) $(x \rightarrowtail y) \rightarrowtail ((y \rightarrowtail z) \rightarrowtail (x \rightarrowtail z)) = 1$,

(I4) $(x \rightarrowtail y) \rightarrowtail y = (y \rightarrowtail x) \rightarrowtail x$,

(I5) $((x \rightarrowtail y) \rightarrowtail (y \rightarrowtail x)) \rightarrowtail (y \rightarrowtail x) = 1$,

(I6) $((x \rightarrowtail (x \rightarrowtail y)) \rightarrowtail x) \rightarrowtail x = 1$ (See also [2,3,4,5,6]).

We denote by I_3 the variety of the I_3-algebras .

Next, we give the most simple and important example of an I_3-algebra.

Let $T = \{0,1/2,1\}$ and $\rightarrowtail$ be the operation defined by means of the table :

$\rightarrowtail$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

Then $(T,\rightarrowtail,1) \in I_3$.

If $A \in I_3$, for all $x,y \in A$ we define the operation $\longrightarrow$ by means of the formula :

(I7) $x \longrightarrow y = x \rightarrowtail (x \rightarrowtail y)$.

The following properties are valid in every I_3-algebra and they are proved in [10] :

(I8) $x \rightarrowtail x = 1$,

(I9) $x \rightarrowtail 1 = 1$,

(I10) The relation $\leq$ defined by $x \leq y$ if and only if $x \rightarrowtail y = 1$ is a partial order on A.

(I11) $x \leq y$ implies $y \rightarrowtail z \leq x \rightarrowtail z$,

(I12) $(A,\leq)$ is a join semi-lattice and for all $x,y \in A$, the element $x \vee y = (x \rightarrowtail y) \rightarrowtail y$ is the supremum of x, y .

(I13) $x \rightarrowtail (y \rightarrowtail z) = y \rightarrowtail (x \rightarrowtail z)$,

(I14) $(x \rightarrowtail y) \rightarrowtail ((z \rightarrowtail x) \rightarrowtail (z \rightarrowtail y)) = 1$, (I15) $x \leq y$ implies $z \rightarrowtail x \leq z \rightarrowtail y$,

(I16) $x \rightarrow (y \rightarrow x) = 1$,

(I17) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,

(I18) $((x \rightarrow y) \rightarrow x) \rightarrow x = 1$,

(I19) $1 \rightarrow x = x$.

If $A \in I_3$ and there exists an element $0 \in A$ such that

(I20) $0 \leq x$, for all $x \in A$, (i.e. A is bounded)

Then

(I21) $(A,\wedge,\vee,\sim,\nabla,1)$ is a three-valued Lukasiewicz algebra where $\sim x = x \rightarrowtail 0$, $x \wedge y = \sim(\sim x \vee \sim y)$ [8,9,10] and it verifies $x \rightarrowtail y = (\nabla\sim x \vee y) \wedge (\nabla y \vee \sim x)$ ([10]) .

If $\mathcal{A}=(A,\rightarrowtail,1) \in I_3$, we denote by $L(\mathcal{A})$ (or L(A)) the alg%bra $(A,\wedge,\vee,\sim,\nabla,1)$ described in (I21) .

Let $A \in I_3$. $D \subseteq A$ is a deductive system (d.s.) of A if it satisfies:

(D1) $1 \in D$,

(D2) $x, x \rightarrowtail y \in D$ imply $y \in D$.

Let $\mathbb{D}(A)$ be the set of all d.s. of A .

By [10] (See also [4]) we know that (D2) is equivalent to:

(D'2) $x, x \rightarrow y \in D$ imply $y \in D$.

We denote by $Con_{I_3}(A)$ the set of all I_3-congruences of A . If $R \in Con_{I_3}(A)$, x_R represents the equivalence class of x , $x \in A$ and $q_R : A \rightarrow A/R$ defined by $q_R(x) = x_R$ is the canonical epimorphism.

Then :

$Con_{I_3}(A) = \{ R(D) : D \in \mathbb{D}(A) \}$, where

$R(D) = \{(x,y) \in A^2 : x \rightarrowtail y , y \rightarrowtail x \in D \}$. Furthermore, for each $R \in Con_{I_3}(A)$ we have that $D = 1_R \in \mathbb{D}(A)$ and $R = R(D)$.

Let $Hom_{I_3}(A,B)$ ($Epi_{I_3}(A,B)$) be the set of all the I_3-homomorphisms (I_3-epimorphisms) from A into B (from A onto B) .

If $h \in Hom_{I_3}(A,B)$ then the set $N(h) = \{ x \in A : h(x) = 1 \}$, called kernel of h , has the following properties :

(H1) $N(h) \in \mathbb{D}(A)$,

(H2) $(x,y) \in R(N(h))$ if and only if $h(x) = h(y)$,

(H3) $A/N(h)$ and $h(A)$ are isomorphic I_3-algebras

(i.e. $A/N(h) \cong h(A)$) .

We denote by $\mathbb{E}(A)$ the set of all maximal d.s. (m.d.s.) of A.

Then it holds :

(M1) $\bigcap \{ M : M \in \mathbb{E}(A) \} = \{1\}$,

(M2) For each $M \in \mathbb{E}(A)$, $A/M \cong S$, where S is a non trivial I_3-subalgebra of T.

Next , we consider the 3-SLPC which is an extension of the 3-ILPC because it is obtained by adding the connective $\wedge$ and the axiom schemes :

(SC1) $(x \wedge y) \rightarrowtail x$,

(SC2) $(x \wedge y) \rightarrowtail y$,

(SC3) $x \rightarrowtail (y \rightarrowtail (x \wedge y))$,

(SC4) $((x \rightarrowtail y) \wedge (x \rightarrowtail z)) \rightarrowtail (x \rightarrowtail (z \wedge y))$.

We denote by F_s and T_s the sets of all the formulas of the 3-SLPC and all the formulas which can be obtained from RO,R1,C1,...,C5,SC1,...,SC4, respectively .

If α , $\beta \in F_s$ we write $\alpha \equiv_s \beta$ if $\alpha \rightarrowtail \beta$, $\beta \rightarrowtail \alpha \in T_s$.

Then , we have :

(SR1) $\dfrac{x \rightarrowtail y,\ x \rightarrowtail z}{x \rightarrowtail (z \wedge y)}$:

(1) $(x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail y) \wedge (x \rightarrowtail z))$ [SC3] ,

(2) $(x \rightarrowtail y) \wedge (x \rightarrowtail z)$ [(1),hip,R1] ,

(3) $((x \rightarrowtail y) \wedge (x \rightarrowtail z)) \rightarrowtail (x \rightarrowtail (z \wedge y))$ [SC4] ,

(4) $x \rightarrowtail (z \wedge y)$ [(3),(2),R1] .

(SC5) $(x \rightarrowtail (y \wedge z)) \rightarrowtail ((x \rightarrowtail z) \wedge (x \rightarrowtail y))$:

(1) $(y \wedge z) \rightarrowtail z$ [SC2] ,

(2) $(x \rightarrowtail (y \wedge z)) \rightarrowtail (x \rightarrowtail z)$ [(1),R4] ,

(3) $(y \wedge z) \rightarrowtail y$ [SC1] ,

(4) $(x \rightarrowtail (y \wedge z)) \rightarrowtail (x \rightarrowtail y)$ [(3),R4] ,

(5) $(x \rightarrowtail (y \wedge z)) \rightarrowtail ((x \rightarrowtail z) \wedge (x \rightarrowtail y))$ [(2),(4),SR1],

(SC6) $(x \rightarrowtail y) \rightarrowtail ((x \wedge z) \rightarrowtail (y \wedge z))$:

(1) $(x \wedge z) \rightarrowtail x$ [SC1] ,

(2) $(x \rightarrowtail y) \rightarrowtail ((x \wedge z) \rightarrowtail y)$ [(1),R3] ,

(3) $(x \wedge z) \rightarrowtail z$ [SC2] ,

(4) $(x \rightarrowtail y) \rightarrowtail ((x \wedge z) \rightarrowtail z)$ [(3),R2] ,

(5) $(x \rightarrowtail y) \rightarrowtail (((x \wedge z) \rightarrowtail y) \wedge ((x \wedge z) \rightarrowtail z))$ [(2),(4),SR1] ,

(6) $(((x \wedge z) \rightarrowtail y) \wedge ((x \wedge z) \rightarrowtail z)) \rightarrowtail ((x \wedge z) \rightarrowtail (y \wedge z))$ [SC4] ,

(7) $((x \rightarrowtail y) \rightarrowtail (((x \wedge z) \rightarrowtail ((x \wedge z) \rightarrowtail z))) \rightarrowtail$

$((x \rightarrowtail y) \rightarrowtail ((x \wedge z) \rightarrowtail (y \wedge z)))$ [(6),R4] ,

(8) $(x \rightarrowtail y) \rightarrowtail ((x \wedge z) \rightarrowtail (y \wedge z))$ [(7),(5),R1] .

(SC7) $(x \wedge y) \rightarrowtail (y \wedge x)$:

(1) $(x \wedge y) \rightarrowtail x$ [SC1] ,

(2) $(x \wedge y) \rightarrowtail y$ [SC2] ,

(3) $(x \wedge y) \rightarrowtail (y \wedge x)$ [(1),(2),SR1] .

(SC8) $(x \rightarrowtail y) \rightarrowtail ((z \wedge x) \rightarrowtail (z \wedge y))$:

(1) $(x \rightarrowtail y) \rightarrowtail ((x \wedge z) \rightarrowtail (y \wedge z))$ [SC6] ,

(2) $(x \wedge z) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail (y \wedge z))$ [(1),R5] ,

(3) $((z \wedge x) \rightarrowtail (x \wedge z)) \rightarrowtail ((z \wedge x) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail$

$(y \wedge z)))$ [(2),R4] ,

(4) $(z \wedge x) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail (y \wedge z))$ [(3),SC7,R1] ,

(5) $(x \rightarrowtail y) \rightarrowtail ((z \wedge x) \rightarrowtail (y \wedge z))$ [(4),R5] ,

(6) $(y \wedge z) \rightarrowtail (z \wedge y)$ [SC7] ,

(7) $((z \wedge x) \rightarrowtail (y \wedge z)) \rightarrowtail ((z \wedge x) \rightarrowtail (z \wedge y))$[(6),R4] ,

(8) $((x \rightarrowtail y) \rightarrowtail ((z \wedge x) \rightarrowtail (y \wedge z)) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail$

$((z \wedge x) \rightarrowtail (z \wedge y)))$ [(7),R4] ,

(9) $(x \rightarrowtail y) \rightarrowtail ((z \wedge x) \rightarrowtail (z \wedge y))$ [(8),(5),R1] .

From the above results it follows at once that $\equiv_s$ is a congruence on F_s and the Lindenbaum algebra $(F_s/\equiv_s, \rightarrowtail, \wedge, 1)$ (where $1 = T_s$) satisfies :

(1°) $(F_s/\equiv_s, \rightarrowtail, 1) \in Ia$, (2°) The following identities are verified :

(S1) $(\alpha \wedge \beta) \rightarrowtail \alpha = 1$,

(S2) $\alpha \rightarrowtail (\beta \wedge \gamma) = (\alpha \rightarrowtail \gamma) \wedge (\alpha \rightarrowtail \beta)$.

2. SI_3-ALGEBRAS

2.1. **DEFINITION.** A SI_3-algebra is an algebra $(A,\rightarrowtail,\wedge,1)$ (A, as short) of type (2,2,0) where $(A,\rightarrowtail,1) \in I_3$ and the following identities hold :

(S1) $(x \wedge y) \rightarrowtail y = 1$,

(S2) $x \rightarrowtail (y \wedge z) = (x \rightarrowtail z) \wedge (x \rightarrowtail y)$.

We denote by SI_3 the variety of the SI_3-algebras.

2.2. **EXAMPLES.**

(1°) Let $(T,\rightarrowtail,\wedge,1)$, where $(T,\rightarrowtail,1)$ is the I_3-algebra indicated in 1 , and $\wedge$ is defined by : $x \wedge y = 0$ for all $x,y \in T$. Then T verifies (S1) but not (S2), because :

$1/2 \rightarrowtail (1 \wedge 1) = 1/2 \rightarrowtail 0 = 1/2$ and $(1/2 \rightarrowtail 1) \wedge (1/2 \rightarrowtail 1) = 0$.

(2°) Let $(T,\rightarrowtail,\wedge,1)$, where $(T,\rightarrowtail,1)$ is the I_3-algebra of 1, and $\wedge$ is defined by the table :

$\wedge$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

Then T satisfies (S2) but not (S1) because :

$(1/2 \wedge 0) \rightarrowtail 0 = 1/2 \rightarrowtail 0 = 1/2 \neq 1$

(3°) Let $T^* = (T, \rightarrowtail, \wedge, 1)$, where $(T, \rightarrowtail, 1)$ is the Is-algebra of 1, and $\wedge$ is defined by the table :

$\wedge$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

Then T^* is an SIs-algebra. The non trivial SIs-subalgebras of T^* are $B^* = \{0,1\}$ and $L^* = \{1/2, 1\}$. Moreover $B^* \cong L^*$.

2.3. LEMMA. For every $A \in$ SIs it holds:

(S3) $1 = 1 \wedge 1$,

(S4) $y = 1 \wedge y$,

(S5) $x \wedge y = y \wedge x$,

(S6) $x \wedge y \leq x$,

(S7) $x \wedge y \leq y$,

(S8) $z \leq x$, $z \leq y$ imply $z \leq x \wedge y$,

(S9) $(A, \leq)$ is a meet semi-lattice where the infimun of x, y is $x \wedge y$,

(S10) $(A, \vee, \wedge, 1)$, where $\vee$ is the operation determined by the formula (I12) , is a distributive lattice with last element 1 ,

(S11) $(x \rightarrowtail y) \rightarrowtail ((x \wedge z) \rightarrowtail (y \wedge z)) = 1$.

Proof.

(S3) $1 = $ (I8) $(1 \wedge 1) \rightarrowtail (1 \wedge 1) = $ (S2) $((1 \wedge 1) \rightarrowtail 1) \wedge ((1 \wedge 1)$ $\rightarrowtail 1) = $(S1) $1 \wedge 1$.

(S4) (1) $y \rightarrowtail (1 \wedge y) = $ (S2) $(y \rightarrowtail y) \wedge (y \rightarrowtail 1) = $ (I8,I9) $=$ $1 \wedge 1 = $ (S3) 1 . From (1) and I10 .

(2) $y \leq 1 \wedge y$. By S1 and I10 (3) $1 \wedge y \leq y$. From (2) and (3) $y = 1 \wedge y$.

(S5) $x \wedge y = $ (I1) $1 \rightarrowtail (x \wedge y) = $ (S2) $(1 \rightarrowtail y) \wedge (1 \rightarrowtail x) = $ (I1) $y \wedge x$.

(S6) $1 = $ (S1) $(y \wedge x) \rightarrowtail x = $ (S5) $(x \wedge y) \rightarrowtail x$, then by I10 $x \wedge y \leq x$.

(S7) It follows from (S1) and I10 .

(S8) Let $x,y,z \in A$ be such that (1) $z \rightarrowtail x = 1$, (2) $z \rightarrowtail y = 1$, then $1 = $ (S3) $1 \wedge 1 = $ ((1),(2)) $(z \rightarrowtail x) \wedge (z \rightarrowtail y) = $ (S2,S6) $z \rightarrowtail (x \wedge y)$, hence by I10 $z \leq x \wedge y$.

(S9) It follows from (S6), (S7) and (S8) .

(S10) We shall prove that the cancelation law holds :

(C.L.) $x \wedge y = x \wedge z$, $x \vee y = x \vee z$ imply $y = z$.

Indeed , let $x,y,z \in A$ be such that :

(1) $x \wedge y = x \wedge z$, (2) $x \vee y = x \vee z$, then

(3) $x \rightarrowtail y = (x \rightarrowtail y) \wedge 1 = $ (I8) $(x \rightarrowtail y) \wedge (x \rightarrowtail x) = $ (S2) $x \rightarrowtail (x \wedge y) = $ ((1)) $x \rightarrowtail (x \wedge z) = $ (S2,I8,S4) $x \rightarrowtail z$.

On the other hand , $1 = $ (I12,I10) $y \rightarrowtail (x \vee y) = $ ((2)) $y \rightarrowtail (x \vee z) = $ (I12) $y \rightarrowtail ((x \rightarrowtail z) \rightarrowtail z) = $ (I13) $(x \rightarrowtail z) \rightarrowtail (y \rightarrowtail z)$. By I10 $x \rightarrowtail z \leq y \rightarrowtail z$, so by (3) we have

(4) $x \rightarrowtail y \leq y \rightarrowtail z$.

Furthermore $y \rightarrowtail x =$ (S4,S5) $(y \rightarrowtail x) \wedge 1 =$ (I8) $(y \rightarrowtail x) \wedge (x \rightarrowtail x) =$ (S2) $y \rightarrowtail (y \wedge x) =$ ((1),S5) $y \rightarrowtail (x \wedge z) =$ (S2) $(y \rightarrowtail z) \wedge (y \rightarrowtail x)$, and so (5) $y \rightarrowtail x \leq y \rightarrowtail z$. Then $1 =$ (I5,I12) $(x \rightarrowtail y) \vee (y \rightarrowtail x) =$ ((4),(5)) $y \rightarrowtail z$, hence by I10 (6) $y \leq z$. In a similar way we prove (7) $z \leq y$. From (6) and (7) $y = z$.

(S11) $(x \rightarrowtail y) \rightarrowtail ((x \wedge z) \rightarrowtail (y \wedge z)) =$ (S2) $(x \rightarrowtail y) \rightarrowtail$ $(((x \wedge z) \rightarrowtail z) \wedge ((x \wedge z) \rightarrowtail y)) =$ (S1,S4) $(x \rightarrowtail y) \rightarrowtail$ $((x \wedge z) \rightarrowtail y) =$ (I13) $(x \wedge z) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail y) =$ (I12) $(x \wedge z) \rightarrowtail (x \vee y) = 1$.

It is easy to see that :

2.4. LEMMA. Let $(A,\rightarrowtail,\wedge,1) \in$ SIs and $(A,\rightarrowtail,1)$ be bounded with first element 0 . Then, it holds :

$x \wedge y = {\sim}({\sim}x \vee {\sim}y) = (((x \rightarrowtail 0) \rightarrowtail (y \rightarrowtail 0)) \rightarrowtail (y \rightarrowtail 0)) \rightarrowtail 0$.

2.5. EXAMPLE. Let T^* be the SIs-algebra indicated in 2.2(3°), N be the set of the positive integers and T^{*N} be the set of all functions from N into T^* pointwise algebrized .For each $f \in T^{*N}$ we denote by A_f the set $\{ i \in N : f(i) \neq 1\}$. Let $A = \{ f \in T^{*N} : |A_f| < \omega \}$. Then it is easy to see that A is a SIs-subalgebra of T^{*N}, so $A \in$ SIs and A does not have first element .

If $\mathcal{A} = (A,\rightarrowtail,\wedge,1) \in$ SIs then we represent by $\mathcal{A}^R$ the reduct $(A,\rightarrowtail,1)$ and when there is no doubt we write A^R instead of $\mathcal{A}^R$.

2.6. LEMMA. If $A \in SI_3$ then $Con_{SI_3}(A) = Con_{I_3}(A^R)$.

Proof. It is clear that $Con_{SI_3}(A) \subseteq Con_{I_3}(A^R)$.

If $R \in Con_{I_3}(A^R)$ then there exists $D \in \mathbb{D}(A)$ such that $R = R(D)$. If $(x,y) \in R$ then (1) $x \rightarrowtail y \in D$, (2) $y \rightarrowtail x \in D$. From (1), (S11), D1, and D2 we have (3) $(x \wedge z) \rightarrowtail (y \wedge z) \in D$. In the same way we prove that (4) $(y \wedge z) \rightarrowtail (x \wedge z) \in D$. By (3) and (4) , (5) $(x \wedge z , y \wedge z) \in R$. By (5) and (S5) ,(6)$(z \wedge x , z \wedge y) \in R$. Hence, from (5) and (6) it results that $R \in Con_{SI_3}(A)$. Therefore $Con_{I_3}(A^R) \subseteq Con_{SI_3}(A)$.

2.7. REMARKS. Since the SI_3-congruences of a SI_3-algebra A are determined by the d.s. of A and the operator $\longrightarrow$ verifies the properties (I16),...,(I19) then taking into account (D'2) and some results of A. Monteiro [11] (See also [12]) we can state that:

(1°) Every proper d.s. of A is the intersection of m.d.s. of A . In particular {1} is the intersection of all the m.d.s. of A .

(2°) In any SI_3-algebra A the following conditions are equivalent

(a) A/M is a simple SI_3-algebra,

(b) $M \in \mathbb{E}(A)$,

(c) If $a \in A-M$ and $b \in A$ then $a \longrightarrow b \in M$.

(3°) The SI_3-algebras are semi-simples in the sense of the following theorem.

2.8. THEOREM. Every non trival SI_3-algebra A is a subdirect product of simple SI_3-algebras.

Proof.

Let $P = \prod_{M \in \mathbb{E}(A)} A/M$ and $\varphi : A \longrightarrow P$ be the mapping defined by

$\varphi(f) = F$, where $F(M) = q_M(f)$, for all $M \in \mathbb{E}(A)$.

Since $q_M : A \longrightarrow A/M$ is the canonical epimorphism, φ is a SI_3-homomorphism and by 2.7(1) φ is injective .

2.9. THEOREM. Let $A \in SI_3$ be non trivial. Then the following conditions are equivalent

(1°) A is simple ,

(2°) $A \cong S$, where S is a non trival SI_3-subalgebra of T^* .

Proof.

(1°) $\Longrightarrow$ (2°) If $A \in SI_3$ is simple , then it has more than one element . By 2.6 and the results indicated in 1 it follows that $\mathbb{D}(A) = \{1\}$. Then A^R is a simple I_3-algebra, so $A^R \cong S^R$, where S^R is an I_3-subalgebra of T . Finally , taking into account that $(A,\wedge,\vee)$ is a distributive lattice we obtain that $A \cong S$, where S is a non trivial SI_3-subalgebra of T^* .

(2°) $\Longrightarrow$ (1°) It is clear that T^* and all their non trivial SI_3-subalgebras are simple, so if A verifies (2°) then A is simple.

2.10. REMARK. If $A \in SI_3$ is finite then the application φ of 2.8

is onto. Indeed, since A is finite it has first element. Then we can consider the Lukasiewicz algebra L(A) of (I21) and apply the known results for these algebras.

3. SI$_3$-ALGEBRAS WITH A FINITE SET OF FREE GENERATORS

Now , we are going to determine the structure of the SI$_3$-algebra with n free generators, n positive integer.

3.1. DEFINITION. If c is a positive cardinal number, we say that S(c) is SI$_3$-algebra with c free generators if :

(1) S(c) has a set of generators G such that $|G| = c$,

(2) Any function f from G into a SI$_3$-algebra A can be extended to a SI$_3$-homomorphism $h : S(c) \longrightarrow A$.

Since the notion of SI$_3$-algebra is equationally definable, a result of G. Birkhoff [1] allows us to assert that for any cardinal $c > 0$ there exists S(c) and it is unique up to isomorphisms . Moreover the SI$_3$-homomorphism of definition 3.1 is unique.

If we consider in the 3-SLPC a set of propositional variables of cardinal c , $c > 0$ then the Lindenbaum algebra mentioned in 1. is a SI$_3$-algebra with c free generators.

Let $G = \{g_1, g_2, \ldots, g_n\}$ a set of free generators of S(c) . We shall denote by T^G the set of all functions from G into $T = \{0, 1/2, 1\}$.

Since each function $f \in T^G$ can be extended to a unique

$h \in Hom_{SI_3}(S(n),T^*)$ such that $h/G = f$, the correspondence $f \longrightarrow h$ is bijection between T^G and $Hom_{SI_3}(S(n),T^*)$. Hence $Hom_{SI_3}(S(n),T^*)$ is finite.

3.2. LEMMA. The map $\varphi : Hom_{SI_3}(S(n),T^*) \longrightarrow \mathbb{E}(S(n))$ defined by $\varphi(h) = N(h)$ is onto .

Proof.

Let $M \in \mathbb{E}(S(n))$ and q_M: $S(n) \longrightarrow S(n)/M$ be the canonical epimorphism. From 2.7.(2°) and 2.9 there exists a SI_3-homomorphism $i : S(n)/M \longrightarrow T^*$. Then $h = i \circ q_M \in Hom_{SI_3}(S(n),T^*)$ and $\varphi(h) = M$, so φ is onto .

S(n) is a subdirect product of finite algebras S(n)/M, $M \in \mathbb{E}(S(n))$ Then, taking into account 3.2 and 2.9 we have :

3.3. COROLLARY. If n is a positive integer, then S(n) is finite and $S(n) \cong \prod_{M \in \mathbb{E}(S(n))} S(n)/M$.

Let $\mathbb{E}_1 = \{ M \in \mathbb{E}(S(n)) : |S(n)/M| = 2 \}$ and

$\mathbb{E}_2 = \{ M \in \mathbb{E}(S(n)) : |S(n)/M| = 3 \}$. Then $\{\mathbb{E}_1,\mathbb{E}_2\}$ is a partition of $\mathbb{E}$. Taking into account 2.2(3°),2.7(2°) and 2.9 we can write :

$$S(n) \cong L^{|\mathbb{E}_1|} \times T^{|\mathbb{E}_2|} \quad (I) .$$

Now, we prove :

3.4. LEMMA. For each non trivial $h \in Hom_{SI_3}(S(n),T^*)$ there exists $h' \in Hom_{SI_3}(S(n),T^*)$ which verifies :

(1°) $h'(S(n)) = \{1/2,1\} = L^*$,

(2°) $N(h') = N(h)$.

Proof.Let $h \in Hom_{SI_3}(S(n),T^*)$ be non trivial and suppose that $h(S(n)) = \{0,1\} = B$.

Then defining $\alpha : B \longrightarrow L$ in the following way : $\alpha(0) = 1/2$, $\alpha(1) = 1$ we have that $h' = \alpha \circ h$ verifies (1°) and (2°) .

Let $A = \{ h \in Hom_{SI_3}(S(n),T^*) : h(S(n)) = L^* \}, B = Epi_{SI_3}(S(n),T^*)$

Since T^* has non trivial automorphism, from 3.4. it results :

$$|\mathbb{E}_1| = |A| \quad (II) \quad ,$$

$$|\mathbb{E}_2| = |B| \quad (III).$$

On the other hand , let $A' = \{ f \in T^G : (f(G)) = L^* \}$ and $B' = \{ f \in T^G : (f(G)) = T^* \}$, where $(f(G))$ is the SI_3-subalgebra of T generated by $f(G)$. Then we have :

$$|A| = |A'| \quad (IV) \quad ,$$

$$|B| = |B'| \quad (V) \quad .$$

Furthermore,

$A' = \{ f \in T^G : f(G) \leq \{1/2,1\} \leq T$ y $1/2 \in f(G) \}$.

Let $B'_1 = \{ f \in B' : f(G) = \{0,1/2\}\}$ and

$B'_2 = \{ f \in B' : f(G) = \{0,1/2,1\} \}$. Then $\{B'_1,B'_2\}$ is a partition of B'.

Finally we have

$|A'| = 2^n - 1 \quad (VI)$,

$|B'_1| = 2^n - 1$,

$$|B'_2| = \sum_{i=0}^{3-1} (-1)^i \binom{3}{i} (3-i)^n = 3^n - 3.2^n + 3 ,$$

$$|B'| = (2^n - 1) + (3^n - 3.2^n + 3) = 3^n - 2^{n+1} + 2 \qquad \text{(VII)}$$

From (I),(II),...,(VII) we obtain :

3.6. THEOREM. The SI3-algebra S(n) with n free generators, n integer , n > 0 verifies :

(1°) $S(n) \cong L^{*2^n-1} \times T^{*3^n - 2^{n+1} + 2}$,

(2°) $|S(n)| = 2^{2^n-1} . 3^{3^n - 2^{n+1} + 2}$.

REFERENCES

[01] G.BIRKHOFF, *Lattice theory*, 3rd ed., Amer. Math. Soc., Coll Pu. ,25,Providence, 1967).

[02] D.DIAZ, y A.V.FIGALLO, *Dos conjuntos de axiomas para las álgebras de Lukasiewicz trivalentes*, Cuadernos del Instituto de Matemática, Serie A, N° 3, 1985, Univ. Nac. de San Juan, San Juan, Argentina.

[03] A.V. FIGALLO, *(I_3-∇)-Algebras*, Rev. Colombiana de Matemáticas , 17(1983), 105-116 .

[04] A.V. FIGALLO, *$I\Delta_3$-Algebras*, Rep. on Math. Logic, 24 (1990), 3 - 16.

[05] A. V. FIGALLO, I_{n+1}*-Algebras con operadores,* Tésis presentada en la U.N. del Sur , Bahía Blanca , 1989.

[06] L. ITURRIOZ et O. RUEDA, *Algèbres implicativas trivalentes de Lukasiewicz libres,* Discrete Math. , 18 (1977), 35-44 .

[07] S. Mc CALL and R. K. MEYER, *Pure three-valued Lukasiewiczian implication,* Jour . Symb. logic, 31(1966) .

[08] G. C. MOISIL, *Essais sur les logiques non-chrysippiennes,* Ed. Academiei, Bucarest, 1972.

[09] A. MONTEIRO, *Sur la dèfinition des algèbres de Lukasiewicz trivalentes,* Bull. Math. Soc. Sc. Math. Phys., R.P. Roum.,7(55), 1-2(1963).

[10] A. MONTEIRO, *Algebras implicativas trivalentes de Lukasiewicz,* lectures given at the Univ. Nac. del Sur, Bahia Blanca, Argentina, 1968.

[11] A. MONTEIRO, *La semi-simplicitè des algèbres de Boole topologiques et les systemes deductifs,* Rev. de la Unión Mat. Argentina, 25 (1971) .

[12] A. MONTEIRO, *Sur les algèbres de Heyting Symetriques.*

[13] A. ROSE, *Formalisation du calcul propositionnel implicatif a m valeurs de Lukasiewicz,* C.R. ACAD . Sei. París, 243(1956), 1263-1264.

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