U–Brouwerian semilattices

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Abstract

The variety of U–Brouwerian semilattices (or UBS–algebras) is introduced as a generalization of Brouwerian semilattices (P. Köhler, *Brouwerian semilattices*, Trans. Amer. Math. Soc., 268 (1981), 103–126). A UBS–algebra is an algebra $\langle A, \wedge, \rightarrow, \forall, 1\rangle$ of type $(2,2,1,0)$ such that $\langle A, \wedge, \rightarrow, 1\rangle$ is a Brouwerian semilattice and $\forall$ satisfies the identities: $\forall 1 = 1$, $x \wedge \forall x = \forall x$, $\forall\forall x = \forall x$, $\forall(x \wedge y) = \forall x \wedge \forall y$, $\forall(\forall x \rightarrow \forall y) = \forall x \rightarrow \forall y$.

The congruences are determined and it is shown that the principal ones are equationally definable. Furthermore, it is demonstrated that the variety of UBS–algebras is arithmetical.

Finally, the subdirectly irreducible UBS–algebras, as well as the simple and the semisimple ones are characterized.

Key words and phrases. Brouwerian semilattice. Heyting algebras. Congruences. Subdirectly irreducible algebras

AMS subject classification (1997). Primary 06A12; Secondary 03B55, 03G25.

Introduction

In 1957, A. Monteiro and O. Varsavsky [10] considered a generalization of the monadic Boolean algebras introduced by P. Halmos [5] and defined the monadic Heyting algebras as triples $(L, \exists, \forall)$, where L is a Heyting algebra, $\exists$ is a quantifier on L [3, p. 185] and $\forall$ is the dual of an additive clousure operator [1, p. 47], called existential quantifier and universal quantifier, respectively which also satisfy the following identities: $\forall\exists x = \exists x$ and $\exists\forall x = \forall x$.

In 1963, W. C. Nemitz [11] started the study of the implicative semilattices from a purely algebraic point of view. An interest in these algebras also arose from the field of Logic since they form the algebraic counterpart of the fragment of intuitionistic propositional calculus, whose only connectives are the implication and the conjunction. The implicative semilattices have also been studied by other authors under different names: Brouwerian semilattices by Köhler [6], Hertz algebras by Porta [13], generalized Curry algebras by A. Monteiro [9], etc.. A detailed description of them can be found in [4], [6], [9] and [11].

On the other hand, it is well known that every finite implicative semilattice is a Heyting algebra, but in general it may not admit of an underlying lattice structure. Therefore, it is not possible to define the notion of existential quantifier on these algebras. Nevertheless, it is possible to extend them by adding a particular universal quantifier and this is what motivated us to consider what we have called U-Brouwerian semilattices.

Section 1 of this paper contains a concise summary of the results of the implicative semilattice theory which are necessary for the development of the part that follows. In section 2, we introduce the variety of implicative U-semilattices; we also show that the principal congruences are equationally definable and that the variety is arithmetical. In section 3, we characterize the subdirectly irreducible algebras and the simple ones of this variety. Finally, in section 4, by using a result obtained by A. Monteiro [8], we demonstrate that the semisimple algebras form a subvariety.

1 Preliminaries

A Brouwerian semilattice (or BS-algebra) is an algebra $\langle A, \wedge, \rightarrow, 1\rangle$ of type $(2,2,0)$, which satisfies the following identities:

(B1) $x \rightarrow x = 1$,

(B2) $(x \rightarrow y) \wedge y = y$,

(B3) $x \wedge (x \rightarrow y) = x \wedge y$,

(B4) $x \rightarrow (y \wedge z) = (x \rightarrow z) \wedge (x \rightarrow y)$.

The variety of BS–algebras will be denoted by **BS**.

It is well known that every BS–algebra is a meet–semilattice with last element 1. The following properties will be frequently used in this paper. For any BS–algebra A and $x, y, z \in A$, the following identities are fulfilled :

(B5) $1 \to x = x$,

(B6) $x \to (y \to z) = (x \wedge y) \to z = (x \to y) \to (x \to z)$,

(B7) $x = x \wedge ((x \to y) \to y)$,

(B8) $((x \to y) \to y) \wedge (x \to y) = y$,

(B9) $x \leq y$ if and only if $x \to y = 1$,

(B10) $x \leq y$ implies $z \to x \leq z \to y$, $y \to z \leq x \to z$,

(B11) $x \to (y \to z) = y \to (x \to z)$.

In [11], it was proved that the congruence relations of a BS–algebra are in a one–to–one correspondence with their filters. More precisely, if F is a filter of a BS–algebra A, then the relation

$$R(F) = \{(x, y) \in A \times A : (x \to y) \wedge (y \to x) \in F\}$$

is a congruence relation on A and the mapping $F \longrightarrow R(F)$ is an isomorphism from the lattice $\mathcal{F}(A)$ of all the filters of A onto the lattice $Con(A)$ of all the congruences of A, the inverse of which is given by $\Theta \longrightarrow [1]_\Theta$, where $[1]_\Theta$ denotes the congruence class of 1 modulo Θ.

2 UBS–algebras

Definition 2.1 *An algebra $\langle A, \wedge, \to, \forall, 1\rangle$ of type $(2, 2, 1, 0)$ is a U–Brouwerian semilattice (or UBS–algebra) if $\langle A, \wedge, \to, 1\rangle$ is a BS–algebra and the following properties are verified:*

(U1) $\forall 1 = 1$,

(U2) $x \wedge \forall x = \forall x$,

(U3) $\forall\forall x = \forall x$,

(U4) $\forall(x \wedge y) = \forall x \wedge \forall y$,

(U5) $\forall(\forall x \rightarrow \forall y) = \forall x \rightarrow \forall y$.

We shall denote the variety of UBS–algebras by **UBS** and the elements of **UBS** simply by A or by $(A, \forall)$, as usual.

U1 and U3 can be easily demonstrated from the other axioms. Besides, each of the axioms U2, U4 and U5 is independent of the rest. Indeed,

Independence of U2: Let $A \in$ **BS** whose Hasse diagram is indicated in Figure 1 and where $\forall a = \forall 1 = 1$ and $\forall 0 = 0$. Then, U4 and U5 are verified but U2 is not valid because $a \wedge \forall a = a \neq 1 = \forall a$.

Independence of U4: Let A be the BS–algebra indicated in Figure 2, where $\forall x = x$ if $x \neq c$ and $\forall c = a$. Hence U2 and U5 hold but U4 does not, because $\forall(b \wedge c) = \forall b = b \neq 0 = \forall b \wedge \forall c$.

Independence of U5: Let $A \in$ **BS**, whose Hasse diagram is indicated in Figure 3, and $\forall 0 = \forall a = 0$, $\forall b = \forall d = b$ and $\forall x = x$ if $x = c, 1$. Then U2 and U4 are valid, but U5 is not, because $\forall(\forall c \rightarrow \forall d) = \forall(c \rightarrow d) = b \neq d = \forall c \rightarrow \forall d$.

Fig. 1

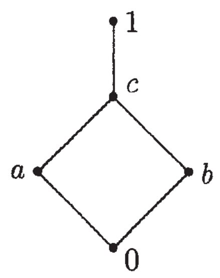

Fig. 2

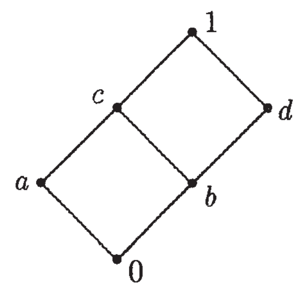

Fig. 3

It is important to point out that the universal quantifiers on BS–algebras are determined by their range in the following sense:

Proposition 2.1 *If $\forall$ is a universal quantifier on a BS-algebra A, then the range of $\forall$, $\forall A$, is a subalgebra of A such that, for all $x \in A$, $\forall x$ is the largest element in the set $(x] \cap \forall A$, where $(x] = \{a \in A : a \leq x\}$.*

Proposition 2.2 *If S is a subalgebra of a BS-algebra A such that there exists the largest element of $(x] \cap S$ for all $x \in A$, then $(A, \forall)$ is a UBS-algebra, such that $\forall A = S$, where $\forall x$ is the largest element of the set $(x] \cap S$.*

Proof.

(U2) It follows from the fact that $\forall x \in (x] \cap S$.

(U4) It can be seen without any difficulty that (1) if $x, y \in A$ and $x \leq y$, then $\forall x \leq \forall y$. Let $z = \forall x \wedge \forall y$. By U2, we have that $z \leq x$ and $z \leq y$. Furthermore, since $\forall x$, $\forall y \in S$ we get that $z \in S$. Then $z \in (x \wedge y] \cap S$ and, as a consequence, $z \leq \forall(x \wedge y)$. The other inequality follows immediately from (1).

(U5) By U2, $\forall(\forall x \rightarrow \forall y) \leq \forall x \rightarrow \forall y$. On the other hand, since $\forall x$, $\forall y \in S$ we have that $\forall x \rightarrow \forall y \in S$. Therefore, $\forall x \rightarrow \forall y \in (\forall x \rightarrow \forall y] \cap S$ and from this it results that $\forall x \rightarrow \forall y \leq \forall(\forall x \rightarrow \forall y)$. □

Next, we shall characterize the congruences in **UBS** similarly to that indicated in [11].

Definition 2.2 *Let $A \in$ **UBS**. A U-filter F of A is a filter such that, if $x \in F$, it implies $\forall x \in F$.*

We shall denote by $\mathcal{F}_U(A)$ the family of all U-filters of A.

Theorem 2.1 *If $A \in$ **UBS**, then $Con(A) = \{R(F) : F \in \mathcal{F}_U(A)\}$.*

Proof. We only prove that $R(F)$ is compatible with $\forall$. If $(x, y) \in R(F)$, then $(x \rightarrow y) \wedge (y \rightarrow x) \in F$. Hence, from Definition 2.2, U4 and B3, we have that $\forall((x \rightarrow y) \wedge (y \rightarrow x)) = \forall(x \rightarrow y) \wedge \forall(y \rightarrow x) \leq (\forall x \rightarrow \forall y) \wedge (\forall y \rightarrow \forall x) \in F$. □

On the other hand, in this variety the principal congruences are equationally definable. More precisely,

Proposition 2.3 *Let $A \in$ **UBS** and $a, b \in A$. Then $\Theta(a, b) = \{(x, y) \in A \times A : \forall((a \rightarrow b) \wedge (b \rightarrow a)) \leq (x \rightarrow y) \wedge (y \rightarrow x)\}$ is the congruence generated by (a, b).*

Proof. Let $F = [\forall((a \to b) \wedge (b \to a)))$ be the filter generated by $\forall((a \to b) \wedge (b \to a))$. Then F is a U-filter and $R(F) = \{(x,y) \in A \times A : \forall((a \to b) \wedge (b \to a)) \leq (x \to y) \wedge (y \to x)\}$. Moreover, from U2 we have that $(a,b) \in R(F)$. On the other hand, let (1) $\alpha \in Con(A)$ such that (2) $(a,b) \in \alpha$, hence $R(F) \subseteq \alpha$. Indeed, let $(x,y) \in R(F)$. Consequently $z = \forall((a \to b) \wedge (b \to a)) \leq (x \to y) \wedge (y \to x)$ and by B3 we obtain that (3) $z \wedge x \leq y$ and (4) $z \wedge y \leq x$. From (1) and (2) we get that $(z,1) \in \alpha$, and so from (3) and (4) it follows that $(x \to y, y \to x) \in \alpha$. Therefore, from B2 and B3 it results that $(x,y) \in \alpha$. Hence $\Theta(a,b) = R(F)$. □

From Proposition 2.3 we conclude that **UBS** has the congruence extension property.

Lemma 2.1 *In* **UBS** *the following identities are verified:*

$$m(x,z,z) = m(x,y,x) = m(z,z,x) = x,$$

where $m(x,y,z) = (\forall(x \to y) \to z) \wedge (\forall(z \to y) \to x) \wedge (\forall(z \to x) \to x)$.

Proof.

(i) $m(x,z,z) = x$: From B1, B5, U1 and B2 we have that $m(x,z,z) = (\forall(x \to z) \to z) \wedge x \wedge (\forall(z \to x) \to x) = (\forall(x \to z) \to z) \wedge x$. Since $x \leq \forall(x \to z) \to z$, then (i) holds.

(ii) $m(x,y,x) = x$: It is an immediate consequence of B1, U1, B5 and B2.

(iii) $m(z,z,x) = x$: It is an immediate consequence of B1, U1 and B5. □

Theorem 2.2 *The variety* **UBS** *is arithmetical.*

Proof. It follows immediately from Lemma 2.1 and [12]. □

3 Subdirectly irreducible and simple algebras

Next, we shall characterize the subdirectly irreducible algebras as well as the simple algebras of this variety.

Lemma 3.1 *Let* $A \in$ **UBS**. *Then the following conditions are equivalent:*

(i) *A is subdirectly irreducible,*

(ii) *there is* $F_0 \in \mathcal{F}_U(A) \setminus \{1\}$, *such that* $F_0 \subseteq F$ *for all* $F \in \mathcal{F}_U(A) \setminus \{1\}$.

Theorem 3.1 *Let* $A \in$ **UBS**. *Then the following conditions are equivalent:*

(i) A *is subdirectly irreducible,*

(ii) $\forall A \setminus \{1\}$ *has last element.*

Proof. (i)⇒(ii): By Lemma 3.1, there is $F_0 \in \mathcal{F}_U(A) \setminus \{1\}$, such that $F_0 \subseteq F$ for all $F \in \mathcal{F}_U(A) \setminus \{1\}$. Since $F_0 \neq \{1\}$, there is $p \in F_0$, $p \neq 1$. From $\forall p \neq 1$ and $\forall p \in F_0$ it follows that $[\forall p) \subseteq F_0$ and, consequently, (1) $[\forall p) = F_0$. On the other hand, if $\forall x \in \forall A \setminus \{1\}$, then $[\forall x) \in \mathcal{F}_U(A) \setminus \{1\}$ and by (1) we get that $[\forall p) \subseteq [\forall x)$. Hence, $\forall p$ is the last element of $\forall A \setminus \{1\}$.

(ii)⇒(i): Let $F_0 = [\forall p)$, where $\forall p$ is the last element of $\forall A \setminus \{1\}$. Then $F_0 \in \mathcal{F}_U(A) \setminus \{1\}$. Furthermore, if $F \in \mathcal{F}_U(A) \setminus \{1\}$, then there is $x \in F$, $x \neq 1$. Therefore, $\forall x \in \forall A \setminus \{1\}$ from which it follows that $F_0 \subseteq F$. Then, by Lemma 3.1 we conclude the proof. □

Proposition 3.1 *Let* $A \in$ **UBS** *with more than one element. If* A *is simple then it has first element.*

Proof. Let $a \in A$, $a \neq 1$. Hence $\{1\} \neq [\forall a) \in \mathcal{F}_U(A)$ and since A is simple, we have that $[\forall a) = A$ and so $\forall a \leq x$ for all $x \in A$. □

Theorem 3.2 gives a characterization of the simple **UBS**–algebras.

Theorem 3.2 *Let* $A \in$ **UBS** *with more than one element. Then the following conditions are equivalent:*

(i) A *is simple,*

(ii) $\forall A = \{0, 1\}$.

Proof. (i)⇒(ii): Let $x \in \forall A$. Then $[\forall x) \in \mathcal{F}_U(A)$ and by (i) we have that $[\forall x) = \{1\}$ or $[\forall x) = A$. Hence $\forall A = \{0, 1\}$.

(ii)⇒(i): Let $F \in \mathcal{F}_U(A) \setminus \{1\}$, then there is $x \in F$, $x \neq 1$. Since $\forall x \in F \setminus \{1\}$ we get that $0 \in F$, from which it follows that $F = A$. □

4 Semisimple algebras

Our next task will be to demonstrate that semisimple UBS–algebras form a variety.

The following result will be used in the subsequent parts of this section.

Lemma 4.1 *In* ***UBS*** *the following identity holds true:*

(U6) $\forall(\forall x \rightarrow y) = \forall x \rightarrow \forall y$.

Proof. By U5, U2 and B10 we have that $\forall x \rightarrow \forall y = \forall(\forall x \rightarrow \forall y) \leq \forall(\forall x \rightarrow y)$. On the other hand, by U3, U4 and B3 it follows that $\forall(\forall x \rightarrow y) \wedge \forall x = \forall(\forall x \wedge y) \leq \forall y$ and so $\forall(\forall x \rightarrow y) \leq \forall x \rightarrow \forall y$. □

Taking into account a well known result obtained by A. Monteiro [8] and in order to characterize semisimple algebras, we define a new binary operation $\Rightarrow$ on a UBS–algebra A by means of the formula: $x \Rightarrow y = \forall x \rightarrow y$ and we call it weak implication.

Lemma 4.2 *The weak implication verifies the following properties:*

(I1) $x \Rightarrow x = 1$,

(I2) $x \Rightarrow (y \Rightarrow x) = 1$,

(I3) $(x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x \Rightarrow y) \Rightarrow (x \Rightarrow z)) = 1$,

(I4) $1 \Rightarrow x = x$,

(I5) $x \Rightarrow \forall x = 1$,

(I6) $x \leq y$ *implies* $x \Rightarrow y = 1$.

Proof. We only prove
(I3) By the definition of $\Rightarrow$ and U6 we have that $(x \Rightarrow y) \Rightarrow (x \Rightarrow z) = \forall(\forall x \rightarrow y) \rightarrow (\forall x \rightarrow z) = (\forall x \rightarrow \forall y) \rightarrow (\forall x \rightarrow z) = \forall x \rightarrow (\forall y \rightarrow z) = x \Rightarrow (y \Rightarrow z)$. Then by I1 we conclude the proof. □

Definition 4.1 *Let* $A \in$ **UBS**. *A subset* D *of* A *is a U–deductive system (U–d.s.) if* D *verifies:*

(D1) $1 \in D$,

(D2) $x,\ x \to y \in D$ *imply* $y \in D$,

(D3) $x \in D$ *implies* $\forall x \in D$.

It is easy to see that the notions of U–filter and U–d.s. are equivalent.

Definition 4.2 *Let* $A \in \boldsymbol{UBS}$. *A subset* D *of* A *is a weak deductive system (w.d.s.) if* D *satisfies* D1 *and*

(D′2) $x,\ x \Rightarrow y \in D$ *imply* $y \in D$.

Lemma 4.3 *Let* $A \in \boldsymbol{UBS}$ *and* $D \subseteq A$. *Then the following properties are equivalent:*

(i) D *is a* U*–d.s.,*

(ii) D *is a w.d.s..*

Proof. (i)⇒(ii): Suppose that $x,\ x \Rightarrow y = \forall x \to y \in D$. Then by D3, we have that $\forall x \in D$ and so by D2 we get that $y \in D$.

(ii)⇒(i): It is simple to see that D3 follows from I5 and D′2. Suppose now that (1) $x,\ x \to y \in D$. Since $x \to y \leq x \Rightarrow y$, by I6 we have that $(x \to y) \Rightarrow (x \Rightarrow y) \in D$, and from (1) and D′2 we get that $y \in D$. □

Now, we are ready to determine semisimple UBS–algebras.

Lemma 4.4 *Let* $A \in \boldsymbol{UBS}$ *and* $x, y \in A$. *Then the following identities are equivalent:*

(i) $(\forall x \to \forall y) \to \forall x = \forall x$,

(ii) $((\forall x \to \forall y) \to \forall x) \to x = 1$,

(iii) $((x \Rightarrow y) \Rightarrow x) \Rightarrow x = 1$.

Theorem 4.1 *Let* $A \in \boldsymbol{UBS}$ *non–trivial. Then the following conditions are equivalent:*

(i) *A is a subdirect product of simple UBS–algebras,*

(ii) *A satisfies the identity* $(\forall x \to \forall y) \to \forall x = \forall x$,

(iii) $\forall A$ *is a Tarski algebra.*

Proof. It is an immediate consequence of [8] and Lemmas 4.1, 4.2, 4.3 and 4.4. □

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