Free Algebras in Some Subvarieties of Ockham Algebras

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Abstract

In this paper we provide a construction of free Ockham algebras over a poset. We also investigate free objects in the Berman classes $\mathbf{P}_{m,n}$, and in the class **MS** of Morgan-Stone algebras and their generalisation MS_n-algebras.

1 Introduction

An Ockham algebra is an algebra $(A; \vee, \wedge, f, 0, 1)$ of type $(2,2,1,0,0)$ in which $(A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and f is a dual lattice homomorphism, i.e., $f(0) = 1$, $f(1) = 0$, and the equations

(1) $$f(x \wedge y) = f(x) \vee f(y)$$

(2) $$f(x \vee y) = f(x) \wedge f(y)$$

hold identically.

An Ockham algebra $(A; \vee, \wedge, f, 0, 1)$ will often be denoted by the simpler notation $(A; f)$.

The class of Ockham algebras is a variety, and will be denoted by $\mathcal{O}$. They were introduced by J. Berman [2] in a short but very important paper in 1977.

Subvarieties of $\mathcal{O}$ of paramount importance are the so called *Berman varieties* $\mathbf{P}_{m,n}$, $m > n \geq 0$, $m - n$ even (see [11],[2]). These are obtained by placing restrictions on the dual endomorphism f. Precisely, we define $\mathbf{P}_{m,n}$ to be the subclass of $\mathcal{O}$ obtained by adjoining the equation

(3) $$f^m(x) = f^n(x).$$

The smallest Berman class is the class $\mathbf{P}_{2,0}$ of de Morgan algebras. The class $\mathbf{P}_{3,1}$ contains the class of de Morgan algebras and the class of Stone algebras as well.

We consider also subvarieties $\mathbf{P}_{m,n}$, with $m > n \geq 0$ and $m-n$ odd, given equationally within $\mathcal{O}$ by

(4) $$f^m(x) \wedge f^n(x) = 0 \quad \text{and} \quad f^m(x) \vee f^n(x) = 1$$

Among other subvarieties of $\mathcal{O}$ that have been studied, we bring out the variety **MS** of MS-algebras, introduced by T. Blyth and J. Varlet as a common generalization of de Morgan algebras and Stone algebras. An MS-algebra is an Ockham algebra $(A; f)$ in which $x \leq f^2(x)$, for every $x \in A$ [4]. The class **MS** is a subclass of $\mathbf{P}_{3,1}$. As a natural generalization, M. Ramalho and M. Sequeira [12] considered in 1987 more generally the subvarieties of $\mathcal{O}$ defined by $x \leq f^{2n}(x)$, for $n \geq 1$. These subvarieties will be denoted $\mathbf{MS}_n$, and their elements will be called MS_n-algebras. So, MS_n-algebras are Ockham algebras $(A; f)$ in which f^{2n} is a closure operator on A.

Section 2 is devoted to the determination of the algebraic structure of the free Ockham algebra over a poset. Sections 3 and 4 deal entirely with the subvarieties $\mathbf{P_{m,n}}$ whereas section 5 is devoted to MS_n-algebras.

Some of the results contained in this paper are generalisations of Golberg's results in [7], but our techniques are completely different.

2 Free Ockham algebras over a poset

The sole aim of this section is to derive a characterization of free Ockham algebras over a poset given in Theorem 2.6.

For any Ockham algebra A and a subset X in A, $OS(X)$ and $SL(X)$ respectively denote the Ockham subalgebra and the (distributive) sublattice generated by X.

Definition 2.1 *Let I be a poset. An Ockham algebra FO(I) is called* **free** *over I if the following conditions are satisfied:*
(A) $I \subset FO(I)$ and $OS(I) = FO(I)$.
(B) Let f be an order-preserving map from I into an Ockham algebra A. Then there exists a homomorphism h from FO(I) into A such that $h|I = f$.

It is easy to see ([9]) that if $FO(I)$ exists, it is unique up to isomorphism and the homomorphism h in (B) is also unique. Throughout this paper the following remark will be an important tool.

Remark 2.2 *Let I_i, $i = 1, \ldots, n$, be posets and $I_i \subset D$ with $D \in \mathbf{D}_{01}$. Let $h_i : I_i \longrightarrow A$ be an order-reversing map, where $A \in \mathbf{D}_{0,1}$. For the order dual A^* of A, let $\alpha : A \longrightarrow A^*$ be the natural anti-isomorphism. If, for all $i = 1, \ldots, n$, the maps $\alpha \circ h_i$ can be extended to a homomorphism $H : D \longrightarrow A^*$ such that $H|I_i = \alpha \circ h_i$ for every $i = 1, \ldots, n$, then there exists a dual lattice homomorphism $H^* : D \longrightarrow A$ such that $H^*|I_i = h_i$. In fact, we put $H^* = \alpha^{-1} \circ H$. It is easy to see that $H^*|I_i = h_i$ and that H^* is a dual homomorphism. In particular, if D is the free distributive lattice $L(I)$ over a poset I and $h : I \longrightarrow A$ is an order-reversing map, then we can extend h to a dual homomorphism $H^* : L(I) \longrightarrow A$.*

Let I be a poset and I^* its order dual. Let

$$G = \sum_{i \geq 0} I_i, \tag{5}$$

where $I_i \cong I$ if i is even, $I_i \cong I^*$ if i is odd and $\sum$ is the cardinal sum of posets. Let $\alpha_i : I_i \longrightarrow I_{i+1}$ be the natural anti-isomorphism. Let

$$\mathbf{f} : G \longrightarrow G \tag{6}$$

be the map defined as follows: if $g \in I_i$ for some $i \geq 0$, then $\mathbf{f}(g) = \alpha_i(g)$. It is clear that for $x \in I_i$, there exists $x_0 \in I_0$ such that $x = f^i(x_0)$.

Lemma 2.3 **f** *is an order-reversing map.*

Proof For $x, y \in G$, if $x \leq y$, it is clear that x and y lie in the same I_i $(i \geq 0)$. Hence $\mathbf{f}(x) = \alpha_i(x) \geq \alpha_i(y) = \mathbf{f}(y)$. □

Let $L(G)$ be the free distributive lattice over G. A construction of $L(G)$ is developed in [9], and when G is finite

(7)
$$L(G) \cong \mathbf{2}^{[\mathbf{2}^{[G]}]},$$

where $\mathbf{2}$ is the two element chain and $\mathbf{2}^{[X]}$ is the distributive lattice of all order-preserving maps from the poset X to $\mathbf{2}$.

By 2.2 and 2.3, $\mathbf{f}$ can be extended to a dual homomorphism $\mathbf{F} : L(G) \to L(G)$. Then $(L(G); \mathbf{F})$ is an Ockham algebra.

We are now going to point out some properties that we shall need to establish an isomorphism between $(L(G); \mathbf{F})$ and $FO(I)$.

Lemma 2.4 $(L(G); \mathbf{F}) = OS(I_0)$.

Proof Let $x \in I_i$, then $x \in \mathbf{f}^i(I_0)$. Consequently $x \in OS(I_0)$ whenever $x \in G$. Since $L(G)$ is free over G, then $(L(G); \mathbf{F}) = SL(G) \subseteq OS(I_0)$. Hence $(L(G), \mathbf{F}) = OS(I_0)$. □

Let f be the dual homomorphism corresponding to $FO(I)$.

Lemma 2.5 *Let* $Q = \bigcup_{i \geq 0} f^i(I)$. *Then* $SL(Q) = FO(I)$.

Proof The set $\{x \in SL(Q) : f(x) \in SL(Q)\}$ is a sublattice of $SL(Q)$ containing Q, so for every $x \in SL(Q)$, $f(x) \in SL(Q)$. Then $SL(Q)$ is closed under f, that is, $SL(Q)$ is an Ockham subalgebra of $FO(I)$. Since $I \subseteq SL(Q)$, then $SL(Q) = FO(I)$. □

Theorem 2.6 $FO(I) \cong (L(G); \mathbf{F})$.

Proof Let $h : I \longrightarrow I_0 \subseteq (L(G); \mathbf{F})$ be the order-isomorphism identity. Then h can be extended to a homomorphism of Ockham algebras $H : FO(I) \longrightarrow (L(G); \mathbf{F})$. By 2.4, $H(FO(I)) = H(OS(I)) = OS(H(I)) = OS(I_0) = (L(G); \mathbf{F})$. Hence H is an epimorphism.

We consider now
$$k : G \longrightarrow Q,$$
defined as follows: If $x \in I_0$, we put $k(x) = h^{-1}(x)$. If $x \in I_i$, then $x \in \mathbf{F}^i(x_0)$, with $x_0 \in I_0$. Then we put $k(x) = f^i(h^{-1}(x_0))$. It is not difficult to prove that k is an order-preserving map, and then, it can be extended to a lattice homomorphism $K : (L(G); \mathbf{F}) \longrightarrow FO(I)$. Moreover, K is an Ockham homomorphism. Let us check that $K.H = Id_{FO(I)}$. If $x \in I$, then $K.H(x) = K(h(x)) = k(h(x)) = x$. Now, the only homomorphism extending the identity is the identity Ockham homomorphism $Id_{FO(I)}$. Then $K.H = Id_{FO(I)}$. Hence H is also a monomorphism and therefore $FO(I) \cong (L(G); \mathbf{F})$. □

3 Free $P_{m,n}$-Ockham algebras over a poset with $m-n$ even

Now we investigate Berman varieties $\mathbf{P}_{m,n}$. The definition of free $P_{m,n}$-Ockham algebra $FO_{m,n}(I)$ over a poset I is analogous to Definition 1.1. In this case we consider the poset

(8) $$G = \sum_{i=0}^{m-1} I_i,$$

where $I_i \cong I$ if i is even and $I_i \cong I^*$ if i is odd, and, as before, $\alpha_i : I_i \to I_{i+1}$ is the natural anti-isomorphism for $i < m-1$. Let

(9) $$\mathbf{f} : G \longrightarrow G$$

be the map defined as follows: if $g \in I_i$ for some $0 \leq i < m-1$, then we put $\mathbf{f}(g) = \alpha_i(g)$. If $g \in I_{m-1}$, then $g = \alpha_{m-2} \circ \ldots \alpha_1 \circ \alpha_0(x)$ with $x \in I_0$, and we put $\mathbf{f}(g) = \alpha_{n-1} \circ \ldots \alpha_1 \circ \alpha_0(x)$. It is clear that, for $x_0 \in I_0$, $f^i(x_0) \in I_i$, $1 \leq i < m$ and $f^m(x_0) = f^n(x_0)$. In addition, for $x \in I_i$, there exists $x_0 \in I_0$ such that $f^i(x_0) = x$.

Lemma 3.1 $\mathbf{f}$ *is an order-reversing map and verifies* $\mathbf{f}^m(g) = \mathbf{f}^n(g)$ *for all* $g \in G$.

Proof Let $x, y \in G$, $x \leq y$. Clearly, if x and y are comparable then they lie in the same I_i $(i \geq 0)$. If $i < m-1$ then $\mathbf{f}$ is an order-reversing map being that so is α_i. Let $x, y \in I_{m-1}$, $x \leq y$. We may assume that $m-1$ is even and n is odd. Let $x_0, y_0 \in I_0$ such that $y = \alpha_{m-2} \circ \ldots \alpha_1 \circ \alpha_0(y_0)$ and $x = \alpha_{m-2} \circ \ldots \alpha_1 \circ \alpha_0(x_0)$. Since $m-2$ is odd, then $\alpha_{m-2} \circ \ldots \alpha_1 \circ \alpha_0$ is an order isomorphism. And thus $x_0 \leq y_0$. Then $f^m(y_0) \leq f^n(y_0)$ since n is odd. So $f(y) \leq f(x)$. Analogous, if $m-1$ is odd and n is even. This completes the proof of the first part of the lemma. Let $x \in I_i$, then $x = \mathbf{f}^i(x_0)$ with $x_0 \in I_0$. Therefore

$$\mathbf{f}^m(x) = \mathbf{f}^m(\mathbf{f}^i(x_0)) = \mathbf{f}^i(\mathbf{f}^m(x_0)) = \mathbf{f}^i(\mathbf{f}^n(x_0)) = \mathbf{f}^n(\mathbf{f}^i(x_0)) = \mathbf{f}^n(x).$$

□

By Remark 2.2 and the preceding Lemma we can extend $\mathbf{f}$ to a dual homomorphism

(10) $$\mathbf{F_{m,n}} : L(G) \longrightarrow L(G).$$

It is not difficult to prove the following Lemma.

Lemma 3.2 $\mathbf{F}^m_{\mathbf{m,n}}(x) = \mathbf{F}^n_{\mathbf{m,n}}(x)$, *for all* $x \in L(G)$.

As a consequence we have that $(L(G); \mathbf{F_{m,n}}) \in P_{m,n}$.

Theorem 3.3 $FO_{m,n}(I) \cong (L(G); \mathbf{F_{m,n}})$.

Proof Analogous to Theorem 2.6. □

The following Lemma provides some known properties of powers of sets, and can be found in [8].

Lemma 3.4 *Let A, B y C be posets. Then*
(i) $A^{[B+C]} \cong A^{[B]} \times A^{[C]}$.
(ii) $(A^{[B]})^{[C]} \cong A^{[B\times C]}$.
(iii) $(A \times B)^{[C]} \cong A^{[C]} \times B^{[C]}$.
(iv)

$$\prod_{i=1}^{n} C_i^* = (\prod_{i=1}^{n} C_i)^*,$$

with C_i posets, for all $i = 1, 2, \ldots, n-1$.

Corollary 3.5 *Let* **I** *be a* **finite** *poset. Then*

$$FO_{m,n}(I) \cong \mathbf{2}^{[\prod_{i=0}^{m-1} L_i\ ]},$$

where L_i is the distributive lattice with the set of its join irreducible elements isomorphic to I, if i is odd, and I^* if i is even.

Proof In [8] one can see that every finite distributive lattice L is isomorphic to $\mathbf{2}^{[J(L)^*]}$, where $J(L)$ is the set of join irreducible elements in L. On the other hand, the free distributive lattice over a poset I is isomorphic to $\mathbf{2}^{[\mathbf{2}^{[I]}]}$. Consequently, by Lemma 3.4, if we put $I = G$, we have the thesis. □

Corollary 3.6 *Let $FO_{m,n}(s)$ be the free Ockham algebra with s generators, s a finite positive cardinal number. Then*

$$FO_{m,n}(s) \cong L(m.s),$$

with $L(m.s)$ the free distributive lattice with $m.s$ generators.

Proof Immediate from Corollary 3.5 and Theorem 3.3. □

This Corollary was proved by M. Goldberg in [7] with different techniques.

4 Free $P_{m,n}$-Ockham algebras over a poset with $m-n$ odd

It is easy to see that $P_{m,n} \subseteq P_{2m-n,n}$. This fact will allow us to use some results of the preceding sections.

Lemma 4.1 *Let $FO_{m,n}(I)$ be the free $P_{m,n}$ Ockham algebra over a poset I. Then*

$$SL\left(\bigcup_{i=0}^{2m-n-1} f^i(I)\right) = FO_{m,n}(I)$$

Proof Analogous to Lemma 2.5. □

Consider now the sets O_1 and O_2 defined as follows:

(11) $$O_1 = \sum_{i=0}^{n-1} I_i,\ O_2 = \sum_{i=n}^{m-1} I_i,$$

where $I_i \cong I$, if i is even, and $I_i \cong I^*$, if i is odd.

Let $\mathbf{L}_1 = L(O_1)$ be the free distributive lattice over the poset O_1, and let $\mathbf{B}_1 = B(O_2)$ be the free Boolean algebra over the poset O_2. The construction of the free Boolean algebra $B(I)$ over a poset I is analogous to the construction of the free distributive lattice over a poset [9], and for the finite case we have that

(12) $$B(I) \cong \mathbf{2}^{|\mathbf{2}^{[I]}|},$$

i.e., all functions from $\mathbf{2}^{[I]}$ to $\mathbf{2}$. In the sequel we will adopt the "coproduct convention" developed in [1]. Let

(13) $$\mathbf{L} = \mathbf{L}_1 * \mathbf{B}_1,$$

where "$*$" is the coproduct in $\mathbf{D}_{01}$. Let

(14) $$f_1 : O_1 \longrightarrow \mathbf{L},$$

be a map defined as follows: if $x \in O_1$, then $x \in I_i$ for some $i < n$. Then we put $f_1(x) = \alpha_i(x)$. Let

(15) $$f_2 : O_2 \longrightarrow \mathbf{L},$$

be a map defined as follows: If $x \in O_2$ therefore $x \in I_i$ for some $n \leq i \leq m-1$. If $i < m-1$, then we put $f_2(x) = \alpha_i(x)$. If $x \in I_{m-1}$ then $x = \alpha_{m-2} \circ \ldots \circ \alpha_n(x_n)$ with $x_n \in I_n$. In this case we put $f_2(x) = \overline{x_n}$, where $\overline{x_n}$ is the Boolean complement of x_n in $\mathbf{B}_1$. It is not difficult to prove that f_1 and f_2 are order-reversing maps. Hence, by Remark 2.2, f_1 and f_2 can be extended to dual homomorphisms

(16) $$\mathrm{f}_1 : \mathbf{L}_1 \longrightarrow \mathbf{L}$$

and

(17) $$\mathrm{f}_2 : \mathbf{B}_1 \longrightarrow \mathbf{L}.$$

We note that f_2 **preserves complementation**. By Remark 2.2, f_1 and f_2 can be extended to a dual homomorphism

(18) $$\mathrm{f}_3 : \mathbf{L} \longrightarrow \mathbf{L}.$$

Note that $\mathrm{f}_3|\mathbf{B}_1 = \mathrm{f}_2$, then f_3 preserves complements in $\mathbf{B}_1$.

If B is a Boolean algebra, denote by $BS(X)$ the Boolean subalgebra generated by X in B.

Lemma 4.2 f_3 *verifies the equations* (4) *for all* $x \in \mathbf{L}$.

Proof Let $L_0 = \{O_1, O_2, \overline{O_2}\} \cup \{0,1\}$, where $\overline{O_2} = \{\overline{x} : x \in O_2\}$, and define by induction $L_s = \{x \wedge y, x \in L_i, y \in L_j, i+j = s-1\} \cup \{x \vee y, x \in L_i, y \in L_j, i+j = s-1\}$. Let $A = \bigcup_{s \geq 0} L_s$. Since $SL(O_2 \cup \overline{O_2}) = \mathbf{B}_1$ and $SL(O_1) = \mathbf{L}_1$, then $SL(O_1 \cup O_2 \cup \overline{O_2}) = SL(\mathbf{B}_1 \cup \mathbf{L}_1) = \mathbf{L}$. Therefore $A = \mathbf{L}$. We claim that the equations (4) are valid for all L_s, which will be readily seen by induction on s. Let $x \in L_0$. If $x \in \{0,1\}$, as m and n have different parity, then the equations (4) are trivially verified. Let $x \in O_1$. Then $x \in I_i$, for $0 \leq i \leq n-1$, and $x = \mathbf{f}_3^i(x_0)$, with $x_0 \in I_0$. Therefore

$$\mathbf{f}_3^n(x) \wedge \mathbf{f}_3^m(x) = \mathbf{f}_3^n(x) \wedge \mathbf{f}_3^m(\mathbf{f}_3^i(x_0)) = \mathbf{f}_3^n(x) \wedge \mathbf{f}_3^i(\mathbf{f}_3^{m-n}(\mathbf{f}_3^n(x_0)))) =$$

$$\mathbf{f}_3^n(x) \wedge \mathbf{f}_3^i(\mathbf{f}_3(\mathbf{f}_3^{m-n-1}(\mathbf{f}_3^n(x_0))))) = \mathbf{f}_3^n(x) \wedge \mathbf{f}_3^i(\overline{\mathbf{f}_3^n(x_0)}).$$

Since $\mathbf{f}_3|\mathbf{B}_1$ preserves complements, it follows that

$$\mathbf{f}_3^n(x) \wedge \mathbf{f}_3^i(\overline{\mathbf{f}_3^n(x_0)}) = \mathbf{f}_3^n(x) \wedge \overline{\mathbf{f}_3^i(\mathbf{f}_3^n(x_0))} = \mathbf{f}_3^n(x) \wedge \overline{\mathbf{f}_3^n(\mathbf{f}_3^i(x_0))} = \mathbf{f}_3^n(x) \wedge \overline{\mathbf{f}_3^n(x)} = 0.$$

For the other equation of (4) the proof is analogous. Let $x \in O_2$. Then $x \in I_i$, $n \leq i \leq m-1$, and $x = \mathbf{f}_3^{i-n}(x_n)$, $x_n \in \overline{I_n}$. Therefore

$$\mathbf{f}_3^n(x) \wedge \mathbf{f}_3^m(x) = \mathbf{f}_3^n(x) \wedge \mathbf{f}_3^m(\mathbf{f}_3^{i-n}(x_n)) = \mathbf{f}_3^n(x) \wedge \mathbf{f}_3^i(\mathbf{f}_3(\mathbf{f}_3^{m-1-n}(x_n))) = \mathbf{f}_3^n(x) \wedge \mathbf{f}_3^i(\overline{x_n}),$$

and since $\mathbf{f}_3|\mathbf{B}_1$ preserves complements, then we have

$$\mathbf{f}_3^n(x) \wedge \mathbf{f}_3^i(\overline{x_n}) = \mathbf{f}_3^n(x) \wedge \overline{\mathbf{f}_3^i(x_n)} = \mathbf{f}_3^n(x) \wedge \overline{\mathbf{f}_3^n(\mathbf{f}_3^{i-n}(x_n))} = \mathbf{f}_3^n(x) \wedge \overline{\mathbf{f}_3^n(x)} = 0.$$

The proof is analogous for the other equation of (4). If $x \in \overline{O_2}$ the proof is analogous to the preceding case. Hence the equations of (4) are verified in L_0. Suppose that the equations of (4) are verified in L_k, with $k < s$. Let $z \in L_s$. Then we have two posibilities:
1. $z = x \wedge y$, with $x \in L_i, y \in L_j, i+j = s-1$. By inductive hipotesis the equations of (4) hold for x and y. Then if we suppose that n is even (m odd), we have

$$\mathbf{f}_3^n(z) \wedge \mathbf{f}_3^m(z) = \mathbf{f}_3^n(x \wedge y) \wedge \mathbf{f}_3^m(x \wedge y) = \mathbf{f}_3^n(x) \wedge \mathbf{f}_3^n(y) \wedge (\mathbf{f}_3^m(x) \vee \mathbf{f}_3^m(y)) =$$

$$\mathbf{f}_3^n(x) \wedge \mathbf{f}_3^n(y) \wedge (\overline{\mathbf{f}_3^n(x)} \vee \overline{\mathbf{f}_3^n(y)}) = \mathbf{f}_3^n(x) \wedge \mathbf{f}_3^n(y) \wedge (\overline{\mathbf{f}_3^n(x) \wedge \mathbf{f}_3^n(y)}) = 0.$$

The proof is analogous in the case that n is odd.
2. $z = x \vee y$, with $x \in L_i, y \in L_j, i+j = s-1$. Analogous to item 1. □

By the preceding Lemma we can conclude that $(\mathbf{L}; \mathbf{f}_3) \in P_{m,n}$.

Lemma 4.3 *I_0 is a generating set of* $(\mathbf{L}; \mathbf{f}_3)$ *as Ockham algebra.*

Proof Let $x \in O_1 \cup O_2$. Then $x = \mathbf{f}_3^i(x_0)$ with $x_0 \in I_0$. Therefore $O_1 \cup O_2 \subseteq OS(I_0)$. Let $x \in \overline{O_2}$. Then $x = \overline{\mathbf{f}_3^k(x_n)}$, with $0 \leq k \leq m-n-1$ and $x_n \in I_n$. Moreover $x_n = \mathbf{f}_3^n(x_0)$, with $x_0 \in I_0$. Since $\mathbf{f}_3|\mathbf{B}_1$ preserves complements, it follows that

$$x = \overline{\mathbf{f}_3^k(x_n)} = \mathbf{f}_3^k(\overline{x_n}) = \mathbf{f}_3^k(\overline{\mathbf{f}_3^n(x_0)}) = \mathbf{f}_3^k(\mathbf{f}_3^m(x_0)).$$

Consequently $\overline{O_2} \subseteq OS(I_0)$. Therefore $\mathbf{L} = SL(O_1 \cup O_2 \cup \overline{O_2}) \subseteq OS(I_0)$. Hence $\mathbf{L} = OS(I_0)$. □

Theorem 4.4

$$(\mathbf{L};\mathbf{f}_3) \cong FO_{m,n}(I)$$

Proof Let $\alpha : I_0 \to I$, be the identity isomorphism. Let f be the dual homomorphism corresponding to $FO_{m,n}(I)$ and let

$$h_1 : O_1 \to \bigcup_{i=0}^{n-1} f^i(I) \tag{19}$$

be defined as follows: Let $x \in I_i$, $x = \mathbf{f}_3^i(x_0)$ with $x_0 \in I_0$. Then we put $h_1(x) = f^i(\alpha(x_0))$. h_1 can be extended to a homomorphism of distributive lattices

$$H_1 : \mathbf{L}_1 \longrightarrow FO_{m,n}(I). \tag{20}$$

Let

$$h_2 : O_2 \to \bigcup_{i=n}^{m-1} f^i(I) \tag{21}$$

be defined as follows: If $x \in I_i$, then $x = \mathbf{f}_3^i(x_0)$ with $x_0 \in I_0$. Therefore we put $h_2(x) = f^i(\alpha(x_0))$. Then h_2 can be extended to a homomorphism of Boolean algebras

$$H_2 : \mathbf{B}_1 \longrightarrow B(FO_{m,n}(I)) \subseteq FO_{m,n}(I). \tag{22}$$

Hence H_1, H_2 can be extended to a homomorphism of distributive lattices

$$H : \mathbf{L} \longrightarrow FO_{m,n}(I). \tag{23}$$

We claim that H is onto. In fact, it is easy to see that

$$\bigcup_{i=1}^{m-1} f^i(I) \subseteq H(\mathbf{L}).$$

By the definition of h_2, we also have that

$$\bigcup_{i=n}^{m-1} \overline{f^i(I)} \subseteq h_2(\mathbf{B}_1),$$

moreover

$$\bigcup_{i=m}^{2m-n-1} f^i(I) = \bigcup_{i=n}^{m-1} \overline{f^i(I)}.$$

Hence

$$\bigcup_{i=0}^{2m-n-1} f^i(I) \subseteq H(\mathbf{L}).$$

Consequently

$$FO_{m,n} = SL\left(\bigcup_{i=0}^{2m-n-1} f^i(I)\right) \subseteq H(\mathbf{L}).$$

Let $k : I \to I_0$ be the identity. k can be extended to an Ockham homomorphism

$$K : FO_{m,n}(I) \longrightarrow \mathbf{L}.$$

By lemma 4.3, K is an epimorphism, since $OS(I_0) = \mathbf{L}$. As in Theorem 2.6, it can be proved that $H = K^{-1}$. Hence K is an isomorphism. □

Theorem 4.5 *Let I be a* **finite** *poset. Then*

$$FO_{m,n} \cong \prod_{i=1}^{N} \mathbf{2}^{[\mathbf{2}^{[M]}]},$$

with $N = \left|\mathbf{2}^{[\sum_{i=n}^{m-1} I_i]}\right|$, *and* $M = \sum_{i=o}^{n-1} I_i$.

Proof By the preceding theorem

$$\mathbf{B}_1 \cong B\left(\sum_{i=n}^{m-1} I_i\right)$$

and

$$\mathbf{L}_1 \cong L\left(\sum_{i=0}^{n-1} I_i\right) \cong \mathbf{2}^{\left[\mathbf{2}^{[\sum_{i=0}^{n-1} I_i]}\right]},$$

therefore by [8]

$$\mathbf{L} = \mathbf{B}_1 * \mathbf{L}_1 = (\mathbf{L}_1)^{[J(\mathbf{B}_1)]}.$$

Since $J(\mathbf{B}_1)$ is an antichain and

$$|J(\mathbf{B}_1)| = \left|\mathbf{2}^{[\sum_{i=n}^{m-1} I_i]}\right|,$$

the stated is clear. □

Corollary 4.6 *Let $FO_{m,n}(r)$ be the free Ockham algebra with r generators, r a finite positive cardinal number. Then*

$$FO_{m,n}(r) \cong \prod_{i=1}^{2^{(m-n).r}} L(r.n).$$

The preceding Corollary was proved by M. Goldberg in [7] for the particular case $n = m - 1$.

5 A Construction of free MS_n–Algebras over a poset

In this section we deal with MS_n-algebras.

Given an MS_n-algebra D and a subset X in D, $S(X)$ denote the MS_n-subalgebra generated by X. The definition of free MS_n–algebra $F_n(I)$ over a poset I, is analogous to Definition 1.1.

Let I be a poset and I^* its order dual. Let

(24) $$G = \overline{I_0} + \sum_{i=1}^{2n-1} I_i,$$

where $I_i \cong I$ if i is even, $I_i \cong I^*$ if i is odd, $0 \leq i \leq 2n-1$, $\overline{I_0} \cong I_0 \times \mathbf{2}$ with $\mathbf{2}$ the 2-element chain, and $\sum$ is the cardinal sum of posets. Let $\alpha_i : I_i \to I_{i+1}$ be the natural anti-isomorphism, and $p : \overline{I_0} \to I_0$, the projection $p((g_0, u)) = g_0$, $g_0 \in I_0$, $u \in \mathbf{2}$. Let

$$\mathbf{f} : G \longrightarrow G \tag{25}$$

be the map defined as follows: If $g \in \overline{I_0}$ then $g = (g_0, u)$, then $\mathbf{f}(g) = \alpha_0(p(g)) = \alpha_0(g_0)$. If $g \in I_i$, $1 \leq i \leq 2n-2$, $\mathbf{f}(g) = \alpha_i(g)$. If $g \in I_{2n-1}$ then $g = \alpha_{2n-2}(\ldots \alpha_1(\alpha_0(p((g_0,u))))\ldots)$, then we put $\mathbf{f}(g) = (g_0, 1)$.

It is clear that, for $x \in \overline{I_0}$, $\mathbf{f}^i(x) \in I_i$, $1 \leq i \leq 2n-1$ and $\mathbf{f}^{2n}(x) \geq x$. In addition, for $y \in I_i$, there exists $(y_0, 0) \in \overline{I_0} \times 0$ such that $y = \mathbf{f}^i((y_0,0))$. If $y = (y_0, 1) \in I_0 \times 1$, then $y = \mathbf{f}((y_0, 0))$.

Lemma 5.1 $\mathbf{f}$ *is an order-reversing map.*

Proof Let $x, y \in G$, $x \leq y$. It is clear that if x and y are comparable then they lie in the same I_i $(i \geq 1)$ or $x, y \in \overline{I_0}$. Then we have three cases:

1. If $1 \leq i \leq 2n-2$ then $\mathbf{f}(x) = \alpha_i(x) \geq \alpha_i(y) = \mathbf{f}(y)$.
2. If $i = 0$ then $x = (x_1, x_2) \leq y = (y_1, y_2)$. Hence $\mathbf{f}(x) = \alpha_0(x_0) \geq \alpha_0(y_0) = \mathbf{f}(y)$.
3. If $i = 2n-1$ then $x = \alpha_{2n-2} \circ \ldots \alpha_1 \circ \alpha_0(x_0, 0)$ and $y = \alpha_{2n-2} \circ \ldots \alpha_1 \circ \alpha_0(y_0, 0)$. Since α_i is an order-reversing map, then $(x_0, 0) \geq (y_0, 0)$. Hence $f(x) = (x_0, 1) \geq (y_0, 1) = f(y)$.

□

Lemma 5.2 $f^{2n}(g) \geq g$ *for all* $g \in G$.

Proof Let $g \in \overline{I_0}$. Then by the definition of f we have that $f^{2n}(g) \geq g$. Let $g \in I_i$, with $i \geq 1$. Then $g = f^i((x,0))$ with $(x, 0) \in \overline{I_0}$. Then

$$f^{2n}(x) = f^{2n} f^i((x,0)) = f^i f^{2n}((x,0)) = f^i((x,1)) = f^{i-1} f((x,1)) =$$

$$= f^i \alpha_0 \circ p((x,1)) = f^i \alpha_0 \circ p((x,0)) = f^i((x,0)) = x.$$

□

By 2.2 and 5.1, $\mathbf{f}$ can be extended to a dual homomorphism $\mathbf{F} : L(G) \to L(G)$. Then $(L(G); \mathbf{F})$ is an Ockham algebra.

Lemma 5.3 $\mathbf{F}^{2n}(x) \geq x$ *for all* $x \in (L(G); \mathbf{F})$.

Proof This follows immediately on noting that the set $\{x \in L(G) : \mathbf{F}^{2n}(x) \geq x\}$ is a sublattice of $L(G)$ containing G. So $\mathbf{F}^{2n}(x) \geq x$ for all $x \in L(G)$. □

By the preceding Lemma, $(L(G); \mathbf{F})$ is an MS_n-algebra.

The following property of $(L(G); \mathbf{F})$ will be used to establish an isomorphism between $(L(G); \mathbf{F})$ and $F_n(I)$.

Lemma 5.4 $(L(G); \mathbf{F}) = S(I_0 \times \{0\})$.

Proof Let $x \in I_i$ and $i \geq 1$, then $x = \mathbf{f}^i((x,0))$. If $(x,1) \in I_0 \times \{1\}$ then $\mathbf{F}^{2n}((x,0)) = (x,1)$ with $(x,0) \in I_0 \times \{0\}$. Consequently $x \in S(I_0 \times \{0\}))$ whenever $x \in G$. Since $L(G)$ is free over G, then $(L(G);\mathbf{F}) = SL(G) \subseteq S(I_0 \times \{0\}))$. Hence $(L(G);\mathbf{F}) = S(I_0 \times \{0\}))$. □

Let f be the dual homomorphism corresponding to $F_n(I)$, the free MS_n-algebra over I.

Lemma 5.5 *Let $Q = \bigcup_{i \geq 0} f^i(I)$. Then $SL(Q) = F_n(I)$.*

Proof The set $\{x \in SL(Q) : f(x) \in SL(Q)\}$ is a sublattice of $SL(Q)$ containing Q. So $SL(Q)$ is closed under f, and therefore, $SL(Q)$ is an MS_n-subalgebra of $F_n(I)$ containing I. Thus $SL(Q) = F_n(I)$. □

Theorem 5.6 $F_n(I) \cong (L(G);\mathbf{F})$.

Proof Let $h : I \longrightarrow I_0 \times \{0\} \subseteq (L(G);\mathbf{F})$ be the natural order-isomorphism. Then h can be extended to a homomorphism of MS_n algebras $H : FO(I) \longrightarrow (L(G);\mathbf{F})$. By 5.4, $H(F_n(I)) = H(S(I)) = S(H(I)) = S(I_0 \times \{0\}) = (L(G);\mathbf{F})$. Hence H is an epimorphism.

We consider now

$$k : G \longrightarrow Q,$$

defined as follows: If $x \in I_0 \times \{0\}$, we put $k(x) = h^{-1}(x)$. If $x \in I_i$, $1 \leq i \leq 2n-1$, then $x = \mathbf{F}^i((x_0,0))$, with $x_0 \in I_0$. Then $k(x) = f^i(h^{-1}(x_0))$. If $x \in I_0 \times \{1\}$, then $x = \mathbf{F}^{2n}((x_0,0))$. In this case $k(x) = f^i(h^{-1}(x_0))$. k is order-preserving. Indeed, for x, $y \in G$, suppose $x \leq y$. Let us consider the case $x = (x_0,0)$, $y = (y_0,1) \in I_0 \times \{1\}$, $x_0 \leq y_0$, the other cases being easy to check. Thus, $k(x) = h^{-1}((x_0,0))$, $k(y) = f^{2n}(h^{-1}((y_0,0)))$. From $(x_0,0) \leq (y_0,0)$ we have $h^{-1}((x_0,0)) \leq h^{-1}((y_0,0))$. and since f^{2n} is a closure operator, $h^{-1}((y_0,0)) \leq f^{2n}(h^{-1}((y_0,0)))$. Therefore $k(x) \leq k(y)$. Let $K : (L(G),\mathbf{F}) \longrightarrow FO(I)$ be the extension of k. It is easy to check that $K.H$ is the identity over I, and then $H.K$ is the identity map. Therefore H is an isomorphism. □

Corollary 5.7 *Let I be a* **finite** *poset. Then*

$$F_n(I) \cong \mathbf{2}^{\left[L_0^{[2]} \times \prod_{i=1}^{2n-1} L_i\right]},$$

where L_i is the distributive lattice with the set of its join irreducible elements isomorphic to I, if i is even, and I^* if i is odd.

Proof It is known that every finite distributive lattice L is isomorphic to $\mathbf{2}^{[J(L)^*]}$, where $J(L)$ is the set of join irreducible elements in L, and that the free distributive lattice over a poset I is isomorphic to $\mathbf{2}^{[\mathbf{2}^{[I]}]}$. Consequently, if we put $I = G$, by Lemma 3.4

$$F_n(I) \cong \mathbf{2}^{\left[\mathbf{2}^{\left[\overline{I_0} + \sum_{i=1}^{2n-1} I_i\right]}\right]} \cong \mathbf{2}^{\left[\mathbf{2}^{[I_0 \times \mathbf{2}]} \times \mathbf{2}^{\left[\sum_{i=1}^{2n-1} I_i\right]}\right]} \cong \mathbf{2}^{\left[L_0^{[2]} \times \prod_{i=1}^{2n-1} L_i\right]}.$$

□

Let r be a finite positive cardinal number, and $F_n(r)$ the free MS_n-algebra with r generators.

Corollary 5.8

$$F_n(r) \cong \mathbf{2}^{[\mathbf{3}^r \times \mathbf{2}^{r(2n-1)}]}.$$

Proof Here, the poset of free generators is an antichain; then, by Corollary 5.7 and Lemma 3.4, the result is immediate. □

Observe that

$$\Pi(F_n(r)) \cong \mathbf{3}^r \times \mathbf{2}^{r(2n-1)},$$

where $\Pi(F_n(r))$ is the set of join irreducible elements of $F_n(r)$.

Corollary 5.9 *Let $\mathcal{F}(r)$ the free MS-algebra with r generators. Then*

$$\mathcal{F}(r) \cong \mathbf{2}^{[\mathbf{3}^r \times \mathbf{2}^r]}$$

Proof Immediate from Corollary 5.8. □

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