A NOTE ON THE CONVEX HULL OF SELF-SIMILAR SETS

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ABSTRACT: *We show sufficient conditions for the convex hull of a self-similar set to be a polyhedron (theorem 1), and we exhibit an example showing that these conditions are in a certain sense optimal (example 1). In theorem 3 we investigate the boundary of a (general) self-similar set in* R^2.

Let A be a subset of R^n; $C(A)$ will denote its convex hull. Let $Y_i : R^n \longrightarrow R^n$, $i=1,\dots,L$, be mappings such that

(0) $$Y_i(x) = c_i A_i x + \alpha_i$$

where c_i is a real number such that $0 < c_i < 1$, α_i is a vector in R^n and A_i is a (real) ortogonal matrix with determinant one, i.e., $A_i \in O_n^*$, the group of rigid rotations in R^n.

By K a self-similar set we will understand the unique compact set K of R^n such that

$$K = \bigcup_{i=1}^{L} Y_i(K)$$

By different reasons (see [PA]) it is useful to have a knowledge of the convex hull of a self-similar set K (notice that $C(K)$ is compact). For $C(K)$ a polyhedron we will write $C(K) = C(\{p_1, \dots, p_m\})$, where the points p_i, $i=1,\dots,m$, are a minimal set of generators.

The following theorem states that in certain cases $C(K)$ is a polyhedron and shows how to calculate the points that generate it.

THEOREM 1. Let K be a self-similar set of R^n. Then

a) if $C(K)$ is a polyhedron, i.e., $C(K)=C(\{p_1,\dots,p_m\})$, then for any j, $1\le j\le m$, there exist indices $i_1, \dots, i_{k_2}$, such that

(1) $$p_j = Y_{i_1}(\dots(Y_{i_{k_1}}(q))\dots) \quad \text{where}$$

$$q = Y_{i_{k_1+1}}(\dots(Y_{i_{k_2}}(q))\dots) \quad \text{and}$$

$$1 \le i_1, \ \dots \ , i_{k_1}, i_{k_1+1}, \ \dots \ , i_{k_2} \le L$$

(k_1 may be zero : $p_j = q$ a fixed point of $Y_{i_1}(\dots(Y_{i_{k_2}}(.))\dots).$)

b) if the subgroup generated by $\{A_i\}$ in O_n^* is finite of order μ then $\mathcal{C}(K)$ is a polyhedron and is equal to the convex hull of all possible points p_j as **in (1)** with $1 \le k_2 \le \mu$.

PROOF. Observe that, if K is a self-similar set, then by linearity we have

$$(2) \qquad \mathcal{C}(K) = \mathcal{C}\Big(\bigcup_{i=1}^{L} \mathcal{C}(Y_i(K)) \Big) ,$$
$$\mathcal{C}(Y_i(K)) = Y_i(\mathcal{C}(K))$$

a) if $\mathcal{C}(K)$ is a polyhedron as stated then the points p_j must be extremal points of $\mathcal{C}(K)$: there do not exist points p,q, belonging to $\mathcal{C}(K)$, $p,q \neq p_j$, such that $p_j \in [p,q]$ (see [EGG]). Fix j. Using (2) we have that $p_j \in Y_{i_1}(\mathcal{C}(K))$ for some index i_1. Since $Y_{i_1}(\mathcal{C}(K))$ is a polyhedron contained in $\mathcal{C}(K)$ and p_j is an extremal point of $\mathcal{C}(K)$, there exists p_k such that $Y_{i_1}(p_k) = p_j$. Repeating this process we obtain a sequence of points such that

$$(3) \qquad \dots \xrightarrow{Y_{i_2}} p_k \xrightarrow{Y_{i_1}} p_j$$

But there are only a finite number of points p_i . Therefore (3) may be rewritten as

$$\dots \longrightarrow p_r = q \xrightarrow{Y_{i_{k_2}}} \dots \xrightarrow{Y_{i_{k_1+1}}} p_r = q \xrightarrow{Y_{i_{k_1}}} \dots \xrightarrow{Y_{i_2}} p_k \xrightarrow{Y_{i_1}} p_j$$

and a) follows.

b) We repeat, up to some point, the argument used in a). Let p_0 be an extremal point of $\mathcal{C}(K)$. By (2) there exists an index i_1 such that $p_0 \in Y_{i_1}(\mathcal{C}(K))$. Set $p_1 = Y_{i_1}^{-1}(p_0)$. Then p_1 is an extremal point of $\mathcal{C}(K)$. Repeat the process with p_1 and so on. Then we have a sequence of **extremal** points such that

(4) $$\cdots \xrightarrow{Y_{i_3}} p_2 \xrightarrow{Y_{i_2}} p_1 \xrightarrow{Y_{i_1}} p_o$$

Since the mappings are such that $Y_j(x) = c_j A_j x + a_j$, $j = 1,\ldots,L$, and the subgroup generated by $\{A_j\}$ is finite of order μ, we have that given the sequence $i_1, i_2, \ldots$, there exist indices $0 \le k_1 < k_2 \le \mu$ such that

$$A_{i_1} \cdot A_{i_2} \cdot \ldots \cdot A_{i_{k_1}} = A_{i_1} \cdot A_{i_2} \cdot \ldots \cdot A_{i_{k_2}}, \text{ so}$$

$$A_{i_{k_1+1}} \cdot \ldots \cdot A_{i_{k_2}} = I = \text{identity matrix.}$$

Therefore $Y_{i_{k_1+1}}(\ldots Y_{i_{k_2}}(x) \ldots) := Y(x) = cx + a$

where $0 < c < 1$ (there is **no** rotation).

Suppose $p_{k_1} \neq p_{k_2}$. Then, since $Y(p_{k_2}) = p_{k_1}$, we have

$$p_{k_1} = p_{k_2} . c/(1+c) + Y(p_{k_1}) . 1/(1+c)$$

This means, $p_{k_1} \in [p_{k_2}, Y(p_{k_1})]$ being $p_{k_1} \neq p_{k_2}, Y(p_{k_1})$.

Therefore p_{k_1} is not an extremal point, a contradiction.

Therefore $p_{k_1} = p_{k_2}$ and we have proved that given an extremal point p_o there exist indices $0 \le k_1 < k_2 \le \mu$ such that

$$p_o = Y_{i_1}(\ldots Y_{i_{k_1}}(q) \ldots)$$

$$q = Y_{i_{k_1+1}}(\ldots Y_{i_{k_2}}(q) \ldots).$$

Consequently the number of extremal points is finite. It is a well-known fact that this means that $C(K)$ is a polyhedron (see [EGG]). ∎

A consequence of theorem 1 b) is that if we have mappings $Y_j(z) : R^2 \longrightarrow R^2$, $j=1,\ldots,L$, such that

(5) $$Y_j(z) = c_j e^{i2\pi\theta_j} z + a_j$$

or

$$\text{(6)} \qquad Y_j(z) = c_j e^{i2\pi\theta_j}\,\bar{z} + \alpha_j$$

where $0 < c_j < 1$, θ_j is rational and z, α_j, are complex then $\mathcal{C}(K)$ must be a polygon. This fact is proved as follows: if all mappings are as in (5) then theorem 1 b) applies directly. If there is a mapping as in (6) then for each mapping Y_j as in (5), associate a mapping $\tilde{Y}_j: R^3 \longrightarrow R^3$ in the following way (here $\beta_j = 2\pi\theta_j$)

$$\tilde{Y}_j((x,y,z)) = c_j \begin{vmatrix} \cos\beta_j & -\sin\beta_j & 0 \\ \sin\beta_j & \cos\beta_j & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} + \begin{vmatrix} \mathrm{Re}\,\alpha_j \\ \mathrm{Im}\,\alpha_j \\ 0 \end{vmatrix}$$

and for each Y_j as (6) we associate $\tilde{Y}_j: R^3 \longrightarrow R^3$ as

$$\tilde{Y}_j((x,y,z)) = c_j \begin{vmatrix} \cos\beta_j & \sin\beta_j & 0 \\ \sin\beta_j & -\cos\beta_j & 0 \\ 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} + \begin{vmatrix} \mathrm{Re}\,\alpha_j \\ \mathrm{Im}\,\alpha_j \\ 0 \end{vmatrix}$$

Observe that in any case $\tilde{Y}_j((x,y,0)) = Y_j(x+iy)$ and that $\tilde{K}$, the self-similar set associated with $\{\tilde{Y}_j\}$, is contained in $R^2 = \{(x,y,0)\}$. Therefore $\tilde{K} = K$. But theorem 1 b) applies for $\tilde{K}$. This finishes the proof.

The following example in R^2 shows that we should not expect $\mathcal{C}(K)$ be always a polyhedron.

Example 1: Set

$$Y_1(z) = z\, e^{i2\pi^2}/2 \quad , \qquad Y_2(z) = zi/2 + 1 - i/2$$

and let K be the self-similar set associated with these mappings. Then

THEOREM 2. $\mathcal{C}(K)$ is not a polygon.

PROOF. First observe that for the above mappings we have $Y_1(0) = 0$, $Y_2(1) = 1$. Therefore $0,1 \in K$ (see [FAL]) and therefore $Y_1^i(1)$, $Y_2^i(0) \in K$ for $i = 1,2,\ldots$.

It is not difficult to see that

$$(7) \qquad 0 \in \operatorname{int} \mathcal{C}\left(\bigcup_{i=1}^{\infty} \{ Y_1^i(1) \} \right) \subset \operatorname{int} \mathcal{C}(K)$$

$$1 \in \operatorname{int} \mathcal{C}\left(\bigcup_{i=1}^{\infty} \{ Y_2^i(0) \} \right) \subset \operatorname{int} \mathcal{C}(K)$$

Suppose $\mathcal{C}(K)$ is a polygon, i.e. $\mathcal{C}(K) = \mathcal{C}(\{p_1, \ldots, p_m\})$ with p_j ordered clockwise in $\mathcal{C}(K)$.

By theorem 1 a) we know that each point p_j is obtained as

$$(8) \qquad p_j = Y_{i_1}(\ldots Y_{i_{k_1}}(q)\ldots),$$

$$q = Y_{i_{k_1+1}}(\ldots Y_{i_{k_2}}(q)\ldots) \quad \text{and} \quad 1 \le i_1, \ldots, i_{k_1}, i_{k_1+1}, \ldots, i_{k_2} \le 2$$

If q is not an extremal point then by (8) p_j is not an extremal point. Thus q is a point p_i for some i, say $q=p_1$, and by (8)

$$p_1 = Y_{i_{k_1+1}}(\ldots(Y_{i_{k_2}}(p_1))\ldots)$$

If $i_{k_1+1} = \ldots = i_{k_2}$ then p_1 would be 0 or 1. By (7) this can not happen. Therefore $k_2 - k_1 \ge 2$ and there exists at least one index $i_j = 1$ with $k_1 < j \le k_2$. Then we must have

$$Y_{i_{k_1+1}}(\ \ldots\ Y_{i_{k_2}}(z)\ \ldots\ ) := Y(z) = c\, e^{i2\pi\theta} z + \alpha$$

with $0 < c < 1$, θ **irrational**, α a complex number.

By (2), if we set $T = \mathcal{C}(\{p_m, p_1, p_2\})$ then $Y(T) \subset \mathcal{C}(K)$. But θ is not 2π times an integer and $Y(p_1) = p_1$. Hence, the triangle $\mathcal{C}(\{ Y(p_m), Y(p_1), Y(p_2)\})$ is not contained in $\mathcal{C}(\{p_m, p_1, p_2\})$ and $Y(p_m) \notin \mathcal{C}(K)$ or $Y(p_2) \notin \mathcal{C}(K)$, which is absurd. ∎

The following theorem shows that the extremal points of a self-similar set in R^2 can be divided into two classes.

Let $Y_j(z) = c_j e^{i2\pi\theta_j} z + \alpha_j$, $j = 1,\ldots,L$, be mappings of R^2 such that $0< c_j< 1$, α_j complex and θ_j real. Let K be the self-similar set associated with $\{Y_j\}$ and suppose that $C(K)$ is not a polygon.

Let p_o be an extremal point of $C(K)$ and define $\theta_{p_o}(C(K)) := \{ \sup \theta \ / \ \theta$ is the angle between segments $[p_o,a]$, $[p_o,b]$, with $a,b \in C(K) \}$. It is clear that $0 < \theta_{p_o}(C(K)) \leq \pi$.

For p_o there are only two possibilities :
p_o is locally linear, i.e., there exist points c,d, such that p_o, c, d, are not colinear and $[p_o,c],[p_o,d] \subset \partial C(K)$ (therefore $\theta_{p_o}(C(K))$ = angle between $[p_o,c]$ and $[p_o,d]$),
or p_o is not locally linear.

THEOREM 3. In the above conditions we have

a) if p_o is not locally linear then $\theta_{p_o}(C(K)) = \pi$

b) if p_o is locally linear then

$$p_o = Y_{i_1}(\ldots(Y_{i_{k_1}}(q))\ldots) \quad \text{where}$$

$$q = Y_{i_{k_1+1}}(\ldots(Y_{i_{k_2}}(q))\ldots) \quad \text{and}$$

$$1 \leq i_1, \ \ldots \ , i_{k_1}, i_{k_1+1} \ \ldots \ , i_{k_2} \leq L$$

PROOF. a) Suppose p_o is such that $\theta_{p_o}(C(K)) \leq \pi - \epsilon$ with ϵ positive and small. As p_o is not locally linear one can find a sequence p_{o_i} such that

I) $p_{o_i} \longrightarrow p_o$; p_o, p_{o_i} are extremal points of $C(K)$ and $p_{o_i} \neq p_o$ for all i.

Using (2) one can extract a subsequence of p_{o_i} such that $p_{o_{i_k}}, p_o \in Y_{j_1}(C(K))$ for all k, for some index j_1. Set $p_1 = Y_{j_1}^{-1}(p_o)$, $p_{1k} = Y_{j_1}^{-1}(p_{o_{i_k}})$. Therefore, p_1, p_{1k} satisfy condition

I) and $\theta_{p_1}(\mathcal{C}(K)) \leq \theta_{p_o}(\mathcal{C}(K))$. Repeating the process one has

$$\cdots \longrightarrow p_2 \xrightarrow{Y_{j_2}} p_1 \xrightarrow{Y_{j_1}} p_o$$

where $Y_{j_s}(p_{si})$ for all i is a subsequence of $p_{(s-1)i}$ and all points involved are extremal points. Also

(9) $$\cdots \leq \theta_{p_1}(\mathcal{C}(K)) \leq \theta_{p_o}(\mathcal{C}(K)) \leq \pi - \epsilon$$

From (9) one gets $p_{k_1} = p_{k_2}$ for some pair $k_1 = k_2$ (if all p_i were different, take the polygon $\mathcal{C}(\{p_o, \ldots, p_m\})$ whose sum of interior angles, because of (9), would be less than or equal to $\theta_{p_o}(\mathcal{C}(K)) + \ldots + \theta_{p_m}(\mathcal{C}(K)) \leq (m+1)(\pi - \epsilon)$. But the sum of the interior angles is $(m+1)\pi - 2\pi$. If m is great enough we get a contradiction).

Therefore

$$\cdots \longrightarrow p_{k_2} = q \xrightarrow{Y_{j_{k_2}}} \cdots \longrightarrow p_{k_1} = q \xrightarrow{Y_{j_{k_1}}} \cdots \longrightarrow p_1 \xrightarrow{Y_{j_1}} p_o$$

Set $Y(x) := Y_{j_{k_1+1}}(\ \ldots\ (Y_{j_{k_2}}(x))\ \ldots\ )$. Then $Y(q)=q$ and we have

$\theta_q(\mathcal{C}(K)) = \theta_q(Y(\mathcal{C}(K)))$. As $Y(\mathcal{C}(K)) \subset \mathcal{C}(K)$ the map $Y(x)$ must be a contraction plus a translation (there is **no** rotation).

But $Y(p_{k_2 1}) = p_{k_1 i}$ for some i and therefore $p_{k_1 i} \in [q, p_{k_2 1}]$ being $p_{k_2 1}$ and q different from $p_{k_1 i}$. Therefore $p_{k_1 i}$ is not an extremal point. This proves a).

b) By arguments used before one can prove that there is a sequence of extremal points and mappings such that

$$\cdots \longrightarrow p_2 \xrightarrow{Y_{j_2}} p_1 \xrightarrow{Y_{j_1}} p_o \text{ and } \cdots \leq \theta_{p_1}(\mathcal{C}(K)) \leq \theta_{p_o}(\mathcal{C}(K)) \leq \pi - \epsilon$$

As in a) the above conditions give that $p_{k_2} = p_{k_1}$ with $k_1 < k_2$ and b) follows. ∎

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