THE PRESCRIBED MEAN CURVATURE EQUATION WITH CONSTANT BOUNDARY VALUES

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1. Introduction

We consider the Dirichlet problem with constant boundary value $c \in I\!R^3$ in the unit disc $B = \{(u,v) \in I\!R^2; u^2 + v^2 < 1\}$ for a vector function $X : \overline{B} \to I\!R^3$ which satisfies the equation of prescribed mean curvature

$$(\text{Dir}) \begin{cases} \Delta X = 2H(X) X_u \wedge X_v & \text{in } B \\ X = c & \text{on } \partial B \end{cases}$$

where $X_u = \dfrac{\partial X}{\partial u}$, $X_v = \dfrac{\partial X}{\partial v}$, "$\wedge$" denotes the exterior product in $I\!R^3$ and $H : I\!R^3 \to I\!R$ is a given continuous function.

It is known that for $H = H_0 \in I\!R$, the only weak solution to (Dir) is c.[**6**]

For variable H and fixed $c \neq 0$ in $I\!R^3$, we prove first that there is a class of functions H such that c is still the only weak solution to (Dir). Then, for another class, that there are at least two solutions.

2. Notations

We denote $W^{1,p}(B, I\!R^3)$ the usual Sobolev spaces [**1**] and $H^1(B, I\!R^3) = W^{1,2}(B, I\!R^3)$.

For $X \in H^1(B, I\!R^3)$, $\|X\|_{L^2(\partial B, I\!R^3)} = \left(\int_{\partial B} |\text{Tr } X|^2\right)^{\frac{1}{2}}$ where $\text{Tr} : H^1(B, I\!R^3) \to L^2(\partial B, I\!R^3)$ is the usual trace operator [**1**] and for $Y \in L^\infty(U, I\!R^n)$ we denote

$$\|Y\|_\infty = \operatorname*{ess\,sup}_{w \in U} |Y(w)|.$$

When H is bounded we call $X \in H^1(B, I\!R^3)$ a weak solution of (Dir) if for every $\varphi \in C_0^1(B, I\!R^3)$

$$\begin{cases} \int_B \left(\nabla X \cdot \nabla \varphi + 2H(X) X_u \wedge X_v \cdot \varphi\right) = 0 \\ X \in c + H_0^1(B, I\!R^3) \end{cases}$$

where $H_0^1(B, \mathbb{R}^3) =$ adh $_{H^1} C_0^1(B, \mathbb{R}^3)$

We will obtain weak solutions as critical points of $D_H(X) = D(X) + 2V(X)$ with $D(X) = \frac{1}{2}\int_B |\nabla X|^2$ the Dirichlet integral and $V(X) = \frac{1}{3}\int_B Q(X) \cdot X_u \wedge X_v$ the Hildebrandt volume[**3**].

Finally, we denote

$$dD_H(X)(\varphi) = \lim_{t \to 0} \left[\frac{D_H(X + t\varphi) - D_H(X)}{t} \right]$$

wherever this limit exists (resp. $dV(X)(\varphi)$).

3. A uniqueness type theorem

Theorem 1: *Consider $H : \mathbb{R}^3 \to \mathbb{R}$ continuous and bounded such that the associated vector field Q satisfies*

(1) $Q \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$, $\dfrac{\partial Q_i}{\partial \xi_j} \in L^\infty(\mathbb{R}^3)$ *for* $i \neq j$.

If $c \in \mathbb{R}^3$ *verifies*

(2) $\sup_{\xi \in \mathbb{R}^3} |H(\xi)(\xi - c)| < 1$

Then the only weak solution of (Dir) in $W^{1,\infty}(B; \mathbb{R}^3)$ *is* c.

Proof: (1) ensures that D_H is well defined in $W^{1,\infty}(B, \mathbb{R}^3)$ and that

$$dV(X)(\varphi) = 3 \int_B H(X) X_u \wedge X_v \cdot \varphi$$

for $\varphi \in C_0^1(B, \mathbb{R}^3)$. As $H(X) \in L^\infty(B)$ and $X_u \wedge X_v \in L^\infty(B; \mathbb{R}^3)$, this equality holds for $\varphi \in H_0^1(B; \mathbb{R}^3)$, so $dD_H(X)(X - c) = 0$. But

$$\begin{aligned} dD_H(X)(X - c) &= \int_B |\nabla X|^2 + 2H(X) X_u \wedge X_v \cdot (X - c) \geq \\ &\geq 2D(X)(1 - \|H(X)(X - c)\|_\infty) \end{aligned}$$

So $D(X) = 0$ and $X = c$ on ∂B. Hence $X = c$ on B. □

4. Nonuniqueness in the Dirichlet problem with constant boundary values

We will give c's en $I\!R^3$ and H's in $C^1(I\!R^3)$ such that (Dir) has at least two weak solutions. For this purpose we will use the following lemma and proposition.

Lemma 1: *Let $H \in C^1(I\!R^3) \cap W^{1,\infty}(I\!R^3)$. If the function Q associated to H satisfies (1), then given $X_0 \in H^1(B, I\!R^3)$ such that $H(X_0) = 0$ and $\nabla H(X_0) = 0$, we have that $dV(X_0)(\varphi) = 0$ and $d^2V(X_0)(\varphi, \psi) = 0$ for $\varphi, \psi \in C_0^\infty(B, I\!R^3)$.*

Proof: $dV(X_0)(\varphi) = 0$ holds trivially. The directional derivative of $dD_H(\,\cdot\,)(\varphi)$ at X in the direction Ψ is given by

$$d^2V(X)(\varphi,\psi) = \int_B (\nabla H(X)\cdot\psi) X_u \wedge X_v \cdot \varphi + \int_B (\nabla H(X)\cdot X_v)\psi_u \wedge \varphi \cdot X$$
$$+ \int_B (\nabla H(X)\cdot X_u)\varphi \wedge \psi_v \cdot X + \int_B H(X)(\psi_u \wedge \varphi_v + \varphi_u \wedge \psi_v)\cdot X$$

From $H(X_0) = 0$, $\nabla H(X_0) = 0$ in B, we have that $d^2V(X_0)(\varphi,\psi) = 0$ and the proof is complete. □

For $c \in I\!R^3$, $H \in C^1(I\!R^3)$ with $0 < H_0 = \|H\|_\infty < \infty$ and $k > 0$ in $I\!R$, we define

$$M_k = \left\{ X \in c + H_0^1(B, I\!R^3);\ \|X - c\|_\infty \le \frac{1}{H_0},\ \|\nabla(X-c)\|_\infty \le k \right\}$$

and denote ρ the slope of D_H in M_k, i.e.

$$\rho(X) = \sup_{Y \in M_k} dD_H(X)(X - Y). \qquad [5]$$

Proposition 1: *Any $X \in M_k$ with slope $\rho(X) = 0$ is a weak solution of* (Dir).

Proof: If $\rho(X) = 0$, it is known that $dD_H(X)(X-c) < 0$ or X is a weak solution of (Dir) (Lemma 1 [4]). But

$$dD_H(X)(X-c) \ge \int_B \left[|\nabla X|^2 - 2H_0|X-c||X_u \wedge X_v|\right] \ge 0.$$

□

Now, we build c's, H's and k's such that c is a local minimum of D_H in $c + H_0^1(B, I\!R^3)$ and $D_H(\overline{X}) < D_H(c)$ for some $\overline{X} \in M_k$. Hence by Theorem 3 in [4], there exist $X \in M_k, X \neq c$,with slope $\rho(X) = 0$, so X is another weak solution by Proposition 1.

Let $c = (0, a, 0)$, with $a \in I\!R$ and let $H \in C^1(I\!R^3)$ be such that

$$H(\xi) = \begin{cases} H_0 & \text{if } \xi_1^2 + \xi_2^2 \leq R^2 \text{ and } \varepsilon_1 \leq \xi_3 \leq \varepsilon_2 \\ 0 & \text{if } \xi_1^2 + \xi_2^2 > (R + \varepsilon_3)^2 \text{ or } \xi_3 \notin (\varepsilon_1 - \varepsilon_3, \varepsilon_2 + \varepsilon_3) \end{cases}$$

For a convenient choice of $H_0 \neq 0$ in $I\!R$, ε_1, ε_2, ε_3 positive, the element $\overline{X}(r, \alpha) = ((1 - r^2)r^2 \int_0^\alpha \sin^2 t dt$, a ,$1 - r^2)$ (in polar coordinates) is in M_k for $k > \|\nabla \overline{X}\|_\infty$ and $D_H(\overline{X}) < D_H(c) = 0$ [4]. Finally it follows easily that c is a local minimum in $c + H_0^1(B; I\!R^3)$ from lemma 1.

Remarks

i) If X_1 is an unstable critical point of D_H in M_k, then $\|H(X_1)(X_1 - c)\|_\infty = 1$,because if we suppose that $\|H(X_1)(X_1 - c)\|_\infty < 1$, as in Theorem 1 we have that

$$\begin{aligned} 0 &= dD_H(X_1)(X_1 - c) \\ &= \int_B [\nabla X_1 \cdot \nabla (X_1 - c) + 2H(X_1) X_{1u} \wedge X_{1v} \cdot (X_1 - c)] \\ &\geq 2D(X_1)(1 - \|H(X_1)(X_1 - c)\|_\infty). \end{aligned}$$

Hence, we deduce that $D(X_1) = 0$ and $X_1 = c$.

A contradiction.

ii) Given $c \neq 0$ in $I\!R^3$, we can choose $|H_0|$ small enough in our example of H to satisfy the assumptions of theorem 1.

References

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