De Morgan Algebras with an additional operation

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Abstract

De Morgan E–algebras $\langle A, \vee, \wedge, \sim, h, 0, 1\rangle$ of type $(2, 2, 1, 1, 0, 0)$ such that $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra and h is a De Morgan endomorphism are defined. The dual category and the lattice of congruences is given.

Finally, the class of k–cyclic De Morgan E–algebras are considered and a method for obtaining the free algebra over an ordered set are determined. [1]

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1 Preliminaries

Throughout this paper **L** denotes the variety of $(0, 1)$–distributive lattices. The category of L–algebras and L–homomorphisms will be denoted by $\mathcal{L}$.

General references for concepts and results on distributive lattices used in this paper are the books [1] and [2].

Recall that a totally order–disconnected topological space is a triple $(Y, \tau, \leq)$ such that $(Y, \leq)$ is a poset, (Y, τ) is a topological space and given x, y in Y, with $x \not\leq y$, there exists a clopen (i.e. closed and open) increasing set V such that $x \in V$

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and $y \notin V$.

A *Priestley space* (or P–space) is a compact totally order–disconnected topological space.

We shall denote by $\mathbf{P}$ the class of P–spaces and by $\boldsymbol{\mathcal{P}}$ the category whose objects are the elements of $\mathbf{P}$ and whose morphisms are the order–preserving continuous mappings (or P– mappings).

As usual, we are going to denote the objects in $\boldsymbol{\mathcal{P}}$ by its underlying set Y.

For each $A \in \mathbf{L}$ we represent by $\boldsymbol{X}(A)$ the set of prime filters of A. Ordering $\boldsymbol{X}(A)$ by inclusion and with the topology τ having as a subbasis the sets of the form

(i) $\quad \sigma_A(a) = \{P \in \boldsymbol{X}(A) : a \in P\}$ and $\boldsymbol{X}(A) \setminus \sigma_A(a)$, for each $a \in A$,

we have that $P_r(A) = (\boldsymbol{X}(A), \mathcal{T}, \subseteq) \in \mathbf{P}$.

If $Y \in \mathbf{P}$ and $\boldsymbol{D}(Y)$ is the set of increasing and τ–clopen subsets of Y, then $\mathcal{L}(Y) = \langle \boldsymbol{D}(Y), \cap, \cup, \emptyset, Y, \rangle \in \mathbf{L}$.

In [9,10] it is proved that the category $\boldsymbol{\mathcal{L}}$ is dually equivalent to $\boldsymbol{\mathcal{P}}$.

A *De Morgan algebra* (or M–algebra) is a pair $(A, \sim)$, where $A \in \mathbf{L}$ and $\sim$ is a unary operation satisfying the following identities:

$$\sim\sim x = x, \quad \sim (x \vee y) = \sim x \wedge \sim y.$$

We shall denote by $\mathbf{M}$ the variety of M–algebras and by $\boldsymbol{\mathcal{M}}$ the category of M–algebras and M–homomorphisms, i.e. De Morgan homomorphisms.

A *De Morgan space* (or mP–space) is a pair (Y, g) such that Y is an object in $\boldsymbol{\mathcal{P}}$ and $g : Y \to Y$ is a decreasing and continuous mapping satisfying $g = g^{-1}$.

Let (Y, g) and (Y', g') be mP–spaces. An mP–*mapping* from (Y, g) into (Y', g') is a P–mapping $f : Y \to Y'$ such that $f \circ g = g' \circ f$.

We shall denote by $\mathbf{mP}$ the class of mP–spaces and by $\boldsymbol{m\mathcal{P}}$ the category of mP–spaces and mP–mappings.

If $(A, \sim) \in \mathbf{M}$ and $g_A : \boldsymbol{X}(A) \to \boldsymbol{X}(A)$ is defined by

(ii) $\quad g_A(P) = \boldsymbol{X}(A) \setminus \{\sim x : x \in P\}$, for every $P \in \boldsymbol{X}(A)$,

then $P_m(A) = (P_r(A), g_A)$ is an mP–space.

If $(Y, g) \in \mathbf{mP}$ and $\sim : \boldsymbol{D}(Y) \to \boldsymbol{D}(Y)$ is defined by

(iii) $\quad \sim (V) = Y \setminus g^{-1}(V)$, for every $V \in \boldsymbol{D}(Y)$, then $\mathcal{M}(Y) = (\mathcal{L}(Y), \sim) \in \mathbf{M}$.

In [4] it is proved that the category $\mathcal{M}$ is dually equivalent to $\boldsymbol{m\mathcal{P}}$.

Furthemore, if $A \in \mathbf{L}$, it is well known that the set $Con_{\mathbf{L}}(A)$ of all congruences of A is determined by the closed subsets of $\boldsymbol{X}(A)$. More precisely, if $\mathcal{C}(\boldsymbol{X}(A))$ is the set of the closed subsets of $\boldsymbol{X}(A)$, then the mapping $\Phi : \mathcal{C}(\boldsymbol{X}(A)) \rightarrow Con_{\mathbf{M}}(A)$ defined by

(iv) $\quad \Phi(Y) = \{(a,b) \in A \times A : \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\}$, for every $Y \in \mathcal{C}(\boldsymbol{X}(A))$,

is an L–antiisomorphism

On the other hand, if $A \in \mathbf{M}$ then the set $Con_{\mathbf{M}}(A)$ of all congruences of A is determined by the involutive closed subsets of $\boldsymbol{X}(A)$. Recall that $Y \subseteq \boldsymbol{X}(A)$ is said to be *involutive* if $g_A(Y) = Y$. More precisely, if $C_I(\boldsymbol{X}(A))$ is the set of involutive closed subsets of $\boldsymbol{X}(A)$, then the mapping $\Phi : C_I(\boldsymbol{X}(A)) \rightarrow Con_{\mathbf{M}}(A)$ indicated in (iv) is an L–antiisomorphism.

An E–*lattice* (see [5]) is a pair (A, h) where $A \in \mathbf{L}$ and h is an L–endomorphism.

We denote by $\mathbf{E}$ the variety of E–lattices and by $\mathcal{E}$ the category of E–lattices and E–homomorphisms, i.e. E–lattice homomorphisms.

An eP–*space* is a pair (Y, α) such that Y is an object in $\mathcal{P}$ and α is a morphism of $\mathcal{P}$.

Let (Y, α) and (Y', α') be eP–spaces. An eP–*mapping* from (Y, α) into (Y', α') is a P–mapping $f : Y \rightarrow Y'$ such that $f \circ \alpha = \alpha' \circ f$.

We shall denote by $\boldsymbol{e\mathcal{P}}$ the category of eP–spaces and eP–mappings.

For each $(A, h) \in \mathbf{E}$ we consider the mapping $\alpha_A : \boldsymbol{X}(A) \rightarrow \boldsymbol{X}(A)$ defined by

(v) $\quad \alpha_A(P) = h^{-1}(P)$, for every $P \in \boldsymbol{X}(A)$,

then $P_e(A) = (P_r(A), \alpha_A)$ is an eP–space.

On the other hand, if (Y, α) is an eP–space and $h : \boldsymbol{D}(Y) \rightarrow \boldsymbol{D}(Y)$ is the mapping defined by

(vi) $\quad h(V) = \alpha^{-1}(V)$, for every $V \in \boldsymbol{D}(Y)$,

then $\mathcal{H}(Y) = (\mathcal{L}(Y), h) \in \mathbf{E}$.

The category $\mathcal{E}$ is dually equivalent to $\boldsymbol{e\mathcal{P}}$([5]).

Furthermore in [5] it is proved that $\theta \in Con_{\mathbf{E}}(A)$ if and only if there exists $Y \subseteq \boldsymbol{X}(A)$, such that it verifies the following properties:

$$P \in Y \text{ implies } h^{-1}(P) \in Y, \quad \theta = \Theta(Y).$$

2 De Morgan E–algebras

Definition 2.1 *A De Morgan* E*–algebra (or* ME*–algebra) is a triple* $(A, h, \sim)$ *where* (A, h) *is an E–algebra,* $(A, \sim)$ *is an* M*–algebra and* h *is an* M*–homomorphism.*

We denote by **ME** the variety of ME–algebras.

Lemma 2.1 *Let* $(A, \sim) \in \mathbf{M}$ *and* $h : A \to A$ *be a map. If it holds that* $h^{-1}(P) \in \boldsymbol{X}(A)$ *and* $g_A(h^{-1}(P)) = h^{-1}(g_A(P))$ *for all* $P \in \boldsymbol{X}(A)$*, then* $(A, h, \sim) \in \mathbf{ME}$.

From the results of Cornish and Fowler [4] which generalized the Priestley duality for De Morgan algebras and the results of Figallo and Monteiro [5], we have the Theorems 2.1 and 2.2 indicated below.

Definition 2.2 *An* meP*–space is a triple* (Y, α, g) *such that* (Y, α) *is an* eP*–space,* (Y, g) *is an* mP*–space and it holds* $g \circ \alpha = \alpha \circ g$.

Definition 2.3 *Let* (Y, α, g) *and* (Y', α', g') *be* meP*–spaces. An* meP*–mapping from* (Y, α, g) *into* (Y', α', g') *is an* eP*–mapping and* mP*–mapping.*

Lemma 2.2 *If* (Y, α, g) *be an* meP*–space then* $\mathcal{H}(Y) = (\mathcal{L}(Y), h, \sim) \in \mathbf{ME}$*, where* h *and* $\sim$ *are defined as in* (vi) *and* (iii) *respectively.*

Lemma 2.3 *If* $(A, h, \sim) \in \mathbf{ME}$ *then* $P_{me}(A) = (P_r(A), \alpha_A, g_A) \in \mathbf{meP}$*, where* α_A *and* g_A *are defined as in* (v) *and* (ii) *respectively.*

We denote by $\boldsymbol{\mathcal{ME}}$ and $\boldsymbol{me\mathcal{P}}$ the categories of ME–algebras with ME–homomorphisms, and the meP–spaces with meP–mappings, respectively.

Theorem 2.1 *The category* $\boldsymbol{\mathcal{ME}}$ *is dually equivalent to the category* $\boldsymbol{me\mathcal{P}}$.

Now we are going to determine the set $Con_{\mathbf{ME}}(A)$ of all congruences of $A \in \mathbf{ME}$.

Lemma 2.4 *Let* $A \in \mathbf{ME}$*,* $Y \subseteq \boldsymbol{X}(A)$ *and* $\Theta(Y)$ *be the relation defined in* (vi)*. Then the following conditions are equivalent:*

(a) $\theta \in Con_{\mathbf{ME}}(A)$,

(b) *there exists* $Y \subseteq \boldsymbol{X}(A)$ *such that:*

(1) $\theta = \Theta(Y)$,

(2) $P \in Y$ *implies* $h^{-1}(P) \in Y$,

(3) $P \in Y$ *implies* $g(P) \in Y$.

Theorem 2.2 *Let* $(A, h) \in \mathbf{ME}$*. Then the lattice* $Con_{\mathbf{ME}}(A)$ *is isomorphic to the dual lattice of involutives* meP*–sets of* $\boldsymbol{X}(A)$.

3 k–cyclic ME–algebras

In this section we study the class of k–cyclic ME–algebras and we give a method to determine the free k–cyclic ME–algebra over a poset which is a generalization of those obtained in [7],[8] and [5].

Let k be a fixed positive integer. We say that (A, h) is a *k–cyclic* E–*algebra* (or E_k–algebra) if it verifies for all $x \in A$, $h^k(x) = x$ ([5]).

Observe that $h^0(x) = x$ and $h^{n+1}(x) = h^n(h(x))$ for all positive integer n.

We shall denote by $\mathbf{E}_k$ the variety of E_k–algebras.

Definition 3.1 $(A, h, \sim) \in \mathbf{ME}$ *is said to be a k–cyclic* ME–*algebra (or* ME_k*–algebra) if* $(A, h) \in \mathbf{E_k}$.

We shall denote by $\mathbf{ME}_k$ the variety of ME_k–algebras.

In what follows if $\mathbf{K}$ is one of the varieties $\mathbf{E_k}$ or $\mathbf{ME_k}$, $A \in \mathbf{K}$ and $Y \subseteq A$, we shall denote by $[Y]_{\mathbf{K}}$ the K–subalgebra of A generated by Y.

Definition 3.2 *Let I be a poset. $L \in \mathbf{K}$ is free over I if the following conditions are satisfied:*

(L1) *There exists an order–isomorphism $f : I \to L$ such that $[f(I)]_{\mathbf{K}} = L$.*

(L2) *If $A \in \mathbf{K}$ and $s : I \to A$ is an increasing mapping, then there exists a K–homomorphism $h : L \to A$ which verifies $h \circ f = s$.*

Construction of L

Let $(I, \leq)$ be a poset. For each non negative integer j let $I_j = \{(x, j) : x \in I\}$. We define an order relation $\preceq$ over I_j in the following way: $(x, j) \preceq (y, j)$ if and only if $x \leq y$.

For each j, $0 \leq j \leq k-1$ let I_j^* be the dual poset of I_j.

Now we consider the set $R = T + T^*$ (cardinal sum,[1]), where $T = I_0 + I_1 + \ldots + I_{k-1}$ and $T^* = I_0^* + I_1^* + \ldots + I_{k-1}^*$. For convenience, in some cases we denote by $\{t^* : t \in T\}$ the set T^*.

Let $\lambda : T \to T^*$, $\beta : R \to R$ be the mappings defined by

$$\lambda(t) = t^*, \ \beta(r) = \begin{cases} \lambda(r), & \text{if } r \in T, \\ \lambda^{-1}(r), & \text{otherwise} \end{cases}$$

Let $B = \{0,1\}$ be the Boolean algebra with $0 < 1$ and $\mathcal{I} = \mathcal{I}(R,B)$ the set of all order preserving functions from R into B. We define on $\mathcal{I}$ the operations $\wedge$, $\vee$ as usual and $\sim$ by means of the formula $(\sim x)(r) = -x(\beta(r))$ for all $x \in \mathcal{I}$ and $r \in R$. Then $(\mathcal{I}, \wedge, \vee, \sim, 0, 1) \in \mathbf{M}$.

Let $\eta_j : I_j \to I_{j+1}$, $\eta_j^* : I_j^* \to I_{j+1}^*$, for all $j \in \{0, \dots, k-2\}$, and $\eta_{k-1} : I_{k-1} \to I_0$, $\eta_{k-1}^* : I_{k-1}^* \to I_0^*$ be the mappings defined by

$$\eta_j((x,j)) = (x, j+1), \ \eta_j^*((x^*,j)) = (x^*, j+1),$$

and

$$\eta_{k-1}(x, k-1) = (x, 0), \ \eta_{k-1}^*(x^*, k-1) = (x^*, 0),$$

If we consider $h : \mathcal{I} \to \mathcal{I}$ defined by

$$(h(x))(r) = x(\mu(r)),$$

where

$$\mu(r) = \begin{cases} \eta_j(r), & \text{if } r \in I_j \\ \eta_j^*(r), & \text{if } r \in I_j^* \end{cases},$$

then it is easy to see that $(\mathcal{I}, h, \sim) \in \mathbf{ME_k}$ and $(\mathcal{P}(\mathcal{I}), H, N) \in \mathbf{ME_k}$, where $\mathcal{P}(\mathcal{I})$ is the set of all subsets of $\mathcal{I}$, $H(Y) = h^{-1}(Y)$ and $N(Y) = \mathcal{I} \setminus \{\sim y : y \in Y\}$, for all $Y \subseteq \mathcal{I}$.

If $\gamma : R \to \mathcal{P}(\mathcal{I})$ is the mapping defined by

$$\gamma(r) = G_r = \{x \in \mathcal{I} : x(\beta^{-1}(r)) = 1\}, \text{ for all } r \in R,$$

and $\mathcal{L} = [\gamma(I_0)]_{\mathbf{ME}_k}$, then $\mathcal{L}$ is the free ME_k–algebra over I_0 (and so it is also free over I).

Indeed, it is easy to see that

(11) $\gamma : I_0 \to \mathcal{L}$ is an order–isomorphism.

On the other hand

(12) $H(G_r) = G_{\mu(r)}$. (see [5])

Now we shall prove that

(13) $N(G_r) = G_{\beta(r)}$.

Indeed, the following conditions are pairwise equivalent:

(1) $x \in N(G_r)$,

(2) $\sim x \notin G_r$,

(3) $(\sim x)(\beta^{-1}(r)) = 0,$

(4) $-x(\beta(\beta^{-1}(r))) = 0,$

(5) $x(\beta^{-1}(\beta(r))) = 1,$

(6) $x \in G_{\beta(r)}.$

Then we have that $N(\gamma(I_0)) = \gamma(\beta(I_0)) = \gamma(I_0^*)$, and so it holds

(14) $\pounds = [\gamma(I_0)]_{\mathbf{ME}_k} = [\gamma(I_0) \cup \gamma(\beta(I_0))]_{\mathbf{E}_k} = [\gamma(T_0)]_{\mathbf{E}_k}$, where $T_0 = I_0 + I_o^*$.

Let $(A, h_A, \sim_A)$ be an arbitrary ME_k–algebra and $s : I_0 \to A$ be an increasing mapping.

Then we define the function $S : T \to A$ by

$$S(r) = h_A^j(s(\mu^{-j}(r))), \text{ for each } r \in I_j, 0 \leq j \leq k-1.$$

Let us consider the function $S' : R \to A$ defined by

$$S'(r) = \begin{cases} S(r), & \text{if } r \in T \\ \sim S(\beta(r)), & \text{otherwise} \end{cases}$$

It is clear that $S'_{|T} = S$ and $S'_{|I_0} = s$.

Since S is an isotone function, by (14) $\pounds$ is the free distributive lattice over T (see [7]), then it holds:

(15) there exists an L–homomorphism $\overline{h} : \pounds \to A$ such that $\overline{h}(\gamma(T)) = S(T)$.

Furthermore $\overline{h} \circ \gamma = S'$ is verified and on the other hand it holds:

(16) $\overline{h}$ is an M–homomorphism.

Indeed, it is holds:

(1) $\overline{h}(N(G_r)) = \overline{h}(G_{\beta(r)}) = \overline{h}(\gamma(\beta(r))) = (\overline{h} \circ \gamma)(\beta(r)) = S'(\beta(r))$

(2) $\sim \overline{h}(G_r) = \sim \overline{h}(\gamma(r)) = \sim (\overline{h} \circ \gamma)(r) = \sim S'(r),$

(3) $S'(\beta(r)) = \sim S'(r)$, because it holds:

(a) if $r \in T$, then $\beta(r) \in T^*$ and

$$S'(\beta(r)) = \sim S(\beta(\beta(r))) = \sim S(r) = \sim S'(r),$$

(b) if $r \in T^*$ then $\beta(r) \in S$ and

$$S'(\beta(r)) = S(\beta(r)) = \sim S(r) = \sim S'(r).$$

Finally, we have

(17) $\overline{h}$ is an ME–homomorphism

Indeed, it is holds

(1) $\overline{h}(H(G_r)) = \overline{h}(G_{\mu(r)})) = \overline{h}(\gamma(\mu(r))) = (\overline{h} \circ \gamma)(\mu(r)) = S'(\mu(r)),$

(2) $h_A(\overline{h}(G_r)) = h_A((\overline{h} \circ \gamma)(r)) = h_A(S'(r)),$

(3) $h_A(S'(r)) = S'(\mu(r))$, because is holds:

(a) if $r \in I_j$, then

$$\begin{aligned} h_A(S'(r)) &= h_A(S(r)) = h_A(h_A^j(s(\mu^{-j}(r)))) \\ &= h_A^{j+1}(s(\mu^{-(j+1)}(\mu(r))))) \\ &= S(\mu(r)) = S'(\mu(r)), \end{aligned}$$

(b) if $r \in I_j^*$, then

$$\begin{aligned} h_A(S'(r)) &= h_A(\sim S(\beta(r))) = \sim (h_A^{j+1}(s(\mu^{-j}(\beta r)))) \\ &= \sim (h_A^{j+1}(s(\mu^{-(j+1)}(\mu(\beta r))))) \\ &= \sim (h_A^{j+1}(s(\mu^{-(j+1)}(\beta(\mu(r))))) \\ &= \sim S(\beta(\mu(r))) = S'(\mu(r)). \end{aligned}$$

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