Introducción a las Álgebras Inclinadas

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BAHÍA BLANCA, ABRIL DE 1995

AN INTRODUCTION TO TILTED ALGEBRAS

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The main objective in representation theory of algebras is the study of categories of modules over algebras. A possible general strategy is to consider a class of well-known algebras and from there to construct another class where it is possible to transport informations from the former class to the new one.

For a given algebra A, let mod$A$ denote the category of the finitely generated right A-modules and ind$A$ be the subcategory of mod$A$ with one representative of each isomorphism class of indecomposable A-module. In general, if A is an algebra and $M \in$ mod$A$, then $B =(\text{End}_A M)$ is also an algebra. Depending on the hypothesis imposed on A and on M, we will have some control over the algebra B constructed as above (or equivalently, over the category mod$B$). For instance, if M is a projective progenerator of mod$A$, and $B =(\text{End}_A M) \cong A^{op}$, then the functor $\text{Hom}_A(M, -)$: mod$A \longrightarrow$ modB^{op} gives an equivalence of the categories mod$A$ and modB^{op} in this case.

Another interesting situation is the following. Let A be a representation-finite algebra, that is, an algebra such that ind$A$ has only finitely many objects. Let M be the sum of one copy of each module of ind$A$. The algebra $B =(\text{End}_A M)$ is called the *Auslander algebra* of A and it has well-known good homological properties (see [4] or [20] for details). In this case, the category mod$A$ is equivalent to the category of projective B-modules.

The situation we shall discuss here is roughly speaking the following. We start with an algebra A and a module T with some special homological properties (T will be called tilting module) and look at $B =$End$T$. In general, the categories mod$A$ and mod$B$ will not be equivalent but we will still have a control over some subcategories of them. Again, depending on extra conditions on A and T, the relations between such subcategories of mod$A$ and mod$B$ will enable us to get important informations on B from A. In a sense, the better situation one can think of is when one starts with a hereditary algebra A. In this situation, for a given tilting A-module, the algebra $B =$End$T$ will be called tilted algebra and each module in mod$B$ can be seen as the image of a module in mod$A$ by some convenient functors.

The idea behind *tilting* algebras goes back to the work of Bernstein-Gelfand-Ponomarev [12], where the so-called Coxeter functors were used to give another proof for Gabriel's theorem (see [21]). A generalization of this procedure was given

The author is partially supported by CNPq

by Auslander-Platzeck-Reiten in [5]. The modules considered by them is nowadays called APR-tilting modules (examples will be given in section 3).

A further generalization were considered by Brenner-Butler in [11]. Finally, a general procedure was given by Happel-Ringel in [24], where the so-called tilting modules and tilted algebras were defined. Since then, much work has been done in the study of tilted algebras and possible generalizations. The literature on tilting theory is nowadays very extensive and the interested reader will have no difficulties to find out more on this subject. A good initial reference is the survey article [1]. In the books [23, 35], the authors discuss some aspects on tilting theory which we shall not consider here. At the end of these notes we give a list of references concerning only the aspects discussed in these notes. We believe it is just the start point for someone interested in knowing more on tilting theory.

These notes is divided into five sections. The first two sections contains standard notions in representation theory of algebras and we shall quickly recall them in order to establish some notations. Section one contains the characterization of algebras as quotients of path algebras while section two contains the basic Auslander-Reiten theory. General references for this part are [7, 20]. Sections three is devoted to the definition of tilting modules and the relation between them and torsion theories. Section four contains a discussion on tilted algebras and some informations on the so-called Auslander-Reiten quivers of them. Section five follows very closely a joint work with I. Assem [2], where some homological properties of tilted algebras were discussed.

By an algebra we shall mean an associative, with unity, basic and indecomposable finite dimensional algebra over a fixed algebraically closed field k. Modules are always finitely generated and for a given algebra A we keep the notations mod$A$ and ind$A$ established above. We shall use basic notions on module theory and homological algebra which can be easily found in textbooks on algebras.

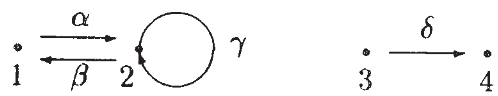
These notes were written based in a series of talks given during the *III Congresso Dr. Antonio A. R. Monteiro* held in Bahia Blanca in April, 1995. The author is very grateful to the organizers, in special to Maria Inés Platzeck, for the kind invitation to participate in such a meeting, and the hospitality given during his stay in Argentina.

1. Algebras given by quivers with relations

1.1. In this section we are mainly interested in describing the finite dimensional basic k-algebras, where k is an algebraically closed field. We shall see that such an algebra is isomorphic to a quotient of a path algebra of a suitable quiver (see definitions below). This construction is nowadays very standard in representation theory of finite dimensional algebras and provide us a handful of examples. We shall only indicate the main steps of this construction, since the details can be found for instance in [7, 13].

1.2. A *quiver* Δ is given by two sets Δ_0 and Δ_1, together with two maps s, e from Δ_1 to Δ_0. The elements of Δ_0 are called *vertices of* Δ, while the elements of Δ_1 are called *arrows of* Δ. For a given arrow $\alpha \in \Delta_1$, the vertices given by $s(\alpha)$ and $e(\alpha)$ are called, respectively, the *start vertex* and the *end vertex* of α, and we denote it by $\alpha\colon s(\alpha) \longrightarrow e(\alpha)$. For simplicity, we normally draw the quivers as shown by the following example.

Example. Δ:



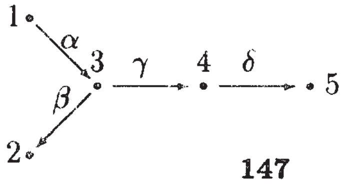
For this quiver, $\Delta_0 = \{1,2,3,4\}$, $\Delta_1 = \{\alpha,\beta,\gamma,\delta\}$ and $s(\alpha) = e(\beta) = 1$, $s(\beta) = e(\alpha) = s(\gamma) = e(\gamma) = 2$, $s(\delta) = 3$ and $e(\delta) = 4$.

Let $\Delta = (\Delta_0, \Delta_1, s, e)$ be a quiver. A *path* γ in Δ is $x_0 \xrightarrow{\alpha_1} x_1 \longrightarrow \cdots \longrightarrow x_{t-1} \xrightarrow{\alpha_t} x_t$, where for each $i = 1, \cdots t$, $\alpha_i \in \Delta_1$. We also denote it by $\gamma = \alpha_t \cdots \alpha_1$. The *length* of a path is the number of arrows in it; in the above case the length of γ is t. By convention, to each vertex a of Δ, it is assigned a *path of length zero* ϵ_a (also called *trivial path*). An *oriented cycle* is a path of length greater or equal to one from a vertex x to itself. A *loop* is an oriented cycle of length one.

Let $x, y \in \Delta_0$. We denote by $x - y$ in case there exists either an arrow $x \longrightarrow y$ or an arrow $y \longrightarrow x$. A *walk* between x and y is given by $x = x_0 - x_1 - \cdots - x_t = y$. The quiver Δ is called *connected* if, for any given two vertices in Δ_0, there is always a walk between them.

1.3. Let Δ be a finite quiver, and k be a (fixed) field. We shall now assign to this pair an algebra $k\Delta$, called the *path algebra* of Δ, in the following way. As a k-vector space, we consider as basis the set of all paths in Δ of length greater or equal to zero (that is, including the trivial paths). To give it an algebra structure, we shall define now a multiplication in this basis. Let $\gamma_1\colon x_1 \longrightarrow \cdots \longrightarrow x_r$ and $\gamma_2\colon y_1 \longrightarrow \cdots \longrightarrow y_s$ be two paths in Δ. If $x_r = y_1$, then we define the product $\gamma_1 \cdot \gamma_2$ to be the path $x_1 \longrightarrow \cdots \longrightarrow x_r = y_1 \longrightarrow \cdots \longrightarrow y_s$ and if $x_r \neq y_1$, then we define the product $\gamma_1 \cdot \gamma_2$ to be zero. With such a multiplication on the elements of its basis, $k\Delta$ is indeed a k-algebra.

Example. Let Δ be the following quiver



The set of paths in Δ is $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \alpha, \beta, \gamma, \delta, \beta\alpha, \gamma\alpha, \delta\gamma, \delta\gamma\alpha\}$. Therefore, $k\Delta$ has dimension 13 as a vector space. As an ilustration, observe that $\alpha \cdot \epsilon_1 = \epsilon_3 \cdot \alpha = \alpha$, $\gamma\alpha \cdot \delta = 0$ and $\delta \cdot \gamma\alpha = \delta\gamma\alpha$.

1.4. The following proposition is straightforward from the above considerations and we leave the proof for the reader. For details we refer to [13].

Proposition. *Let Δ be a finite quiver, k be a field and $A = k\Delta$ be the path algebra of Δ as defined above. Then*

(a) *A is an associative basic k-algebra, not necessarily commutative (see example above).*
(b) *Denote by $\epsilon_1, \cdots, \epsilon_n$ the set of all trivial paths (one for each vertex). Then $\{\epsilon_1, \cdots, \epsilon_n\}$ is a complete set of primitive orthogonal idempotents of A, and $1 = \sum_{i=1}^{n} \epsilon_i$ is the identity of A.*
(c) *The algebra A is finite-dimensional if and only if Δ has no oriented cycles, and in this case, rad$A$ is generated by the set of arrows.*
(d) *The algebra A is indecomposable (as a ring) if and only if Δ is connected.*

1.5. Let Δ be a finite quiver and k be a field. An ideal I of $k\Delta$ is called *admissible* if there exists an n such that $J^2 \subset I \subset J^n$, where J is the ideal generated by all the arrows of Δ. Observe that if I is an admissible ideal of $k\Delta$, then $k\Delta/I$ is always a finite dimensional algebra. The next result (due to Gabriel [21]) gives a characterization of basic finite dimensional k-algebra in case k is algebraically closed.

Theorem. *Let A be a finite dimensional basic k-algebra, where k is an algebraically closed field. Then there exists a quiver Δ_A such that $A \cong k\Delta_A/I$, for some admissible ideal I.*

Sketch of the proof. We shall only indicate how the proof goes and leave the details for the reader (complete proof can be found in [13]). Let A be as in the statement and let $\{e_1, \cdots, e_n\}$ be a complete set of orthogonal and primitive idempotents of A. First, define the quiver Δ_A as follows: Δ_A has n vertices $\epsilon_1, \cdots, \epsilon_n$ in a one-to-one correspondence with the e_i's. Then for each pair ϵ_i, ϵ_j, the number of arrows from ϵ_i to ϵ_j if defined to be the dimension

$$\dim_k e_j \left(\frac{\text{rad}A}{\text{rad}^2 A} \right) e_i$$

Here, the reader has to be convinced that the above quiver is well-defined (up to, of course, some reordering of the vertices).

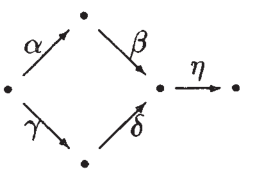
Choose now $\{x_\alpha \in \text{rad}A : \alpha \in (\Delta_A)_1\}$ such that for each pair i, j, the set of classes $S_{ij} = \{x_\alpha + \text{rad}^2 A : s(\alpha) = i \text{ and } e(\alpha) = j\}$ gives a k-basis for $\frac{\text{rad}A}{\text{rad}^2 A}$. Define now a morphism $\phi\colon k\Delta_A \longrightarrow A$ as follows: $\phi(\epsilon_i) = e_i$, $\phi(\alpha) = x_\alpha$, for each $\alpha \in (\Delta_A)_1$, and, for each path $\gamma\colon x_0 \xrightarrow{\alpha_1} x_1 \longrightarrow \cdots \longrightarrow x_{t-1} \xrightarrow{\alpha_t} x_t$, $\phi(\gamma) = x_{\alpha_n} \cdots x_{\alpha_1}$. This defines a morphism of algebras which is indeed an epimorphism (for the later, the hypothesis of basic and k algebraically closed are essential). Moreover, it is not difficult to see that $\ker\phi$ is an admissible ideal of $k\Delta_A$, and hence the result follows because then $A \cong k\Delta_A/\ker\phi$.

The quiver Δ_A is called the *ordinary quiver of* A.

1.6. An important class of algebras we are going to consider is the class of hereditary algebras. Recall that an algebra A is *hereditary* if the radical of A, rad$A$, is a projective A-module. It is not difficult to see that $A = k\Delta/I$ is hereditary if and only if $I = 0$. Moreover, P. Gabriel [22] has shown that a hereditary algebra $A = k\Delta$ is representation-finite (that is, ind$A$ has only finitely many indecomposable nonisomorphic modules) if and only if Δ is a Dynkin quiver.

1.7. Let $A = k\Delta/I$ be an algebra, where Δ is a quiver and I is an admissible ideal of $k\Delta$. As a k-vector space, I has clearly a finite basis, whose elements we shall call *relations*: they are linear combinations of paths of length at least two. Therefore, the algebra A is given by a quiver Δ together with some relations (which generates I). We give an example.

Example. Let A be the algebra given by the quiver Δ



with $\beta\alpha = \delta\gamma$ and $\eta\beta = 0$

This means that $A = \frac{k\Delta}{R}$ where R is the ideal of $k\Delta$ generated by the relations $\beta\alpha - \delta\gamma$ and $\eta\beta$.

1.8. The advantage of describing the algebras as quivers with relations is that this allow us to describe also their finitely generated modules in terms of representations of the corresponding quivers.

Let (Δ, R) be a quiver Δ together with a set of relations R, that is, R consists of linear combinations of paths of length at least two. A (Δ, R)*-representation* is given by $V = ((V_i)_{i\in\Delta_0}, (f_\alpha)_{\alpha\in\Delta_1})$, where for each $i \in \Delta_0$, V_i is a finite dimensional k-vector space, and for each $\alpha \in \Delta_1$, f_α is a linear transformation from $V_{s(\alpha)}$ to $V_{e(\alpha)}$. Moreover, the linear transformations f_α's have to satisfy the relations of R in the following sense. Clearly, for a given path $\gamma\colon x = x_0 \xrightarrow{\alpha_1} x_1 \longrightarrow \cdots \longrightarrow x_{t-1} \xrightarrow{\alpha_t} x_t = y$ in Δ, one can assign a linear transformation $f_\gamma\colon V_x \longrightarrow V_y$ as the composition $f_\gamma = f_{\alpha_t} \cdots f_{\alpha_1}$. Now

we say that $V = ((V_i)_{i\in\Delta_0}, (f_\alpha)_{\alpha\in\Delta_1})$ *satisfies a relation* $\sum \gamma_i \in R$ if the corresponding su
m $\sum f_{\gamma_i}$ is zero, and V *satisfies* R if it satisfies each relation of R.

Let now $V = ((V_i)_{i\in\Delta_0}, (f_\alpha)_{\alpha\in\Delta_1})$ and $W = ((W_i)_{i\in\Delta_0}, (g_\alpha)_{\alpha\in\Delta_1})$ be two (Δ, R)-representations. A (Δ, R)-*morphism* $\phi\colon V \longrightarrow W$ is given by $\phi = (\phi_i)_{i\in\Delta_0}$ such that for each $\alpha\colon i \longrightarrow j$, the following diagram commutes:

$$\begin{array}{ccc} V_i & \xrightarrow{f_\alpha} & V_j \\ \phi_i\downarrow & © & \downarrow\phi_j \\ W_i & \xrightarrow{g_\alpha} & W_j . \end{array} \qquad \text{that is,} \quad \phi_j f_\alpha = g_\alpha \phi_i.$$

The category (Δ, R)-mod is now defined as follows. The objects of (Δ, R)-mod are the (Δ, R)-representations and the morphisms are as defined as above. We leave to the reader the formulation of the notion of direct sum of two objects in (Δ, R)-mod as well as the notion of indecomposability. Also, a morphism $\phi = (\phi_i)_{i\in\Delta_0}\colon V \longrightarrow W$ in (Δ, R)-mod is a monomorphism (respectively, an epimorphism, or an isomorphism) if and only if each ϕ_i is a monomorphism (respectively, an epimorphism or an isomorphism). Observe that the category (Δ, R)-mod satisfies the Krull-Schmidt theorem, and hence each object of (Δ, R)-mod can be written as a (finite) direct sum of indecomposable objects in a uniquely determined way, up to isomorphism. We have the following result.

Theorem. *Let A be an algebra given by a quiver Δ with relations R (that is, $A \cong k\Delta/I$, where I is the ideal generated by R). Then the categories (Δ, R)-mod and A-mod are equivalent*

1.9. Example. Let (Δ, R) be given by $\underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet}$ with $\beta\alpha = 0$.

The following are examples of (Δ, R)-representations: $V_1 = (k \xrightarrow{0} 0 \xrightarrow{0} 0)$, $V_2 = (0 \xrightarrow{0} k \xrightarrow{0} 0)$, $V_3 = (k \xrightarrow{1} k \xrightarrow{0} 0)$, $V_4 = (0 \xrightarrow{0} k \xrightarrow{1} k)$ and $V_5 = (k \xrightarrow{0} k \xrightarrow{1} k)$ (in fact these correspond to all indecomposable modules over the algebra given by (Δ, R)). We have that $\mathrm{Hom}_A(V_1, V_3) = \mathrm{Hom}_A(V_3, V_2) = 0$ but $\mathrm{Hom}_A(V_3, V_1) \cong \mathrm{Hom}_A(V_2, V_3) \cong k$. Observe that $k \xrightarrow{1} k \xrightarrow{1} k$ is not a (Δ, R)-representation because the composition of the linear transformations is nonzero.

2. Auslander-Reiten theory

2.1. We shall recall in this section the basic notions on the so-called Auslander-Reiten theory. The basis for such a theory is the notion of almost split maps and sequences. Besides of course their theoretical importance, these sequences can be used to define the Auslander-Reiten quiver of an algebra A, which records many informations one has in the category ind$A$. Much of the recent investigations on representation theory of algebras is based in the study and description of such quivers. We shall state the main results here without proof, which can be easily found in [5, 6]. Along this section A is an algebra.

2.2. We start with the following definition.

Definition. Let $f\colon X \longrightarrow Y$ be a map in mod$A$.
(a) We say that f is a *minimal left almost split map of* X if

(i) f is not a split monomorphism;
(ii) for each map $g\colon X \longrightarrow M$ which is not a split monomorphism, there exists $\overline{g}\colon Y \longrightarrow M$ such that $\overline{g}f = g$;
(iii) if $hf = f$ for some $h \in \text{End}Y$, then h is an automorphism.

(b) We say that f is a *minimal right almost split map of* Y if

(i) f is not a split epimorphism;
(ii) for each map $g\colon M \longrightarrow Y$ which is not a split epimorphism, there exists $\overline{g}\colon M \longrightarrow X$ such that $f\overline{g} = g$;
(iii) if $fh = f$ for some $h \in \text{End}X$, then h is an automorphism.

Remarks. (i) If P is an indecomposable projective A-module, then the natural inclusion $\iota\colon \text{rad}P \longrightarrow P$ is a minimal right almost split map. Dually, if I is an indecomposable injective A-module, then the natural projection $\pi\colon I \longrightarrow I/\text{soc}I$ is a minimal left almost split map
(ii) If $f\colon X \longrightarrow Y$ is a minimal left (or right) almost split map, then X (respectively, Y) is indecomposable.

2.3. The following result is essential in representation theory.

Theorem (Auslander-Reiten). *Let X be an indecomposable module.*

(a) If X is not a simple projective, then there exists a unique (up to isomorphism) minimal right almost split map $g\colon E \longrightarrow X$ of X. Moreover, if X is not projective, then such a g is an epimorphism and the inclusion map $\iota\colon$ ker$g \longrightarrow E$ is a minimal left almost split map of ker$g$.

(a) If X is not a simple injective, then there exists a unique (up to isomorphism) minimal left almost split map $f\colon X \longrightarrow E$ of X. Moreover, if X is not injective, then such an f is a monomorphism and the projection map $\pi\colon E \longrightarrow$ Cokerg is a minimal right almost split map of Cokerf.

2.4. Let now $X \in \text{ind}A$. Suppose that X is not projective and consider the short exact sequence

$$0 \longrightarrow \ker g \xrightarrow{\iota} E \xrightarrow{g} X \longrightarrow 0 \qquad (*)$$

where g is a minimal right almost split map. This sequence is unique up to isomorphism and it is called the *almost split sequence ending at* X. By (2.2), ker$g$ is an indecomposable module and clearly it is uniquely determined (up to isomorphism) from the choice of X. We shall denote by τX the ker$g$, where g is an minimal right almost split map of X, and call it the *Auslander-Reiten translate* of X. We stress the fact that the maps ι and g of the above sequence $(*)$ have the lifting properties of maps.

Dually, if X is not injective, the (unique up to isomorphism) sequence

$$0 \longrightarrow X \xrightarrow{f} E \xrightarrow{\pi} \text{Coker}f \longrightarrow 0 \qquad (*)$$

where $f\colon X \longrightarrow E$ is a minimal left almost split map, is called the *almost split sequence starting at* X, and we shall denote Coker$f$ by $\tau^{-1}X$. Therefore, τ induces a bijection between the set of all indecomposable modules which are not projective and the set of all indecomposable modules which are not injective. Moreover, if X is an indecomposable module which is not projective, then $\tau^{-1}\tau X \cong X$ and if X is not injective, then $\tau\tau^{-1}X \cong X$.

2.5. The following relation is very useful in the calculations of $\text{Ext}^1_A(-,?)$.

Theorem (Auslander-Reiten formulae.) *Let X, $Y \in A$-mod. Then there exist isomorphisms*

$$Ext^1_A(X,Y) \cong D\underline{Hom}_A(\tau^{-1}Y, X) \cong D\overline{Hom}_A(Y, \tau X)$$

where $\underline{Hom}_A(M,N)$ and $\overline{Hom}_A(M,N)$ denote the set of homomorphisms from M to N which factor through projective and injective, respectively.

Remarks. For a given indecomposable nonprojective module X, τX is isomorphic to D$\text{T}X$, where D $= \text{Hom}_k(-k)$ is the usual duality and Tr is the transpose. Recall that the transpose Tr of a module X is defined as follows. Consider first

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} X \longrightarrow 0$$

the minimal projective presentation of X and apply $\text{Hom}_A(-, A) = (-)^t$ to it. The transpose of X, TrX, is defined to be $\text{Coker}(p_1^t)$. Dually, if X is an indecomposable noninjective module, then $\tau^{-1}X$ is isomorphic to TrDX.

2.6. Another important type of maps are the so-called irreducible maps. We shall recall their definition and relate it to the previous notions.

Definition. A morphism $f\colon X \longrightarrow Y$ is called *irreducible* if

(i) f does not split;
(ii) if $f = gh$, then either g is a split monomorphism or h is a split epimorphism.

Remark. If $f\colon X \longrightarrow Y$ is an irreducible map, then f is either a monomorphism or an epimorphism. In fact, just consider the decomposition $f = gh$, where $h\colon X \longrightarrow Imf$ is the natural projection and $g\colon Imf \longrightarrow Y$ is the natural inclusion. Since f is irreducible, then either g is a split epimorphism, and therefore f is an epimorphism, or h is a split monomorphism and then f is a monomorphism.

2.7. The next result relates almost split maps and irreducible maps.

Proposition. *Let $f\colon X \longrightarrow Y$ be a morphism in modA.*

(a) *If Y is indecomposable, then f is an irreducible map if and only if there exists $f'\colon X' \longrightarrow Y$ such that $(f, f')\colon X \oplus X' \longrightarrow Y$ is a minimal right almost split map.*

(b) *If X is indecomposable, then f is an irreducible map if and only if there exists $f'\colon X \longrightarrow Y'$ such that $(f, f')^t\colon X \longrightarrow Y \oplus Y'$ is a minimal left almost split map.*

Remark. Let $f\colon X \longrightarrow Y$ be a map in ind$A$. It is not difficult to see that f is irreducible if and only if $f \in \operatorname{rad}(X, Y) \setminus \operatorname{rad}^2(X, Y)$.

2.8. We shall now define the important notion of Auslander-Reiten quiver Γ_A of an algebra A. The vertices $(\Gamma_A)_0$ of Γ_A are in a one-to-one correspondence with the isomorphism classes of indecomposable A-modules. For each $X \in \operatorname{ind}A$, let $[X]$ denote the corresponding vertex in $(\Gamma_A)_0$. Now, for $X, Y \in \operatorname{ind}A$, the number of arrows from $[X]$ to $[Y]$ is defined to be the dimension of the vector space $\operatorname{rad}(X, Y)/\operatorname{rad}^2(X, Y)$.

It follows from the discussion above that the structure of Γ_A is intimately related with the almost split sequences and the Auslander-Reiten translations. Observe in particular that:

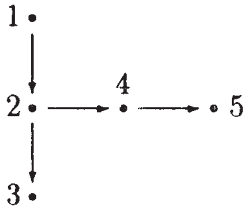
(a) Γ_A has no loops. Otherwise, there would exist an irreducible map from a vertex to itself, which is not possible.

(b) Γ_A is locally finite, that is, for each vertex $[X]$, there exist only finitely many vertices linked with $[X]$ by an arrow.

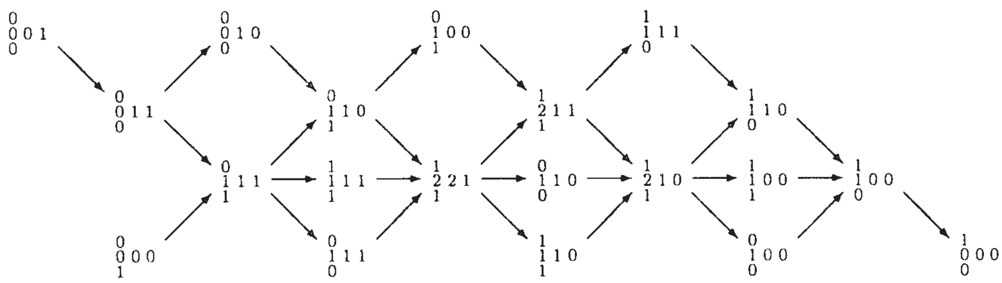
(c) The Auslander-Reiten translations induce a bijection, also denoted by τ, between a subset $\overline{\Gamma}$ of $(\Gamma_A)_0$ (formed by all vertices corresponding to nonprojective modules) and $\underline{\Gamma}$ of $(\Gamma_A)_0$ (formed by all vertices corresponding to noninjective modules) such that for each $[X] \in \overline{\Gamma}$, the number of arrows from a vertex $[Y]$ to $[X]$ equals the number of arrows from $\tau[X]$ to $[Y]$. A such a bijection is called *translation in* Γ_A.

A quiver Δ is called a *translation quiver* if it satisfies conditions (a) and (b), and it has a translation defined on it. Therefore, the Auslander-Reiten quiver of an algebra is a translation quiver. In general, Γ_A is not a connected quiver. In fact, it has been conjectured that Γ_A is connected if and only if A is representation-finite. A *sectional path* in Γ_A is a path $x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_t$ where for each $i = 2, \cdots, t$, $\tau x_i \neq x_{i-1}$.

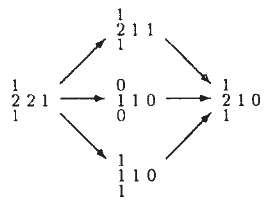
2.9. Examples. (a) Let $k\Delta$, where Δ is the quiver $\mathbf{D}_5$ with the following orientation



We shall indicate each A-module by the dimension vector, that is, by the dimension of the vector spaces at each vertex. With this notation Γ_A is the following quiver



It is not difficult to read from the above quiver the almost split sequences and the translations. For instance



is an almost split sequence and clearly $\tau(\substack{1\\2\,1\,0\\1}) = \substack{1\\2\,2\,1\\1}$.

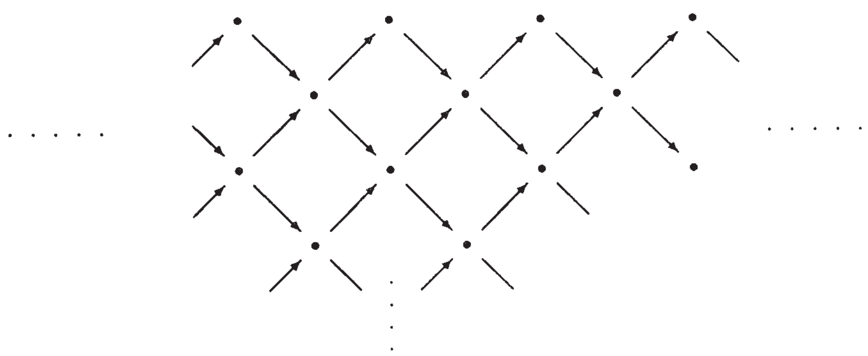
(b) Let $A = k\Delta$ be a hereditary algebra, where Δ is a connected quiver without oriented cycles. The structure of the Auslander-Reiten quiver of A is by now well-understrood (see [7]). There exists a component $\mathcal{P}$ containing all the indecomposable projective A-modules, called *postprojective component*, satisfying the following: (i) if

$X \in \mathcal{P}$, then $\tau^n X$ is projective, for some positive integer n; and (ii) $\mathcal{P}$ has no oriented cycles.

Dually, all the indecomposable injective A-modules lie in a *preinjective component* $\mathcal{I}$, that is, a component satisfying (i′) if $X \in \mathcal{I}$, then $\tau^{-n} X$ is injective, for some positive integer n; and (ii) $\mathcal{I}$ has no oriented cycles.

If A is representation-finite (or equivalently, if Δ is a Dynkin quiver), then the postprojective and the preinjective components coincide and it is the unique component of Γ_A (see the example (a) above). In case A is representation-infinite, then $\mathcal{P}$ and $\mathcal{I}$ are distinct and there are infinitely many other components (all of them with neither projective nor injective). These components can be of two types:

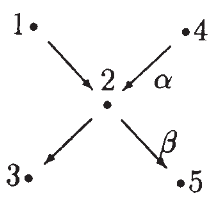
- either of the form ${}_{\infty}$, that is,



(infinite in three directions)

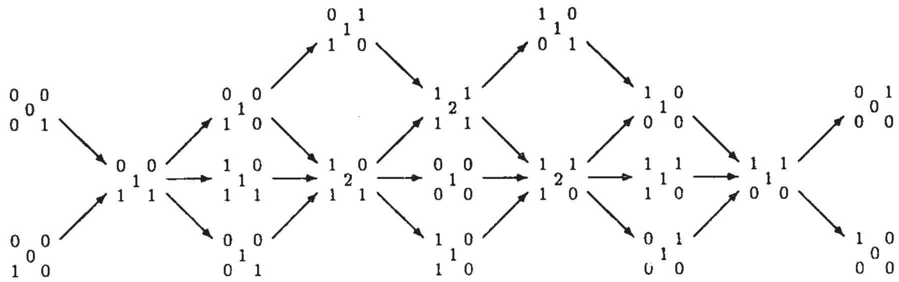
- or of type ${}_{\infty}/(\tau^r)$, for some r, that is $\tau^r X \cong X$ for each X in this component. Such a component is called a *tube of rank r*. If Γ_A contains tubes, then all but finitely many of them have rank 1.

(c) Let A be given by the quiver



with $\beta\alpha = 0$

Observe that A is not hereditary. The Auslander-Reiten quiver of A is



We shall agree to identify indecomposable A-modules with the corresponding vertices in Γ_A.

3. Tilting modules and torsion theory

3.1. Let A be an algebra. In this section we shall discuss the notion of tilting A-module T_A and see how the categories mod$A$ and mod(End(T_A)) are related. Such a module will induce a torsion theory in mod$A$ which, by the important theorem of Brenner-Butler, is related to a torsion theory in mod(End(T_A)).

3.2. We start with the following definition.

Definition. Let A be an algebra. An A-module T_A is called a *tilting module* provided:

(T1) $\mathrm{pd}_A T \leq 1$;
(T2) $\mathrm{Ext}^1_A(T,T) = 0$;
(T3) There exists a short exact sequence $0 \longrightarrow A_A \longrightarrow T'_A \longrightarrow T''_A \longrightarrow 0$, where T'_A and T''_A belong to add$T$.

Remarks. (a) Under hypothesis (T1) and (T2), the condition (T3) is equivalent to:

(T3′) The number of nonisomorphic indecomposable summands of T equals the number of nonisomorphic simple A-modules (= rank of the Grothendieck group $\mathrm{K}_0(A)$ of A).

(b) If A is hereditary, then the condition (T1) is satisfied naturally. We shall concentrate in this situation in the next section.

3.3. Examples. Let A be an algebra and $A_A = P_1^{n_1} \oplus \cdots \oplus P_t^{n_t}$ be its decomposition into indecomposable modules. Since we are assuming that A is basic, then $n_i = 1$, for each $i = 1, \cdots, t$.
(1) The module $T = P_1 \oplus \cdots \oplus P_t$ is clearly a tilting module.
(2) Suppose now that A is hereditary and let $T = DA = \mathrm{Hom}_k(A,k)$. Then, T is the sum of the indecomposable injective A-modules. Therefore, $\mathrm{Ext}^1_A(T,T) = 0$. Consider now the following short exact sequence

$$0 \longrightarrow A_A \stackrel{\iota}{\longrightarrow} T_1 \longrightarrow Coker(\iota) \longrightarrow 0$$

where ι is the injective envelope of A. Then $T_1 \in \text{add}T$, and since $\text{Coker}\iota$ is a quotient of T_1, it also belongs to $\text{add}T$. Therefore, T is a tilting module.

(3) Let A be an algebra and suppose there exists a simple projective noninjective module S_1. Let $A = S_1 \oplus P_2 \oplus \cdots \oplus P_t$ be the decomposition of A into indecomposable projective modules and consider the module

$$T^{(1)} \doteq \tau^{-1}S_1 \oplus (\oplus_{j=2}^{t} P_j)$$

We shall show that $T^{(1)}$ is a tilting module. Consider the almost split sequence starting at S_1

$$0 \longrightarrow S_1 \longrightarrow E \longrightarrow \tau^{-1}S_1 \longrightarrow 0$$

It is not difficult to see that E is a projective A-module (in fact the summands of E are those indecomposable projective modules P_a, whenever there is an arrow $a \longrightarrow 1$). The above sequence shows (T1) and (T3). To show (T2), just observe, using (2.5), that

$$\text{Ext}_A^1(T^{(1)}, T^{(1)}) \cong \text{DHom}_A(T^{(1)}, \tau T^{(1)}) \cong \text{DHom}_A(T^{(1)}, S_1) = 0$$

because S_1 is a simple projective. The tilting module constructed above is called *APR-tilting* because its construction is due to Auslander-Platzeck-Reiten [5]. It generalizes the reflection functors of Bernstein-Gelfand-Ponomarev [12].

(4) The APR-tilting modules were generalized by Brenner-Butler [11] in the following way. Let A be an algebra and S_1 be a simple A-module such that $\text{pd}\tau^{-1}S_1 \leq 1$ and $\text{Ext}_A^1(S_1, S_1) = 0$. Then the module $T = \tau^{-1}S_1 \oplus (\oplus_{j\neq 1} P_j)$ is a tilting module (where P_1 is the projective cover of S_1). Conditions (T1) and (T3′) are clearly satisfied. The claim follows from the following:

$$\text{Ext}_A^1(\tau^{-1}S_1, P_j) \cong \text{DHom}_A(P_j, S_1) = 0 \quad \text{if } j \neq 1$$

$$\text{and} \quad \text{Ext}_A^1(\tau^{-1}S_1, \tau^{-1}S_1) \cong \text{DHom}_A(\tau^{-1}S_1, S_1) = 0.$$

3.4. Before we go on, let us show that a module M satisfying conditions (T1) and (T2) (called a *partial tilting module*) can be completed to a tilting module. This result is due to Bongartz [9].

Lemma (Bongartz). *Let M be a module over the algebra A satisfying properties (T1) and (T2) of definition (3.2). Then there exists an A-module X such that $M \oplus X$ is a tilting module.*

Proof. Let $\alpha_1, \cdots, \alpha_s$ be a k-basis of $\text{Ext}_A^1(M, A)$, and consider the short exact sequence

$$0 \longrightarrow A \longrightarrow X \longrightarrow M^s \longrightarrow 0 \qquad (*)$$

defined as the pushout of the exact sequence $\oplus_{i=1}^{s}\alpha_i$ along the diagonal map $A^s \longrightarrow A$. We shall show that $M \oplus X$ is a tilting module. For this, we shall only prove that

$\mathrm{Ext}^1_A(M\oplus X, M\oplus X) = 0$ and $\mathrm{pd}_A X \leq 1$, the sequence $(*)$ above giving the condition (T3). Observe that $\mathrm{pd}_A X \leq 1$ because $\mathrm{pd}_A M \leq 1$ (see [10]). Apply $\mathrm{Hom}_A(M, -)$ to $(*)$:

$$\cdots \mathrm{Hom}_A(M, M^s) \xrightarrow{\phi} \mathrm{Ext}^1_A(M, A) \longrightarrow \mathrm{Ext}^1_A(M, X) \longrightarrow 0$$

Since, by construction, ϕ is an epimorphism, we infer that $\mathrm{Ext}^1_A(M, X) = 0$. Applying $\mathrm{Hom}_A(-, M)$ and $\mathrm{Hom}_A(-, X)$ to $(*)$, we get, respectively

$$0 = \mathrm{Ext}^1_A(M^s, M) \longrightarrow \mathrm{Ext}^1_A(X, M) \longrightarrow \mathrm{Ext}^1_A(A, T) = 0$$

and

$$0 = \mathrm{Ext}^1_A(M^s, X) \longrightarrow \mathrm{Ext}^1_A(X, X) \longrightarrow \mathrm{Ext}^1_A(A, X) = 0$$

Hence, $\mathrm{Ext}^1_A(M \oplus X, M \oplus X) = 0$ and $M \oplus X$ is a tilting module. □

3.5. From now on, let A be an algebra and T_A be a tilting module. We shall see now that T_A induces a torsion theory in mod$A$. We start by recalling such a notion.

Definition. A *torsion theory* in mod$A$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of modules such that (i) $\mathrm{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}$ and $N \in \mathcal{F}$.
(ii) If $\mathrm{Hom}_A(M, X) = 0$ for all $X \in \mathcal{F}$, then $M \in \mathcal{T}$.
(iii) If $\mathrm{Hom}_A(X, N) = 0$ for all $X \in \mathcal{T}$, then $M \in \mathcal{F}$.
The class $\mathcal{T}$ is called the *torsion class* of this theory and its elements are called *torsion modules*, while $\mathcal{F}$ is called the *torsion-free class* and its elements are called *torsion-free modules.*

Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in mod$A$. For each $M \in$ mod$A$, define $\mathrm{t}M$ as the sum of all submodules of M which belong to $\mathcal{T}$. The submodule $\mathrm{t}M$ is called the *trace* of $\mathcal{T}$ in M. Clearly, $\mathrm{t}(\mathrm{t}(M)) \in \mathcal{T}$ and $M/\mathrm{t}M \in \mathcal{F}$.

Remarks. (i) Each of $\mathcal{T}$, $\mathcal{F}$, and t uniquely determines the others (see [18]).
(ii) Each simple A-module is either torsion or torsion-free.

3.6. Define now $\mathcal{T}(T_A)$ to be the set

$$\mathrm{Gen}(T_A) = \{X \in \mathrm{mod}A : \text{ there exists an epimorphism } T^d \longrightarrow X \text{ for some } d\}$$

and

$$\mathcal{F}(T_A) = \{M_A : \mathrm{Hom}_A(T, M) = 0\}$$

Proposition. *The pair $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ is a torsion theory in modA.*

Proof. We shall show that $\mathrm{Gen}(T_A)$ is a torsion theory class. It is not difficult to see that this will imply that $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ is a torsion theory in mod$A$. By [18], a class of modules $\mathcal{X}$ is a torsion class if (and only if) $\mathcal{X}$ is closed under images, direct sums and extensions. Clearly, $\mathrm{Gen}(T_A)$ is closed under images and direct sums. To show that it is also closed under extensions, we shall first show that $\mathrm{Ext}^1_A(T,M)=0$ for each $M \in \mathrm{Gen}(T_A)$. In fact, if $M \in \mathrm{Gen}(T_A)$, then, by definition, there exists an epimorphism $T^m \longrightarrow M$, for some m. By (T1), it induces an epimorphism

$$\mathrm{Ext}^1_A(T,T^m) \longrightarrow \mathrm{Ext}^1_A(T,M) \longrightarrow 0$$

Since $\mathrm{Ext}^1_A(T,T^m)=0$ (by (T2)), we conclude that $\mathrm{Ext}^1_A(T,M)=0$ as required. Let now $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence with M' and M'' in $\mathrm{Gen}(T)$, and apply $\mathrm{Hom}_A(T,-)$ to it

$$0 \longrightarrow \mathrm{Hom}_A(T,M') \longrightarrow \mathrm{Hom}_A(T,M) \longrightarrow \mathrm{Hom}_A(T,M'') \longrightarrow \mathrm{Ext}^1_A(T,M')=0$$

Let $B=\mathrm{End}T$ and apply $-\otimes_B T$ to obtain the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathrm{Hom}_A(T,M')\otimes_B T & \longrightarrow & \mathrm{Hom}_A(T,M)\otimes_B T & \longrightarrow & \mathrm{Hom}_A(T,M'')\otimes_B T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon_{M'}} & & \downarrow{\scriptstyle \epsilon_M} & & \downarrow{\scriptstyle \epsilon_{M''}} & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

where ϵ_X is the evaluation map of X. We leave as exercise to show that $\epsilon_{M'}$ and $\epsilon_{M''}$ are surjective. Thus, ϵ_M is also surjective by the Five Lemma. Clearly now, $M \in \mathrm{Gen}(T_A)$ and the result is proven. □

3.7. The next result gives another description of $\mathcal{T}(T)$.

Lemma. *For a tilting module T,*

$$\mathcal{T}(T) = Gen(T_A) = \{M : Ext^1_A(T,M)=0\}$$

Proof. We have seen in the proof of (3.6) that $\mathrm{Ext}^1_A(T,M)=0$ for each $M \in \mathrm{Gen}(T_A)$. Therefore,

$$\mathcal{T}(T) = \mathrm{Gen}(T_A) \subset \{M : \mathrm{Ext}^1_A(T,M)=0\}.$$

Let now M such that $\mathrm{Ext}^1_A(T,M)=0$, and consider the short exact sequence

$$0 \longrightarrow tM \longrightarrow M \longrightarrow \frac{M}{tM} \longrightarrow 0$$

Applying $\mathrm{Hom}_A(T,-)$ to it, and using (T1), we conclude that there exists an epimorphism

$$\mathrm{Ext}^1_A(T,M) \longrightarrow \mathrm{Ext}^1_A(T,\frac{M}{tM}) \longrightarrow 0$$

and so $\mathrm{Ext}^1_A(T, \frac{M}{\mathrm{t}M}) = 0$ because $\mathrm{Ext}^1_A(T, M) = 0$.
Applying now $\mathrm{Hom}_A(-, \frac{M}{\mathrm{t}M})$ to the sequence given by (T3), we get

$$0 = \mathrm{Hom}_A(T', \frac{M}{\mathrm{t}M}) \longrightarrow \mathrm{Hom}_A(A, \frac{M}{\mathrm{t}M}) \longrightarrow \mathrm{Ext}^1_A(T'', \frac{M}{\mathrm{t}M}) = 0$$

By above, we conclude that $0 = \mathrm{Hom}_A(A, \frac{M}{\mathrm{t}M}) \cong \frac{M}{\mathrm{t}M}$. Therefore, $M = \mathrm{t}M \in \mathcal{T}(T)$ and the lemma is proven. □

3.8. We shall keep from now on the following convention. For a given tilting module T, the torsion theory induced by T is the pair $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ as defined above.

Remark. Any injective A-module belongs to $\mathcal{T}(T_A)$.

Example. Let A be an algebra such that there exists a simple projective noninjective module S_1 and let $T^{(1)} = \tau^{-1}S_1 \oplus (\oplus_{j=2}^{t} P_j)$ be an APR-tilting module (3.3). Observe that $M \in \mathcal{T}(T)$ if and only if

$$0 = \mathrm{Ext}^1_A(T, M) = D\mathrm{Hom}_A(M, \tau T) = D\mathrm{Hom}_A(M, S_1)$$

Then $\mathrm{ind}\mathcal{T}(T) = \mathrm{ind}A \setminus \{S_1\}$. It is not difficult to see that $\mathrm{ind}\mathcal{F}(T) = \{S_1\}$ (in the terminology of (4.4), we would say that $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ is a splliting torsion theory).

3.9. We shall now state the important Brenner-Butler theorem. However, we shall not prove it here and refer to [11, 24] for a proof.

Theorem. *Let A be an algebra, T_A be a tilting module, and $B = End_A T$. Then*

(i) *${}_BT$ is a tilting B-module and $A \cong End_B T$.*

(ii) *The categories*

$$\mathcal{T}(T_A) = \{M_A : Ext^1_A(T, M) = 0\} \quad and \quad \mathcal{Y}(T_A) = \{N_B : Tor_1^B(N, T) = 0\}$$

are equivalent, the equivalences giving by the functors $Hom_A(T, -)$ and $- \otimes_B T$.

(iii) *The categories*

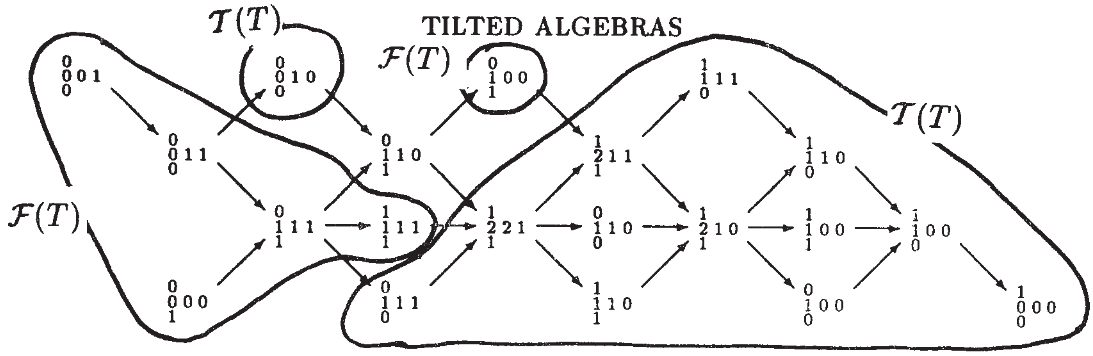
$$\mathcal{F}(T_A) = \{M_A : Hom_A(T, M) = 0\} \quad and \quad \mathcal{X}(T_A) = \{N_B : N \otimes_B T = 0\}$$

are equivalent, the equivalences giving by the functors $Ext^1_A(T, -)$ and $Tor_1^B(-, T)$.

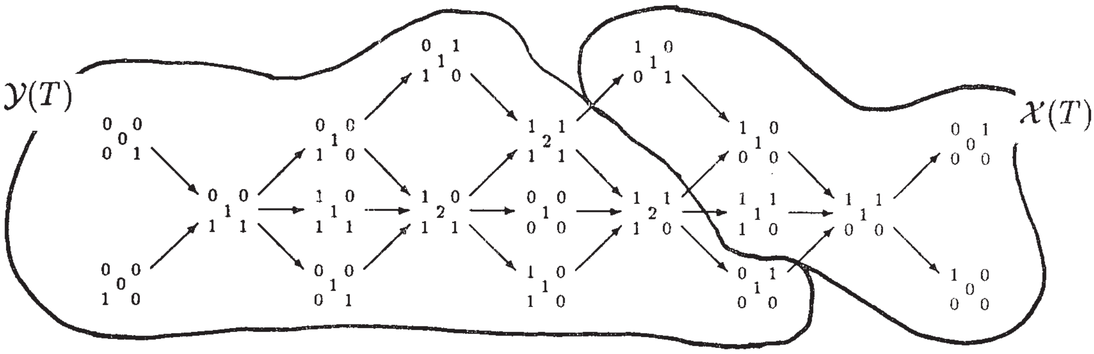
3.10. Example. Let A be the algebra given in example (2.9)(a) and let

$$T = \begin{smallmatrix} 0 & & \\ 1 & 1 & 0 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} 1 & & \\ 2 & 2 & 1 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} 0 & & \\ 1 & 1 & 1 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} 1 & & \\ 1 & 1 & 1 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} 0 & & \\ 0 & 1 & 0 \\ 0 & & \end{smallmatrix}$$

be a tilting module. We leave to the reader to check that it is really a tilting module (use (2.5) and the fact that for a representation-finite algebra, each morphism, which is not an isomorphism, is a sum of composite of irreducible maps). We indicate below the classes $\mathcal{T}(T)$ and $\mathcal{F}(T)$.



Consider now B =End(T). It is not difficult to see that B is the algebra given by the example (2.9)(c). Therefore Γ_B is



where $\mathcal{X}(T)$ and $\mathcal{Y}(T)$ are indicated .

3.11. Remarks. There is a generalization of the notion of tilting modules due to Miyashita [32]. An A-module T is called a *generalized tilting module* provided:

(i) pd$T < \infty$;

(ii) $\mathrm{Ext}^i_A(T,T) = 0$ for all $i > 0$.

(iii) There exists a long exact sequence

$$0 \longrightarrow_A A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_m \longrightarrow 0$$

with $T_i \in$add(T), for all $i = 0, 1, \cdots, m$.

Let T be a generalized tilting module. In general, the relations between mod$A$ and mod(End$_A T$)^{op} are not as good as in the situation when pd$T \leq 1$, which is the situation we have been considering. We also observe that the Bongartz's lemma (3.4) does not hold in this generalization as observed by Rickard-Schofield in [33]. We refer also to [14, 16, 27] for more on this problem.

4. Tilted algebras

4.1. As we saw in the last section, a tilting module T over an algebra A induces an equivalence between subcategories of mod$A$ and mod(End$_A T$) given by the Brenner-Butler theorem. In this section we shall study a particular case where the union of the subcategories $\mathcal{X}$ and $\mathcal{Y}$ (in the notation of the Brenner-Butler theorem) gives the whole category mod($\text{End}_A T$). This will occur, for instance, when A is hereditary. In fact, we shall concentrate in this last situation.

Definition. Let $A = k\Delta$ be a hereditary algebra and T be a tilting module over A. The algebra $B = (\text{End}_A T)$ is called *tilted algebra.* The *type of* B is the underlying graph $\overline{\Delta}$ of Δ. Recall that the underlying graph of a quiver Δ is a graph with the same set of vertices of Δ and where the arrows are changed by edges.

Example. Let A be the algebra given in example(2.9)(c). We have seen at the end of last section that A is tilted of type $\mathbf{D}_5$.

4.2. We shall first investigate some homological properties of tilted algebras. We start with the following result.

Proposition. *Let T be a tilting module over a hereditary algebra $A = k\Delta$, $B = End_A(T)$, and M be an A-module.*

(a) *If $M \in \mathcal{T}(T)$, then $pd_B Hom_A(T, M) \leq 1$.*
(b) *If $M \in \mathcal{F}(T)$, then $id_B Ext_A^1(T, M) \leq 1$.*

Proof. We shall prove only (a) since the proof of (b) is similar.
(a) If $\text{pd}_A M = 0$, then M is a projective and since it belongs to $\mathcal{T}(T)$ we infer that $M \in \text{add}(T)$. Hence $\text{Hom}_A(T, M)$ is projective (or equivalently, $\text{pd}_B\text{Hom}_A(T, M) = 0$). Assume now that $\text{pd}_A M = 1$. Observe that there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow T_0 \longrightarrow M \longrightarrow 0 \qquad (*)$$

with $T_0 \in \text{add}T$ and $K \in \mathcal{T}(T)$. Applying $\text{Hom}_A(T, -)$ to $(*)$, we get

$$0 \longrightarrow \text{Hom}_A(T, K) \longrightarrow \text{Hom}_A(T, T_0) \longrightarrow \text{Hom}_A(T, M) \longrightarrow \text{Ext}_A^1(T, K) = 0$$

It suffices to show that $\text{Hom}_A(T, K)$ is projective, or equivalently, that $K \in \text{add}T$. Let now $N \in \mathcal{T}(T)$ and apply $\text{Hom}_A(-, N)$ to $(*)$.

$$0 = \text{Ext}_A^1(T_0, N) \longrightarrow \text{Ext}_A^1(K, N) \longrightarrow \text{Ext}_A^2(M, N) = 0$$

Therefore, $\text{Ext}_A^1(K, -)|_{\mathcal{T}(T)} = 0$, that is, K is Ext-projective in $\mathcal{T}(T)$ and so $K \in \text{add}T$. Hence $\text{pd}_B\text{Hom}_A(T, M) \leq 1$, as required. □

Corollary. *Let B be a tilted algebra. Then for each $X \in indB$ either $pd_B X \leq 1$ or $id_B X \leq 1$.*

4.3. Theorem. *The global dimension of a tilted algebra is at most two.*

Proof. Let $A = k\Delta$ be a hereditary algebra, T be a tilting A-module and $B = \text{End}T$. We shall prove that gl.dim $B \leq 2$. Let $N_B \in B$-mod and consider the short exact sequence

$$0 \longrightarrow Z \longrightarrow P(N) \xrightarrow{\pi} N \longrightarrow 0$$

where $\pi\colon P(N) \longrightarrow N$ is the projective cover of N. Since $P(N) \in \mathcal{Y}(T)$, we infer that $Z \in \mathcal{Y}(T)$. By the Brenner-Butler theorem, there exists a module $M_A \in \mathcal{T}(T)$ such that $Z = \text{Hom}_A(T, M)$, and by the above, $\text{pd}_B Z \leq 1$. Therefore, $\text{pd}_B N \leq 2$ and the theorem is proven. □

4.4. Let $B = \text{End}T$ be a tilted algebra. The next result will show that each B-module belongs to one of the subcategories $\mathcal{X}(T)$ or $\mathcal{Y}(T)$. We say that a tilting module T over an (arbitrary) algebra A is *splitting* provided the torsion theory $(\mathcal{X}(T), \mathcal{Y}(T))$ splits, that is, each $X \in \text{ind}(\text{End}_A T)$ belongs either to $\mathcal{X}(T)$ or to $\mathcal{Y}(T)$.

Proposition. *Any tilting module over a hereditary algebra is splitting.*

Proof. By [23](III.4.12), a tilting module splits if and only if for each $M \in \mathcal{F}(T)$, $\text{id}_A M \leq 1$. Clearly, this last condition is satisfied for hereditary algebras, giving the desired result. □

This result has as consequence that the representation type of a tilted algebra is not more complicated than the representation type of the hereditary algebra it came from. In fact, it can be much simplier. For instance, let $A = k\Delta$ be a hereditary algebra, where Δ is an Euclidean quiver. Therefore, A is representation-infinite and Γ_A is described in (2.9)(b). Let now T be a tilting module with summands in both the postprojective and the preinjective components of Γ_A. Then $B = \text{End}T$ is representation-finite (see [25]).

4.5. We shall now investigate some facts concerning the Auslander-Reiten quiver of a tilted algebra. For this, we shall see how is the behavior of the almost split sequences under the tilting process. Also, we shall see how the type of a tilted algebra can be read of from its Auslander-Reiten quiver through the notion of complete slices.

In order to study the relation between almost split sequences in A and in B, we need the following lemma.

Lemma (Connecting lemma). *Let T be a tilting module over a hereditary algebra $A = k\Delta$ and $B = \text{End}_A T$. Let P_S and I_S be respectively the indecomposable projective and the indecomposable injective A-modules associated with the simple S. Then*

$$\tau^{-1}\text{Hom}_A(T, I) \cong \text{Ext}^1_A(T, P).$$

In particular, $P_S \in \text{add}(T)$ if and only if $\text{Hom}_A(T, I_S)$ is an injective B-module.

Proof. Consider the short exact sequence

$$0 \longrightarrow P_A \longrightarrow T_0 \xrightarrow{g} T_1 \longrightarrow 0$$

with T_0, $T_1 \in$ add (T) (from (T3)). Apply $\mathrm{Hom}_A(-T)$ to $(*)$ to get

$$0 \longrightarrow \mathrm{Hom}_A(T_1, T) \xrightarrow{(g,T)} \mathrm{Hom}_A(T_0, T) \longrightarrow \mathrm{Hom}_A(P, T) \longrightarrow \mathrm{Ext}_A^1(T_1, T) = 0$$

which is clearly a projective resolution for $\mathrm{Hom}_A(P,_B T)$ Applying now $\mathrm{Hom}_A(T, -)$ to $(*)$ yields

$$0 \longrightarrow \mathrm{Hom}_A(T, P) \longrightarrow \mathrm{Hom}_A(T, T_0) \longrightarrow \mathrm{Hom}_A(T, T_1) \xrightarrow{(T,g)} \mathrm{Ext}_A^1(T, P) \longrightarrow 0$$

By the definition of the transpose, we infer that $\mathrm{TrHom}_A(P, T) = \mathrm{Ext}_A^1(T, P)$ (see (2.5)). However, $\mathrm{Hom}_A(P, T) \cong \mathrm{DHom}_A(T, I)$. Then $\mathrm{Ext}_A^1(T, P) \cong \mathrm{TrDHom}_A(P, T) = \tau^{-1}\mathrm{Hom}_A(T, I)$ as required. $\square$

4.6. Theorem. *Let $B = End_A T$ be a tilted algebra. Then every almost split sequence in modB either lies completely in $\mathcal{X}(T)$, or lies completely in $\mathcal{Y}(T)$, or is of the form*

$$0 \longrightarrow Hom_A(T, I) \longrightarrow Hom_A(T, I/S) \oplus Ext_A^1(T, radP) \longrightarrow Ext_A^1(T, P) \longrightarrow 0$$

where P is an indecomposable projective A-module not in addT, S is its simple top and I the injective envelope of S.

Proof. Let $0 \longrightarrow \tau X \longrightarrow E \longrightarrow X \longrightarrow 0$ be an almost split sequence in mod$B$. By (4.4) we know that the torsion theory $(\mathcal{X}(T), \mathcal{Y}(T))$ splits. We shall analise the following possibilties:

(i) If $\tau X \in \mathcal{X}(T)$, then the sequence lies entirely in $\mathcal{X}(T)$.

(ii) If $X \in \mathcal{Y}(T)$, then the sequence lies entirely in $\mathcal{Y}(T)$.

(iii) Suppose $\tau X \in \mathcal{Y}(T)$ and $X \in \mathcal{X}(T)$, and let $M = \tau X \otimes_B T$. Denote by I the injective envelope of M. Therefore,

$$\mathrm{Ext}_A^1(I/M, M) \cong \mathrm{Ext}_B^1(\mathrm{Hom}_A(T, I/M), \mathrm{Hom}_A(T, M)) \cong$$

$$\cong \mathrm{Ext}_B^1(\mathrm{Hom}_A(T, I/M), \tau X) \cong \mathrm{D}\ \underline{\mathrm{Hom}_B}(\tau^{-1}\tau X, \mathrm{Hom}_A(T, I/M)) = 0$$

because $M \in \mathcal{T}(T)$ and $\mathrm{Hom}_A(T, I/M) \in \mathcal{Y}(T)$. Hence, M is a direct summand of I, in fact $I = M$. Then I is indecomposable and $\mathrm{Hom}_A(T, I) \cong \tau X$. By the lemma above, $X \cong \mathrm{Ext}_A^1(T, P)$, where P is the projective cover of soc$I$. Observe that $P \notin$ add(T). Therefore, the almost split sequence above is of the form

$$0 \longrightarrow \mathrm{Hom}_A(T, I) \longrightarrow E \longrightarrow \mathrm{Ext}_A^1(T, P) \longrightarrow 0$$

and by (4.5) the canonical sequence for E in $(\mathcal{X}(T), \mathcal{Y}(T))$ is

$$0 \longrightarrow \mathrm{Ext}_A^1(T, \mathrm{rad}P) \longrightarrow E \longrightarrow \mathrm{Hom}_A(T, I(M)/S) \longrightarrow 0$$

Since the torsion theory splits, we infer that $E \cong \mathrm{Hom}_A(T, I(M)/S) \oplus \mathrm{Ext}_A^1(T, \mathrm{rad}P)$ and this proves the result. □

4.7. We shall see now that the fact that an algebra is tilted can be read of from its Auslander-Reiten quiver. In fact, if B is a tilted algebra of type Δ, then Γ_B contain a connected sectional subquiver whose underlying graph is isomorphic to Δ. We call such a subquiver a complete slice. Let us define it more formally.

Definition. A class Σ of modules in ind$A$ is called a *complete slice* if:

(i) The sum $X = \oplus_{M \in \Sigma} M$ is sincere, that is $\mathrm{Hom}_A(P, X) \neq 0$ for any projective A-module P.

(ii) If $X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t$ is a sequence of nonisomorphisms in mod$A$, with $X_0, X_t \in \Sigma$, then $X_i \in \Sigma$ for all $0 < i < m$.

(iii) Let $0 \longrightarrow \tau Y \longrightarrow E \longrightarrow Y \longrightarrow 0$ be an almost split sequence. Then at most one of Y and τY lies in Σ. Moreover, if an indecomposable summand of E lies in Σ, then either Y or τY lie in Σ

The module X is called the *slice module* in Σ.

4.8. The next result relates complete slices and tilted algebras. For a proof we refer to [24, 35].

Theorem. *Let A be a hereditary algebra, $T \in modA$ a tilting module and $B = End_A T$ the corresponding tilted algebra.*

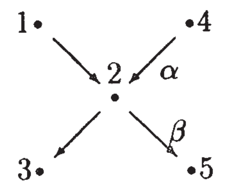
(i) *The class $\{Hom_A(T, I)$: I is an indecomposable injective A-module $\}$ is a complete slice in modB.*

(ii) *If Σ is a complete slice in modB, then $X_B = \oplus_{M \in \Sigma} M$ is a tilting module and $A \cong End_A X$ (and Σ is isomorphic to a complete slice of the previous form).*

(iii) *There exists at most two connected components of Γ_B containing complete slices. Moreover, Γ_B has exactly two connected components containing complete slices if and only if either all summands of T are postprojective or all summands of T are preinjective.*

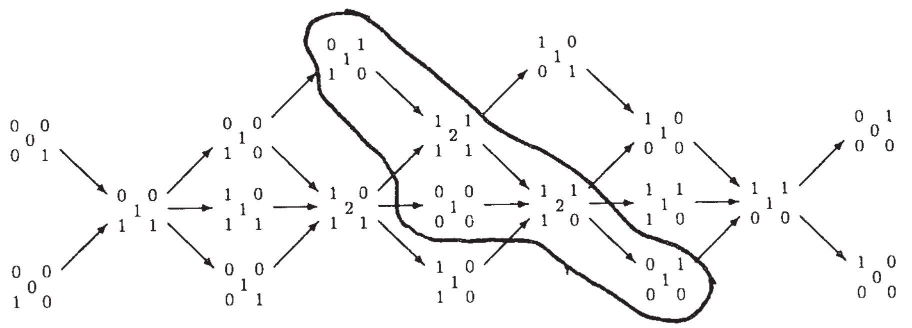
Let B be a tilted algebra. A connected component of Γ_B containing a complete slice is called *connecting component* (it contains the connecting sequences of (4.5)). Also, a tilted algebra with two distinct connecting components is called *concealed.* Concealed algebras have been extensively studied and play an important role in the theory nowadays. We refer to [35] for more details.

4.9. Example. Let B be the algebra given by



with $\beta\alpha = 0$

We have seen in (4.1) that B is a tilted algebra of type $\mathbf{D}_5$. It is not difficult to find a complete slice in Γ_B. Take the one marked below



Also, we can recover the tilting module T used to get B using (4.8) above. Indeed, we have

$$\mathrm{Hom}_A(T, I_1) = \begin{smallmatrix}0&&1\\&1&\\0&&0\end{smallmatrix} \quad \mathrm{Hom}_A(T, I_2) = \begin{smallmatrix}1&&1\\&2&\\1&&0\end{smallmatrix} \quad \mathrm{Hom}_A(T, I_3) = \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix}$$

$$\mathrm{Hom}_A(T, I_4) = \begin{smallmatrix}1&&1\\&2&\\1&&1\end{smallmatrix} \qquad \mathrm{Hom}_A(T, I_5) = \begin{smallmatrix}0&&1\\&1&\\1&&0\end{smallmatrix}$$

and then

$$T = \begin{smallmatrix}0&&\\1&1&0\\0&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\2&2&1\\1&&\end{smallmatrix} \oplus \begin{smallmatrix}0&&\\1&1&1\\0&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\1&1&1\\0&&\end{smallmatrix} \oplus \begin{smallmatrix}0&&\\0&1&0\\0&&\end{smallmatrix}$$

Observe that the above T is the one we used in example (3.9). If one choose another complete slice, one gets a different module. Observe that one can find, in this example, complete slices in any possible orientation of $\mathbf{D}_5$. We observe that this in not a general fact: the algebra C of (1.9) is a tilted algebra of type $\mathbf{A}_3$ and there is only one complete slice in Γ_C.

The structure of the Auslander-Reiten quiver of a tilted algebra is by now well known. We refer to [19, 28, 29, 30, 34, 35] for details.

4.10. Remarks. (1) We have seen that if A is a tilted algebra then gl.dim$A \leq 2$ and for each $X \in \text{ind}A$, pd$X \leq 1$ or $idX \leq 1$. These properties does not characterize tilted algebras. In [26], Happel-Reiten-Smalø introduced the notion of *quasitilted algebra*, that is an algebra A with gl.dim$A \leq 2$ and for each $X \in \text{ind}A$, pd$X \leq 1$ or $idX \leq 1$ and have shown several examples of quasitilted algebras which are not tilted. We also mention [15, 17] for more information on the Auslander-Reiten quiver of a quasitilted algebra.

(2) The tilting procedure can be iterated to get others interesting classes of algebras. We refer to [1] for an account on this.

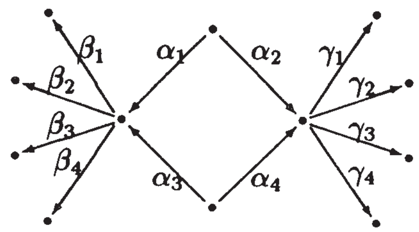
5. Complete slices and homological properties

5.1. The contents of this section follows very closely the paper [3], where in a joint work with I. Assem we have considered the study of some homological properties in relation to the position of a complete slice in the Auslander-Reiten quiver of a tilted algebra.

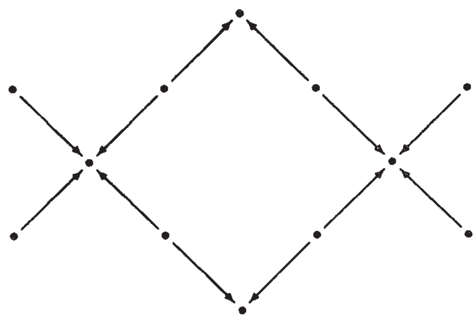
Let now A be a tilted algebra. We define the *left type* of A as follows. If A has a complete slice in a postprojective component, then the left type of A is defined to be the empty graph. Otherwise, A has a unique connecting component Γ which is not postprojective. If Γ contains no projective module (so that every module in Γ is left stable, that is $\tau^n M \neq 0$ for all $n \geq 0$ and all $M \in \Gamma$), we define the left type of A to be the type of the tilted algebra A, as defined above. Suppose Γ contains a projective module. Let Σ be the subsection of Γ consisting of the left stable modules M in Γ such that there exists a path in Γ of length at least one from M to some projective, and any such path is sectional. Since Σ is generally not connected, we can write it as $\Sigma_1 \cup \cdots \cup \Sigma_t$, where each Σ_i is a connected component of Σ. Then $\Sigma = \Sigma_1 \cup \cdots \cup$ gma$_t$ will be called *the left extremal subsection of* A, and its underlying graph $\overline{\Sigma}$ will be called the left type of A. Observe that, since Γ contains a complete slice, no injective module is a predecessor of Σ.

Dually, we define the *right type* of A as follows. If A has a complete slice in a preinjective component, the right type of A is defined to be the empty graph. Otherwise, A has a unique connecting component Γ which is not preinjective. If Γ contains no injective module (so that every module in Γ is right stable, that is $\tau^n M \neq 0$ for all $n \leq 0$ and all $M \in \Gamma$), we define the right type of A to be the type of the tilted algebra A. If Γ contains an injective module, let Σ be the subsection of Γ consisting of the right stable modules M in Γ such that there exists a path in Γ of length at least one from some injective to M, and any such path is sectional. Let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$, where each Σ_i is a connected component of Σ. Then $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$ will be called *the right extremal subsection of* A, and its underlying graph $\overline{\Sigma}$ will be called the r ht type of A. Observe that, since Γ contains a complete slice, no projective mod : is a successor of Σ.

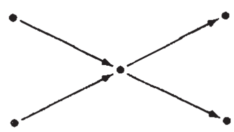
Example. We borrow this example from [28](5). Let A be given by the quiver



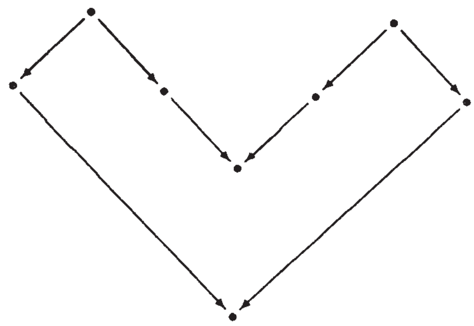
bound by $\alpha_1\beta_i = 0 = \alpha_2\gamma_i$ for $i = 1,2,3$ and $\alpha_3\beta_j = 0 = \alpha_4\gamma_j$ for $j = 2,3,4$. Then A is tilted, and its type is the underlying graph of the following representing slice



The left type of A equals the disjoint union of two copies of $\tilde{D}_4$, and the left extremal subsection equals the disjoint union of two copies of the quiver



The right type of A is equal to $\tilde{A}_7$, and the right extremal subsection is given by the quiver



5.2. The following lemma records the most immediate properties of the left and right types of a tilted algebra.

Lemma. *Let A be a tilted algebra.*

(a) *The left and the right types of A are empty graphs if and only if A is representation-finite or concealed.*

(b) *The left (or right) type of A equals the type of A if and only if A has a unique connecting component containing no projective (or no injective, respectively).*

Proof. (a) The left type of A is empty if and only if A has a complete slice in a postprojective component Γ, and the right type of A is empty if and only if A has a complete slice in a preinjective component Γ'. If $\Gamma = \Gamma'$, then A is representation-finite. If $\Gamma \neq \Gamma'$, then A is concealed. The converse is obvious.

(b) Follows directly from the definition. □

5.3. Of more immediate concern to us, however, is the relation between the left and right types and the left and right end algebras as defined in [28]. We record it in the following lemma.

Lemma. *Let A be a representation-infinite algebra which is tilted but not concealed.*

(a) *Each connected component of the left extremal subsection is a complete slice in the connecting component without projective modules of the Auslander-Reiten quiver of a connected component of the left end algebra ${}_{\infty}A$. In particular, the left type of A equals the type of ${}_{\infty}A$ as a tilted algebra.*

(b) *Each connected component of the right extremal subsection is a complete slice in the connecting component without injective modules of the Auslander-Reiten quiver of a connected component of the right end algebra A_{∞}. In particular, the right type of A equals the type of A_{∞} as a tilted algebra.*

Proof. Under the stated hypothesis, A has a unique connecting component. The statement then follows from the description of mod$A$ as given in [28], and the respective definitions. □

5.4. The following lemma is a direct consequence of the homological properties of tilting modules discussed earlier.

Lemma. *Let A be a tilted algebra and Σ be a complete slice in Γ_A.*

(a) *If M is an indecomposable successor of Σ, then $id_A M \leq 1$.*
(b) *If M is an indecomposable predecessor of Σ, then $pd_A M \leq 1$.*

Proof. We shall only prove (a), since the proof of (b) is dual.
Let $T = \oplus\{N : N \in \Sigma\}$. Assume $\mathrm{id}_A M > 1$ for some indecomposable successor M of Σ. Then $\mathrm{Hom}_A(\tau^{-1}M, A) \neq 0$ (by [35](2.4)(1*) p.74). Let P_A be an indecomposable projective module such that $\mathrm{Hom}_A(\tau^{-1}M, P) \neq 0$. Since M is a successor of Σ, so are $\tau^{-1}M$ and P. Hence P belongs to the subcategory of all A-modules generated by T so that there exist $m > 0$ and an epimorphism $T^{(m)} \longrightarrow P$ which must split, because P is projective. Hence P is a direct summand of T, thus belongs to Σ. We have obtained a sequence of non-zero non-isomorphisms $U \to \cdots \to M \to * \to \tau^{-1}M \to P$ with both U, P on Σ, which is not sectional, a contradiction to the fact that Σ is a complete slice. □

5.5. As a consequence of this lemma and the fact that the global dimension of a tilted algebra is at most two, we have:

Corollary. *A tilted algebra A is representation-finite if and only if $pd_A M = 2$ and $id_A M = 2$ for almost all M in ind$A$.*

Proof. Since the necessity is obvious, let us show the sufficiency. Since A is tilted, there exists a complete slice Σ in Γ_A. By (5.4), Σ has only finitely many predecessors and only finitely many successors. Hence A is representation-finite. □

5.6. The following result was obtained in [3](3.4), and independently in [38].

Proposition. *A representation-infinite algebra A is concealed if and only if $pdM = 1$ and $idM = 1$ for almost all M in ind$A$.*

5.7. We shall now define the reduced right and left types of a representation-infinite tilted algebra A.

Let A be a representation-infinite tilted algebra. We first define the *reduced left type of A*. If A has a postprojective component containing a complete slice or if the unique connecting component of A contains no projective module, we define the reduced left type of A to be equal to the left type of A, that is, respectively, the empty graph and the type of A. Assume that the unique connecting component of A is not postprojective but contains projectives, and let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$ be the

left extremal subsection of A. We define the reduced left extremal subsection of A to be $\Sigma'_1 \cup \cdots \cup \Sigma'_t$ where, for each i, Σ'_i is the full (convex) subquiver of Σ_i obtained by deleting all the sinks. The reduced left type of A is then the underlying graph $\overline{\Sigma}'_1 \cup \cdots \cup \overline{\Sigma}'_t$ of the reduced left extremal subsection. Observe that the sinks of Σ_i correspond to radical
summands of projective A-modules.

Likewise, we define the *reduced right type of A*. If A has a preinjective component containing a complete slice or if the unique connecting component of A contains no injective module, we define the reduced right type of A to be equal to the right type of A, that is, respectively, the empty graph and the type of A. Assume that the unique connecting component of A is not preinjective but contains injectives, and let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$ be the right extremal subsection of A. We define the reduced right extremal subsection of A to be $\Sigma'_1 \cup \cdots \cup \Sigma'_t$ where, for each i, Σ'_i is the full (convex) subquiver of Σ_i obtained by deleting all the sources. The reduced right type of A is then the underlying graph $\overline{\Sigma}'_1 \cup \cdots \cup \overline{\Sigma}'_t$ of the reduced right extremal subsection . Observe that the sources of Σ_i correspond to socle factors of injective A-modules.

Since A is representation-infinite, the reduced left (or right) type of A is empty if and only if so is the left (or right, respectively) type of A.

For instance, in the example in (5.1), the reduced left extremal subsection is the disjoint union of two copies of the quiver

while the reduced right extremal subsection is the disjoint union of two copies of the quiver

In particular, both the reduced right and left types of A are disjoint unions of Dynkin graphs, so that A satisfies the conditions of our main theorem in this section, which we shall now prove.

Theorem. *Let A be a representation-infinite algebra which is tilted but not concealed, and Σ be a complete slice in Γ_A.*

(a) *$pdM = 2$ for almost all indecomposable successors M of Σ if and only if the reduced right type of A is empty or a disjoint union of Dynkin graphs.*
(b) *$idM = 2$ for almost all indecomposable predecessors M of Σ if and only if the reduced left type of A is empty or a disjoint union of Dynkin graphs.*

Proof. We shall only prove (a), since the proof of (b) is dual.
(i) Proof of sufficiency. Since A is not concealed, there exists only one component

Γ containing complete slices. Now, if the reduced right type is empty then Γ is preinjective, Σ has at most finitely many non-isomorphic indecomposable successors, and clearly, the projective dimension of almost all of them equals 2. Suppose now that the reduced right type of A is a non-empty disjoint union of Dynkin graphs, which, is equivalent, in our case, to say that Γ is not a preinjective component. Since the global dimension of A is at most 2 and $\mathrm{pd}_A M > 1$ if and only if $\mathrm{Hom}_A(DA, \tau M) \neq 0$ (by [35](2.4)(1) p.74), it suffices to prove that $\mathrm{Hom}_A(DA, N) \neq 0$ for almost all indecomposable successors N of Σ. Let thus $A_\infty = A_1 \times \cdots \times A_t$, where the A_i are connected tilted algebras. For each i, let Σ_i be the right extremal subsection in the connecting component of Γ_{A_i} as constructed in (5.1), so that the right type of A equals $\overline{\Sigma}_1 \cup \cdots \cup \overline{\Sigma}_t$ (by (5.3)). Let $T = \oplus\{M : M \in \Sigma_i\}$, considered as an A_i-module. Then $H = \mathrm{End}(T_{A_i})$ is a hereditary algebra and, for each source S in Σ_i, the H-module $S' = \mathrm{Hom}_{A_i}(T, S)$ is simple projective.
Let U denote the direct sum of all sources in Σ_i and set $U' = \mathrm{Hom}_{A_i}(T, U)$. Since U' is the direct sum of simple projective H-modules, we can write $U' = eH$ for some non-zero idempotent $e \in H$. The hereditary algebra $H' = \mathrm{End}(1-e)H$ has for type the full convex subquiver Σ_i' of Σ_i obtained by dropping the summands of U. That is, Σ_i' is the (disjoint union of the) component(s) of the reduced type corresponding to Σ_i, hence $\overline{\Sigma}_i'$ is a (disjoint union of) Dynkin graph(s), by hypothesis. Consequently, H' is representation-finite. This implies that $\mathrm{Hom}_H(U', X) \neq 0$ for almost all indecomposable H-modules X.
Let $(\mathcal{T}, \mathcal{F})$ denote the torsion theory induced by T in $\mathrm{mod}A_i$. It is easy to see that $\mathcal{F}$ consists of the proper predecessors of Σ_i in the connecting component of Γ_{A_i}, hence contains only finitely many non-isomorphic indecomposables. On the other hand, by the Brenner-Butler theorem

$$\mathrm{Hom}_{A_i}(U, M) \cong \mathrm{Hom}_H(U', \mathrm{Hom}_{A_i}(T, M))$$

for any M in $\mathcal{T}$. This implies that $\mathrm{Hom}_{A_i}(U, M) \neq 0$ for almost all M in $\mathrm{ind}A_i$.
Since, for any indecomposable summand S of U, there exists an indecomposable injective A-module I and an irreducible epimorphism $I \longrightarrow S$, we deduce that $\mathrm{Hom}_A(DA, N) \neq 0$ for almost all N in $\mathrm{ind}A_i$. This being true for each i, we infer that $\mathrm{Hom}_A(DA, N) \neq 0$ for almost all N in $\mathrm{ind}A_\infty$. Since almost all indecomposable successors of Σ are A_∞-modules, this completes the proof of the sufficiency.
(*ii*) Proof of necessity. Suppose that A is a representation-infinite tilted algebra and let Σ be a complete slice in a connecting component Γ of Γ_A such that $\mathrm{pd}_A M = 2$ for almost all indecomposable successors M of Σ. If Γ is a preinjective component then the reduced right type of A is the empty graph and we are done. Suppose from now on that Γ is not a preinjective component and let $\Sigma_1 \cup \cdots \cup \Sigma_t$ denote the right extremal subsection of A, where each Σ_i is connected. We must show that the reduced right extremal subsection $\Sigma_1' \cup \cdots \cup \Sigma_t'$ is such that, for each $1 \leq i \leq t$, $\overline{\Sigma}_i'$ is a (disjoint union of) Dynkin graph(s). Assume that, for some i, $\overline{\Sigma}_i'$ is not a (disjoint

union of) Dynkin graph(s) and let H' be the endomorphism algebra of the module $\oplus\{M : M \in \Sigma_i'\}$. Then H' is a representation-infinite full convex subcategory of the (hereditary) endomorphism algebra H of the module $\oplus\{N : N \in \Sigma_i\}$. By definition (and (5.3)), the types of A_i (which is the connected component of A_∞ having Σ_i as a complete slice in its connecting component) and of H coincide and equal $\overline{\Sigma}_i$. Moreover, there are infinitely many non-isomorphic indecomposable A_i-modules $(L_\lambda)_{\lambda\in\Lambda}$ such that $\mathrm{Hom}_A(S_1 \oplus \cdots \oplus S_m, L_\lambda) = 0$ for all $\lambda \in \Lambda$, where $S_1, \cdots, S_m$ are all the sources in Σ_i. We claim that this implies $\mathrm{Hom}_A(\mathrm{D}A, L_\lambda) = 0$ for all $\lambda \in \Lambda$. Indeed, let I be an indecomposable injective A-module and consider the left minimal almost split morphism $f \colon I \to K$. If K has no summand in $\mathrm{mod}A_i$, then clearly $\mathrm{Hom}_A(I, L_\lambda) = 0$ since any non-zero morphism would factor through f, an absurdity. If K has a summand in $\mathrm{mod}A_i$, it must be one of $S_1, \cdots, S_m$. Thus, $\mathrm{Hom}_A(I, L_\lambda) \neq 0$ implies $\mathrm{Hom}_A(S_j, L_\lambda) \neq 0$ for some $1 \leq j \leq m$, a contradiction to our assumption on the family $(L_\lambda)_{\lambda\in\Lambda}$. This shows that we indeed have $\mathrm{Hom}_A(\mathrm{D}A, L_\lambda) = 0$ for all $\lambda \in \Lambda$, or, equivalently, that $\mathrm{pd}_A(\tau^{-1}L_\lambda) \leq 1$ for all $\lambda \in \Lambda$, a contradiction to our hypothesis. □

Corollary. *Let A be a representation-infinite tilted algebra.*

(a) *If A has a complete slice in a postprojective component then $\mathrm{pd}_A M = 2$ for almost all M in $\mathrm{ind}A$ if and only if the reduced right type of A is a disjoint union of Dynkin graphs.*

(b) *If A has a complete slice in a preinjective component then $\mathrm{id}_A M = 2$ for almost all M in $\mathrm{ind}A$ if and only if the reduced left type of A is a disjoint union of Dynkin graphs.*

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