THE MULTIPLICATIVE DISTRIBUTIONAL PRODUCT OF $L^j\left\{(m^2+P\pm io)^{-l-1+j}\right\}\cdot K^k\{\delta(x)\}$ AND OTHERS

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Abstract. *This note was inspired in the paper due to Cheng Lin Zhi and Li Chen Kuan (cf.* **[1]***).*

Here, we evaluate several cumbersome n*-dimensional multiplicative distributional products, such as* $L^j\left\{(P+io)^{-m-1+j}\right\}\cdot L^k\{\delta\}$,
$L^jP^{-m-1+j}\cdot K^k\{\delta\}$, $K^j\left\{(m^2+P)^{-l-1+j}\right\}\cdot K^k\{\delta\}$, *where* L^j *is the* n*-dimensional ultrahyperbolic operator, iterated* j*-times (*j *integer* ≥ 1*), and* K^j *is the* n*-dimensional ultrahyperbolic Klein-Gordon operator, iterated* j*-times, (*j *integer* ≥ 1*), cf. (II,1;1) and (II,3;1), respectively.*

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I.1. Definitions

We begin with some definitions. Let $x=(x_1,x_2,\ldots,x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Consider a nondegenerate quadratic form in n variables of the form

$$P=P(x)=x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2, \qquad \text{(I,1;1)}$$

where $n=p+q$.

The distributions $(P\pm io)^\lambda$ are defined by

$$(P\pm io)^\lambda=\lim_{\varepsilon\to 0}(P\pm i\varepsilon|x|^2)^\lambda, \qquad \text{(I,1;2)}$$

where $\varepsilon>0$, $|x|^2=x_1^2+\cdots+x_n^2$, $\lambda\in\mathbb{C}$.

$$(P+io)^{-k}=Pf(P)^{-k}\mp\frac{i\Pi(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(P). \qquad \text{(I,1;2')}$$

The distributions $(m^2+P\pm io)^\lambda$ are defined in analogue manner as the distributions $(P\pm io)^\lambda$. Let us put (cf. [**2**], p. 289).

$$(m^2+P\pm io)^\lambda=\lim_{\varepsilon\to 0}\left(m^2+P\pm i\varepsilon|x|^2\right)^\lambda, \qquad \text{(I,1;3)}$$

where ε is an arbitrary positive number, and m is a positive real number.

It is useful to state an equivalent definition of the distributions $(m^2+P\pm io)^\lambda$.

In this definition appear the distributions

$$(m^2+P)_+^\lambda=\begin{cases}(m^2+P)^\lambda & \text{if } (m^2+P)\geq 0,\\ 0 & \text{if } (m^2+P)<0.\end{cases} \qquad \text{(I,1;4)}$$

$$(m^2+P)_-^\lambda=\begin{cases}0 & \text{if } (m^2+P)<0,\\ (-m^2-P)^\lambda & \text{if } (m^2+P)\geq 0.\end{cases} \qquad \text{(I,1;5)}$$

We can prove, without difficulty that the following formula is valid (cf. [**3**], p. 566).

$$(m^2+P\pm io)^\lambda=(m^2+P)_+^\lambda+e^{\pm i\pi\lambda}(m^2+P)_-^\lambda. \qquad \text{(I,1;6)}$$

From this formula we conclude immediately that

$$(m^2+P+io)^\lambda=(m^2+P-io)^\lambda=(m^2+P)^\lambda, \qquad \text{(I,1;7)}$$

when $\lambda=k=$ positive integer.

We observe that $(m^2 + P \pm io)^\lambda$ are entire distributional functions of λ. This is the principal difference between the distributions, formally analogue, $(P \pm io)^\lambda$ which have poles at the point $\lambda = -\frac{n}{2} - k$, $k = 0, 1, \ldots$.

It can proved (cf. [3],p. 573, formula (2.14) and p. 575, formula (3.5)) that

$$(m^2 + P \pm io)^{-k} = Pf(m^2 + P)^{-k} \mp \frac{i\Pi(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(m^2 + P),$$
$$k = 0, 1, \ldots. \tag{I,1;8}$$

This formula is a multidimensional analogue of the well-known unidimensional formula (cf. [2], pp. 96 and 97).

We prove (cf. [4], p. 23, Theorem 1) the

Theorem 1. *The following formula is true for every* $\lambda, \mu \in \mathbb{C}$ *and* $m^2 \neq 0$:

$$(m^2 + P \pm io)^\lambda \cdot (m^2 + P \pm io)^\mu = (m^2 + P \pm io)^{\lambda+\mu}. \tag{I,1;9}$$

We observe that the following formulas are valid

$$[\text{sgn}(m^2 + P)]\, |m^2 + P|^\lambda = (m^2 + P)_+^\lambda - (m^2 + P)_-^\lambda, \tag{I,1;10}$$

and

$$|m^2 + P|^\lambda = (m^2 + P)_+^\lambda + (m^2 + P)_-^\lambda. \tag{I,1;11}$$

We shall define (cf. [2], p. 294)

$$\delta^{(k-1)}(m^2 + P) = \left.\frac{(m^2 + P)_+^\lambda}{\Gamma(\lambda + 1)}\right|_{\lambda=-k}. \tag{I,1;12}$$

II.1. The multiplicative product of $L^j\left\{P^{-m-1+j}\right\} \cdot L^k\{\delta\}$

Let us define the n-dimensional ultrahyperbolic operator, iterated j-times, (j integer ≥ 1).

$$L^j = \left\{\frac{\partial^2}{\partial x_1} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right\}^j, \tag{II,1;1}$$

where $n = p + q$.

We have (cf. [2], p. 272)

$$LP^{\lambda+1} = 2(\lambda + 1)(2\lambda + n)P^\lambda. \tag{II,1;2}$$

Iterating it j-times, we arrive at

$$\begin{aligned} L^j P^{\lambda+j} &= 2^{2j}(\lambda + 1)\ldots(\lambda + j)(\lambda + \frac{n}{2})\ldots(\lambda + \frac{n}{2} + j - 1)P^j \\ &= \frac{2^{2j}\Gamma(-\lambda)\Gamma(1 - \lambda - \frac{n}{2})}{\Gamma(-\lambda - j)\Gamma(-\lambda - \frac{n}{2} - j + 1)}P^\lambda. \end{aligned} \tag{II,1;3}$$

From (II,1;3), putting $\lambda = -m-1$, we obtain

$$L^j\left\{P^{-m-1+j}\right\} \cdot L^k\{\delta\} = \\ = \frac{2^{2j}\Gamma(m+1)\Gamma(m-\frac{n}{2}+2)}{\Gamma(m-j+1)\Gamma(m-\frac{n}{2}-j+2)} \cdot \left(P^{-m-1} \cdot L^k\{\delta\}\right). \tag{II,1;4}$$

Taking into account the formula (II,11), p. 6 of [5], we arrive at

$$L^j\{P^{-m-1+j}\} \cdot L^k\{\delta\} = \\ = \frac{2^{2j}\Gamma(m+1)\Gamma(m-\frac{n}{2}+2)}{\Gamma(m-j+1)\Gamma(m-\frac{n}{2}+j+2)} \cdot \left(P^{-m-1} \cdot L^k\{\delta\}\right) \\ = \begin{cases} 0 & \text{if } \frac{n}{2} \le m+1, \ n \text{ even}, \\ A_{j,m,n,k} L^{k+m+1}\{\delta\} & \text{if } \frac{n}{2} \le m+1, \ n \text{ odd}; \end{cases} \tag{II,1;5}$$

where

$$A_{j,m,n,k} = \\ = \frac{2^{2j} m!k!\Gamma\left(\frac{n}{2}+k\right)\Gamma\left(m-\frac{n}{2}+2\right)}{4^{m+1}\Gamma(m-j+1)\Gamma\left(m-\frac{n}{2}-j+2\right)\Gamma(m+k+j+1)\Gamma\left(k+m+1+\frac{n}{2}\right)}. \tag{II,1;6}$$

By putting $j=1$ and $k=0$ in the above formula, we have

$$L\left\{P^{-m}\right\} \cdot \delta = \\ = \begin{cases} \dfrac{2m!(2m-n+2)}{4^{m+1}(m+1)! \cdot \Gamma\left(m+1+\frac{n}{2}\right)} L^{m+1}\{\delta\}, & \text{if } \frac{n}{2} \le m+1, \ n \text{ odd}, \\ 0 & \text{if } \frac{n}{2} \le m+1, \ n \text{ even}. \end{cases} \tag{II,1;7}$$

II.2. The multiplicative product of $L^j\left\{(P+io)^{-m-1+j}\right\} L^k\{\delta\}$

We have,

$$L^j\left\{(P+io)^{\lambda+j}\right\} = 2^{2j}(\lambda+1)\ldots(\lambda+j)(\lambda+\frac{n}{2})\ldots(\lambda+\frac{n}{2}+j-1) \cdot (P+io)^{\lambda} \\ = \frac{2^{2j}\Gamma(-\lambda)\Gamma(-\lambda-\frac{n}{2}+1)}{\Gamma(-\lambda-j)\Gamma(-\lambda-\frac{n}{2}-j+1)}(P+io)^{\lambda}. \tag{II,2;1}$$

Putting $\lambda = -m-1$ in (II,2;1), we obtain

$$L^j\left\{(P+io)^{-m-1+j}\right\} \cdot L^k\{\delta\} = \\ = \frac{2^{2j} m!\Gamma(m-\frac{n}{2}+2)}{\Gamma(m-j+1)\Gamma\left(m-\frac{n}{2}+j+2\right)} \cdot \left\{(P+io)^{-m-1} \cdot L^k\{\delta\}\right\} \\ = \begin{cases} 0, & \text{if } \frac{n}{2} \le m+1, \\ A_{j,m,n,k} L^{k+m+1}\{\delta\}, & \text{if } \frac{n}{2} \le m+1, \ n \text{ odd}; \end{cases} \tag{II,2;2}$$

where $A_{j,m,n,k}$ is defined by (II,1;6), and $(P+io)^{-m-1}\cdot L^k\{\delta\}$ is given by (I,8) and (I,9) of [5], p. 3.

By putting $j=1$ and $k=0$ in (II,2;2), we arrive at

$$L\left\{(P+io)^{-m}\right\}\cdot\delta = A_{1,m,n,0}\cdot L^{m+1}\{\delta\} \text{ if } \frac{n}{2}\leq m+1,\ n \text{ odd.} \tag{II,2;3}$$

Formula (II,2;3) coincides with (II,1;6).

Finally, we have that

$$L\left\{P^{-m}\right\}\cdot\delta = L(P+io)^{-m}\cdot\delta. \tag{II,2;4}$$

II.3. The formula $L^j\left\{P^{-m-1+j}\right\}\cdot K^k\{\delta\}$

We can generalize the formula $L^j\left\{P^{-m-1+j}\right\}\cdot L^k\{\delta\}$. Effectively, we know that

$$K^k = \left\{L-a^2\right\}^k = \sum_{v=0}^{k}\binom{k}{v}(-a^2)^{k-v}L^v. \tag{II,3;1}$$

Therefore, we have

$$\begin{aligned} &L^j\left\{P^{-m-1+j}\right\}\cdot K^k\{\delta\} = \\ &= L^j\left\{P^{-m-1+j}\right\}\cdot\sum_{v=0}^{k}\binom{k}{v}(-a^2)^{k-v}L^v\{\delta\} \\ &= \sum_{v=0}^{k}\binom{k}{v}(-a^2)^{k-v}\left\{L^j\left(P^{-m-1+j}\right)\cdot L^v\{\delta\}\right\} \\ &= \begin{cases} 0 & \text{if } \frac{n}{2}\leq m+1,\ n \text{ even,} \\ \sum_{v=1}^{k}\binom{k}{v}(-a^2)^{k-v}A_{j,m,n,v}L^{v+m+1}\{\delta\}, & \text{if } \frac{n}{2}\leq m+1,\ n \text{ odd.} \end{cases} \end{aligned} \tag{II,3;2}$$

Otherwise, we have

$$\begin{aligned} &K^j\left\{P^{-m-1+j}\right\}\cdot K^k\{\delta\} = \\ &= \sum_{l=0}^{j}\binom{j}{l}(-a^2)^{j-l}\left\{L^l\{P\}^{-m-1+j}\cdot K^k\{\delta\}\right\} \\ &= \sum_{l=0}^{j}\binom{j}{l}(-a^2)^{j-l}\sum_{v=0}^{k}\binom{k}{v}(-a^2)^{k-v}A_{j,m,n,v}\cdot L^{v+m+1}\{\delta\}. \end{aligned} \tag{II,3;3}$$

III.1. The product of $L^j\left\{(m^2+P)^{\lambda+j}\right\}\cdot L^k\{\delta\}$

We know (cf. ..., page ..., formula (...)) that

$$(m^2+P)^\lambda=\sum_{v\geq 0}a_{v,\lambda}(m^2)^v P^{\lambda-v}\quad \text{if } P>m^2, \tag{III,1;1}$$

where

$$a_{v,\lambda}=\frac{(-1)^v\Gamma(-\lambda+v)}{\Gamma(-\lambda)}. \tag{III,1;2}$$

We obtain

$$\begin{aligned}&L^j\left\{(m^2+P)^{\lambda+j}\right\}\cdot L^k\{\delta\}=\\&=\sum_{v\geq 0}a_{v,\lambda}(m^2)^v L^j\left\{P^{\lambda-v+j}\right\}\cdot L^k\{\delta\}.\end{aligned} \tag{III,1;3}$$

Putting in the above formula $\lambda=-l-1$, we have

$$\begin{aligned}&L^j\left\{(m^2+P)^{-l-1+j}\right\}\cdot L^k\{\delta\}=\\&=\sum_{v\geq 0}a_{v,l}(m^2)^v L^j\left\{P^{-l-1-v+j}\right\}\cdot L^k\{\delta\}\\&=\sum_{v\geq 0}a_{v,l}(m^2)^v A_{j,l+v,n}\cdot L^{k+l+v+1}\{\delta\},\end{aligned} \tag{III,1;4}$$

where

$$\begin{aligned}&A_{j,l+v,n}=\\&=\frac{2^{2j}(l+v)\,!k\,!\Gamma\left(l+v-\frac{n}{2}+2\right)\Gamma\left(\frac{n}{2}+k\right)}{2^{2(l+v+1)}(l+v-j)\,!(k+l+v+1)\,!\Gamma\left(l+v-\frac{n}{2}-j+2\right)\Gamma\left(l+v+\frac{n}{2}+k+1\right)}.\end{aligned} \tag{III,1;5}$$

Making $m^2=0$ in (III,1;4), we obtain the formula (II,1;5).

III.2. The multiplicative product $L^j\left\{(m^2+P)^{-l-1+j}\right\}\cdot K^k\{\delta(x)\}$

We have

$$\begin{aligned}&L^j\left\{(m^2+P)^{-l-1+j}\right\}\cdot K^k\{\delta(x)\}=\\&=L^j\left\{(m^2+P)^{-l-1+j}\right\}\cdot\sum_{v=0}^{k}\binom{k}{v}(m^2)^{k-v}L^v\{\delta\}\\&=\sum_{v=0}^{k}\binom{k}{v}(m^2)^{k-v}\cdot\left\{L^j\{m^2+P\}^{-l-1+j}\cdot L^v\{\delta\}\right\}.\end{aligned} \tag{III,2;1}$$

Substituing (III,1;4) into (III,2;1), we obtain the following formula

$$L^j\left\{(m^2+P)^{-l-1+j}\right\}\cdot K^k\{\delta(x)\}=$$
$$=\sum_{v=0}^{k}\binom{k}{v}(m^2)^{k-v}\cdot\sum_{p\geq 0}a_{p,l}(m^2)^p\cdot A_{j,l+p,n}\cdot L^{k+l+p-1}\{\delta\}. \qquad \text{(III,2;2)}$$

III.3. The multiplicative product $K^j\left\{(m^2+P)^{-l-1+j}\right\}\cdot K^k\{\delta\}$

Let us put

$$K^j\left\{(m^2+P)^{-l-1+j}\right\}\cdot K^k\{\delta\}=$$
$$=\sum_{r=0}^{j}\binom{j}{r}(-m^2)^{j-r}\left\{L^r(m^2+P)^{-l-1+j}\right\}\cdot K^r\{\delta\}. \qquad \text{(III,3;1)}$$

Substituing (III,2;2) into (III,3;1), we shall obtain the final result of the above product.

IV.1. The multiplicative product of $L^j\left\{(m^2+p\pm io)^{-l-1+j}\right\}\cdot L^k\{\delta\}$

Taking into account the following formula (cf. [7], page 6, formula (I,1;23))

$$(m^2+P\pm io)^\lambda=\sum_{v=0}^{\infty}\frac{(m^2)^v}{v!}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)}(P\pm io)^{\lambda-v}, \qquad \text{(IV,1;1)}$$

if $P\geq m^2$.

Also, we know that

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)}=\frac{(-1)^v\Gamma(-\lambda+v)}{\Gamma(-\lambda)}. \qquad \text{(IV,1;2)}$$

Subtituing (IV,1;2) into (IV,1;1), we arrive at

$$(m^2+P\pm io)^\lambda=\sum_{v=0}^{\infty}a_{v,\lambda}(m^2)^v(P\pm io)^{\lambda-v}, \qquad \text{(IV,1;3)}$$

where

$$a_{v,\lambda}=\frac{(-1)^v\Gamma(-\lambda+v)}{v!\Gamma(-\lambda)}. \qquad \text{(IV,1;4)}$$

By similar reasons of the development of the above formulas, we obtain

$$L^j\left\{(m^2+P\pm io)^{\lambda+j}\right\}=\sum_{v=0}^{\infty}a_{v,\lambda}(m^2)^v L^j\left\{(P\pm io)^{\lambda-v+j}\right\}. \qquad \text{(IV,1;5)}$$

Then, we have

$$
\begin{aligned}
& L^j\left\{(m^2+P\pm io)^{-l-1+j}\right\}\cdot L^k\{\delta\}= \\
&=\sum_{v\geq 0} a_{v,l} L^j\left\{(P\pm io)^{-l-1-v+j}\right\}\cdot L^k\{\delta\} \\
&=\sum_{v\geq 0} a_{v,l} b_{j,l+v,n}\cdot c(l+v+1,n,k)\cdot L^{k+v+l+1}\{\delta\}.
\end{aligned}
\qquad \text{(IV,1;6)}
$$

Here

$$
b_{j,l+v,n}\cdot c(l+v+1,n,k)=A_{j,l+v,n}, \qquad \text{(IV,1;7)}
$$

where $A_{j,l+v,n}$ is defined by (III,1;5).

IV.2. The multiplicative product of $L^j\left\{(m^2+P\pm io)^{-l-1+j}\right\}\cdot K^k\{\delta\}$

Therefore, by using (IV,1;6), we obtain

$$
\begin{aligned}
& L^j\left\{(m^2+P\pm io)^{-l-1+j}\right\}\cdot K^k\{\delta\}= \\
&= L^j\left\{(m^2+P\pm io)^{-l-1+j}\right\}\cdot \sum_{v=0}^{k}\binom{k}{v}(m^2)^{k-v}\cdot L^v\{\delta\} \\
&=\sum_{v=0}^{k}\binom{k}{v}(m^2)^{k-v}\cdot L^j\left\{(m^2+P\pm io)^{-l-1+j}\right\}\cdot L^v\{\delta\}.
\end{aligned}
\qquad \text{(IV,1;8)}
$$

Finally, taking into account (IV,1;8) we arrive at

$$
\begin{aligned}
& K^j\left\{(m^2+P\pm io)^{-l+j-1}\right\}\cdot K^k\{\delta\}= \\
&=\sum_{r=0}^{j}\binom{j}{r}(-m^2)^{j-r}\left\{L^r(m^2+P\pm io)^{-l-1+j}\cdot K^k\{\delta\}\right\}.
\end{aligned}
\qquad \text{(IV,1;9)}
$$

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