COMBINATORIAL SUMS AND SERIES FOR THE RIEMANN-HURWITZ FUNCTION (II)

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ABSTRACT. This paper deals with several combinatorial sums. These sums give, in the limit case, formulae for the Riemann-Hurwitz function among others. As an example we quote the following formula for Euler's constant γ:

$$1-\gamma=\sum_{N=1}^{\infty}\sum_{n=2^N+1}^{\infty}\frac{1}{2n(2n-1)4^{n-1}}\left(\frac{3n}{2}-\frac{1}{4}\right)+$$

$$\sum_{N=1}^{\infty}\sum_{n=1}^{2^N}(-1)^{n+1}\frac{\binom{2n}{n}}{2n(2n-1)4^{n-1}\binom{2^N+n}{n}\binom{2^N}{n}}(2^{N-1}+\frac{1}{4})$$

Our method of proof is a generalization of Apéry's proof ([1]).

§0 INTRODUCTION

We show below a method by which several interesting combinatorial sums can be given. The method gives identities involving sums of type $\sum_{n=1}^{N}\frac{1}{(n+x)^i}$, $\sum_{n=1}^{N}\frac{(-1)^n}{(n+x)^i}$, and therefore letting $N\to\infty$, sums for the Riemann-Hurwitz function (here $i\in\mathbf{N}$).

The author explored in [9] some of these sums in the case $N=\infty$. For example it is proved there that

$$\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{15}{4}\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}^2}+\sum_{n=1}^{\infty}\frac{1}{n^3(2n-1)\binom{2n}{n}^2} \tag{1.0}$$

We also recall Apéry's formulae $\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}$ which was used by him in his striking proof of the irrationality of $\zeta(3)$.

This paper is organized as follows: theorem 1 is our main theorem, corollary 1 is an (almost) immediate consequence of theorem 1.

We remark two interesting features of these series or combinatorial identities: first, simply by differentiating them with respect to x we obtain new identities. Second, the convergence is fast.

For an eager reader we recommend to read first the proof of a) below were the method is explained and then jump to the proof of any other formulae. The method is a generalization of Apéry's method [1].

§1 SOME FORMULAE

Recall the following notation $(x)_0 = 1$ and $(x)_n = (x+n-1)(x+n-2)...x$. We define $\psi(x,k) = (x+k)(x+k-1)\ldots(x-k+1)$. Observe that $\psi(n+x,n) = \frac{(x)_{2n+1}}{x}$. Our main result is

Theorem 1. *The following formulae hold:*

a)

$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{(n+x)} = \sum_{n=1}^{N} \frac{(2n-2)!}{4^{n-1}\psi(n+x,n)}\left\{\frac{3}{2}n+\frac{1}{2}x-\frac{1}{4}\right\}+$$

$$\underbrace{+\sum_{n=1}^{N}(-1)^{n+N}\frac{(2n-2)!}{4^{n-1}\psi(N+x,n)}\left\{\frac{N+x}{2}+\frac{1}{4}\right\}}$$

b)

$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{(n+x)^2} =$$

$$= \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!^2.(2n-2)!}{\psi^2(n+x,n)}\left\{(2n+x)^2-\frac{(n+x)^2}{2}-\frac{(n+x)}{2}-\frac{(2n-3n^2)}{2}\right\}+$$

$$\underbrace{+\sum_{n=1}^{N}(-1)^{N-1}.\frac{(n-1)!^2(2n-2)!}{\psi^2(N+x,n)}\left\{\frac{(N+x)^2}{2}+\frac{(N+x)}{2}+\frac{(2n-3n^2)}{2}\right\}}$$

c)

If $A(n,k) = \frac{k^2(-5+50k-195k^2+366k^3-334k^4+120k^5)}{4(3-16k+16k^2)} -$

$$-\frac{k(2-20k+70k^2-95k^3+44k^4)n}{6-32k+32k^2}+\frac{(1-10k+10k^2+100k^3-190k^4+88k^5)n^2}{4(-3+16k-16k^2)}-$$

$$-\frac{5(k-1)kn^3}{(3-4k)}-\frac{5(-1-2k+2k^2)n^4}{4(3-4k)}+\frac{3n^5}{6-8k}+\frac{n^6}{6-8k} \text{ then}$$

$$\sum_{n=1}^{N}\frac{1}{(n+x)^3} = \sum_{n=1}^{N}\frac{(2n-2)!^6}{(4n-4)!\psi^4(n+x,n)}\left\{(n+x)(2n+x)^4 - A(n+x,n)\right\}+$$

$$\underbrace{+\sum_{n=1}^{N}\frac{(2n-2)!^6}{(4n-4)!\psi^4(N+x,n)}.A(N+x,n)}$$

d)

$$\sum_{n=1}^{N}\frac{(-1)^{n-1}}{(n+x)}=$$

$$=\sum_{n=1}^{N}(-1)^{n-1}\frac{(2n-2)!^3}{2^{4(n-1)}(n-1)!^2\psi^2(n+x,n)}\cdot\Big\{(n+x)(2n+x)^2-\frac{(n+x)^3}{2}-\frac{3}{4}(n+x)^2-$$

$$-\frac{3}{2}(n+x)(n-n^2)-\frac{(-1+6n-6n^2)}{8}\Big\}+$$

$$+\underbrace{\sum_{n=1}^{N}(-1)^{N-1}\cdot\frac{(2n-2)!^3}{2^{4(n-1)}.(n-1)!^2\psi^2(N+x,n)}\Big\{\frac{(N+x)^3}{2}+}$$

$$\underbrace{\frac{3}{4}(N+x)^2+\frac{3}{2}(N+x)(n-n^2)+\frac{(-1+6n-6n^2)}{8}\Big\}}$$

e)

$$\text{If } A(n,k)=\frac{k(-3+30k-110k^2+155k^3-74k^4)}{6(5-36k+36k^2)}+$$

$$+\frac{(1-10k+10k^2+160k^3-310k^4+148k^5)n}{6(-5+36k-36k^2)}+\frac{5(1-k)}{5-6k}kn^2-\frac{5(-1-2k+2k^2)n^3}{3(5-6k)}+$$

$$\frac{5n^4}{2(5-6k)}+\frac{n^5}{5-6k}\text{ then}$$

$$\sum_{n=1}^{N}\frac{1}{(n+x)^2}=\sum_{n=1}^{N}(-1)^{n-1}\frac{(2n-2)!^6(3n-3)!}{(n-1)!^3(6n-6)!\psi^3(n+x,n)}\cdot\Big\{(n+x)(2n+x)^3-$$

$$-A(n+x,n)\Big\}+\underbrace{\sum_{n=1}^{N}(-1)^{n-1}\frac{(2n-2)!^6(3n-3)!}{(n-1)!^3(6n-6)!\psi^3(N+x,n)}A(N+x,n)}$$

f)

$$\sum_{n=1}^{N}\frac{(-1)^{n-1}}{(n+x)^2}=$$

$$=\sum_{n=1}^{N}\frac{(2n-2)!^3}{\psi^3(n+x,n)}\cdot\Big\{(n+x)(2n+x)^3-\frac{(n+x)^4}{2}-(n+x)^3-$$

$$-3(1-n)n(n+x)^2-(-1+6n-6n^2)\frac{(n+x)}{2}-\frac{n(-3+18n-34n^2+21n^3)}{2}\Big\}+$$

$$+\sum_{n=1}^{N}(-1)^{n+N}\cdot\frac{(2n-2)!^3}{\psi^3(N+x,n)}\Big\{\frac{(N+x)^4}{2}+(N+x)^3+$$

$$+3(1-n)n(N+x)^2+(-1+6n-6n^2)\frac{(N+x)}{2}+\frac{n(-3+18n-34n^2+21n^3)}{2}\Big\}$$

g)

If $\binom{P(k)}{Q(k)}=(-1)^kM(k)M(k-1)\dots M(1)\binom{1}{1}$; $\binom{P(0)}{Q(0)}=\binom{1}{1}$ where $M(k)$ is *the* 2×2 *matrix* $\begin{pmatrix} \alpha_1(k)+k^6 & \alpha_3(k) \\ \alpha_2(k)-3k^4 & \alpha_4(k)+k^6 \end{pmatrix}$ and $\alpha_i(k),\ i=1,\dots,4,\ B_5^I(n,k)$, $B_5^{II}(n,k)$ *are the polynomials and rational functions defined in 10) of the Appendix, then*

$$\sum_{n=1}^{N}(-1)^{n-1}\left(\frac{1}{(n+x)}+\frac{1}{(n+x)^3}\right)=$$

$$=\sum_{n=1}^{N}\frac{(-1)^{n-1}P(n-1)}{\psi^3(n+x,n)}\Big((2n+x)^3-B_5^I(n+x,n)\Big)+$$

$$+\sum_{n=1}^{N}\frac{(-1)^{n-1}Q(n-1)}{\psi^3(n+x,n)}\Big((2n+x)^3(n+x)^2-B_5^{II}(n+x,n)\Big)+$$

$$+\sum_{n=1}^{N}\frac{(-1)^{N-1}P(n-1)}{\psi^3(N+x,n)}B_5^I(N+x,n)+$$

$$+\sum_{n=1}^{N}\frac{(-1)^{N-1}Q(n-1)}{\psi^3(N+x,n)}B_5^{II}(N+x,n)$$

h)

$$\sum_{n=1}^{N}\frac{1}{(n+x)(n+y)}=\sum_{n=1}^{N}(-1)^{n-1}\frac{(1^2-(x-y)^2)\dots((n-1)^2-(x-y)^2)}{\psi(n+x,n)}\left\{\frac{(2n+x)}{(n+y)}+\right.$$

$$\left.+\frac{1}{(2n-1)}(n+x+\tfrac{1}{2})+\frac{(x-y)}{2n}\right\}+$$

$$+\underbrace{\sum_{n=1}^{N}(-1)^{n-1}\frac{(1^2-(x-y)^2)\dots((n-1)^2-(x-y)^2)}{\psi(N+x,n)}}$$

$$\underbrace{\left\{\frac{1}{2(1-2n)}+\frac{N+x}{(1-2n)}-\frac{(x-y)}{2n}\right\}}$$

Proof of a) First we deduce a 'mother' formula from which others will be derived. This is, $i=1,2,\dots$

$$\text{(1.1)}\qquad \frac{1}{x^i}-\frac{b_1\dots b_K}{(x(x+a_1)\dots(x+a_K))^i}=$$

$$=\sum_{k=1}^{K}\frac{b_1\dots b_{k-1}}{(x(x+a_1)\dots(x+a_k))^i}\left(x^i+\binom{i}{1}x^{i-1}a_k+\dots+\binom{i}{1}x^1a_k^{i-1}+a_k^i-b_k\right)$$

This follows by writting $A_k=\frac{b_1\dots b_k}{(x(x+a_1)\dots(x+a_k))^i}$; $A_0=\frac{1}{x^i}$ where the left of (1.1) is A_0-A_K and each summand on the right is $A_{k-1}-A_k$.

Substitute in (1.1) x by $(n+x)^2$, a_k by $-k^2$ and $K=n-1$. The value of b_k will be given later depending on each formula.Thus we get

$$(1.2)\qquad \frac{1}{(n+x)^{2i}}-\frac{b_1\dots b_{n-1}}{\Big((n+x)\phi(n+x,n-1)\Big)^i}=$$

$$=\sum_{k=1}^{n-1}\frac{b_1\dots b_{k-1}}{\Big((n+x)\phi(n+x,k)\Big)^i}\left((n+x)^{2i}-\binom{i}{1}k^2(n+x)^{2i-2}+\dots+(-k^2)^i-b_k\right)$$

where $\phi(x,k)\overset{def}{=}(x+k)(x+k-1)\dots(x-k)$; $\sum_{k=1}^{0}\overset{def}{=}0$ and we understand $b_1\dots b_{k-1}=1$ if $k=1$. Now let $i=2$ in (1.2), $b_k=k^4$, multiply (1.2) by $(n+x)$ and add from $n=1$ to N. This gives

$$(1.3)\qquad \sum_{n=1}^{N}\frac{1}{(n+x)^3}-\sum_{n=1}^{N}\frac{(n-1)!^4}{(n+x)\phi(n+x,n-1)^2}=$$

$$\sum_{n=1}^{N}\sum_{k=1}^{n-1}\underbrace{\frac{(k-1)!^4}{\phi(n+x,k)^2}\Big((n+x)^3-2k^2(n+x)\Big)}=$$

Now let (*) $\epsilon_{n,k}(x):=\frac{(k-1)!^4\left[\frac{2(n+x)^2+2(n+x)+k}{4(1-2k)}+\frac{k}{2}\right]}{\psi^2(n+x,k)}$ (recall $\psi(x,k)=(x+k)(x+k-1)\dots(x-k+1)$). Then $\epsilon_{n,k}(x)-\epsilon_{n-1,k}(x)=$ the bracket of (1.3). Thus the last double sum of (1.3) is equal to

$$(1.4)\qquad =\sum_{n=1}^{N}\sum_{k=1}^{n-1}\epsilon_{n,k}(x)-\epsilon_{n-1,k}(x)=\sum_{k=1}^{N}\epsilon_{N,k}(x)-\sum_{k=1}^{N}\epsilon_{k,k}(x)$$

and therefore

$$\sum_{n=1}^{N}\frac{1}{(n+x)^3}=\sum_{n=1}^{N}\left\{\frac{n^2}{(n+x)}+\frac{5}{2}n+x-\frac{n+2(n+x)+2(n+x)^2}{4(1-2n)}\right\}\frac{(n-1)!^4}{\psi^2(n+x,n)}+$$

(1.5)

$$\underbrace{\sum_{n=1}^{N}\frac{(n-1)!^4\left[\frac{2(N+x)^2+2(N+x)+n}{4(1-2n)}+\frac{n}{2}\right]}{\psi^2(N+x,n)}}$$

From this we get formula (1.0). To show this verify that if $0 \leqslant x \leqslant 1$ then

$$(1.6) \qquad \sum_{n=1}^{N} \frac{(n-1)!^4}{\psi^2(N+x,n)} \leqslant \sum_{n=1}^{N} \frac{(n-1)!^4}{\psi^2(N,n)} = \sum_{n=1}^{N} \frac{1}{\binom{N+n}{n}^2 \binom{N}{n}^2 n^4} = \mathcal{O}\left(\frac{1}{N^4}\right)$$

and thus the bracket of (1.5) tends to zero as N tends to infinity. Putting $x = 0$ in (1.5) we get (1.0) after some elementary simplification.

A more systematic way to find expressions like $\epsilon_{n,k}(x)$ is needed, so we put things in a more clear and abstract view.

Let n, k be variables, j an integer, $p(n,k)$ a function and $A(n,k)$ a solution of

$$(1.7) \qquad A(n,k)(n-k)^j - A(n-1,k)(n+k)^j = p(n,k)$$

then if we define

$$(1.8) \qquad \epsilon_{n,k}(x) = \frac{A(n+x,k)}{\psi^j(n+x,k)}$$

then

$$(1.9) \qquad \epsilon_{n,k}(x) - \epsilon_{n-1,k}(x) = \frac{p(n+x,k)}{\phi^j(n+x,k)}$$

In the Appendix we have put several functions $p(n,k)$ with the corresponding solutions $A(n,k)$ of (1.7). The A's stand for solutions of type (1.7) and the B's stand for solutions of the equation

$$(1.10) \qquad B(n,k)(n-k)^j + B(n-1,k)(n+k)^j = p(n,k)$$

where if we define

$$(1.11) \qquad f_{n,k}(x) = (-1)^{n-1} \frac{B(n+x,k)}{\psi^j(n+x,k)}$$

then

$$(1.12) \qquad f_{n,k}(x) - f_{n-1,k}(x) = (-1)^{n-1} \frac{p(n+x,k)}{\phi^j(n+x,k)}$$

Moreover we see that the equation is linear, i.e., if $A(n,k)$ is a solution of (1.7) and $\mathcal{A}(n,k)$ is a solution of $\mathcal{A}(n,k)(n-k)^j - \mathcal{A}(n-1,k)(n+k)^j = q(n,k)$, then $\tilde{A}(n,k) := r(k)A(n,k) + s(k)\mathcal{A}(n,k)$ satisfies $\tilde{A}(n,k)(n-k)^j - \tilde{A}(n-1,k)(n+k)^j = r(k)p(n,k) + s(k)q(n,k)$. Therefore, for example, if one finds solutions of (1.7) with $p(n,k)$ equal to 1, n, n^2, ... , then one has , by linearity, the solution for any $p(n,k)$, a polynomial in n with coefficients functions of k. As with $A(n,k)$, the same discussion of linearity applies for solutions of (1.10) i.e., for the $B(n,k)$'s. One should remark that the particular formula (*) above is obtained using solutions 4) and 6) of the Appendix.

Now we prove a) of theorem 1. In (1.2) let $i=1$, $b_k = k(\frac{1}{2}-k)$, multiply (1.2) by $(n+x)(-1)^{n-1}$ and add from $n=1$ to N. This gives $b_1 \dots b_{k-1} = (-1)^{k-1}\frac{(2k-2)!}{4^{k-1}}$ and

$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{(n+x)} - \sum_{n=1}^{N} (-1)^{n-1} \frac{b_1 \dots b_{n-1}}{\phi(n+x,n-1)} =$$

$$(1.13) \qquad = \sum_{n=1}^{N} \underbrace{\sum_{k=1}^{n-1} \frac{(-1)^{n-1}}{\phi(n+x,k)} b_1 \dots b_{k-1} \left\{ (n+x)^2 - \frac{k}{2} \right\}}$$

Thus $\underbrace{}$ $= f_{n,k}(x) - f_{n-1,k}(x)$ if we define $f_{n,k}(x)$ as (1.11) with $j=1$, $B(n,k) = b_1 \dots b_{k-1}.B_1(n,k)$ and $B_1(n,k)$ is 2) of the Appendix. Thus (1.13) is equal to $\sum_{n=1}^{N} \sum_{k=1}^{n-1} f_{n,k}(x) - f_{n-1,k}(x) = \sum_{k=1}^{N} f_{N,k}(x) - \sum_{k=1}^{N} f_{k,k}(x)$. This gives a) after some rearrengement.

The proofs of the remaining formulae of theorem 1 follow these lines. ■

Proof of b) Let in (1.2) $i=2$, $b_k = 4k^3(k-\frac{1}{2})$ (thus $b_1 \dots b_{k-1} = (2k-2)!(k-1)!^2$); multiply (1.2) by $(-1)^{n-1}(n+x)^2$ and add from $n=1$ to N. This gives

$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{(n+x)^2} - \sum_{n=1}^{N} \frac{(-1)^{n-1} b_1 \dots b_{n-1}}{\phi^2(n+x,n-1)} =$$

(1.14)

$$= \sum_{n=1}^{N} \underbrace{\sum_{k=1}^{n-1} \frac{(-1)^{n-1} b_1 \dots b_{k-1}}{\phi^2(n+x,k)} \left\{ (n+x)^4 - 2k^2(n+x)^2 + 2k^3 - 3k^4 \right\}}$$

Let $f_{n,k}(x)$ as (1.11) with $j=2$, $B(n,k) = b_1 \dots b_{k-1}.B_2(n,k)$, $B_2(n,k)$ is 5) of the Appendix. Thus the bracket of (1.14) is equal to $f_{n,k}(x) - f_{n-1,k}(x)$ and (1.14) is equal to $\sum_{k=1}^{N} f_{N,k}(x) - \sum_{k=1}^{N} f_{k,k}(x)$. In this way we get formula b). ■

Proof of c) Let $i=4$, in (1.2). Multiply it by $(n+x)^5$, $b_k = \frac{8(2k-1)^5.k^5}{(3-16k+16k^2)}$ (and thus $b_1 \dots b_{k-1} = \frac{(2k-2)!^6}{(4k-4)!}$) and add from $n=1$ to N. We get

$$\sum_{n=1}^{N} \frac{1}{(n+x)^3} - \sum_{n=1}^{N} \frac{b_1 \dots b_{n-1}}{\phi^4(n+x,n-1)}(n+x) =$$

$$\sum_{n=1}^{N} \sum_{k=1}^{n-1} \underbrace{\frac{b_1 \dots b_{k-1}}{\phi^4(n+x,k)} \left\{ (n+x)^9 - 4k^2(n+x)^7 + 6k^4(n+x)^5 - 4k^6(n+x)^3 + \right.}$$

$$+(k^8 - b_k)(n+x)\Big\}$$

Now let $\epsilon_{n,k}(x)$ as (1.8) with $j = 4$, $A(n,k) = b_1 \ldots b_{k-1} A_6(n,k)$ with $A_6(n,k)$ defined in 11) of Appendix. Now the above bracket is $\underbrace{}{} = \epsilon_{n,k}(x) - \epsilon_{n-1,k}(x)$.

Thus the above double sum transforms into $\sum_{k=1}^{N} \epsilon_{N,k}(x) - \sum_{k=1}^{N} \epsilon_{k,k}(x)$. In this way we get formula c). ■

Proof of d) Let $i = 2$ in (1.2), multiply it by $(-1)^{n-1}(n+x)^3$ and add from $n = 1$ to N.

Thus

$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{(n+x)} - \sum_{n=1}^{N} (-1)^{n-1} \frac{b_1 \ldots b_{n-1}(n+x)}{\phi^2(n+x, n-1)} =$$

$$\sum_{n=1}^{N} \sum_{k=1}^{n-1} \underbrace{(-1)^{n-1} \frac{b_1 \ldots b_{k-1}}{\phi^2(n+x,k)} \cdot \Big\{(n+x)^5 - 2k^2(n+x)^3 + (n+x)(k^4 - b_k)\Big\}}$$

Let $b_k = \frac{k(2k-1)^3}{2}$ (thus $b_1 \ldots b_{k-1} = \frac{(2k-2)!^3}{2^{4(k-1)}(k-1)!^2}$) and using (1.11) $f_{n,k}(x) = (-1)^{n-1} b_1 \ldots b_{k-1} \frac{B_3(n+x,k)}{\psi^2(n+x,k)}$ with $B_3(n,k)$ defined in 7) of the Appendix. Then the above bracket is equal to $f_{n,k}(x) - f_{n-1,k}(x)$. Proceeding as before we get d). ■

Proof of e) Let $i = 3$ in (1.2). Multiply it by $(n+x)^4$ and add from $n - 1$ to N. Thus

$$\sum_{n=1}^{N} \frac{1}{(n+x)^2} - \sum_{n=1}^{N} \frac{b_1 \ldots b_{n-1}}{\phi^3(n+x, n-1)} (n+x) =$$

(1.15)

$$\sum_{n=1}^{N} \sum_{k=1}^{n-1} \underbrace{\frac{b_1 \ldots b_{k-1}}{\phi^3(n+x,k)} \cdot \Big\{(n+x)^7 - 3k^2(n+x)^5 + 3k^4(n+x)^3 + (-k^6 - b_k)(n+x)\Big\}}$$

Let $b_k = -\frac{8k^3(2k-1)^5}{3(5-36k+36k^2)}$ (thus $b_1 \ldots b_{k-1} = (-1)^{k-1} \frac{(2k-2)!^6.(3k-3)!}{(k-1)!^3.(6k-6)!}$) and using (1.8) $\epsilon_{n,k}(x) = b_1 \ldots b_{k-1} \cdot \frac{A_5(n+x,k)}{\psi^3(n+x,k)}$ with $A_5(n,k)$ defined in 8) of the Appendix. Thus the above bracket is $\underbrace{}{} = \epsilon_{n,k}(x) - \epsilon_{n-1,k}(x)$. In this way we get e). ■

Proof of f) In the same way as we derived (1.15) but multiplying everything by $(-1)^{n-1}(n+x)^4$, we get

$$\sum_{n=1}^{N}\frac{(-1)^{n-1}}{(n+x)^2}-\sum_{n=1}^{N}(-1)^{n-1}\frac{b_1\dots b_{n-1}}{\phi^3(n+x,n-1)}(n+x)=$$

$$\sum_{n=1}^{N}\sum_{k=1}^{n-1}\underbrace{\frac{(-1)^{n-1}b_1\dots b_{k-1}\Big\{(n+x)^7-3k^2(n+x)^5+3k^4(n+x)^3+(-k^6-b_k)(n+x)\Big\}}{\phi^3(n+x,k)}}$$

Let $b_k=8(1-2k)^3.k^3$ (thus $b_1\dots b_{k-1}=(-1)^{k-1}(2k-2)!^3$) and (using (1.11)) $f_{n,k}(x)=(-1)^{n-1}b_1\dots b_{k-1}\frac{B_4(n+x,k)}{\psi^3(n+x,k)}$; $B_4(n,k)$ is 9) of the Appendix. This gives f). ■

Proof of g) Let in (1.2) $i=3$, multiply it by $(-1)^{n-1}(n+x)^3$, put p_k instead of b_k (p_k to be especified later) and add from $n=1$ to N. To this formula add the following: in (1.2) let $i=3$, multiply it by $(-1)^{n-1}(n+x)^5$ put q_k insted of b_k and add from $n=1$ to N. All this gives

$$\sum_{n=1}^{N}(-1)^{n-1}\left(\frac{1}{n+x}+\frac{1}{(n+x)^3}\right)-\sum_{n=1}^{N}(-1)^{n-1}\frac{((n+x)^2q_1\dots q_{n-1}+p_1\dots p_{n-1})}{\phi^3(n+x,n-1)}=$$

$$\sum_{n=1}^{N}\sum_{k=1}^{n-1}\underbrace{\frac{(-1)^{n-1}}{\phi^3(n+x,k)}\Big\{p_1\dots p_{k-1}\Big((n+x)^6-3k^2(n+x)^4+3k^4(n+x)^2+(-k^6-p_k)\Big)}$$

$$\underbrace{+q_1\dots q_{k-1}\Big((n+x)^8-3k^2(n+x)^6+3k^4(n+x)^4+(-k^6-q_k)(n+x)^2\Big)\Big\}}$$

But $\underbrace{\quad}=f_{n,k}(x)-f_{n-1,k}(x)$ if (using (1.11)) we take $f_{n,k}(x)=$ $=\frac{(-1)^{n-1}}{\psi^3(n+x,k)}\{p_1\dots p_{k-1}B_5^I(n+x,k)+q_1\dots q_{k-1}B_5^{II}(n+x,k)\}$, where $B_5^I(n,k)$, $B_5^{II}(n,k)$ and $\alpha_1(k),\dots,\alpha_4(k)$ are given in 10) of the Appendix, and p_k, q_k are such that they satisfy

(i) $p_1\dots p_{k-1}.\alpha_1(k)+q_1\dots q_{k-1}.\alpha_3(k)=p_1\dots p_{k-1}(-k^6-p_k)$

(ii) $p_1\dots p_{k-1}.\alpha_2(k)+q_1\dots q_{k-1}.\alpha_4(k)=p_1\dots p_{k-1}.3k^4+q_1\dots q_{k-1}.(-k^6-q_k)$

Everything works if p_k, $q_k\neq 0$ for all k (recall $p_1\dots p_{k-1}=q_1\dots q_{k-1}=1$ if $k=1$). We rewrite i) ii). Let $P(k)=p_1\dots p_k$; $Q(k)=q_1\dots q_k$, then i) ii) are equivalent to

$$\begin{pmatrix}P(k)\\Q(k)\end{pmatrix}=-\begin{pmatrix}\alpha_1(k)+k^6 & \alpha_3(k)\\ \alpha_2(k)-3k^4 & \alpha_4(k)+k^6\end{pmatrix}\begin{pmatrix}P(k-1)\\Q(k-1)\end{pmatrix}=$$
$$=-M(k)\begin{pmatrix}P(k-1)\\Q(k-1)\end{pmatrix}$$

and $P(0) = Q(0) = 1$. So $\binom{P(k)}{Q(k)} = (-1)^k M(k).M(k-1)\dots M(1)\binom{1}{1}$ and we need to prove $P(k), Q(k) \neq 0$ for all k (this implies $p_k,\ q_k \neq 0$ for all k). To prove this last assertion observe that

$$M(k) = k^6 \begin{pmatrix} 1 + \frac{3\beta_1(k)}{k} & \frac{-21k}{2} + \frac{47\beta_3(k)}{2} \\ \frac{-21}{k^2} + \frac{21\beta_2(k)}{k^3} & 64 - \frac{327\beta_4(k)}{2k} \end{pmatrix}$$

and $|\beta_i(k)| \leqslant 1$ for $i = 1, \dots, 4$ and all $1 \leqslant k$ (this can be proved easily but tediously, we omit details). Let the open quadrants of the plane be counted clockwise i.e. $\binom{c}{d}$ with $c > 0,\ d > 0$ is the first quadrant, $\binom{c}{d}$ with $c > 0,\ d < 0$ is the second quadrant, etc. Then using the fact that $\binom{1}{0}$, $\binom{0}{-1}$ are transformed to the fourth open quadrant by $-M(k)$ if $4 \leqslant k$, we see that $-M(k)$ transforms the second open quadrant into the fourth open quadrant and viceversa if $4 \leqslant k$.

One checks that $P(k), Q(k) \neq 0$ for $1 \leqslant k \leqslant 4$ and that $P(4) > 0, Q(4) < 0$ respectively. Thus $P(k), Q(k) \neq 0$ for all k. ■

Proof of h) In (1.2) put $i = 1$, multiply by $\frac{(n+x)}{(n+y)}$, put $b_k = (x-y)^2 - k^2$ and add from $n = 1$ to N. Thus

$$\sum_{n=1}^{N} \frac{1}{(n+x)(n+y)} - \sum_{n=1}^{N} \frac{b_1 \dots b_{n-1}}{(n+y)\phi(n+x, n-1)} = \underbrace{\sum_{n=1}^{N}\sum_{k=1}^{n-1} \frac{b_1 \dots b_{k-1}}{\phi(n+x, k)}(n + 2x - y)}$$

Let $\epsilon_{n,k}(x,y) = b_1 \dots b_{k-1} \cdot \frac{\left(\frac{1}{2(1-2k)} + \frac{n+x}{(1-2k)} + (x-y)(-\frac{1}{2k})\right)}{\psi(n+x,k)}$ (that is, we have used solutions 0),1),$A_0(n,k)$,$A_1(n,k)$ of the Appendix and (1.8)). Thus $\underbrace{\quad} = \epsilon_{n,k}(x,y) - \epsilon_{n-1,k}(x,y)$. From this one gets h). ■

Corollary 1.

A) In formulae a), b), c), d), e), h) one can substitute N by ∞ and the formulae remain true if the sums underbracketed are suppressed. For example for a)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+x)} = \sum_{n=1}^{\infty} \frac{(2n-2)!}{4^{n-1}\psi(n+x,n)}\left(\frac{3}{2}n + \frac{1}{2}x - \frac{1}{4}\right)$$

B) If $f(x,n) = \frac{3x}{2n} + 2 + \frac{n+x+\frac{1}{2}}{2n-1}$, γ is Euler's constant and $\Gamma(x)$ is the gamma function then

$$\pi cotg(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^{n-1}(-x)_n(x)_n\left(\frac{f(x,n)}{(x)_{2n+1}} - \frac{f(-x,n)}{(-x)_{2n+1}}\right)$$

and

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)} = -\gamma + \sum_{n=1}^{\infty} (-1)^n \frac{(-x)_n(x)_n}{(x)_{2n+1}} f(x,n)$$

C)

$$1-\gamma=\sum_{N=1}^{\infty}\sum_{n=2^N+1}^{\infty}\frac{1}{2n(2n-1)4^{n-1}}\left(\frac{3n}{2}-\frac{1}{4}\right)+$$

$$+\sum_{N=1}^{\infty}\sum_{n=1}^{2^N}(-1)^{n+1}\frac{\binom{2n}{n}}{2n(2n-1)4^{n-1}\binom{2^N+n}{n}\binom{2^N}{n}}(2^{N-1}+\frac{1}{4})$$

D)

$$1-\gamma=\sum_{N=1}^{\infty}N\left\{\sum_{n=2^N+1}^{2^{N+1}}\frac{1}{2n(2n-1)4^{n-1}}\left(\frac{3n}{2}-\frac{1}{4}\right)\right\}+$$

$$+\sum_{N=1}^{\infty}N\left\{\sum_{n=1}^{2^{N+1}}(-1)^{n}\frac{\binom{2n}{n}}{2n(2n-1)4^{n-1}\binom{2^{N+1}+n}{n}\binom{2^{N+1}}{n}}(2^{N}+\frac{1}{4})\right\}-$$

$$-\sum_{N=1}^{\infty}N\left\{\sum_{n=1}^{2^{N}}(-1)^{n}\frac{\binom{2n}{n}}{2n(2n-1)4^{n-1}\binom{2^{N}+n}{n}\binom{2^{N}}{n}}(2^{N-1}+\frac{1}{4})\right\}$$

E) $\frac{1}{3}-\frac{\pi}{9\sqrt{3}}-\frac{log2}{9}=\sum_{n=1}^{\infty}(\frac{9}{4})^{n-1}\frac{2n-2!}{(6n+1)(6n-2)(6n-5)\ldots4}(\frac{3n}{2}-\frac{1}{12})$

F) $Catalan=1+\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-256)^n}{\binom{2n}{n}\binom{4n}{2n}^2}\frac{5n^2-\frac{1}{8}}{n^3(2n-1)(4n+1)^2}$

G) $\zeta(3)=\frac{5}{16}\sum_{n=1}^{\infty}\frac{1}{\binom{4n}{2n}}\frac{(1-6n+4n^2+24n^3)}{n^3(2n-1)^3}$

H) If $p(n,N)=\frac{1}{2}(-34n^3+21n^4-6n^2(N^2+N-3)+n(-3+6N+6N^2)+N(-1+2N^2+N^3))$ *then*

$$\sum_{n=1}^{N}\frac{1}{n^2}\left((-1)^{n-1}+\frac{1-4n}{4(1-2n)^2}\right)=\sum_{n=1}^{N}(-1)^{N+n}\frac{\binom{2n}{n}^3}{8n^3(2n-1)^3\binom{n+N}{n}^3\binom{N}{n}^3}p(n,N)$$

I) $\zeta(2)=\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\binom{2n}{n}^2}{\binom{6n}{3n}\binom{3n}{2n}}\frac{(1-6n-8n^2+112n^3)}{n^2(2n-1)^3}$

J) Let $S_i(n)=\sum_{j=1}^{2n}\frac{1}{j^i}$ *then*

$$\zeta(3)=\frac{3}{8}\sum_{1}^{\infty}(-1)^{n-1}\frac{\binom{2n}{n}^2}{\binom{6n}{3n}\binom{3n}{2n}}\frac{1}{n^3(2n-1)^5}\Bigg((1-2n)^2n(1-6n-8n^2+112n^3)S_1(n)-$$

$$-\frac{1-10n-80n^2+1240n^3-4240n^4+4288n^5}{12})\Bigg)$$

and

$$\zeta(5)=-\frac{1}{64}\sum_{1}^{\infty}(-1)^{n-1}\frac{\binom{2n}{n}^2}{\binom{6n}{3n}\binom{3n}{2n}}\frac{(6n-1)}{n^3(2n-1)^5}\Bigg(\frac{2(1-2n)^2n(1-6n-8n^2+112n^3)}{3(6n-1)}(-27S_1(n)^3-$$

$$27S_1(n)S_2(n)-6S_3(n))+\frac{1-10n-80n^2+1240n^3-4240n^4+4288n^5}{6(6n-1)}(27S_1(n)^2+9S_2(n))-$$

$$36n(5-35n+54n^2)S_1(n)+2(5-65n+146n^2)\Bigg)$$

Proof of A) We prove A) only for formula e). For the other formulae the proofs are very similar and we omit them. We will use proposition 1 which is given below. Let $0 \leqslant x \leqslant 1$. If we denote $\underbrace{\quad}$ the formula underbracketed of e) we have $|\underbrace{\quad}| \leqslant c_1 N^5 \sum_{n=1}^{N} \frac{(2n-2)!^6(3n-3)!}{(n-1)!^3(6n-6)!\psi^3(N+x,n)} \leqslant c_1 N^5 \sum_{n=1}^{N} \frac{(2n-2)!^6(3n-3)!}{(n-1)!^3(6n-6)!\psi^3(N,n)} =$ $c_1 N^5 \sum_{n=1}^{N} \frac{(2n-2)!^6(3n-3)!}{(n-1)!^9(6n-6)!\binom{N+n}{n}^3\binom{N}{n}^3 n^6} = c_1 N^5 \sum_{n=1}^{N} \frac{\binom{2n-2}{n-1}^5}{\binom{3n-3}{2n-2}\binom{6n-6}{3n-3}} \frac{1}{\binom{N+n}{n}^3\binom{N}{n}^3 n^6}$ $\leqslant c_2 N^5 \sum_{n=1}^{N} \frac{\alpha^n}{\binom{N+n}{n}^3\binom{N}{n}^3 n^6}$ with $\alpha < 64$, using the fact that $\binom{2n}{n} \sim 4^n$, $\binom{3n}{2n} \sim (\frac{27}{4})^n$ as $n \to \infty$. Now use proposition 1 to show that the last formula tends to zero.

From this A) follows for $0 \leqslant x \leqslant 1$. By analytic continuation it follows for all x.■

Proof of B) Recall the well-known expansions

$$\pi cotg(\pi x) = \frac{1}{x} + \sum_{\substack{n\in Z\\ n\neq 0}} \left(\frac{1}{x+n} - \frac{1}{n}\right) = \frac{1}{x} - x\left(\sum_{n=1}^{\infty} \frac{1}{n(n+x)} + \sum_{n=1}^{\infty} \frac{1}{n(n-x)}\right)$$

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)} = -\gamma + \sum_{m=0}^{\infty} \left(\frac{1}{m+1} - \frac{1}{x+m+1}\right) = -\gamma + x\sum_{n=1}^{\infty} \frac{1}{n(n+x)}$$

and use formula h) of theorem 1 with $y = 0$ and A) of this corollary.■

Proof of C) and D) We have if M is even $\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{M}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{M}\right) - \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{M}\right) = 0$. Thus we get $\frac{1}{M/2+1} + \frac{1}{M/2+2} + \cdots + \frac{1}{M} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{M}$. Put $M = 2^N$ and add the above equality for $N = 1, 2, \ldots, N_0$ getting (recall $log\, 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$)

$$\sum_{i=1}^{2^{N_o}} \frac{1}{i} - 1 = \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(1 - \frac{1}{2} + \cdots - \frac{1}{2^{N_o}}\right) =$$

$$= N_o.log\, 2 - \left(\frac{1}{3} - \frac{1}{4} + \ldots\right) - \left(\frac{1}{5} - \frac{1}{6} + \frac{1}{7} \cdots\right) - \left(\frac{1}{2^{N_o}+1} - \frac{1}{2^{N_o}+2} + \ldots\right)$$

Thus

$$\gamma - 1 = lim_{N_o \to \infty}\left(\sum_{i=1}^{2^{N_o}} \frac{1}{i} - N_o log\, 2\right) - 1 = -\sum_{N=1}^{\infty} \underbrace{\sum_{n=2^N+1}^{\infty} \frac{(-1)^{n-1}}{n}}_{} =$$

(1.16)

$$= -\sum_{N=1}^{\infty} N \underbrace{\left(\sum_{n=2^N+1}^{2^{N+1}} \frac{(-1)^{n-1}}{n}\right)}_{}$$

Also from a) of theorem 1 one has putting $x = 0$

$$\sum_{n=N_0}^{N} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} - \sum_{n=1}^{N_0-1} \frac{(-1)^{n-1}}{n} =$$

$$= \sum_{n=N_0}^{N} \frac{1}{2n(2n-1)4^{n-1}}\left(\frac{3n}{2} - \frac{1}{4}\right) + \sum_{n=1}^{N}(-1)^{n+N} \frac{\binom{2n}{n}}{2n(2n-1)4^{n-1}\binom{N+n}{n}\binom{N}{n}}\left(\frac{N}{2} + \frac{1}{4}\right) -$$

(1.17)

$$- \sum_{n=1}^{N_0-1} (-1)^{n+N_0-1} \frac{\binom{2n}{n}}{2n(2n-1)4^{n-1}\binom{N_0-1+n}{n}\binom{N_0-1}{n}}\left(\frac{N_0-1}{2} + \frac{1}{4}\right)$$

Thus letting $N \to \infty$ we get

$$\sum_{n=N_0}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=N_0}^{\infty} \frac{1}{2n(2n-1)4^{n-1}}\left(\frac{3n}{2} - \frac{1}{4}\right) -$$

(1.18)

$$- \sum_{n=1}^{N_0-1} (-1)^{n+N_0-1} \frac{\binom{2n}{n}}{2n(2n-1)4^{n-1}\binom{N_0-1+n}{n}\binom{N_0-1}{n}}\left(\frac{N_0-1}{2} + \frac{1}{4}\right)$$

Now use (1.17) in the last formula of (1.16) for the sum underbracketed to get D).Use (1.18) for the other sum underbracketed of (1.16) to get C).■

Proof of E) ... J) All these formulae are evaluations of the formulae of theorem 1 with $N = \infty$ (except H). E) is a) with $x = \frac{1}{3}$. F) is b) with $x = \frac{1}{2}$. G) is c) with $x = 0$. H) is f) with $x = 0$. I) is e) with $x = 0$. To get J) differentiate e) one and three times and put $x = 0$. ■

Proposition 1. *Let $m = 1, 2, 3, \ldots$ and $0 < \alpha < 4$ then there exists a constant $c = c(\alpha, m)$ depending only on α and m such that*

$$\sum_{n=1}^{N} \frac{\alpha^{n.m}}{\binom{N}{n}^m \binom{N+n}{n}^m n^{2m}} \leqslant c\frac{log^m N}{N^{2m}}$$

if $2 \leqslant N$.

Proof: Let $a_{n,N}(\alpha) = \frac{\alpha^n}{\binom{N}{n}\binom{N+n}{n}n^2}$. From the fact that if $a_{n,N}(\alpha) \geqslant 0$, $\sum_{n=1}^{N} a_{n,N}(\alpha)^m \leqslant (\sum_{n=1}^{N} a_{n,N}(\alpha))^m$, it is enough to prove the above proposition for $m = 1$. We do this first in a weak form, ie., we prove first that $\sum_{n=1}^{N} a_{n,N}(\alpha) \leqslant$

$c(\alpha) log\, N$, with $c(\alpha)$ depending only on α. Using the integral representation of the beta function one has

$$\sum_{n=1}^{N} \frac{\alpha^n}{\binom{N}{n}\binom{N+n}{n} n^2} = \sum_{n=1}^{N} \int_0^1 \int_0^1 t_1^N (1-t_1)^{n-1} t_2^{N-n} (1-t_2)^{n-1} \alpha^n dt_1 dt_2 =$$

$= \alpha \int_0^1 \int_0^1 t_1^N t_2^{N-1} \frac{1-(\frac{\alpha(1-t_1)(1-t_2)}{t_2})^N}{1-(\frac{\alpha(1-t_1)(1-t_2)}{t_2})} dt_1 dt_2$. Integrating first in t_2 and making the change of variable $t_2 = \frac{\alpha(1-t_1)}{\alpha(1-t_1)+\tau_2}$ (i.e $\tau_2 = \frac{\alpha(1-t_1)(1-t_2)}{t_2}$) one has

$$\alpha^{N+1} \int_0^1 (t_1(1-t_1))^N \Big(\int_0^\infty \frac{1-\tau_2^N}{1-\tau_2} \frac{1}{(\alpha(1-t_1)+\tau_2)^{N+1}} d\tau_2 \Big) dt_1$$

We separate this last integral $\int_0^\infty = \int_0^1 + \int_1^\infty$. But

$$\int_1^\infty \leqslant \int_1^2 \frac{1-\tau_2^N}{1-\tau_2} \frac{d\tau_2}{\tau_2^{N+1}} + \int_2^\infty \frac{1-\tau_2^N}{1-\tau_2} \frac{d\tau_2}{\tau_2^{N+1}} \leqslant c_1 N + c_2$$

with c_1 and c_2 independient of N. Thus

$$\alpha^{N+1} \int_0^1 (t_1(1-t_1))^N \Big(\int_1^\infty \dots d\tau_2 \Big) dt_1 \leqslant \frac{\alpha^{N+1}}{4^N} (c_1 N + c_2)$$

because the maximun of $t_1(1-t_1)$ is in $t_1 = 1/2$.

Thus it remains to estimate

$$\alpha^{N+1} \int_0^1 \int_0^1 (t_1(1-t_1))^N \frac{1-\tau_2^N}{1-\tau_2} \frac{1}{(\alpha(1-t_1)+\tau_2)^{N+1}} d\tau_2 dt_1 =$$

$$= \alpha^{N+1} \int_0^1 \frac{1-\tau_2^N}{1-\tau_2} \Big(\int_0^1 \Big(\frac{t_1(1-t_1)}{\alpha(1-t_1)+\tau_2} \Big)^N \frac{1}{(\alpha(1-t_1)+\tau_2)} dt_1 \Big) d\tau_2 \leqslant$$

(1.19)

$$\leqslant \alpha^{N+1} \int_0^1 \frac{1-\tau_2^N}{1-\tau_2} \Big(\max_{t_1 \in [0,1]} \frac{t_1(1-t_1)}{\alpha(1-t_1)+\tau_2} \Big)^N \frac{log(1+\frac{\alpha}{\tau_2})}{\alpha} d\tau_2 =$$

But the maximun of $\frac{t_1(1-t_1)}{\alpha(1-t_1)+\tau_2}$ for $t_1 \in [0,1]$ is given at $t_1 = \frac{\tau_2}{\alpha}\Big(1+\frac{\alpha}{\tau_2}-\sqrt{1+\frac{\alpha}{\tau_2}}\Big)$.

Thus $\Big(\max_{t_1 \in [0,1]} \frac{t_1(1-t_1)}{\alpha(1-t_1)+\tau_2} \Big)^N = \frac{1}{\alpha^N} \Big[\frac{(\sqrt{1+T_2}-1)^2}{T_2} \Big]^N$ if $T_2 = \frac{\alpha}{\tau_2}$. If in the last integral of (1.19) one makes the change of variable $T_2 = \frac{\alpha}{\tau_2}$ and puts $\frac{1-\tau_2^N}{1-\tau_2} = 1+\tau_2+\dots+\tau_2^{N-1}$ one gets

$$= \alpha \sum_{j=0}^{N-1} \int_\alpha^{+\infty} log(1+T_2) . \Big[\frac{(\sqrt{1+T_2}-1)^2}{T_2} \Big]^N . \frac{\alpha^j}{T_2^{j+2}} dT_2 \leqslant$$

$$\leqslant \alpha \sum_{j=0}^{N-1} \alpha^j \int_\alpha^{+\infty} \frac{log(1+T_2)}{T_2^{j+2}} dT_2 \leqslant c_3(\alpha) log N$$

for $N \geqslant 2$ and $c_3(\alpha)$ depending on α only. All this gives $\sum_{n=1}^{N} a_{n,N}(\alpha) \leqslant c(\alpha)logN$ for $N \geqslant 2$. Now we prove the 'strong' form of proposition 1 for $m = 1$. Let α, β two numbers such $0 < \alpha < \beta < 4$, $\gamma := \frac{\alpha}{\beta}(< 1)$ and N_0 = nearest integer to $-\frac{logN^2}{log\gamma}$. Thus using the weaker form we get

$$\sum_{n=1}^{N} a_{n,N}(\alpha) = \sum_{n=1}^{N_0} a_{n,N}(\alpha) + \sum_{n=N_0+1}^{N} a_{n,N}(\beta).\gamma^n \leqslant$$

$$\leqslant \sum_{n=1}^{N_0} a_{n,N}(\alpha) + c(\beta).logN.\gamma^{N_0+1} \leqslant \sum_{n=1}^{N_0} a_{n,N}(\alpha) + c(\beta)\frac{logN}{N^2}$$

Using the fact that $\frac{a_{n,N}(\alpha)}{a_{n-1,N}(\alpha)} = \frac{\alpha(n-1)^2}{(N-n+1)(N+n)} < 1$ if $1 \leqslant n \leqslant N_0$ and N is big enough, one sees that $a_{n,N}(\alpha)$ is decreasing in n for $1 \leqslant n \leqslant N_0$ if N is great enough. Thus

$$\sum_{n=1}^{N_0} a_{n,N}(\alpha) \leqslant N_0.a_{1,N}(\alpha) = \frac{N_0\alpha}{N(N+1)} \leqslant c_4(\alpha).\frac{logN}{N^2}$$

This proves proposition 1.■

Final Remarks.

Note that formula e) of theorem 1 with $N = \infty$ is the most rapidly convergent series of all stated formulae.

APPENDIX

0) $j = 1$, $\quad p(n,k) = 1$, $\quad A_0(n,k) = -\frac{1}{2k}$

1) $j = 1$, $\quad p(n,k) = n$, $\quad A_1(n,k) = \frac{1}{2(1-2k)} + \frac{n}{1-2k}$

2) $j = 1$, $\quad p(n,k) = n^2 - k/2$, $\quad B_1(n,k) = n/2 + 1/4$

3) $j = 1$, $\quad p(n,k) = n^2$, $\quad A_2(n,k) = \frac{n}{2(1-k)} + \frac{n^2}{2(1-k)}$

4) $j = 2$, $\quad p(n,k) = n$, $\quad A_3(n,k) = -\frac{1}{4k}$

5) $j = 2$, $\quad p(n,k) = n^4 - 2k^2n^2 + 2k^3 - 3k^4$,
$B_2(n,k) = \frac{2k-3k^2}{2} + \frac{n}{2} + \frac{n^2}{2}$

6) $j = 2$, $\quad p(n,k) = n^3$, $\quad A_4(n,k) = \frac{(k+2n+2n^2)}{4(1-2k)}$

7) $j = 2, \qquad p(n,k) = n^5 - 2k^2n^3 + n(k^4 - b_k),$

$$B_3(n,k) = \frac{n^3}{2} + \frac{3n^2}{4} + \frac{3(k-k^2)n}{2} + \frac{(-1+6k-6k^2)}{8}$$

here $b_k = k(2k-1)^3/2$.

8) $j = 3, \qquad p(n,k) = n^7 - 3k^2n^5 + 3k^4n^3 + (-k^6 - b_k)n$

$$A_5(n,k) = \frac{k(-3+30k-110k^2+155k^3-74k^4)}{6(5-36k+36k^2)} +$$
$$\frac{(1-10k+10k^2+160k^3-310k^4+148k^5)n}{6(-5+36k-36k^2)}$$
$$+\frac{5(1-k)}{5-6k}kn^2 - \frac{5(-1-2k+2k^2)n^3}{3(5-6k)} + \frac{5n^4}{2(5-6k)} + \frac{n^5}{5-6k}$$

here $b_k = -\frac{8k^3(2k-1)^5}{3(5-36k+36k^2)}$.

9) $j = 3, \qquad p(n,k) = n^7 - 3k^2n^5 + 3k^4n^3 + n(-k^6 - b_k),$

$$B_4(n,k) = \frac{n^4}{2} + n^3 + 3(k-k^2)n^2 +$$
$$+\frac{(-1+6k-6k^2)n}{2} + \frac{k(-3+18k-34k^2+21k^3)}{2}$$

here $b_k = 8(1-2k)^3k^3$.

10) $j = 3, \qquad p(n,k) = n^6 - 3k^2n^4 + n^2\alpha_2(k) + \alpha_1(k),$

$$B_5^I(n,k) = \frac{(-1+9k-12k^2)}{8} + \frac{3(3-4k)kn}{4} + \frac{3n^2}{4} + \frac{n^3}{2}$$

here $\alpha_1(k) = \frac{k^3(1-9k+12k^2)}{4}, \alpha_2(k) = \frac{3k(1-9k+28k^2-24k^3)}{4}$

$j = 3, \qquad p(n,k) = n^8 - 3k^2n^6 + 3k^4n^4 + \alpha_4(k)n^2 + \alpha_3(k)$

$$B_5^{II}(n,k) = \frac{4-45k+147k^2-188k^3+84k^4}{16} + \frac{k(-30+135k-188k^2+84k^3)n}{8} +$$
$$+\frac{(-10+45k-36k^2)n^2}{8} + \frac{3(5-4k)kn^3}{4} + \frac{5n^4}{4} + \frac{n^5}{2}$$

here $\alpha_3(k) = \frac{k^3(-4+45k-147k^2+188k^3-84k^4)}{8}$,
$\alpha_4(k) = \frac{k(-12+135k-601k^2+1284k^3-1308k^4+504k^5)}{8}$.

11) $j = 4, \qquad p(n,k) = n^9 - 4k^2n^7 + 6k^4n^5 - 4k^6n^3 + (k^8 - b_k)n,$

$$A_6(n,k) = \frac{k^2(-5+50k-195k^2+366k^3-334k^4+120k^5)}{4(3-16k+16k^2)} -$$
$$-\frac{k(2-20k+70k^2-95k^3+44k^4)n}{6-32k+32k^2} + \frac{(1-10k+10k^2+100k^3-190k^4+88k^5)n^2}{4(-3+16k-16k^2)} -$$
$$-\frac{5(k-1)kn^3}{(3-4k)} - \frac{5(-1-2k+2k^2)n^4}{4(3-4k)} + \frac{3n^5}{6-8k} + \frac{n^6}{6-8k}$$

here $b_k = \frac{8(2k-1)^5k^5}{3-16k+16k^2}$

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