ON AXIOMS AND SOME PROPERTIES OF MONADIC FOUR-VALUED MODAL ALGEBRAS[1]

Alicia Ziliani

Departamento de Matemática, Universidad Nacional del Sur, 8000 - Bahía Blanca, Argentina.

Abstract

Four-valued modal algebras were introduced by A. Monteiro in 1978 as a generalization of the three-valued Lukasiewicz algebras [8] and they were studied by I. Loureiro [5,6] (also see [3,4]). In this paper we define monadic four-valued modal algebras and we give a set of independent axioms for them. We study the congruences and homomorphisms, showing that monadic four-valued modal algebras are semisimple and finally we characterize the simple algebras.

Our results generalize those obtained by L. Monteiro [10] for monadic three - valued Lukasiewicz algebras.

1 Preliminary definitions and properties

General references for concepts and results on distributive lattices and universal algebra used in this paper are the books [1] and [2].

Four - valued modal algebras have been defined by A. Monteiro in 1978, in the following way:

Definition 1.1 *A four - valued modal algebra* $(A, \wedge, \vee, \sim, \nabla, 1)$ *is an algebra of type* $(2,2,1,1,0)$ *which satisfies the following axioms:*

A1) $x \wedge (x \vee y) = x$, A2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,

A3) $\sim \sim x = x$, A4) $\sim (x \wedge y) = \sim x \vee \sim y$,

A5) $\sim x \vee \nabla x = 1$, A6) $\sim x \wedge \nabla x = x \wedge \sim x$.

From the definition, it follows that A is a distributive lattice [12] and a De Morgan algebra ([1,8]). For more details on four - valued modal algebras we lead the readers to [5,6] (see also [3,4]).

Definition 1.2 *A monadic four - valued modal algebra (or* MTM *- algebra)* $(A, \wedge, \vee, \sim, \nabla, \exists, 1)$ *is an algebra of type* $(2,2,1,1,1,0)$ *such that* $(A, \wedge, \vee, \sim, \nabla, 1)$ *is a four - valued modal algebra and* $\exists$ *is a unary operator on* A *(called* ***existential quantifier****) which satis -*

[1]Some of the results of this paper were presented at the Annual Meeting of the Unión Matemática Argentina (October, 1988) [13].

fies the following equations:

E1) $x \wedge \exists x = x$,

E2) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,

E3) $\nabla \exists x = \exists \nabla x$,

E4) $\Delta \exists x = \exists \Delta x$, *where* $\Delta x = \sim \nabla \sim x$,

E5) $\exists \sim \exists x = \sim \exists x$.

If A satisfies the axiom $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, then we get a monadic three-valued Lukasiewicz algebra [10].

We proceed to consider some examples of **MTM**-algebras.

Examples 1.1

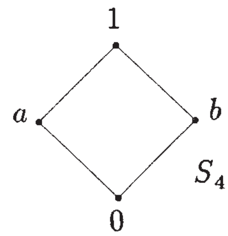
x	$\sim x$	∇x	$\exists x$
0	1	0	0
1	0	1	1

S_2

x	$\sim x$	∇x	$\exists x$
0	1	0	0
c	c	1	1
1	0	1	1

S_3

x	$\sim x$	∇x	$\exists x$
0	1	0	0
a	a	1	a
b	b	1	b
1	0	1	1



S_4

We shall denote by **MTM** the variety of monadic four-valued modal algebras.

We have proved that A1,...,A6,E1,E2,E5 and $\exists \sim \nabla \sim x = \sim \nabla \sim \exists x$ are mutually independent axioms for an **MTM**-algebra.

Lemma 1.1 *If* $A \in$ **MTM** *then it holds:*

E6) $x \leq \exists x$,

E7) $\exists 1 = 1$,

E8) $\exists 0 = 0$,

E9) $\exists\exists x = \exists x$,

E10) $x \leq y$ *implies* $\exists x \leq \exists y$,

E11) $\sim x \vee \nabla \exists x = 1$,

E12) $\exists x \vee \nabla \sim x = 1$,

E13) $\exists(x \vee y) = \exists x \vee \exists y$,

E14) *The set* $K(A) = \{x \in A: \exists x = x\}$ *of the invariant elements of* A *is a monadic four-valued modal subalgebra of* A.

In what follows, for any $A \in \mathbf{MTM}$, let $B(A)$ be the set of boolean elements of A, and $I(A) = \{x \in A\colon \nabla x = x\} = \{x \in A\colon \Delta x = x\} = \{x \in B(A)\colon -x = \sim x\}$ ([6]), where $-x$ denotes the boolean complement of x, $x \in B(A)$.

In the following lemmas we collect some results that we shall use in the subsequents parts of this paper.

Lemma 1.2 *If $A \in \mathbf{MTM}$ and $K(A) \simeq S_2$ then A is a Boolean algebra, where $-x = \sim x$, for all $x \in A$.*

Proof. Assume that there exists $x \in A$ such that $x \wedge \sim x \neq 0$. By hypothesis, it follows that $\exists\,(x \wedge \sim x) = 1$. Therefore $1 = \Delta \exists\,(x \wedge \sim x) = \exists(\Delta x \wedge \Delta \sim x) = \exists 0 = 0$, contradiction.
Then $x \wedge \sim x = 0$ for all $x \in A$, and so also $\sim x \vee x = 1$ for all $x \in A$. □

From [6] it is easy to check that

Lemma 1.3 *If $A \in \mathbf{MTM}$ then $I(A)$ is a subalgebra of A.*

Lemma 1.4 *If $A \in \mathbf{MTM}$ then $(I(A), \exists)$ is a monadic Boolean algebra.*

Proof. For all $x \in I(A)$, $x = \nabla x$, then $\sim x = \sim \nabla x$. Therefore $\sim x$ is the boolean complement of x. □

Definition 1.3 *Let $A \in \mathbf{MTM}$, then $c \in A$ is a center of A if $c = \sim c$.*

Remark that $c \in S_3$ and $a, b \in S_4$ are centers.

Lemma 1.5 ([6]) *Let $A \in \mathbf{MTM}$. The following conditions are equivalent:*

(*i*) *c is a center of A,*

(*ii*) *$\nabla c = 1$ and $\Delta c = 0$.*

Lemma 1.6 *Let $A \in \mathbf{MTM}$ and $K(A) \simeq S_3$. Then*

(*i*) *c is not a boolean element of A,*

(*ii*) *c is the unique center of A.*

Proof. (*i*) If c is a boolean element of A there exists $-c \in A$ such that $c \wedge -c = 0$ and $c \vee -c = 1$. Since $\exists c = c$, then it results $c \wedge \exists -c = 0$ and $c \vee \exists -c = 1$. Therefore $\exists -c = -c$ and so $-c \in K(A)$, contradiction.

(*ii*) Let (1) $f \in A$ be a center of A then $\exists f \in \{0, c, 1\}$.
If $\exists f = 0$ then $f = 0$ and so $\sim f \neq f$ which contradicts (1).
If $\exists f = 1$ then $1 = \Delta \exists f = \exists \Delta f$ and so, by (1) and lemma 1.5, it results $1 = 0$, contradiction.

Finally we have (2) $\exists f = c$, hence (3) $f \leq c$. Since c is a center of A, from (1) and (2) we have (4) $c = \sim\exists\sim f \leq f$. From (3) and (4) we obtain $f = c$. □

It follows at once that

Corollary 1.1 *Let $A \in$ MTM and $K(A) \simeq S_3$. Then A is not a Boolean algebra.*

Lemma 1.7 *Let $A \in$ MTM. If $K(A) \simeq S_4$ then a and b are the unique centers of A.*

Proof. Suppose that (1) c is a center of A. Since $\exists c \in K(A)$, we must consider the following cases:

(*i*) If $\exists c = 0$ then $c = 0$, which contradicts (1).

(*ii*) If $\exists c = a$ then we have (2) $c \leq a$ and by (1) $\sim a \leq c$. Since a is a center of A it follows (3) $a \leq c$. From (2) and (3) it results $c = a$.

(*iii*) If $\exists c = b$ similarly as (*ii*) we have $c = b$.

(*iv*) If $\exists c = 1$ then (4) $\Delta\exists c = 1$. On the other hand, from lemma 1.5 we have (5) $\Delta\exists c = \exists\Delta c = 0$. From (4) and (5) it results $0 = 1$, contradiction.

By (*ii*) and (*iii*) we get $c = a$ or $c = b$. □

The unary operation $\forall x = \sim\exists\sim x$ defined on an MTM-algebra A is called universal quantifier, and it fulfil the dual properties of the existential quantifier.

Definition 1.4 *Let $h: A \rightarrow B$ be a homomorphism from A into B. The kernel of h is the set $Ker(h) = \{x \in A\colon h(x) = 1\}$.*

Lemma 1.8 *The set $Ker(h)$ has the following properties:*

N1) *$Ker(h)$ is a filter of A (i.e. a filter in the underlying lattice A),*

N2) *if $x \in Ker(h)$ then $\Delta x \in Ker(h)$,*

N3) *if $x \in Ker(h)$ then $\forall x \in Ker(h)$.*

If F is a filter of an MTM-algebra A which verifies conditions N2 and N3 we say that F is a monadic filter (M-filter). If F is a filter verifying N2 is said to be a strong filter (S-filter) (see [6]).

If F is an M-filter of an algebra A, then the relation: $x \equiv y \ (mod\ F)$ is and only if there exists $f \in F$ such that $x \wedge f = y \wedge f$, is a congruence. If $x \in A$, $|x|$ denotes the congruence class containing x, and A/F denotes the quotient algebra, where the operations are defined as usual: $|x| \wedge |y| = |x \wedge y|$, $|x| \vee |y| = |x \vee y|$, $\sim|x| = |\sim x|$, $\nabla|x| = |\nabla x|$, $\exists|x| = |\exists x|$. The function $q: A \rightarrow A/F$ defined by $q(x) = |x|$ is an epimorphism such that $Ker(q) = F$.

2 Weak implication and deductive systems

We define a new binary operation $\Rightarrow$ on an MTM-algebra A, called weak implication, as follows:

$$x \Rightarrow y = \nabla \sim \forall x \vee y.$$

It is not hard to prove that

Lemma 2.1 *The weak implication has the following properties:*

W1) $x \Rightarrow x = 1$,

W2) $x \Rightarrow (y \Rightarrow x) = 1$,

W3) $(x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x \Rightarrow y) \Rightarrow (x \Rightarrow z)) = 1$,

W4) $((x \Rightarrow y) \Rightarrow x) \Rightarrow x = 1$,

W5) $1 \Rightarrow x = x$,

W6) $1 \Rightarrow x = 1$ *implies* $x = 1$,

W7) $x \Rightarrow \Delta x = 1$,

W8) $x \Rightarrow \forall x = 1$,

W9) $x \Rightarrow (x \wedge y) = x \Rightarrow y$,

W10) $x \leq y$ *implies* $z \Rightarrow x \leq z \Rightarrow y$,

W11) $x \leq y$ *implies* $x \Rightarrow y = 1$.

Definition 2.1 *A set $D \subseteq A$ is a deductive system (d.s.) if it verifies:*

D1) $1 \in D$,

D2) *if* $x, x \Rightarrow y \in D$ *then* $y \in D$.

D is a proper d.s. if $D \neq A$.

The next lemma gives the relationship between deductive systems and M-filters of an algebra A.

Lemma 2.2 *Let $A \in$ **MTM** and $D \subseteq A$. The following conditions are equivalent:*

(*i*) *D is a d.s.,*

(*ii*) *D is an M-filter.*

Proof. $(i) \Rightarrow (ii)$: From (i) and definition 2.1 we have: (1) $1 \in D$.

Suppose now that $x, y \in D$ then from W2 and W9 we have $1 = y \Rightarrow (x \Rightarrow y) = y \Rightarrow (x \Rightarrow (x \wedge y))$. Therefore $y \Rightarrow (x \Rightarrow (x \wedge y)) \in D$ and from (i) and definition 2.1 we get $x \wedge y \in D$. Hence we obtain: (2) if $x, y \in D$ then $x \wedge y \in D$.

Assume that $x \in D$, $y \in A$ and $x \leq y$. Then from W11 and definition 2.1 we obtain that $y \in D$. So we have: (3) if $x \in D$, $y \in A$ and $x \leq y$ then $y \in D$.

From (1), (2) and (3) it follows that D is a filter of A. By W7, W8, (i) and definition 2.1 we get $\Delta x, \forall x \in D$ for all $x \in D$. Then D is an M-filter of A.

(ii)$\Rightarrow$(i): We only check D2. Let $x, y \in A$ such that $x, x \Rightarrow y \in D$. From N2 and N3 we have that Δx, $\Delta(x \Rightarrow y)$, $\Delta \forall x \in D$. Then by N1 it follows that:

$$\begin{aligned} \Delta x \wedge \Delta \forall x \wedge \Delta(x \Rightarrow y) &= \Delta x \wedge \Delta \forall x \wedge (\nabla \sim \forall x \vee \Delta y) \\ &= (\Delta x \wedge \Delta \forall x \wedge \sim \Delta \forall x) \vee (\Delta x \wedge \Delta \forall x \wedge \Delta y) \\ &= \Delta x \wedge \Delta \forall x \wedge \Delta y \in D. \end{aligned}$$

Since $\Delta x \wedge \Delta \forall x \wedge \Delta y \leq y$, by N1 we have that $y \in D$. □

Corollary 2.1 *Let $A \in$ MTM and $D \subseteq A$. The following conditions are equivalent:*

(i) *D is a proper d.s. of A,*

(ii) *D is the kernel of a homomorphism over A.*

The family of all deductive systems of A ordered by set-theoretical inclusion, is upper inductive. Then, by Zorn's lemma, any proper d.s. is contained in a maximal d.s

Taking into account W2, W3, W4, W5 and the results due to A. Monteiro [9] we have that any proper d.s. of an MTM-algebra A is an intersection of maximal deductive systems of A.

Then, by well known results of universal algebra we get:

Theorem 2.1 *Any non trivial algebra A is a subdirect product of the family $\{A/M\}_{M \in \mathcal{S}(A)}$, where $\mathcal{S}(A)$ is the sets of all maximal deductive systems of A.*

Let $A \in$ **MTM**, $H \subseteq A$ and $a \in A$. We shall denote by $[H)$ and $[H, a)$ respectively the d.s. of A generated by H and $H \cup \{a\}$.

From W2, W3 and [9] we have that $[H) = \{x \in A$: there exist $h_1, \ldots, h_k \in H$ such that $h_1 \Rightarrow (h_2 \Rightarrow \ldots (h_k \Rightarrow x) \ldots) = 1\}$ and $[H, a) = \{x \in A : a \Rightarrow x \in [H)\}$.

Recall that if X is a non-empty subset of a distributive lattice R with 0 and 1, then the filter $F(X)$ generated by X is the set of all elements $y \in R$ such that there exist elements $x_1, x_2, \ldots, x_n \in X$ such that $x_1 \wedge x_2 \wedge \ldots \wedge x_n \leq y$. It is well known that if X verifies the property: $x, y \in X$ implies $x \wedge y \in X$, then $F(X) = \{y \in R$: there exists $z \in X$ with $z \leq y\}$.

If $X = \emptyset$, then $F(\emptyset) = \{1\}$. If $X = \{a\}$ we write $F(a)$ instead of $F(\{a\})$. $F(a)$ is called a principal filter. If R is finite, every filter is principal.

Lemma 2.3 *If $A \in \mathbf{MTM}$, $H \subseteq A$ then $[H) = F(\forall \triangle H)$.*

Proof. We shall prove that $F(\forall \triangle H)$ is an $\mathbf{M}$-filter of A. Indeed, if $x \in F(\forall \triangle H)$ then there exists $\forall \triangle h_1, \ldots, \forall \triangle h_k \in \forall \triangle H$ such that $\forall \triangle h_1 \wedge \ldots \wedge \forall \triangle h_k \leq x$. So $\forall \triangle h_1 \wedge \ldots \wedge \forall \triangle h_k \leq \forall \triangle x$ and hence $\forall x$ and $\triangle x$ belong to $F(\forall \triangle H)$. Furthermore $H \subseteq F(\forall \triangle H)$ because $\forall \triangle h \leq h$ for all $h \in H$. Then $[H) \subseteq F(\forall \triangle H)$.

Conversely, it is easy to see that $[H)$ is a filter of A. Furthermore $\forall \triangle H \subseteq [H)$. Indeed, if $h \in H$ then from W7 and W8 we have $\forall \triangle h \in [H)$. Hence $F(\forall \triangle H) \subseteq [H)$. □

Corollary 2.2 *If $A \in \mathbf{MTM}$, $a \in A$ and D is a d.s. of A then $[D, a) = F(D, \forall \triangle a)$.*

Now we are going to indicate a characterization of maximal d.s. of A.

Lemma 2.4 *Let $A \in \mathbf{MTM}$ and $M \subseteq A$ be a d.s.. The following conditions are equivalent:*

(*i*) *M is maximal,*

(*ii*) *if $a \notin M$ then there exists $m \in M$ such that $\forall \triangle a \wedge m = 0$,*

(*iii*) *if $\forall \triangle a \vee b \in M$ then $a \in M$ or $b \in M$,*

(*iv*) *if $a \notin M$ then $\nabla \sim \forall a \in M$,*

(*v*) *if $a \notin M$ and $b \in A$ then $a \Rightarrow b \in M$.*

Proof. (*i*)⇒(*ii*): If $\forall \triangle a \wedge m \neq 0$, for all $m \in M$ then $[M, a)$ is a proper d.s. of A and $M \subset [M, a)$, contradiction.

(*ii*)⇒(*iii*): Assume that $a \notin M$ then by (*ii*) there exists $m \in M$ such that (1) $\forall \triangle a \wedge m = 0$. Since $\forall \triangle a \vee b \in M$, from (1) we have that $(\forall \triangle a \vee b) \wedge m = b \wedge m \in M$. Hence $b \in M$.

(*iii*)⇒(*iv*): Since $\forall \triangle a \vee \nabla \sim \forall a = 1 \in M$ and by hypothesis $a \notin M$ we have that $\nabla \sim \forall a \in M$.

(*iv*)⇒(*v*): Obvious.

(*v*)⇒(*i*): Suppose that M is not maximal then there exists a maximal d.s. M' such that $M \subset M' \subset A$. Let $a \in M' \backslash M$ and $b \in A \backslash M'$. Then by hypothesis $a \Rightarrow b \in M \subset M'$ and so $b \in M'$, contradiction. □

3 Simple algebras

Since the homomorphic images of an MTM-algebra A are the algebras A/D, where D is a d.s. of A, we have:

Lemma 3.1 *If A is an MTM-algebra then the following conditions are equivalent:*

(*i*) *A is simple,*

(*ii*) *$\{1\}$ and A are the only deductive systems of A.*

Let $A \in \mathbf{MTM}$. We shall denote by $I(K) = \{x \in A\colon \nabla x = x = \exists x\} = \{x \in A\colon \Delta x = x = \forall x\}$.

It is easy to see that $I(K) = I(A) \cap K(A)$ is a subalgebra of A. Furhtermore $I(K)$ is a Boolean algebra.

The proofs of the following lemmas is routine:

Lemma 3.2 *$F(a)$ is d.s. of an MTM - algebra A if and only if $a \in I(K)$.*

Lemma 3.3 *If M is a d.s. of an MTM - algebra A, then A/M is simple if and only if M is maximal.*

Lemma 3.4 *$F(a)$ is a maximal d.s. of an MTM - algebra A if and only if a is an atom of $I(K)$.*

Corollary 3.1 *a is an atom of $I(K)$ if and only if $A/F(a)$ is a simple algebra.*

We now give the relationship between deductive systems in an algebra A, S - filters in $K(A)$, M - filters in $I(A)$ and filters in $I(K)$.

Let $\mathfrak{D}, \mathfrak{S}, \mathcal{M}$ and $\mathfrak{F}$ respectively denote the set of all deductive systems in an algebra A, the set of all S - filters in $K(A)$, the set of all M - filters in $I(A)$ and the set of all filters in $I(K)$.

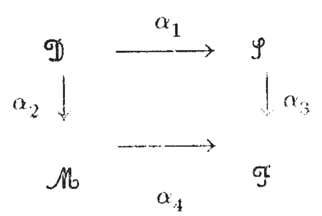
Consider the following functions

$\alpha_1\colon \mathfrak{D} \to \mathfrak{S}, \qquad \alpha_1(D) = D \cap K(A),$

$\alpha_2\colon \mathfrak{D} \to \mathcal{M}, \qquad \alpha_2(D) = D \cap I(A),$

$\alpha_3\colon \mathfrak{S} \to \mathfrak{F}, \qquad \alpha_3(F) = F \cap I(K),$

$\alpha_4\colon \mathcal{M} \to \mathfrak{F}, \qquad \alpha_4(F) = F \cap I(K).$



Then we can prove

Lemma 3.5 *If we order the sets $\mathfrak{D}, \mathfrak{S}, \mathcal{M}$ and $\mathfrak{F}$ by inclusion then α_1, α_2, α_3 and α_4 are order isomorphisms and the diagram indicated above commutes.*

Theorem 3.1 *Let A be a non trivial* MTM*-algebra. The following conditions are equivalent:*

(*i*) A *is simple,*

(*ii*) $\forall \triangle a = 0$ *for all* $a \in A$, $a \neq 1$,

(*iii*) $I(K) \simeq S_2$,

(*iv*) $K(A) \simeq S_2$, $K(A) \simeq S_3$ *or* $K(A) \simeq S_4$.

Proof. (*i*)⇒(*ii*): It follows from the hypothesis and lemma 2.4.

(*ii*)⇒(*iii*): Let $a \in I(K)$, $a \neq 1$. By (*ii*) $0 = \forall \triangle a$. Then $a = 0$, contradiction.

(*iii*)⇒(*iv*): $I(K)$ is a simple Boolean algebra. Then taking into account lemma 3.5 we conclude that $K(A)$ is a simple four-valued modal algebra. Therefore by [6] we have the proof.

(*iv*)⇒(*i*): It is an inmediate consequence of lemma 3.5 and the hypothesis. □

The main result of this section is the following theorem.

Theorem 3.2 *Let A be a simple* MTM*-algebra. Then*

(*i*) $K(A) \simeq S_2$ *implies* $A \simeq S_2^{\alpha}$,

(*ii*) $K(A) \simeq S_3$ *implies* $A \simeq S_3^{\beta}$,

(*iii*) $K(A) \simeq S_4$ *implies* $A \simeq S_4$,

where α, β are non negative cardinals.

Proof. Since A is a four-valued modal algebra it is known [6] that A is isomorphic to a subalgebra of $S_2^{\alpha} \times S_3^{\beta} \times S_4^{\gamma}$ that is, there exists a monomorphism $\psi: A \to S_2^{\alpha} \times S_3^{\beta} \times S_4^{\gamma}$.

(*i*) From the hypothesis and lemma 1.2 $\psi(A)$ is a Boolean algebra such that $-x = \sim x$, for all $x \in \psi(A)$. Then $S_3^{\beta} \cap \psi(A) = \emptyset$ and $S_4^{\gamma} \cap \psi(A) = \emptyset$. Hence $\psi(A) = S_2^{\delta}$, $\delta \leq \alpha$.

(*ii*) By the hypothesis and lemma 1.6 $\psi(A)$ has a center, then $\psi(A) \cap S_2^{\alpha} = \emptyset$. If $S_4^{\gamma} \cap \psi(A) \neq \emptyset$ then there exist $c_1, c_2 \in \psi(A)$ such that $c_1 \neq c_2$ and c_1, c_2 centers of $\psi(A)$, which contradicts lemma 1.6. Therefore $\psi(A) = S_3^{\eta}$, $\eta \leq \beta$.

(*iii*) If $S_2^{\alpha} \cap \psi(A) \neq \emptyset$ then $\psi(A)$ has no center, which contradicts lemma 1.7. If $S_3^{\beta} \cap \psi(A) \neq \emptyset$, there exist $c_1, c_2 \in \psi(A) \backslash B(\psi(A))$ such that $c_1 \neq c_2$ and c_1, c_2 centers of A which contradicts lemma 1.7. Therefore $\psi(A) = S_4^{\rho}$, $\rho \leq \gamma$. If $\rho > 1$ similarly we get a contradiction. Hence $\psi(A) = S_4$. □

Acknowledgement

The author would like to thank Dr. A. V. Figallo who suggested the problem and was a valuable guide throughout the preparation of this paper.

References

[1] R. Balbes and Ph. Dwinger, *Distributive lattices.* University of Missouri Press (1974).

[2] S. Burris and H.P. Sankappanavar, *A course in Universal Algebra.* Graduate Texts in Mathematics 78, Springer, Berlin (1981).

[3] A. Figallo, *Notes on generalized N - lattices.* Revista de la Unión Matemática Argentina 35 (1990), 61 - 65.

[4] A. Figallo, *On the congruence in four - valued modal algebras.* Portugaliae Math. 49 (1992), 249 - 261.

[5] I. Loureiro, *Axiomatisation et propriétés des algèbres modales tétravalentes.* C.R. Acad. Sc. Paris, t. 295 (22 novembre 1982), Serie I, 555 - 557

[6] I. Loureiro, *Algebras modais tetravalentes.* Doctoral thesis. Faculdade de Ciencias de Lisboa, 1983.

[7] A. Monteiro, *Sur la définition des algèbres de Lukasiewicz trivalentes.* Bull. Math. Soc. Sci. Math. Phys., R.P. Roum 7 (55) (1963), 3 - 12.

[8] A. Monteiro. *Matrices de Morgan caractéristiques pour le calcul propositionnel classique.* An. Acad. Brasil. Ciên. 52 (1960), 1 - 7.

[9] A. Monteiro, *Sur les algèbres de Heyting symétriques.* Portugaliae Math. 39 (1980), 1 - 237.

[10] L. Monteiro, *Algebras de Lukasiewicz trivalentes monádicas.* Notas de Lógica Matemática 32, (1974). Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca, Argentina.

[11] L. Monteiro, *Axiomes indépendants pour les algèbres de Lukasiewicz trivalentes.* Notas de Lógica Matemática 22, (1964). Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca, Argentina.

[12] M. Sholander, *Postulates for distributive lattices.* Can. J. Math. 3, (1951), 28 - 30.

[13] A. Ziliani, *Algebras tetravalentes modales monádicas.* Revista de la Unión Matemática Argentina 36 (1990), 224.