Representations of Finite Groups on Polynomial Rings

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Abstract. In [1] we gave a realization of the irreducible representations of the symmetric group S_n on the polynomial ring $K[x_1, \ldots, x_n]$, where K is the field of the rational numbers or K is a finite field F_p (here p is a prime number). In this work we show that if G is a finite subgroup of the linear group $GL_n(K)$ and K is a field of characteristic zero, then each simple G-module over K is isomorphic to a G-submodule in the polynomial ring $K[x_1, \ldots, x_n]$. Furthermore, by making use of certain invariant operators in the Weyl algebra, we describe a finite dimensional G-space $\mathcal{N}$ in $K[x_1, \ldots, x_n]$ which contains all the simple G-modules over K.

1. Simple Modules.

Let K be a field of characteristic zero and n a natural number. Let us denominate V the vector space K^n. We will denote by $\mathcal{A}$ the symmetric algebra of the dual space V^*. If we choose a basis $x_1, \ldots, x_n$ of V^*, then the algebra $\mathcal{A}$ is identified with the ring of polynomial functions defined on V. Since K has characteristic zero, we can assume that $\mathcal{A} = K[x_1, \ldots, x_n]$.
Let G be a finite subgroup of $GL_n(K)$.
We have a natural action of G on $\mathcal{A}$ given by:

$$(\sigma \cdot P)(v) = P(\sigma^{-1} \cdot v) \qquad P \in \mathcal{A}, \sigma \in G, v \in V$$

With $K[G]$ we will denote the group algebra of G over K. We can think of $K[G]$ as the K-vector space of the functions $\varphi : G \to K$. Hence, we can define an action of G on $K[G]$ by:

$$(\sigma \cdot \varphi)(\tau) = \varphi(\sigma^{-1}\tau) \quad \varphi \in K[G],\ \sigma, \tau \in G$$

Definition 1.1: An element $v \in V$ is called *regular for G* if $\sigma \cdot v \neq v$ for all $\sigma \in G$, $\sigma \neq 1$. That is to say, v is regular for G if the isotropy subgroup of v is the trivial group.

Proposition 1.2: *If K is an infinite field, then there is a regular element for G in V.*

Proof: Since K is an infinite field, V cannot be a finite union of proper subspaces, so the spaces of fixed points of the elements of G different from 1 do not cover V, hence, there is a regular element for G in V. ∎

Theorem 1.3: *There is a faithful morphism of G-modules* , $\Psi : K[G] \to \mathcal{A}$.

Proof: Since K has characteristic zero, the space V has a regular vector v. Let $G \cdot v$ be the $G-$orbit of v. Let us denote by P a polynomial function on V such that:

$$P(v) = 1 \ , \ \ (\sigma \cdot P)(v) = 0 \ \ \text{if } \sigma \neq 1$$

Such a function always exists since it is possible to build an interpolating polynomial. Let $G \cdot P$ be the $G-$orbit of P in $\mathcal{A}$ and let $\mathcal{S}$ be the subspace of $\mathcal{A}$ spanned by $G \cdot P$. Let us take the function $\phi \in K[G]$ given by :

$$\phi(1) = 1, \ (\sigma \cdot \phi)(1) = \phi\left(\sigma^{-1}\right) = 0 \ \ \sigma \neq 1$$

Then $G \cdot \phi$ is a basis for $K[G]$ and besides, the elements of $G \cdot P$ are linearly independent. If we define:

$$\Psi(\sigma \cdot \phi) = \sigma \cdot P$$

then Ψ extends by linearity to an injective $K-$morphism of $K[G]$ into $\mathcal{A}$, and Ψ verifies:

$$\Psi(\tau \cdot (\sigma \cdot \phi)) = \Psi((\tau\sigma) \cdot \phi) = (\tau\sigma) \cdot P = \tau \cdot (\sigma \cdot P) \ \ \forall \sigma, \tau \in G$$

that is, $\Psi : K[G] \to \mathcal{S}$ is an isomorphism of G-modules. ∎

With the notations of Theorem 1.3 we have:

Corollary 1.4 : *$\mathcal{S}$ contains all the simple left G- modules over K.*

Proof: Since K has characteristic zero, we have that $K[G]$ is semisimple. ∎

2. The space $\mathcal{N}$.

For i in $I_n = \{1, 2, \ldots, n\}$ we put $\partial_i = \dfrac{\partial}{\partial x_i}$ the $i-th$ partial differentiation. If α is a multi-index, that is to say a map $\alpha : I_n \to \mathbb{N}_0$ where $\mathbb{N}_0$ is the set of non-negative integers, we will write:

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \ \text{ and } \ \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

The K-algebra of K-linear operators on $\mathcal{A}$ which is generated by the multiplications by the generators x_i and the derivations ∂_i for $i = 1, 2, ..., n$ is called *the algebra of K-linear differential operators on* $\mathcal{A}$ or *Weyl algebra in n variables over* K and we will denote it by $\mathcal{W}$.

Proposition 2.1. *Each element of $\mathcal{W}$ can be written in a unique way as a finite sum:*

$$\sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta, \ \ c_{\alpha\beta} \in K$$

Where α and β are multiindexes.

Proof: See [3] for a proof of Proposition 2.1. ∎

Definition 2.2. Given a differential operator

$$D = \sum_{\alpha,\beta} c_{\alpha,\beta} x^{\alpha} \partial^{\beta}$$

in $\mathcal{W}$ we define the *degree of* D by:

$$deg\,(D) = \max\left\{\sum_{i=1}^{n} (\alpha_i - \beta_i)\ , c_{\alpha,\beta} \neq 0\right\}$$

The action of G on V^* given by:

$$(\sigma \cdot \varphi)(v) = \varphi(\sigma^{-1} \cdot v) \ \text{ for all } \ \varphi \in V^*,\ \sigma \in G,\ v \in V$$

induces an action of G on $\mathcal{A}$ given by:

$$(\sigma \cdot H)(v) = H(\sigma^{-1} \cdot v) \ \text{ for all } \ H \in \mathcal{A},\ \sigma \in G,\ v \in V$$

Let us remark that since G acts on $\mathcal{A}$ by substitution, this action enables us to interpret an element of G as a ring homomorphism. Hence, we can define an action of G on $End_K(\mathcal{A})$, the ring of K-linear endomorphisms of $\mathcal{A}$ by:

$$\sigma \cdot D = \sigma \circ D \circ \sigma^{-1} \text{ for all } \sigma \in G\ ,\ D \in End_K(\mathcal{A})$$

In particular, since ∂_i is an element of $End_K(\mathcal{A})$, we can evaluate $\sigma \cdot \partial_i$ and we obtain:

$$\begin{aligned} \sigma \cdot \partial_i (H) &= \sigma \left(\partial_i H \left(\textstyle\sum_{j=1}^{n} \lambda_{1j}\, x_j, \ldots, \sum_{j=1}^{n} \lambda_{nj}\, x_j\right)\right) \\ &= \sigma \left(\textstyle\sum_{j=1}^{n} \lambda_{ki}\, \sigma^{-1} \cdot \partial_k (H)\right) = \left(\textstyle\sum_{j=1}^{n} \lambda_{ki}\, \partial_k\right)(H) \end{aligned}$$

That is, $\sigma \cdot \partial_i$ is also an element of $\mathcal{W}$.
Now, let P be an element of $\mathcal{W}$, and let us consider the multiplication by P, putting $P(H) = PH$, for each H in $\mathcal{A}$. Remembering that the action of G on $\mathcal{A}$ is multiplicative, we have:

$$\left(\sigma \circ P \circ \sigma^{-1}\right)(H) = \sigma\left(P\left(\sigma^{-1}(H)\right)\right) = \sigma(P)\,\sigma\left(\left(\sigma^{-1}(H)\right)\right) = (\sigma \cdot P)(H) = \sigma \cdot P(H)$$

Consequently, $\sigma \circ P \circ \sigma^{-1}$ is in fact the multiplication by $\sigma \cdot P$.
Hence, the action of G on $End_K(\mathcal{A})$ is restricted to an action of G on $\mathcal{W}$.
We will denote by $\mathcal{I}$ the subalgebra of invariants of $\mathcal{A}_n(K)$ defined by:

$$\mathcal{I} = \{D \in \mathcal{A}_n(K)\ /\ \sigma \cdot D = D, \forall \sigma \in G\}$$

Let us observe that $D \in \mathcal{I}$, if and only if, D belongs to the centralizer of G in $End_K(\mathcal{A})$.

With $\mathcal{I}^-$ we will denote the subspace of $\mathcal{A}_n(K)$ given by:

$$\mathcal{I}^- = \{D \in \mathcal{I} \,/\, deg(D) \leq -1\}$$

Let $\mathcal{N}$ be the subspace of $\mathcal{A}$ defined by:

$$\mathcal{N} = \left\{P \in \mathcal{A},\, D(P) = 0, \forall D \in \mathcal{I}^-\right\}$$

Then we have:

Theorem 2.3: *$\mathcal{N}$ is finite dimensional, and every simple G-module of $K[G]$ has a copy in $\mathcal{N}$.*

Proof: Let us consider the ring of invariants $K[\partial_1, \ldots, \partial_n]^G \subseteq K[\partial_1, \ldots, \partial_n]$. Since G is finite, $K[\partial_1, \ldots, \partial_n]$ is an integral extension of $K[\partial_1, \ldots, \partial_n]^G$. It follows that for each index i there are operators $D_0, \ldots, D_{m-1} \in K[\partial_1, \ldots, \partial_n]^G$ such that :

$$\partial_i^m + \partial_i^{m-1} D_{m-1} + \cdots + \partial_i^0 D_0 = 0$$

It is clear that each D_i can be chosen being homogeneous of degree $i - m$, that is to say we can assume D_i is in $\mathcal{I}^-$. From the above identity it follows that :

$$\partial_i^m(P) = 0 \text{ for each } P \in \mathcal{N}$$

And this shows that $\mathcal{N}$ is finite dimensional .
Now, let $\mathcal{S}$ be a simple module of $K[G]$ and Ψ as before. Since $\mathcal{S}$ is a simple G-submodule it must be $\Psi(\mathcal{S}) \simeq \mathcal{S}$, and then $\Psi(\mathcal{S})$ is in a homogeneous component of $\mathcal{A}$. Let m in $\mathbb{N}_0$ be the smallest number such that the homogeneous component of degree m of $\mathcal{A}$ contains a submodule $\mathcal{T} \cong \mathcal{S}$. If $\mathcal{T} \nsubseteq \mathcal{N}$, then $m > 0$ and there exists $D \in \mathcal{I}^-$ such that $D(\mathcal{T}) \neq 0$. We can also assume that D is homogeneous of degree $k \leq -1$. Since D is invariant, we have $D(\mathcal{T}) \simeq \mathcal{T} \simeq \mathcal{S}$, and $D(\mathcal{T})$ is in the homogeneous component of degree $m + k \leq m - 1 < m$. But this contradicts the minimality of m.■

Corollary 2.4: *Let us suppose that G acts irreducibly on K^n. Let H be a subgroup of G such that there is a non-trivial linear invariant form for H . Then*

$$\dim(\mathcal{N}) \leq \left(\frac{|G|}{|H|}\right)^n$$

Proof: Let φ be a non-trivial linear invariant form for H, then the G−orbit of φ has at most $\frac{|G|}{|H|} = m$ elements and, from the irreducibility hypothesis, the G−orbit of φ is a system of generators of the dual space $(K^n)^*$.
Let $\varphi = \varphi_1, \ldots, \varphi_m$ the G−orbit of φ. As in the proof of Theorem 2.3, we can infer that:

$$\partial_{\varphi_i}^m(P) = 0 \text{ for each } P \in \mathcal{N}$$

Then:

$$\dim(\mathcal{N}) \leq m^m \leq \left(\frac{|G|}{|H|}\right)^n$$

■

In particular, if $K = \mathbb{R}$, G is a *Coxeter group* of rank n (see [2], [4], [8]) and H is a *Coxeter subgroup* of G of rank $n-1$, the previous inequality holds since in this case there is a linear invariant form for H. However, in this situation it is possible to give an explicit expression for the polynomial interpolator P, as in the proof of Theorem 1.3.

Corollary 2.5: *Let us suppose that G is an irreducible Coxeter group, H a subgroup of G, such that H has index m and rank $n-1$. Let φ be a non-trivial linear invariant form for H. Denoting by $\mathcal{H}_i$ the homogeneous components of $\mathcal{A}$, we have:*

$$\mathcal{N} \subseteq \bigoplus_{i=1}^{\binom{m}{2}} \mathcal{H}_i$$

Proof: We can assume H is the isotropy group of φ, since the isotropy group of φ, is, for a very well-known result, a Coxeter subgroup of G. Let $\varphi_1, ..., \varphi_m$ be the elements of the G−orbit of φ. Let $v \in \mathbb{R}^n$ be such that v is regular for G and it verifies:

$$\varphi_i(v) \neq \varphi_j(v) \text{ if } i \neq j$$

We put:

$$P(w) = \prod_{i<j} (\varphi_i(w) - \varphi_j(v))$$

Then we have $P(v) \neq 0$, and for $\sigma \neq 1$ in G holds:

$$P(\sigma(v)) = \prod_{i<j} (\varphi_i(\sigma(v)) - \varphi_j(v)) = \prod_{i<j} \left(\left(\sigma^{-1}\varphi_i\right)(v) - \varphi_j(v)\right)$$

There is a permutation π of $\{1, ..., m\}$ such that $\sigma^{-1}\varphi_i = \varphi_{\pi(i)}$. If $\pi = 1$, then σ fixes point by point the system of generators $\varphi_1, ..., \varphi_m$, hence $\sigma = 1$. It follows that $\pi \neq 1$, that is to say, there exists a pair i, j with $i < j$ such that $\pi(i) = j$. We conclude that if $\sigma \neq 1$ then $P(\sigma(v)) = 0$ holds.
But on the other hand:

$$deg(P) = \binom{m}{2}$$

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It is possible to show that in the case of the symmetric group of order n, the minimum degree for which the previous contention is valid is in fact $\binom{n}{2}$. We conjecture that for a Coxeter group of G the minimum degree is the number of reflections of G.

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