Equational bases for three-valued Heyting algebras with a quantifier

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Abstract

An equational basis is given for some members of the lattice of subvarieties of three-valued $\mathcal{Q}$-Heyting algebras. For any finite subdirectly irreducible algebra A, an equation which describes the number of order-connected components (o.c.c.'s) of the set $\mathcal{J} = \mathcal{J}(A)$ of join-irreducible elements of A is provided. Then an equation is exhibited characterizing the number of elements of Max $(\mathcal{J}) \setminus$ Min $(\mathcal{J})$. Finally an inductive process is shown to determine the number of maximal non minimal elements in each o.c.c. of $\mathcal{J}$.

1 Introduction and Preliminaries

In this paper we continue the investigation of the variety $\mathcal{Q}_3$ of three-valued Heyting algebras with a quantifier which we started in [3]. Here we present equational bases for some members of the lattice $\Lambda(\mathcal{Q}_3)$ of subvarieties of $\mathcal{Q}_3$. In order to make this paper as self-contained as possible we add some properties and definitions from [3].

A *quantifier* on a Heyting algebra H is a unary operation ∇ satisfying the following conditions, for any $a, b \in H$: (Q_0) $\nabla 0 = 0$, (Q_1) $a \wedge \nabla a = a$, (Q_2) $\nabla(a \wedge \nabla b) = \nabla a \wedge \nabla b$ and (Q_3) $\nabla(a \vee b) = \nabla a \vee \nabla b$.

If ∇ is a quantifier on a Heyting algebra H, then $\nabla 1 = 1$ and $\nabla\nabla a = \nabla a$. In addition, ∇H is a subalgebra of H (see [9]). A Q-Heyting algebra is an algebra (H, ∇) such that H is a Heyting algebra and ∇ is a quantifier on H. Monadic Boolean algebras are the simplest examples of Q-Heyting algebras (see [15]).

The class of Q-Heyting algebras form a variety, which we denote $\mathcal{Q}$.

Among Q-Heyting algebras we single out the subvariety $\mathcal{Q}_3$ of three-valued Q-Heyting algebras. This is the variety of Q-Heyting algebras such that the underlying structure of Heyting algebra is three-valued, that is, the following condition is satisfied: $((a \to c) \to b) \to (((b \to a) \to b) \to b) = 1$. We determined in [3] the simple and subdirectly irreducible algebras and gave a construction of the lattice of subvarieties of $\mathcal{Q}_3$. In this paper we give a complete equational description for each finite join-irreducible member of the lattice of subvarieties of $\mathcal{Q}_3$. It is an open problem to extend these results to

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the general case. However, between the three-valued case and the linear case there is already a large difference in complexity and we have been unable to find a meaningful characterization even for the lattice of subvarieties of the variety of linear Q-Heyting algebras (see [27]).

If X is a Priestley space (see [3] for any undefined terms), then $D(X)$ is the lattice of clopen decreasing subsets of X, and for $f : X \to X'$ a continuous order preserving map, $D(f) : D(X') \to D(X)$ is defined by $D(f)(U) = f^{-1}(U)$, for each $U \in D(X')$. If L is a bounded distributive lattice, then $X(L)$ is the set of prime ideals of L, ordered by set inclusion and with the topology having as a sub-basis the sets $\eta(a) = \{I \in X(L) : a \notin I\}$ and $X(L) \setminus \eta(a)$ for $a \in L$. If $h : L \to L'$ is a $(0,1)$–lattice homomorphism, then $X(h) : X(L') \to X(L)$ is defined by $X(h)(I) = h^{-1}(I)$. The mapping $\eta : L \to D(X(L))$ is a lattice isomorphism, and $\epsilon : X \to X(D(X))$ defined by $\epsilon(x) = \{U \in D(X) : x \notin U\}$ is a homeomorphism and an order isomorphism.

Since Heyting algebras are bounded distributive lattices, the category of Heyting algebras is isomorphic to a subcategory of bounded distributive lattices. A *Heyting space* is a Priestley space $(X, \leq, \tau)$ such that $[Y)$ is clopen for every convex clopen $Y \subseteq X$. For $a \in H$, $\eta(a) \subseteq X$ denote the clopen decreasing set that represents a. If $a, b \in H$ then, under the duality, $a \to b$ corresponds to the clopen decreasing set $X \setminus [\eta(a) \setminus \eta(b))$. If X and X' are Heyting spaces, a (Heyting) morphism is a continuous order-preserving map $\varphi : X \to X'$ for which $\varphi([x)) = [\varphi(x))$.

A *Q-Heyting space* (see [9]) is a pair (X, E) such that X is a Heyting space and E is an equivalence relation on X satisfying the conditions (E_1) $\nabla_E U \in D(X)$ for each $U \in D(X)$, where $\nabla_E U$ is the union of all blocks of E containing an element of U, and (E_2) the blocks of E are closed in X.

The category of Q-Heyting spaces and continuous order-preserving maps and the category of Q-Heyting algebras and homomorphisms are dually equivalent (see [3]).

The following theorem characterizes the dual space of a three-valued Heyting algebra, and it will play a central role in this paper.

Theorem 1.1 [21] *A Heyting algebra H is three-valued if and only if every element of $X(H)$ is either minimal or maximal and for every maximal element J of $X(H)$ there exists a unique minimal element I of $X(H)$ such that $I \leq J$. If* $\mathrm{Min}(X)$ *denotes the set of all minimal elements of $X = X(H)$, then $X = \bigcup_{I \in \mathrm{Min}(X)} C(I)$, where $C(I) = \{J \in X : I \leq J\} = [I)$.*

The quantifier ∇ is called *simple* if $\nabla H = \{0, 1\}$. When ∇H is a 3-element chain, we always denote $\nabla H = \{0, a, 1\}$. Observe that in this case $\neg a = 0$.

It is known that the subdirectly irreducible three-valued Heyting algebras are the 2-element chain **2** and the 3-element chain **3** [21]. The following result holds in $\mathcal{Q}_3$.

Proposition 1.2 [3] *Let $(H, \nabla) \in \mathcal{Q}_3$. If (H, ∇) is subdirectly irreducible, then ∇H is subdirectly irreducible as a Heyting algebra, that is, then ∇ is the simple quantifier or ∇H is a 3-element algebra.*

The following theorem gives a characterization of the simple objects in $\mathcal{Q}_3$.

Theorem 1.3 [3] *Let* $(H, \nabla) \in \mathcal{Q}_3$. (H, ∇) *is simple if and only if* ∇ *is the simple quantifier and* $\mathrm{Min}(X) = X$.

Corollary 1.4 *The simple algebras in* $\mathcal{Q}_3$ *are the simple monadic Boolean algebras.*

For non simple subdirectly irreducible algebras in $\mathcal{Q}_3$ we have the following results.

Theorem 1.5 [3] *The subdirectly irreducible non simple algebras with the simple quantifier in* $\mathcal{Q}_3$ *are the algebras* $B \times \mathbf{3}$, *where* B *is a Boolean algebra and* $\mathbf{3}$ *is the 3-element Heyting algebra.*

As a consequence of thie theorem we obtain the following corollary.

Corollary 1.6 *Let* $(H, \nabla) \in \mathcal{Q}_3$, ∇ *the simple quantifier. Then* (H, ∇) *is a non simple subdirectly irreducible algebra if and only if* $X = \mathrm{Min}(X) \cup \{I\}$, *with* $I \notin \mathrm{Min}(X)$.

Theorem 1.7 [3] *Let* $(H, \nabla) \in \mathcal{Q}_3$ *with* $\nabla H = \{0, a, 1\}$, *and let* (X, E) *be the* Q*-space associated to* (H, ∇). *Then* (H, ∇) *is subdirectly irreducible if and only if* $\mathrm{Min}(X) = \eta(a)$.

Corollary 1.8 [3] (H, ∇) *is non-simple subdirectly irreducible if and only if the lattice of congruences of* H *is isomorphic to* $\mathbf{3}$.

We have characterized the subdirectly irreducible algebras in the variety $\mathcal{Q}_3$. In particular, in the finite case we have:

Theorem 1.9 *Let* H *be a finite algebra in* $\mathcal{Q}_3$.

(1) H *is simple if and only if* H *is an* n*-atom simple monadic Boolean algebra* $D_{n,0}$

(2) H *is subdirectly irreducible but non simple if and only if either*

  (*i*) H *is an algebra* $D_{n,1}$ *of the form* $\mathbf{2}^n \times \mathbf{3}$, *with the simple quantifier, or*

  (*ii*) H *is an algebra* $T_{n,n_1,\ldots,n_r}$ *of the form* $\mathbf{2}^n \times (\{0\} \oplus \mathbf{2}^{n_1}) \times \ldots \times (\{0\} \oplus \mathbf{2}^{n_r})$, $n \geq 0$, $r \geq 1$, $n_i \geq 1$, $1 \leq i \leq r$ *with the quantifier* ∇ *such that* $\nabla T_{n,n_1,\ldots,n_r} = \{0, a, 1\}$ *and* $\eta(a) = \mathrm{Min}(X)$.

The corresponding dual spaces are exhibited in the following figure, where only the underlying partially ordered sets are shown. Because ∇ is the simple quantifier on $D_{n,0}$ and $D_{n,1}$, each of $X(D_{n,0})$ and $X(D_{n,1})$ have only one equivalence class, while $X(T_{n,n_1,\ldots,n_r})$ has two equivalence classes, namely $\mathrm{Min}(X)$ and $X \setminus \mathrm{Min}(X)$.

Next we present some results about the ordering in the set of varieties generated by the algebras $D_{n,0}$, $D_{n,1}$ and $T_{n,n_1,\ldots,n_r}$, or equivalently, in the set of finite subdirectly irreducible algebras of $\mathcal{Q}_3$. These results will be essential in the sequel.

If K is a class of algebras in a variety V, $V(K)$ denotes the subvariety of V generated by K. $\mathbf{Si}(K)$ and $\mathbf{Si}_{fin}(K)$ respectively denote the class of subdirectly irreducible algebras and the class of finite subdirectly irreducible algebras in K. The class of algebras that are homomorphic images of algebras in K will be denoted by $\mathbf{H}(K)$, and the class of algebras that are subalgebras of algebras in K will be denoted by $\mathbf{S}(K)$.

Recall that for A, $B \in \mathbf{Si}_{fin}(K)$, we may define a partial preorder: $A \leq B$ if and only if $A \in \mathbf{H}(\mathbf{S}(B))$, so that $V(A) \leq V(B)$ if and only if $A \leq B$.

Let $\mathcal{D}_{n,0}$, $\mathcal{D}_{n,1}$ and $\mathcal{T}_{n,n_1,\ldots,n_r}$ be the varieties generated by $D_{n,0}$, $D_{n,1}$ and $T_{n,n_1,\ldots,n_r}$, respectively, that is, $\mathcal{D}_{n,0} = V(D_{n,0})$, $\mathcal{D}_{n,1} = V(D_{n,1})$ and $\mathcal{T}_{n,n_1,\ldots,n_r} = V(T_{n,n_1,\ldots,n_r})$.

Theorem 1.10 [3] *The following conditions hold:*

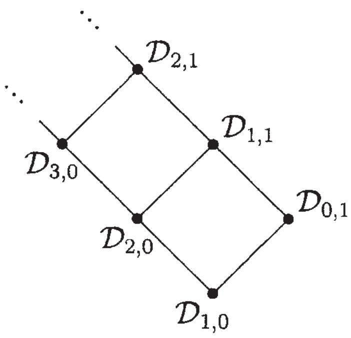
(1) $\mathcal{D}_{n,0} < \mathcal{D}_{n-1,1}$, $n \geq 1$.

(2) $\mathcal{D}_{m,0} \not\leq \mathcal{D}_{n,1}$, $n \geq 0$, $m > n+1$.

(3) $\mathcal{D}_{n,1} \not\leq \mathcal{D}_{m,0}$, $n \geq 0$, $m \geq 1$.

(4) $\mathcal{D}_{m,1} \leq \mathcal{D}_{n,1}$ *if and only if* $m \leq n$.

Consequently, the members of $\mathcal{J}(\Lambda(\mathcal{Q}_3))$ generated by finite subdirectly irreducible algebras with the simple quantifier are ordered according to the following figure:



Corollary 1.11 *The ordered set $\mathbf{Si}_{fin}(\mathcal{Q}_3)$ of all finite subdirectly irreducible algebras in $\mathcal{Q}_3$ with the simple quantifier is isomorphic to $C \times \mathbf{2}$, where C is a chain of type ω.*

Observe that the subclass of $\mathcal{Q}_3$ generated by the subdirectly irreducible algebras with the simple quantifier, that is, generated by the algebras $D_{n,0}$ and $D_{n,1}$, is a subvariety of $\mathcal{Q}_3$. This subvariety is determined by the equation $\nabla x \vee \neg\nabla x = 1$, where $\neg x = x \rightarrow 0$, that is, ∇H is a Boolean algebra.

To complete the ordering between the join-irreducible finitely generated varieties it remains to show the relationship with the varieties of type $\mathcal{T}_{n,n_1,\ldots,n_r}$. This is contained in the following results.

Since the quantifier in $D_{m,1}$, $m \geq 0$, is simple and the image of the quantifier in $T_{n,n_1,\ldots n_r}$, $n \geq 0$, $r \geq 1$, $n_i \geq 1$, $1 \leq i \leq r$, is a 3-element set, we have:

Theorem 1.12 $\mathcal{T}_{n,n_1,\ldots,n_r} \not\leq \mathcal{D}_{m,1}$, $n \geq 0$, $n_i \geq 1$, $1 \leq i \leq r$, $m \geq 0$.

Consider the algebras $T_{n,n_1,\ldots,n_r}$ with $n_1, \ldots, n_r$ such that $n_i \leq n_j$ whenever $i < j$.

It was proved in [3] the following result, where $x \prec y$ means y covers x.

Theorem 1.13 $\mathcal{T}_{m,m_1,\ldots,m_s} \prec \mathcal{T}_{n,n_1,\ldots,n_r}$ *in* $\mathcal{J}(\Lambda(\mathcal{Q}_3))$ *if and only if one of the following conditions hold:*

a) $m = n - 1$, $s = r$, $m_i = n_i$, $1 \leq i \leq r$.

b) $m = n$, $r = s$, *and there exists* $i \in \{1, \ldots, r\}$ *such that* $m_i = n_i - 1$ *and* $m_j = n_j$ *if* $j \neq i$, $1 \leq j \leq r$.

c) $m = n$, $s = r - 1$, *there exist* $i \in \{1, \ldots, s\}$, j, $k \in \{1, \ldots, r\}$ *such that* $m_i = n_j + n_k$ *and relabeling if necessary the sets* $\{m_1, \ldots, m_s\} \setminus \{m_i\}$, $\{n_1, \ldots, n_r\} \setminus \{n_j, n_k\}$, $m_t = n_t$, $1 \leq t \leq r - 2$.

Finally, the following theorem completes the ordering in the set of finitely generated join-irreducible members of the lattice of subvarieties of $\mathcal{Q}_3$.

Theorem 1.14 [3] *The following conditions hold:*

(1) $\mathcal{D}_{n,0} \leq \mathcal{T}_{n-t,1_1,\ldots,1_t}$, $1 \leq t \leq n$, $1_i = 1$, $1 \leq i \leq t$.

(2) $\mathcal{D}_{n,1} \leq \mathcal{T}_{0,1_1,\ldots,1_n,2_{n+1}}$, , $1_i = 1$, $1 \leq i \leq n$, $2_{n+1} = 2$.

(3) $\mathcal{D}_{n,0} \not\leq \mathcal{T}_{n-t,n_1,\ldots,n_r}$, $1 \leq r < t$.

(4) $\mathcal{D}_{n,1} \not\leq \mathcal{T}_{m,m_1,\ldots,m_r}$ *for* $r \leq n$ *and* $D_{n,1} \not\leq \mathcal{T}_{m,1_1,\ldots,1_{n+1}}$.

Observe that $\mathcal{D}_{n,0} \prec \mathcal{T}_{n-t,1_1,\ldots,1_t}$ and $\mathcal{D}_{n,1} \prec \mathcal{T}_{0,1_1,\ldots,1_n,2_{n+1}}$, as it is easily verified.

Corollary 1.15 [3] *The following conditions hold:*

(i) $\mathcal{D}_{n,0} \leq \mathcal{T}_{m,m_1,\ldots,m_r}$ *if and only if* $n \leq m + r$.

(ii) $\mathcal{D}_{n,1} \leq \mathcal{T}_{m,m_1,\ldots,m_r}$ *if and only if (a)* $r \geq n + 2$ *or (b)* $r = n + 1$ *and* $m_r \geq 2$.

2 Equational bases

The aim of this section is to give an equational basis for some members of the lattice of subvarieties of three-valued $\mathcal{Q}$-Heyting algebras. We are going to give first an equation which describes the number of o.c.c's of the set $\mathcal{J} = \mathcal{J}(A)$ of join-irreducible elements of a finite subdirectly irreducible algebra A (that is, the number of atoms of the Boolean algebra of the complemented elements of A). Then we will give an equation which characterizes the number of elements of Max $(\mathcal{J}) \setminus$ Min $(\mathcal{J})$, where Max $(\mathcal{J})$ denotes the set of all maximal elements in $\mathcal{J}$. Finally we will give an inductive process to determine the number of maximal non minimal elements in each o.c.c. of $\mathcal{J}$.

Recall that if $\{A_i\}_{i=1,\ldots,n}$ is a finite set of algebras and $V = V(\{A_i\}_{i=1,\ldots,n})$, an identity holds in V if and only if it holds in any algebra A_i. Also observe that any finite set of equations characterizing a variety of $\mathcal{Q}$-Heyting algebras can be transformed into just one equation of the form $\gamma_V(x_1, \ldots, x_r) = 1$. Let $\gamma_V(x_1, \ldots, x_r) = 1$ denote the equation which characterizes a variety V.

Consider the following terms:

$$P_1^n(x_1, \ldots, x_n) = \left(\bigwedge_{k=1}^{n} \nabla x_k \right) \wedge \left[\left(\bigvee_{1 \leq i < j \leq n} \nabla(x_i \wedge x_j) \right) \vee \left(\bigvee_{i=1}^{n} x_i \right) \right]$$

$$P_2^n(x_1, \ldots, x_n) = \bigwedge_{i=1}^{n} \nabla x_i$$

The following theorem gives equational bases for the subvarieties $\mathcal{D}_{n,0}$ in $\Lambda(\mathcal{Q}_3)$ (see [22]). The element $x \to 0$ is abbreviated by $\neg x$.

Theorem 2.1 *Let $n \geq 1$. Then the subvarieties $\mathcal{D}_{n,0}$ are characterized within $\mathcal{Q}_3$ by the equations*

(1) $\nabla x = x$ *and* $\neg\neg x = x$ *for* $n = 1$

(2) $P_1^n(x_1, \ldots, x_n) = P_2^n(x_1, \ldots, x_n)$ *and* $\neg\neg x = x$, *for* $n > 1$

Proof It is clear that $\mathcal{D}_{1,0}$ satisfies (1). If U is a subvariety such that $U \not\subseteq \mathcal{D}_{1,0}$, then there exists $l > 1$ such that $D_{l,0} \in U$, or there exists $s \geq 0$ such that $D_{s,1} \in U$ or there exist $m, m_1, \ldots, m_r$, $m \geq 0$, $r \geq 1$, $m_i \geq 1$ such that $T_{m,m_1,\ldots,m_r} \in U$. In the first case, there exists $x \in D_{l,0}$, $x \notin \{0,1\}$ such that $\nabla x = 1$. If $D_{s,1} \in U$, then $D_{0,1} \in U$ and there exists $c \in D_{0,1}$ such that $\neg\neg c = \neg(c \to 0) = \neg 0 = 1 \neq c$. If $T_{m,m_1,\ldots,m_r} \in U$ then $T_{0,1} \in U$ and there exists $c \in T_{0,1}$ such that $\neg\neg c \neq c$.

Suppose that $n > 1$ and let us prove that $D_{n,0}$ satisfies the equations in (2). For every $x \in D_{n,0}$, $\neg\neg x = x$. Consider now the equation $P_1^n(x_1, \ldots, x_n) = P_2^n(x_1, \ldots, x_n)$ and $b_1, \ldots, b_n \in D_{n,0}$. If $b_i = 0$ for some i, the equation holds. Suppose that $b_i \neq 0$ for $i = 1, \ldots, n$. Then $P_2^n(b_1, \ldots, b_n) = 1$. If $b_i \wedge b_j = 0$ for every i, j, $i \neq j$, we have that $\bigvee_{i=1}^{n} b_i = 1$, which implies that $P_1^n(b_1, \ldots, b_n) = 1$. If there exist i, j, $i \neq j$, such that $b_i \wedge b_j \neq 0$, then $\nabla(b_i \wedge b_j) = 1$ and consequently $P_1^n(b_1, \ldots, b_n) = 1$.

Let us see now that these equations do not hold in any subvariety $U \not\subseteq \mathcal{D}_{n,0}$. For such subvariety there exists $l > n$ such that $D_{l,0} \in U$, or there exists $s \geq 0$ such that $D_{s,1} \in U$ or there exist $m \geq 0$, $r \geq 1$, $m_i \geq 1$, $i = 1, \ldots, r$ such that $T_{m,m_1,\ldots,m_r} \in U$. In the first case, $D_{n+1,0} \in U$, since $D_{n+1,0} \in \mathbf{H}(\mathbf{S}(D_{l,0}))$. Let $a_1, \ldots, a_{n+1}$ be the atoms of $D_{n+1,0}$. Then $\nabla a_i = 1$ for all i and $a_i \wedge a_j = 0$ for $i \neq j$. Consequently $P_1^n(a_1, \ldots, a_n) = \bigvee_{i=1}^{n} a_i \neq 1$ and $P_2^n(a_1, \ldots, a_n) = 1$. If $D_{s,1} \in U$, with $s \geq 0$, then $D_{0,1} \in U$ and choosing $c \in D_{0,1}$, $c \neq 0, 1$ we have that $\neg\neg c = 1 \neq c$. Similarly, if $T_{m,m_1,\ldots,m_r} \in U$, then $T_{0,1} \in U$ and $\neg\neg c \neq c$, with $c \in T_{0,1} \setminus \{0, 1\}$. □

Observe that if H is an algebra in $\mathcal{Q}_3$ with the simple quantifier, then H satisfies $P_1^n = P_2^n$ if and only if H satisfies that if $a_1, a_2, \ldots, a_n$ are n elements of H different from 0 and pairwise disjoint, then $a_1 \vee a_2 \vee \ldots \vee a_n = 1$.

We will abbreviate the equations contained in Theorem 2.1 by $\gamma_{D_{n,0}} = 1$.

Consider now the following terms (see [22]):

$$Q_1^n(x_1, \ldots, x_{n+2}) = \left(\bigwedge_{k=1}^{n+2} \nabla x_k\right) \wedge \left[\bigvee_{1 \leq i < j \leq n+2} \nabla(x_i \wedge x_j)\right]$$

$$Q_2^n(x_1, \ldots, x_{n+2}) = \bigwedge_{i=1}^{n+2} \nabla x_i$$

Theorem 2.2 *The subvarieties $\mathcal{D}_{n,1}$, $n \geq 0$, are characterized within $\mathcal{Q}_3$ by the equations $Q_1^n(x_1, \ldots, x_{n+2}) = Q_2^n(x_1, \ldots, x_{n+2})$ and $\neg x \vee \nabla x = 1$.*

Proof Let $x \in D_{n,1}$. If $x = 0$ then $\neg x = 1$ and if $x \neq 0$, then $\nabla x = 1$. In both cases $\neg x \vee \nabla x = 1$.

If $b_1, \ldots, b_{n+2} \in D_{n,1}$ and $b_i = 0$ for some i, $1 \leq i \leq n+2$, then $Q_1^n(b_1, \ldots, b_{n+2}) = 0$ and $Q_2^n(b_1, \ldots, b_{n+2}) = 0$. If $b_i \neq 0$ for every i, $1 \leq i \leq n+2$, then $Q_2^n(b_1, \ldots, b_{n+2}) = 1$. Since D_{n+1} has $n+1$ atoms, there exist $i, j \in \{1, \ldots n+2\}$, $i \neq j$ such that $b_i \wedge b_j \neq 0$. Consequently, $Q_1^n(b_1, \ldots, b_{n+2}) = 1$.

Let U be a subvariety such that $U \not\subseteq \mathcal{D}_{n,1}$. Then (i) there exists $l > n+1$ such that $D_{l,0} \in U$ or (ii) there exists $s > n$ such that $D_{s,1} \in U$ or (iii) there exist $n, n_1, \ldots, n_r$, $n \geq 0$, $r \geq 1$, $n_i \geq 1$, $1 \leq i \leq r$ such that $T_{n,n_1,\ldots,n_r} \in U$. In (i), $D_{n+2,0} \in U$, and if $a_1, \ldots, a_{n+2}$ are the atoms of $D_{n+2,0}$, then $Q_1^n(a_1, \ldots, a_{n+2}) = 0$ and $Q_2^n(a_1, \ldots, a_{n+2}) = 1$. In (ii), $D_{n+1,1} \in U$ and a similar argument shows that $Q_1^n(x_1, \ldots, x_{n+2}) \neq Q_2^n(x_1, \ldots, x_{n+2})$. Finally, if $T_{n,n_1,\ldots,n_r} \in U$ then $T_{0,1} \in U$ and $\neg a \vee \nabla a = a \neq 1$. □

Observe that if H is an algebra in $\mathcal{Q}_3$ with the simple quantifier, then H satisfies $Q_1^n = Q_2^n$ if and only if H satisfies that there exists no set $\{a_1, a_2, \ldots, a_{n+2}\}$ of $n+2$ non-zero and pairwise disjoint elements of H.

The equations contained in Theorem 2.2 will be abbreviated by $\gamma_{D_{n,1}} = 1$.

In the rest of this section, H will denote the algebra $T_{n,n_1,\ldots,n_r}$, $n \geq 0$, $r \geq 1$, $n_i \geq 1$, $1 \leq i \leq r$.

Observe that the set $D(H) = \{x \in H : \neg x = 0\}$ of dense elements in H is $[a)$ (recall that a is the unique element in ∇H different from 0 and 1, or equivalently, $\eta(a) = \text{Min}\,(X(H))$.

Consider in the interval $[a)$ the operation $x' = x \to a$, for $x \in [a)$.

Lemma 2.3 $A = ([a), \wedge, \vee, ', a, 1, \nabla)$ *is a simple monadic Boolean algebra.*

Proof If $x \in [a)$, $x \wedge x' = x \wedge (x \to a) = x \wedge a = a$. It is easy to see that $x \vee x' = 1$ and that ∇ is the simple quantifier. □

Since the atoms of A are the elements of $\text{Max}(X(H)) \setminus \text{Min}(X(H))$, A is isomorphic to the the simple monadic Boolean algebra $\mathbf{2}^{n_1+\cdots+n_r}$.

Lemma 2.4 [14] *For every $x \in H$, $x \vee \neg x \in [a)$.*

Proof Immediate, as $x \vee \neg x \in D(H)$. □

Observe that every element $y \in [a)$ is of the form $y = y \vee \neg y$, as $\neg y = 0$.

Consider the following terms, where $x^* = x \vee \neg x$:

$$P_1^s(x_1, \ldots, x_s) = \left(\bigwedge_{k=1}^{s} \nabla x_k^*\right) \wedge \left[\left(\bigvee_{1\le i<j\le s} \nabla(x_i^* \wedge x_j^*)\right) \vee \left(\bigvee_{i=1}^{s} x_i^*\right)\right]$$

$$P_2^s(x_1, \ldots, x_s) = \bigwedge_{i=1}^{s} \nabla x_i^*$$

The following theorem gives an equation which determines the number of elements of $X \setminus \text{Min}(X) = \text{Max}(X) \setminus \text{Min}(X)$.

Theorem 2.5 $n_1+\cdots+n_r \le s$ *if and only if the equation* $P_1^s(x_1, \ldots, x_s) = P_2^s(x_1, \ldots, x_s)$ *holds in H.*

Proof It is a consequence of Lemma 2.3 and Theorem 2.1. □

The equation contained in Theorem 2.5 will be denoted by $\gamma_{max}^s = 1$.

Let $Rg(H) = \{\neg x : x \in H\}$ be the set of regular elements in H. We know that if $x, y \in H$, then $\neg(x \vee y) = \neg x \wedge \neg y$. Since H is three-valued, it follows that H is a Stone algebra, so $\neg(x \wedge y) = \neg x \vee \neg y$. Consequently, $(Rg(H), \wedge, \vee, \neg, 0, 1)$ is the Boolean algebra $B(H)$ of complemented elements of H with $n+r$ atoms. If we define $\overline{\nabla} x = \neg\neg\nabla x$ on $Rg(H)$, then $\overline{\nabla} 0 = 0$ and $\overline{\nabla} x = 1$ for $x \neq 0$. So $(Rg(H), \wedge, \vee, \neg, 0, 1, \overline{\nabla})$ is a simple monadic Boolean algebra.

Consider now

$$T_1^m(x_1, \ldots, x_m) = \left(\bigwedge_{k=1}^{m} \overline{\nabla} \neg x_k\right) \wedge \left[\left(\bigvee_{1\le i<j\le m} \overline{\nabla}(\neg x_i \wedge \neg x_j)\right) \vee \left(\bigvee_{i=1}^{m} \neg x_i\right)\right]$$

$$T_2^m(x_1, \ldots, x_m) = \bigwedge_{i=1}^{m} \overline{\nabla} \neg x_i$$

The following theorem gives an equation which determines the number of o.c.c.'s of X.

Theorem 2.6 *$n + r \leq m$ if and only if the equation $T_1^m(x_1, \ldots, x_m) = T_2^m(x_1, \ldots, x_m)$ holds in H.*

Proof It is a consequence of the previous remarks and Theorem 2.1. □

We will denote $\gamma_{comp}^m = 1$ the equation contained in Theorem 2.6.

Observe that $\gamma_{comp}^m(A) \subseteq Rg(A)$, for any algebra A and if the number of o.c.c.'s of A is greater that m, then there exists $\vec{x} \in A^m$ such that $\gamma_{comp}^m(\vec{x}) < 1$.

Following the proofs of Theorems 2.1, 2.2, 2.5 and 2.6, it is long but computational to check that

1. $\gamma_{comp}^m(D_{n,0}) = \{1\}$ if $n \leq m$,
   $\gamma_{comp}^m(D_{n,1}) = \{1\}$, for $m \geq n+1$, and
   $\gamma_{comp}^m(T_{n,n_1,\ldots,n_r}) = \{1\}$ when $n + r \leq m$.

2. $\gamma_{max}^m(D_{n,0}) = \{1\}$,
   $\gamma_{max}^m(D_{n,1}) = \{1\}$ if $m \geq 2$, and $\gamma_{max}^m(D_{n,1}) = \{(1,1), (1, \frac{1}{2})\}$, for $m = 1$ (recall that $D_{n,1} \cong \mathbf{2}^n \times \mathbf{3}$, $\mathbf{3} = \{0, \frac{1}{2}, 1\}$).
   $\gamma_{max}^m(T_{n,n_1,\ldots,n_r}) = \{1\}$ when $n_1 + \ldots + n_r \leq m$, and
   $\gamma_{max}^m(T_{n,n_1,\ldots,n_r}) \subseteq [a)$ when $n_1 + \ldots + n_r > m$.

Consider the term $t_1(x) = \neg\neg(\neg x \wedge \nabla x)$. Observe that if $a \in A \in \mathcal{Q}_3$, then $t_1(a)$ is a Boolean element in A. This term will be useful in order to determine the equations that describe the partitions of the maximal non-minimal elements. First we consider this term on a subdirectly irreducible algebra.

Lemma 2.7 $1 \notin t_1(T_{n,n_1,\ldots,n_r}) = \{t_1(x) : x \in T_{n,n_1,\ldots,n_r}\}$.

Proof If $x = 0$ then $t_1(0) = \neg\neg(\neg 0 \wedge \nabla 0) = \neg\neg(1 \wedge 0) = \neg\neg 0 = 0$. If $a \leq x$ then $\neg x = 0$. Consequently $t_1(x) = \neg\neg(\neg x \wedge \nabla x) = \neg\neg(0 \wedge 1) = \neg\neg 0 = 0$. Suppose $0 < x < a$. Since $\neg x$ is associated to $[\mathrm{Min}(X) \setminus \eta(x))$ and $\nabla x = a$ is associated to $\mathrm{Min}(X)$, then $\neg x \wedge \nabla x$ is associated to $[\mathrm{Min}(X) \setminus \eta(x)) \cap \eta(a) = \mathrm{Min}(X) \setminus \eta(x)$. Hence $\neg(\neg x \wedge \nabla x)$ is associated to $X \setminus [\mathrm{Min}(X) \setminus \eta(x)))$. Hence $\neg\neg(\neg x \wedge \nabla x) = \neg x$, that is, $t_1(x) = \neg x$. Observe that $\neg x \neq 1$, since $\eta(x) \neq \emptyset$. Finally, suppose that x is incomparable to a. Then $\nabla x = 1$ and $\neg x \wedge \nabla x = \neg x$. So $t_1(x) = \neg x$, and again $\neg x \neq 1$. Therefore $1 \notin t_1(T_{n,n_1\ldots,n_r})$. □

Corollary 2.8 *If A is subdirectly irreducible, then $t_1(A) = B(A) \setminus \{1\}$.*

Proof This is clear for $A = D_{n,0}$ or $A = D_{n,1}$. For $A = T_{n,n_1\ldots,n_r}$, the corollary follows from the proof of the previous lemma. □

For an algebra $A \cong T_{n,n_1,\ldots,n_r}$ we can algebrize the interval $A^1_{x_0} = \{x \wedge t_1(x_0) : x \in A\} \cong [0, t_1(x_0)]$ considering $(A^1_{x_0}, \wedge_1, \vee_1, \rightarrow_1, \nabla_1, 0_1, 1_1)$, where $0_1 = 0$, $1_1 = t_1(x_0)$, $x \rightarrow_1 y = (x \rightarrow y) \wedge t_1(x_0)$, $\wedge_1 = \wedge$, $\vee_1 = \vee$ y $\nabla_1(x) = \nabla x \wedge t_1(x_0)$ for $x, y \in A^1_{x_0}$. Observe that if $t_1(x_0) \neq 0$, $A^1_{x_0}$ is a subdirectly irreducible algebra with less o.c.c's than A. Also observe that the operations in $A^1_{x_0}$ are terms in the language of $\mathcal{Q}_3$.

Consider $\gamma_0(x_0) = \neg(\overline{\nabla} t_1(x_0) \leftrightarrow 0)$.

Lemma 2.9 *For a subdirectly irreducible algebra A and for every $x_0 \in A$, $\gamma_0(x_0) = 1$ when $t_1(x_0) \neq 0$ and $\gamma_0(x_0) = 0$ when $t_1(x_0) = 0$.*

Proof $t_1(x_0)$ is a Boolean element different from 1, for every subdirectly irreducible algebra A. *Since $\overline{\nabla}$ is simple on the Boolean elements of A, if $t_1(x_0) \neq 0$, then $\overline{\nabla} t_1(x_0) = 1$.* Then $\overline{\nabla} t_1(x_0) \leftrightarrow 0 = 1 \leftrightarrow 0 = 0$. Hence $\neg(\overline{\nabla} t_1(x_0) \leftrightarrow 0) = \neg 0 = 1$, and so $\gamma_0(x_0) = 1$. For $t_1(x_0) = 0$, $\overline{\nabla} t_1(x_0) = 0$, thus $\neg(\overline{\nabla} t_1(x_0) \leftrightarrow 0) = \neg(0 \leftrightarrow 0) = \neg 1 = 0$. Hence $\gamma_0(x_0) = 0$. □

In what follows we show an inductive process to determine the number of maximal non minimal elements in each o.c.c. of the poset of join-irreducible elements of a finite subdirectly irreducible algebra.

Case 1. One order-connected component.

If A is subdirectly irreducible and has one o.c.c., then A is isomorphic to $D_{1,0}$ or to $D_{0,1}$ or to T_{0,r_1}. Theorems 2.1 and 2.2 give equations for $D_{1,0}$ and $D_{0,1}$ respectively. For T_{0,r_1} the equations are $\gamma^1_{comp} = 1$ and $\gamma^{r_1}_{max} = 1$.

Case 2. Two order-connected components.

Suppose now that A is subdirectly irreducible and has two order-connected components. Then A is isomorphic to one of the following algebras: $D_{2,0}$, $D_{1,1}$, T_{0,r_1,r_2}, T_{1,r_1}. The cases $A \cong D_{2,0}$ and $A \cong D_{1,1}$ have already been considered in Theorems 2.1 and 2.2.

Suppose that A is isomorphic to T_{0,r_1,r_2}. Consider the following term:

$$\gamma_T(x) = \nabla\neg x \leftrightarrow \overline{\nabla}\neg x.$$

Observe that $\gamma_T(x) = 1$ is equivalent to $\nabla\neg x = \overline{\nabla}\neg x$, and this is equivalent to $\nabla\neg x = \neg\neg\nabla\neg x$.

Lemma 2.10 *The equation $\gamma_T = 1$ holds in a subdirectly irreducible algebra $A \cong T_{k,n_1,\dots,n_r}$ if and only if there exists no $x \in B(A)$ such that $\nabla x = a$, or equivalently, if and only if $k = 0$.*

Proof Suppose that $\gamma_T = 1$ in $A \cong T_{k,n_1,\dots,n_r}$ and $k \neq 0$. Then by Theorem 1.13, $T_{1,1}$ is a subalgebra of A. But there exists $x \in T_{1,1}$ such that $\nabla\neg x = a$, and consequently, $\gamma_T(x) \neq 1$, a contradiction.

Suppose that $k = 0$. Then for every $x \in B(A)$, $x \neq 0$, we have that $\nabla x = 1$. Hence, as $\neg x \in B(A)$ for every $x \in A$, the equation $\nabla\neg x = \neg\neg\nabla\neg x$, holds in A, that is, $\gamma_T(x) = 1$ holds in A. □

Corollary 2.11 *If A is a subdirectly irreducible algebra with the simple quantifier then A satisfies $\gamma_T = 1$*

Proof Observe that $\nabla x = 1$ for every $x \neq 0$, so the second part of the previous proof can be applied. □

As a consequence we have that $A \cong T_{k,n_1,\ldots,n_r}$ satisfies $\gamma_T(x) = 1$ if and only if $k = 0$, that is, if and only if $T_{1,1} \notin \mathbf{S}(A)$.

Now we go back to the case in which A is isomorphic to T_{0,r_1,r_2}. We want to give an equation that characterizes the numbers r_1 and r_2. Consider the following equation, which is a term in the language of $\mathcal{Q}_3$.

$$\gamma_{part}^{r_1,r_2} = \gamma_0(x_0) \leftrightarrow \left[(\gamma_{T_{0,r_1}}^{t_1(x_0)} \wedge \gamma_{T_{0,r_2}}^{\neg t_1(x_0)}) \vee (\gamma_{T_{0,r_1}}^{\neg t_1(x_0)} \wedge \gamma_{T_{0,r_2}}^{t_1(x_0)})\right] = 1, \tag{1}$$

where $\gamma_{T_{0,r_1}}^{t_1(x_0)} = 1$ is the characteristic equation of $[0, t_1(x_0)]$, etc.

Observe that the number of o.c.c's is given by the equation $\gamma_{comp}^2 = 1$ and the number of maximal non-minimal elements $r_1 + r_2$ is given by the equation $\gamma_{max}^{r_1+r_2} = 1$. We claim that the characteristic equation of T_{0,r_1,r_2} is

$$\gamma_{T_{0,r_1,r_2}} = \gamma_T \wedge \gamma_{max}^{r_1+r_2} \wedge \gamma_{comp}^2 \wedge \gamma_{part}^{r_1,r_2} = 1.$$

We have to check first that T_{0,r_1,r_2} satisfies $\gamma_{part}^{r_1,r_2}$. Indeed, let $x_0 \in T_{0,r_1,r_2}$. If $\gamma_0(x_0) = 0$, then $\gamma_{part}^{r_1,r_2}$ trivially holds. As T_{0,r_1,r_2} has two o.c.c.'s, then there exists x_0 such that $t_1(x_0)$ is a Boolean element different from 0. Besides, $[0, t_1(x_0)]$, with the operations given before, is isomorphic to either T_{0,r_1} or T_{0,r_2}. If $[0, t_1(x_0)]$ is isomorphic to T_{0,r_1}, then $[0, \neg t_1(x_0)]$ is isomorphic to T_{0,r_2} and the equation holds. Similarly if $[0, t_1(x_0)]$ is isomorphic to T_{0,r_2}.

Let us see now that if A is a subdirectly irreducible algebra and A satisfies $\gamma_{T_{0,r_1,r_2}}$, then A is a subalgebra of T_{0,r_1,r_2}.

If A satisfies $\gamma_{T_{0,r_1,r_2}}$, then A satisfies γ_{comp}^2, and consequently, A has either one or two o.c.c.'s.

1. Suppose that A has one o.c.c. Then
   (a) $A \cong D_{0,1}$ and thus, by Corollary 1.15, A is a homomorphic image of a subalgebra of T_{0,r_1,r_2}, or
   (b) $A \cong D_{1,0}$ and thus, by Corollary 1.15, A is a homomorphic image of a subalgebra of T_{0,r_1,r_2}, or
   (c) $A \cong T_{0,r}$. Since A also satisfies $\gamma_{max}^{r_1+r_2}$, it follows that $r \leq r_1 + r_2$, and then A is a subalgebra of T_{0,r_1,r_2} , by Theorem 1.13.
2. Suppose that A has two o.c.c's. Then we have the following cases:
   (a) $A \cong D_{2,0}$ and thus A is a homomorphic image of a subalgebra of T_{0,r_1,r_2}, by Corollary 1.15.
   (b) $A \cong D_{1,1}$. Recall that $r_1 \leq r_2$. Then we have to consider
      i. $r_2 \geq 2$. By Corollary 1.15, $D_{1,1}$ is a subalgebra of T_{0,r_1,r_2}.

ii. $r_2 = 1$. Then $r_1 = 1$. Observe that $\gamma_{T_{0,1}}^{t_1(x_0)} = (\gamma_{max}^1)^{t_1(x_0)} \wedge (\gamma_{comp}^1)^{t_1(x_0)}$. Then there exists $x_0 \in D_{1,1}$ such that $[0, t_1(x_0)] \cong D_{0,1}$. Thus, there exists $\vec{x} \in [0, t_1(x_0)]^n$ such that $\gamma_{T_{0,1}}^{t_1(x_0)}(\vec{x}) \leq \frac{1}{2}$, and so $(\gamma_{T_{0,1}}^{t_1(x_0)} \wedge \gamma_{T_{0,1}}^{\neg t_1(x_0)})(\vec{x}) \leq \frac{1}{2}$, that is, there exists $\vec{x} \in D_{0,1}$ such that $\gamma_{part}^{r_1,r_2}(\vec{x}) = 1 \leftrightarrow \frac{1}{2} \neq 1$. Hence A does not satisfy $\gamma_{part}^{r_1,r_2} = 1$, and consequently A does not satisfy $\gamma_{T_{0,r_1,r_2}}$, which is a contradiction. So A is not isomorphic to $D_{1,1}$.

(c) $A \cong T_{0,s_1,s_2}$. Since A satisfies $\gamma_{max}^{r_1+r_2} = 1$, it follows that $s_1 + s_2 \leq r_1 + r_2$. Then there exists $x_0 \in T_{0,s_1,s_2}$ such that $t_1(x_0) \neq 0$. Suppose that

$$\gamma_{T_{0,r_1}}^{t_1(x_0)} \wedge \gamma_{T_{0,r_2}}^{\neg t_1(x_0)} \neq 1 \tag{2}$$

and

$$\gamma_{T_{0,r_1}}^{\neg t_1(x_0)} \wedge \gamma_{T_{0,r_2}}^{t_1(x_0)} \neq 1. \tag{3}$$

In (2), $\gamma_{T_{0,r_1}}^{t_1(x_0)} \neq 1$ or $\gamma_{T_{0,r_2}}^{\neg t_1(x_0)} \neq 1$. Thus, there exists $\vec{x} \in [0, t_1(x_0)]^n$ such that $\gamma_{T_{0,r_1}}^{t_1(x_0)} \wedge \gamma_{T_{0,r_2}}^{\neg t_1(x_0)}(\vec{x}) \leq b$, b a dual atom of $[0, t_1(x_0)]$. In a similar way, in (3), there exists $\vec{x} \in [0, \neg t_1(x_0)]$ such that $\gamma_{T_{0,r_1}}^{\neg t_1(x_0)} \wedge \gamma_{T_{0,r_2}}^{t_1(x_0)}(\vec{x}) \leq b'$, b' a dual atom of $[0, \neg t_1(x_0)]$.
Then $(2) \vee (3) \leq b \vee b' = b'' \neq 1$. Thus $\gamma_0(x_0) \leftrightarrow \gamma_{part}^{t_1(x_0)} = 1 \leftrightarrow b'' = b'' \neq 1$. Then, if $\gamma_{part}^{r_1,r_2} = 1$ it follows that $(2) = 1$ or $(3) = 1$. In both cases $s_1 \leq r_1$ and $s_2 \leq r_2$. Hence A is a subalgebra of T_{0,r_1,r_2}.

(d) $A \cong T_{1,r}$. Then by Lemma 2.10, A does not satisty the equation $\gamma_T(x) = 1$.

In order to study this last case, that is, $A \cong T_{1,r}$, we consider the following term:

$$\beta_T(x, y) = (\nabla\neg x \to (y \vee \neg y)) \;\vee\; (\nabla\neg\neg x \to (y \vee \neg y)).$$

Lemma 2.12 *$\beta_T(x, y) = 1$ holds in $A \cong T_{1,r}$.*

Proof Observe that $B(A) = \{0, 1, a_1, a_2\}$ and the elements $\neg x, \neg\neg x$ are complemented elements in $B(A)$. If $\{\neg x, \neg\neg x\} = \{0, 1\}$, then $\beta_T(x, y) = 1$ holds, as either $(\nabla\neg x \to (y \vee \neg y)) = 1$ or $(\nabla\neg\neg x \to (y \vee \neg y)) = 1$. If $\{\neg x, \neg\neg x\} = \{a_1, a_2\}$, then either $\nabla\neg a_1 = \nabla a_2 = a$ or $\nabla\neg a_2 = \nabla a_1 = a$. Since $a \leq y \vee \neg y$ for every $y \in A$, it follows that $\beta_T = 1$ holds in A. □

Lemma 2.13 *$\beta_T(x, y) = 1$ does not hold in $A \cong T_{0,r_1 r_2}$.*

Proof Observe that $B(A) = \{0, 1, a_1, a_2\}$ and that $\nabla a_1 = \nabla a_2 = 1$. If we choose $x = a_1$ and $y = a$, then

$$\beta_T(x, y) = \beta_T(a_1, a) = (\nabla\neg a_1 \to (a \vee \neg a)) \;\vee\; (\nabla\neg\neg a_1 \to (a \vee \neg a)) =$$

$$(\nabla a_2 \to a) \vee (\nabla a_1 \to a) = (1 \to a) \vee (1 \to a) = (1 \to a) = a \neq 1.$$

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As a consequence of the above results, it follows that the characteristic equation for the variety generated by $T_{1,r}$ is

$$\gamma_{T_{1,r}} = \beta_T \wedge \gamma^r_{max} \wedge \gamma^2_{comp} \wedge \gamma^{1,r}_{part} = 1.$$

For n o.c.c's, this argument can be recursively applied. As in the case of two o.c.c's, the idea is to algebrize the intervals $[0, b]$, where b is a boolean element, $b \neq 1$, (b can be obtained by means of the term $t_1(x)$) as it was done in the remark preceding Lemma 2.9. By this procedure we get algebras with a number of o.c.c's which is less than or equal to $n - 1$.

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