On Vector Valued Smooth Functions

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Abstract

Recently C. Lizama and H. Prado [6] investigated the inversion of Stieltjes transforms of Banach valued functions defined on the positive real line. They worked in the frame of a McBride type space. The consideration of various examples (Laplace, derived, generalized and iterated Stieltjes, Meijer, theta, etc) strongly suggests that it should be possible to invert the general transforms classified in the sense of Hirschman & Widder [4] by means sequences of fractional operators defined in suitable McBride spaces as in the mentioned Stieltjes case. In order to realize this generalization, in this article we review and extend properties of McBride spaces and some fractional integral operators from the scalar to the vector valued case.

1 Introduction

Throughout this article we´ll denote $\mathbb{I}=(0,\ \infty)$ and by $\mathbb{E}$ a complex Banach space. Hirschman & Widder classified the convolution transforms according to the intrinsic *variation diminishing property* into two classes: finite and non finite ones. In particular, the Stieltjes transform belongs to the second class. Let $S_0(\mathbb{I},\ \mathbb{E})$ be the set of C^∞ functions $\varphi : \mathbb{I} \longrightarrow \mathbb{E}$ such that $t^k(d^k/dt^k)\varphi(t) \longrightarrow 0$ as $t \longrightarrow 0+$ or $t \longrightarrow +\infty$ for each non negative integer k. By considering

$$\Sigma_k\left[\varphi\right](t) = (-t)^{k-1}\left[k!(k-2)!\right]^{-1}\left(\frac{d}{dt}\right)^{2k-1}\left[t^k\ \varphi(t)\right],\ k \geq 2,$$

in [6] it is proved that the above differential operators commute with the Stieltjes transform S on $S_0(\mathbb{I},\ \mathbb{E})$ and that $\lim_{k\rightarrow+\infty}\Sigma_k S = Id_{S_0(\mathbb{I},\ \mathbb{E})}$. In particular, $\Sigma_k^{-1} = (-1)^{k-1}\ k!\ (k-2)!\ I_1^{1-k,2k-1}$ on $S_0(\mathbb{I},\ \mathbb{E})$, i.e. the Stieltjes transform may be inverted in terms of certain fractional operators (see Preliminaries - Note 8). As the operators considered by H&W have common intrinsic properties related to their kernels the above situation seems not to be unique. So, instead of analyzing the inversion of convolution transforms as limits of fractional operators, in this article we generalize the McBride spaces in order to study general integral transforms of vector valued functions. This work is part of a recent joint research by the author, C. Lizama and H. Prado.

2 Preliminaries.

1. We consider the McBride spaces

$$\mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E}) = \{\varphi \in C^\infty(\mathbb{I},\ \mathbb{E}) : \qquad \gamma_k^{p,\mu}(\varphi) < +\infty,\ k \in \mathbb{N}_0\},$$

with $1 \leq p < +\infty$, $\mu \in \mathbb{C}$ and $\gamma_k^{p,\mu}(\varphi) = \left[\int_0^{+\infty} \|t^k\ (d^k/dt^k)\,[t^{-\mu}\ \varphi(t)]\|^p\ dt\right]^{1/p}$.
In particular, we also consider

$$\mathbb{F}_{\infty,\mu}(\mathbb{I},\ \mathbb{E}) = \left\{ \varphi \in C^\infty(\mathbb{I},\ \mathbb{E}) : \quad \begin{array}{c} \lim_{t\to 0+}\ t^k\ (d^k/dt^k)\,[t^{-\mu}\ \varphi(t)] = 0, \\ \lim_{t\to +\infty}\ t^k\ (d^k/dt^k)\,[t^{-\mu}\ \varphi(t)] = 0,\ k \in \mathbb{N}_0 \end{array} \right\},$$

and we write $\gamma_k^{\infty,\mu}(\varphi) = \sup\left\{\left\|t^k\ (d^k/dt^k)\,[t^{-\mu}\ \varphi(t)]\right\|, \qquad t > 0\right\}$.

2. For each $\lambda \in \mathbb{C}$ the operator $t^\lambda\ \varphi = t^\lambda\ \varphi(t)$ is a homeomorphism between $\mathbb{F}_{p,\mu}(\mathbb{I},\mathbb{E})$ and $\mathbb{F}_{p,\mu+\lambda}(\mathbb{I},\mathbb{E})$. Likewise, the usual derivative operator d/dt is linear and bounded between $\mathbb{F}_{p,\mu}(\mathbb{I},\mathbb{E})$ and $\mathbb{F}_{p,\mu-1}(\mathbb{I},\mathbb{E})$.

3. We recall that a function $\varphi : \mathbb{I} \to \mathbb{E}$ is said to be Bochner measurable if it is an almost everywhere limit (with respect to the the Lebesgue measure on $\mathbb{I}$) of step functions. This is equivalent to say that φ is essentially separately valued (i. e. there exist a null subset $\mathbb{Z}$ of $\mathbb{I}$ such that $\varphi(\mathbb{I} - \mathbb{Z})$ is a separable subspace of $\mathbb{E}$) and φ is weakly measurable (i. e. $x^*{}_\circ\ \varphi$ is measurable for every $x^* \in \mathbb{E}^*$) [9]. Thus, it is clear that all functions previously considered are measurable.

4. On the other hand, let us consider the space $\mathbb{B}^p(\mathbb{I},\mathbb{E})$ $(1 \leq p \leq \infty)$ of measurable functions $\varphi : \mathbb{I} \to \mathbb{E}$ such that $\|\varphi\| \in \mathbb{L}^p(\mathbb{I})$. In order that $\varphi \in \mathbb{B}^p(\mathbb{I},\mathbb{E})$ it is necessary and sufficient that there exist a sequence of step functions $\{\varphi_n\}_{n\geq 1}$ such that $\|\varphi - \varphi_n\|_p \to 0$, or equivalently if φ is measurable and $\|\varphi\| \in \mathbb{L}^1(\mathbb{I})$[2].

5. The versions of Fubini's and of bounded convergence theorems for Bochner integrals [1], that we shall apply in this article, hold under the following conditions:

Bounded Convergence Theorem
Let $\{\varphi_n\}_{n\geq 1} \subseteq \mathbb{B}^1(\mathbb{I},\ \mathbb{E})$ such that $\|\varphi_n(t)\| \leq g(t)$ for almost all $t > 0$, where $g : \mathbb{I} \to \mathbb{R}$ is integrable. If $\varphi_n \to \varphi$ a. e. then $\varphi \in \mathbb{B}^1(\mathbb{I},\ \mathbb{E})$ and $\int_{\mathbb{I}} \|\varphi_n(t) - \varphi(t)\|\, dt \to 0$. In particular, $\int_{\mathbb{I}} \varphi_n(t)\ dt \to \int_{\mathbb{I}} \varphi(t)\ dt$.

Fubini's Theorem
If $J,\ K$ are measurable subsets of $\mathbb{I}$ and $\varphi \in \mathbb{B}^1(J \times K,\ \mathbb{E})$ the integrals $\int_J \varphi(s,\ t)\ ds$ and $\int_K \varphi(s,\ t)\ dt$ are well defined for almost all $t \in K$ and $s \in J$ respectively. Moreover,

$$t \to \int_J \varphi(s,\ t)\ ds \in \mathbb{B}^1(K,\ \mathbb{E}), \qquad s \to \int_K \varphi(s,\ t)\ dt \in \mathbb{B}^1(J,\ \mathbb{E})$$

and $\int_{J\times K} \varphi(s,\ t)\ ds\ dt = \int_J \left[\int_K \varphi(s,\ t)\ dt\right] ds = \int_K \left[\int_J \varphi(s,\ t)\ ds\right] dt$.

6. Let $(\Omega,\ \Sigma,\ \mu)$ be a finite measure space, $1 \le p < \infty$, $f \in \mathbb{B}^p(\Omega,\mathbb{E})$ and $g \in \mathbb{B}^{p'}(\Omega,\mathbb{E}^*)$. The function $w \to \langle f(w),\ g(w)\rangle$ belongs to $\mathbb{L}^1(\Omega,\mathbb{C})$ and

$$\int_\Omega |\langle f(w),\ g(w)\rangle|\ d\mu(w) \le \|f\|_{\mathbb{B}^p(\Omega,\mathbb{E})}\ \|g\|_{\mathbb{B}^{p'}(\Omega,\mathbb{E}^*)}\,.$$

Therefore the map $f \to \int_\Omega \langle f(w),\ g(w)\rangle\ d\mu(w)$ is linear and bounded on $\mathbb{B}^p(\Omega,\mathbb{E})$. Moreover,

$$\sup\left\{\int_\Omega |\langle f(w),\ g(w)\rangle|\ d\mu(w) : \qquad \|f\|_{\mathbb{B}^p(\Omega,\mathbb{E})} = 1\right\} = \|g\|_{\mathbb{B}^{p'}(\Omega,\mathbb{E}^*)}\,,$$

i.e. $\mathbb{B}^{p'}(\Omega,\mathbb{E}^*)$ is identified isometrically with a closed subspace of $\mathbb{B}^p(\Omega,\mathbb{E})^*$. This identification becomes an isomorphism if and only if $\mathbb{E}^*$ has the *Radon - Nikodym* property with respect to μ. In general, a Banach space $\mathbb{E}$ has this property if for every $\mathbb{E}$ - valued measure G continuous with respect to the σ - algebra Σ there exist $g \in \mathbb{B}^1(\Omega,\mathbb{E})$ such that $G(C) = \int_C g(w)\ d\mu(w)$ whenever $C \in \Sigma$.

7. Let us consider the following Schwartz spaces

$$D_{\mathbb{L}^p}(\mathbb{E}) \;=\; \{\varphi \in C^\infty(\mathbb{R},\mathbb{E}) : \qquad d^k\varphi/dx^k \in \mathbb{B}^p(\mathbb{R},\mathbb{E}),\ k \in \mathbb{N}_0\}\,,\ 1 \le p < \infty,$$

$$\dot{B}(\mathbb{E}) \;=\; \left\{\varphi \in C^\infty(\mathbb{R},\mathbb{E}) : \qquad \lim_{|x|\to+\infty} d^k\varphi/dx^k = 0,\ k \in \mathbb{N}_0\right\},$$

endowed with the topology given by the family of seminorms

$$\nu_k^p(\varphi) = \left\|d^k\varphi/dx^k\right\|_p,\ 1 \le p \le \infty,\ k \in \mathbb{N}_0.$$

The map $(T_{p,\mu}\varphi)(x) = \exp\left[(1/p - \mu)x\right]\varphi(e^x)$ defines a homeomorphism between $\mathbb{F}_{p,\mu}$ and $D_{\mathbb{L}^p}(\mathbb{E})$ if p is finite and between $\mathbb{F}_{\infty,\mu}$ and $\dot{B}(\mathbb{E})$ (the scalar case is proved in [7], the general case follows easily and formally from that case). Since in each case $C_0^\infty(\mathbb{I},\mathbb{E})$ is dense in $\mathbb{F}_{p,\mu}$ these spaces are Fréchet spaces.

8. It will be of special interest for us to study of the following Erdèlyi - Kober type operators in the framework of McBride spaces of $\mathbb{E}$ - valued functions

$$I_m^{\eta,\alpha} \;=\; x^{-m(\eta+\alpha)} \circ I_m^\alpha \circ x^{m\eta},$$

$$K_m^{\eta,\alpha} \;=\; x^{m\eta} \circ K_m^\alpha \circ x^{-m(\eta+\alpha)},$$

where $m > 0,\ \alpha,\ \eta \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0$ and for $x > 0$ and adequate φ's we write

$$I_m^\alpha\varphi(x) \;=\; \int_0^x \frac{(x^m - t^m)^{\alpha-1}}{\Gamma(\alpha)}\ \varphi(t)\ dt^m,$$

$$K_m^\alpha\varphi(x) \;=\; \int_x^\infty \frac{(t^m - x^m)^{\alpha-1}}{\Gamma(\alpha)}\ \varphi(t)\ dt^m.$$

9. If $\mathbb{E} = \mathbb{C}$, $\lambda + \rho + \sigma = 1/p\ ' + 1/r\ '$, $\mathrm{Re}(\sigma) < 1/p\ '$, $f(t)$ and $\phi(z)$ are measurable functions on $\mathbb{I}$ then

$$\left| \int_0^\infty dz \int_0^z \frac{f(t)\ \phi(z)}{t^\sigma\ z^\rho\ (z-t)^\lambda}\ dt \right| \leq K\ \|f\|_p\ \|\phi\|_r\ , \tag{1}$$

or equivalently

$$\left\| \int_0^z \frac{f(t)\ dt}{t^\sigma\ z^\rho\ (z-t)^\lambda} \right\|_{r\ '} \leq K\ \|f\|_p, \tag{2}$$

under each of the following conditions:
(i) $1 \leq p \leq \infty$, $1/p + 1/r = 1$, $\mathrm{Re}(\lambda) < 1$;
(ii) $p > 1$, $r > 1$, $1/p + 1/r > 1$, $0 < \mathrm{Re}(\lambda) \leq 1/p\ ' + 1/r\ '$;
(iii) $1 \leq p \leq \infty, 1 \leq r \leq \infty$, $1/p + 1/r \geq 1$, $\mathrm{Re}(\lambda) < 1/p\ '$.
(iv) $p = 1 \leq r \leq \infty$, $\mathrm{Re}(\lambda) = 0$.
Moreover, if $f(t) \in \mathbb{L}^p(\mathbb{I})$, $\mathrm{Re}(\zeta) > -1/p\ '$, $\mathrm{Re}(\eta) > -1/p$ then the functions $I_1^{\zeta,\alpha} f$ and $K_1^{\eta,\alpha} f$ are defined a. e. on $\mathbb{I}$ and there exists a positive constant $K = K(p,\ q,\ \alpha,\ \eta,\ \zeta)$ such that

$$\left\| z^{1/p-1/q}\ I_1^{\zeta,\alpha} f(z) \right\|_q \leq K\ \|f\|_p, \qquad \left\| z^{1/p-1/q}\ K_1^{\eta,\alpha} f(z) \right\|_q \leq K\ \|f\|_p,$$

if at least one of the following conditions is satisfied:
(v) $1 \leq p \leq \infty$, $p = q$;
(vi) $1 < p < q < \infty$, $1/p - 1/q \leq \mathrm{Re}(\alpha) \leq 1/p$;
(vii) $1 \leq p \leq q \leq \infty$, $\mathrm{Re}(\alpha) > 1/p$;
(viii) $p = 1 \leq q \leq \infty$, $\mathrm{Re}(\alpha) \geq 1$.(c. f. [5])

10. Our goal is to extend the above formula (1) (or its equivalent form (2)) to vector functions. Kober's theorem relies on the known
Schur's Lemma [10].
Let $1 \leq p \leq \infty$, $H(x,y)$ be an homogeneous function of degree -1 and let us assume that $H(x,1)$ and $f(t)$ are measurable functions on $\mathbb{I}$. On writing

$$T_H\varphi(z) = \int_0^\infty H(t,z)\ f(t)\ dt,\ z > 0,$$

then $\|T_H\varphi\|_p \leq K\ \|f\|_p$, where

$$K = K(H,p) = \int_0^\infty |H(x,1)|\, x^{-1/p}\ dx = \int_0^\infty |H(1,y)|\, y^{-1/p\ '} dy \tag{3}$$

In what follows we shall say that a function $H(x,\ y)$ is a *Schur kernel* if it has the above properties and the constant K is finite.

3 Fractional Valued Integrals

Lemma 1 *Let* $1 < p < \infty$, $\varphi : \mathbb{I} \to \mathbb{E}$ *be a measurable function such that* $t^{1/p} \|\varphi(t)\| \in \mathbb{L}^{\infty}(\mathbb{I}, \mathbb{E})$. *Then* $T_H\varphi$ *is measurable.*

Proof.
Since $t \to H(t,1)$ and $z \to H(1,z)$ are locally integrable functions on $\mathbb{I}$, T_H maps measurable step functions into measurable ones. Let $\{g_n\}_{n\geq 1}$ be an increasing sequence of non negative measurable simple functions such that $g_n(t) \uparrow t^{1/p} \|\varphi(t)\|$ for almost all $t > 0$ and let us write

$$\varphi_n(t) = \varphi(t)\ t^{-1/p} \|\varphi(t)\|^{-1} g_n(t)\ \chi_{\{\varphi\neq 0\}}(t),\ t > 0,\ n \geq 0.$$

Hence $\{\varphi_n\}_{n\geq 1}$ is a sequence of measurable simple functions that converges a. e. to φ and $t^{1/p} \|\varphi_n(t)\| = g_n(t) \leq t^{1/p} \|\varphi(t)\|$. Thus, if $z > 0$ we have

$$\begin{aligned} \|H(t,\ z)\ \varphi_n(t)\| &= z^{-1} \left|H(t\ z^{-1},\ 1)\right| \|\varphi_n(t)\| \\ &\leq z^{-1-1/p}\ (t\ z^{-1})^{-1/p} \left|H(t\ z^{-1},\ 1)\right| t^{1/p} \|\varphi(t)\|,\ t > 0. \end{aligned}$$

By the hypothesis, (3) and the bounded convergence theorem we deduce that $t \to H(t,\ z)\ \varphi(t) \in \mathbb{B}^1(\mathbb{I}, \mathbb{E})$ and $T_H\varphi(z) = \lim_{n\to\infty} T_H\varphi_n(z)$ if $z > 0$. In particular, since the limit a. e. of measurable functions is measurable (see [3]) $T_H\varphi$ becomes measurable and the lemma is proved.□

Theorem 2 *Let* $\varphi \in \mathbb{B}^p(\mathbb{I}, \mathbb{E})$, $1 \leq p \leq \infty$. *The function* $T_H\varphi$ *is well defined,* $T_H\varphi \in \mathbb{B}^p(\mathbb{I}, \mathbb{E})$ *and*

$$\|T_H\varphi\|_{\mathbb{B}^p(\mathbb{I},\mathbb{E})} \leq K(H,\ p)\ \ \|\varphi\|_{\mathbb{B}^p(\mathbb{I},\mathbb{E})} \tag{4}$$

if $p = 1, \infty$, *the same conclusion being true for* $1 < p < \infty$ *if* $t^{1/p} \|\varphi(t)\| \in \mathbb{L}^{\infty}(\mathbb{I})$.

Proof.
We note that $t \to H(t,\ z)\varphi(t) = z^{-1}\ H(z^{-1}\ t,\ 1)\varphi(t)$ is measurable for each $z > 0$. Since $\int_0^{\infty} |H(t,\ z)|\, dt = \int_0^{\infty} \left|H(1,\ t^{-1}\ z)\right| t^{-1}\ zd(z^{-1}\ t) = K < \infty$, if $p = \infty$ then

$$\int_0^{\infty} \|H(t,\ z)\varphi(t)\|\, dt \leq K\ \ \|\varphi\|_{\mathbb{B}^{\infty}(\mathbb{I},\mathbb{E})} < \infty,\ z > 0,$$

and the claim follows in this case. If $p = 1$ we can write

$$\int_{\mathbb{I}\times\mathbb{I}} \|H(t,\ z)\varphi(t)\|\, d(t \times z) = K\ (H,1) \int_{\mathbb{I}} \|\varphi(t)\|\, dt, \tag{5}$$

so $\int_0^{\infty} \|H(t,\ z)\varphi(t)\|\, dt < \infty$ and $T_H\varphi(z)$ is defined for almost all $z > 0$. By Fubini's theorem $T_H\varphi$ is measurable. Moreover, if $\{\varphi_n\}_{n\geq 1}$ is a sequence of measurable step functions such that $\int_{\mathbb{I}} \|\varphi(t) - \varphi_n(t)\| \, dt \to 0$ then

$$\int_0^\infty \left\| \int_0^\infty H(t, z) \left[\varphi(t) - \varphi_n(t)\right] dt \right\| dz \le \int_0^\infty \|\varphi(t) - \varphi_n(t)\| \int_0^\infty |H(t, z)| \; dz \; dt$$

$$= K(H, 1) \int_0^\infty \|\varphi(t) - \varphi_n(t)\| \, dt,$$

i. e. $T_H\varphi \in \mathbb{B}^1(\mathbb{I}, \mathbb{E})$ and by (5) the result follows in this case.
If $1 < p < \infty$ we write

$$\int_0^\infty \|H(t, z) \; \varphi(t)\| \, dt =$$

$$= \int_0^\infty \|\varphi(t)\| \left(\frac{t}{z}\right)^{\frac{1}{pp'}} \left|H\left(\frac{t}{z}, 1\right)\right|^{\frac{1}{p}} \left(\frac{t}{z}\right)^{-\frac{1}{pp'}} \left|H\left(\frac{t}{z}, 1\right)\right|^{\frac{1}{p'}} \frac{dt}{z}$$

$$\le z^{-1} \left[\int_0^\infty \|\varphi(t)\|^p \left(\frac{t}{z}\right)^{\frac{1}{p'}} \left|H\left(\frac{t}{z}, 1\right)\right| dt\right]^{\frac{1}{p}} \left[\int_0^\infty \left(\frac{t}{z}\right)^{-\frac{1}{p}} \left|H\left(\frac{t}{z}, 1\right)\right| dt\right]^{\frac{1}{p'}}.$$

By Lemma 2.1 the function $T_H\varphi$ is measurable and

$$\left[\int_0^\infty \left\| \int_0^\infty H(t, z)\varphi(t)dt \right\|^p dz\right]^{1/p} \le K(H,p) \; \|\varphi\|_{\mathbb{B}^p(\mathbb{I},\mathbb{E})} < \infty. \square$$

The next lemma generalizes the case of scalar functions (see [7], Th. 2.2, page 14) and it shows that the conditions of lemma 2.1 are only apparent on McBride spaces.

Lemma 3 *If* $1 \le p \le \infty$, $\mu \in \mathbb{C}$, $\varphi \in \mathbb{F}_{p,\mu}(\mathbb{I}, \mathbb{E})$ *then* $t^{1/p - \mathrm{Re}(\mu)} \; \varphi(t) \in \mathbb{L}^\infty(\mathbb{I}, \mathbb{E})$.

Proof.
Given $x^* \in \mathbb{E}^*$ we have $x^* \circ \varphi \in \mathbb{F}_{p,\mu}(\mathbb{I}, \mathbb{C})$ and $\gamma_k^{p,\mu}(x^* \circ \varphi) \le \|x^*\| \; \gamma_k^{p,\mu}(x^* \circ \varphi)$ for $k \in \mathbb{N}_0$. From [7] we obtain that

$$\sup\{\, t^{1/p - \mathrm{Re}(\mu)} \; |x^*(\,\varphi(t))| : \; t > 0\} < \infty$$

if $x^* \in \mathbb{E}^*$. But $\mathbb{E}$ is identified isometrically with a closed subspace of $\mathbb{E}^{**}$, $\{t^{1/p-\mu} \; \varphi(t)\}_{t>0} \subseteq \mathbb{E}^{**}$ and by the principle of uniform boundedness we deduce that $\sup\{\, t^{1/p - \mathrm{Re}(\mu)} \; \|\varphi(t)\| : \; t > 0\} < \infty$. $\square$

Corollary 4 *Let* $\varphi \in \mathbb{F}_{p,0}$, $1 \le p \le \infty$. *Then* $T_H\varphi$ *is everywhere defined and it is continuous. In particular,* $T_H\varphi(z) \to 0$ *if* $z \to 0^+$ *or* $z \to +\infty$.

Proof.
If $z > 0$ then $\|H(t, 1) \; \varphi(tz)\| = z^{-1/p} \; t^{-1/p} \; |H(t, 1)| \; (tz)^{1/p} \, \|\varphi(tz)\|$ and it is enough to consider (3) and Lemma 2.3. $\square$

Corollary 5 *If $1 \le p \le \infty$ the operator T_H is linear and continuous on $\mathbb{F}_{p,0}$.*

Proof.
Given $\varphi \in \mathbb{F}_{p,0}$, $z > 0$ we will prove that $z\frac{d}{dz}[T_H\varphi](z) = T_H[t\,\varphi'(t)](z)$. In particular, we already know that $t\,\varphi'(t) \in \mathbb{F}_{p,0}$ and $T_H[t\,\varphi'(t)](z) \in \mathbb{E}$. For $0 \le s \le 1$, $0 < \varepsilon < z$, if $0 < |\delta| \le z - \varepsilon$ then $2z \ge |z + \delta\, s| \ge z - |\delta|\, s \ge \varepsilon$. If $\varepsilon \le u \le 2z$ it is easy to see that $t\,\varphi'(tu) \in \mathbb{F}_{p,0}$ and

$$\|t\,\varphi'(tu)\| \le \varepsilon^{-1-1/p}\, t^{-1/p} \left\|t^{1+1/p}\,\varphi'(t)\right\|_{\mathbb{L}^\infty(\mathbb{I},\mathbb{E})},$$

i. e.

$$\left\|\delta^{-1}\left[T_H\varphi(z+\delta) - T_H\varphi(z)\right] - z^{-1}\, T_H\left[t\,\varphi'(t)\right](z)\right\| =$$

$$= \left\|\int_0^\infty t\, H(t,1) \int_0^1 \{\varphi'[t(z+\delta s)] - \varphi'(tz)\}\, ds\, dt\right\|$$

$$\le \int_0^1 \int_0^\infty t\, |H(t,1)|\, \|\varphi'[t(z+\delta s)] - \varphi'(tz)\|\, dt\, ds.$$

But $t\, \|\varphi'[t(z+\delta s)] - \varphi'(tz)\| \le 2\varepsilon^{-1-1/p}\, t^{-1/p}\, \|t^{1+1/p}\,\varphi'(t)\|_{\mathbb{L}^\infty(\mathbb{I},\mathbb{E})}$ and by (2) it will suffice to apply the Bounded Convergence Theorem. Inductively it follows that

$$z^n \frac{d^n}{dz^n}[T_H\varphi](z) = T_H\left[t^n \frac{d^n}{dt^n}\varphi(t)\right](z),\ z > 0. \tag{6}$$

Therefore $T_H\varphi \in C^\infty(\mathbb{I},\mathbb{E})$ and

$$\left(\int_{\mathbb{I}} \left\|z^n \frac{d^n}{dz^n}[T_H\varphi](z)\right\|^p dz\right)^{1/p} \le K(H,\ p)\ \left(\int_{\mathbb{I}} \left\|t^n \frac{d^n}{dt^n}\varphi(t)\right\|^p dt\right)^{1/p},$$

i. e. $T_H\varphi \in \mathbb{F}_{p,0}$ and $\gamma_n^{p,0}(T_H\varphi) \le K(H,\ p)\ \gamma_n^{p,0}(\varphi)$ for $n \in \mathbb{N}_0$. □

Corollary 6 *If $\mathbb{E}^{**}$ has the Radón - Nikodym property and $1 < p \le \infty$ then (4) holds for each $\varphi \in \mathbb{B}^p(\mathbb{I},\ \mathbb{E})$ if and only if*

$$\left|\int_{\mathbb{I}} \left\langle f(z), \int_{\mathbb{I}} H(t,\ z)\varphi(t)dt\right\rangle\ dz\right| \le K\ \|\varphi\|_{\mathbb{B}^p(\mathbb{I},\ \mathbb{E})}\, \|f\|_{\mathbb{B}^{p'}(\mathbb{I},\ \mathbb{E}^*)} \tag{7}$$

for every functional $f \in \mathbb{B}^{p'}(\mathbb{I},\ \mathbb{E}^*)$.

Proof.
Since Lebesgue measure is σ - finite on $\mathbb{I}$, taking into account the Note 6 of Section 1 the space $\mathbb{B}^{p'}(\mathbb{I},\ \mathbb{E}^*)^*$ is isometrically identified with $\mathbb{B}^p(\mathbb{I},\ \mathbb{E}^{**})$. If $\varphi \in \mathbb{B}^p(\mathbb{I},\ \mathbb{E})$ and we asume (4) then $T_H\varphi \in \mathbb{B}^p(\mathbb{I},\ \mathbb{E}) \hookrightarrow \mathbb{B}^p(\mathbb{I},\ \mathbb{E}^{**})$ and

$$\left|\int_{\mathbb{I}} \langle f(z), T_H\varphi(z)\rangle\ dz\right| \le \|T_H\varphi\|_{\mathbb{B}^p(\mathbb{I},\ \mathbb{E})}\ \|f\|_{\mathbb{B}^{p'}(\mathbb{I},\ \mathbb{E}^*)}.$$

On the other hand, if $\varphi \in \mathbb{B}^p(\mathbb{I},\ \mathbb{E})$ by (7) the linear functional $f \to \langle f, T_H\varphi\rangle$ is bounded on $\mathbb{B}^{p\,\prime}(\mathbb{I},\ \mathbb{E}^*)$ and its norm is not greater than $K\ \|\varphi\|_{\mathbb{B}^p(\mathbb{I},\ \mathbb{E})}$. Thus we have $\|T_H\varphi\|_{\mathbb{B}^p(\mathbb{I},\ \mathbb{E}^{**})} \leq K\ \|\varphi\|_{\mathbb{B}^p(\mathbb{I},\ \mathbb{E})}$. But $T_H\varphi \in \mathbb{B}^p(\mathbb{I},\ \mathbb{E})$ and the result follows. □

Corollary 7 *If $1 \leq p \leq \infty$, $\mu \in \mathbb{C}$ and H is a Schur kernel the linear operator $x^\mu\ T_H\ x^{-\mu}$ is bounded from $\mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E})$ into itself.*

Corollary 8 *Let $m > 0$, η, α, $\mu \in \mathbb{C}$, $H_{m,\mu}^{\eta,\alpha} : \mathbb{I}\times\mathbb{I} \to \mathbb{C}$ be given as*

$$H_{m,\mu}^{\eta,\alpha}(t,z) = m\ \frac{(z^m - t^m)^{\alpha-1}}{\Gamma(\alpha)}\ z^{-\mu-m(\eta+\alpha)}\ t^{m(\eta+1)+\mu-1}\chi_{(0,z]}(t).$$

Then $H_m^{\eta,\alpha}$ is a Schur kernel, $I_m^{\eta,\alpha} = x^\mu\ T_{H_{m,\mu}^{\eta,\alpha}}\ x^{-\mu}$ and $I_m^{\eta,\alpha}$ is a bounded linear operator on $\mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E})$ if $\mathrm{Re}(m\eta+\mu) + m > 1/p$ and $\mathrm{Re}(\alpha) > 0$.

Proof.
It is evident that $H_{m,\mu}^{\eta,\alpha}$ is homogeneous of degree -1. Moreover

$$\begin{aligned}|\Gamma(\alpha)| \int_0^\infty \left|H_{m,\mu}^{\eta,\alpha}(t,1)\right| t^{-1/p}\ dx &= m\int_0^1 (1-t^m)^{\mathrm{Re}(\alpha)-1}\ t^{\mathrm{Re}[m(\eta+1)+\mu]-1-1/p}\ dt \\ &= \int_0^1 (1-u)^{\mathrm{Re}(\alpha)-1}\ u^{\mathrm{Re}[\eta+(\mu-1/p)/m]}\ du \\ &= Be\{\mathrm{Re}(\alpha),\ \mathrm{Re}[\eta+1+(\mu-1/p)/m]\} < \infty\end{aligned}$$

if $\mathrm{Re}(m\eta+\mu)+m > 1/p$ and $\mathrm{Re}(\alpha) > 0$. □

Theorem 9 *For each complex α with positive real part $I_m^{\eta,\alpha}$ has a linear continuous Fréchet derivative on $\mathbb{F}_{p,0}(\mathbb{I},\ \mathbb{E})$ if $m\,\mathrm{Re}(\eta) + m > 1/p$ (for the scalar case see [8]).*

Proof.
We shall write $\log w = \ln|w| + i\arg w$ for the determination of the argument so that $-\pi < \arg w < \pi$. Given $\alpha_0 \in \mathbb{C}$ with positive real part, $\varphi \in \mathbb{F}_{p,0}(\mathbb{I},\ \mathbb{E})$ and $z > 0$ we have

$$\frac{I_m^{\eta,\alpha}\varphi(z) - I_m^{\eta,\alpha_0}\varphi(z)}{\alpha-\alpha_0} = \int_0^z \frac{\frac{[1-t^m/z^m]^{\alpha-1}}{\Gamma(\alpha)} - \frac{[1-t^m/z^m]^{\alpha_0-1}}{\Gamma(\alpha_0)}}{\alpha-\alpha_0}\ \frac{\varphi(t)t^{m\eta}dt^m}{z^{m\eta+m}}. \qquad (8)$$

Thus letting $\alpha \to \alpha_0$ we obtain

$$\frac{\partial I_m^{\eta,\alpha}\varphi}{\partial\alpha}(z)\ |_{\alpha=\alpha_0} = -\psi(\alpha_0)\ I_m^{\eta,\alpha_0}\varphi(z) + J_m^{\eta,\alpha_0}\varphi(z), \qquad (9)$$

where

$$J_m^{\eta,\alpha}\varphi(z) = z^{-m(\eta+\alpha)} \int_0^z \frac{(z^m-t^m)^{\alpha-1}}{\Gamma(\alpha)} \log\left[1-\left(\frac{t}{z}\right)^m\right]\ t^{m\eta}\varphi(t)\ dt^m \qquad (10)$$

and $\psi(\alpha) = d[\log\Gamma(\alpha)]/d\alpha$ is the usual Euler's function. If $|1-w|<1$ then

$$(1-w)^{\alpha-1} = (1-w)^{\alpha_0-1}\exp(\alpha-\alpha_0).$$

Therefore there exists an analytic function $A(\alpha)$ and a positive number ε, both independent of w, so that $\mathrm{Re}(\alpha)>0$ if $|\alpha-\alpha_0|\le\varepsilon$ and

$$\begin{aligned}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-w)^{\alpha_0-1}}{\Gamma(\alpha_0)} &= (1-w)^{\alpha_0-1}\left[\frac{\exp(\alpha-\alpha_0)}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha_0)}\right] \\ &= (1-w)^{\alpha_0-1}(\alpha-\alpha_0)A(\alpha).\end{aligned}$$

By the maximum modulus theorem we have

$$\left|\frac{[1-t^m/z^m]^{\alpha-1}/\Gamma(\alpha) - [1-t^m/z^m]^{\alpha_0-1}/\Gamma(\alpha_0)}{\alpha-\alpha_0}\right| \le (1-w)^{\mathrm{Re}(\alpha_0)-1} \max_{|\alpha-\alpha_0|=\varepsilon}|A(\alpha)|. \tag{11}$$

By using (8) we write

$$\left\|\frac{\frac{[1-t^m/z^m]^{\alpha-1}}{\Gamma(\alpha)} - \frac{[1-t^m/z^m]^{\alpha_0-1}}{\Gamma(\alpha_0)}}{\alpha-\alpha_0}\varphi(t)t^{m\eta+m-1}\right\| \le C_1 \frac{t^{m\eta+m-1+\mathrm{Re}(\mu)-1/p}}{(1-t^m/z^m)^{1-\mathrm{Re}(\alpha_0)}} \tag{12}$$

where C_1 is a positive constant determined by (11) and Lemma 2.3. Por (12), the hypothesis and the bounded convergence theorem it is proved (9). Since linear combinations of Schur kernels are Schur kernels we write

$$\left\|z^n\frac{d^n}{dz^n}\left[\frac{I_m^{\eta,\alpha}\varphi(z) - I_m^{\eta,\alpha_0}\varphi(z)}{\alpha-\alpha_0} + \psi(\alpha_0)\,I_m^{\eta,\alpha_0}\varphi(z) - J_m^{\eta,\alpha_0}\varphi(z)\right]\right\|_{\mathbb{B}^p(\mathbb{I},\,\mathbb{E})} =$$

$$\begin{aligned} &= \left\|T_{\left(H_{m,0}^{\eta,\alpha}-H_{m,0}^{\eta,\alpha_0}\right)/(\alpha-\alpha_0)+\psi(\alpha_0)\,H_{m,0}^{\eta,\alpha_0}-K_{m,0}^{\eta,\alpha_0}}\left[z^n\frac{d^n}{dz^n}\varphi(z)\right]\right\|_{\mathbb{B}^p(\mathbb{I},\,\mathbb{E})} \quad (13)\\ &\le \gamma_n^{p,0}(\varphi)\int_0^1\left|\frac{H_{m,0}^{\eta,\alpha}-H_{m,0}^{\eta,\alpha_0}}{\alpha-\alpha_0} + \psi(\alpha_0)\,H_{m,0}^{\eta,\alpha_0} - K_{m,\mu}^{\eta,\alpha_0}\right|(t,1)\,t^{-1/p}dt.\end{aligned}$$

By (11), with the above notation and conditions there exists a positive constant C_2 such that

$$\left|\frac{H_{m,0}^{\eta,\alpha}-H_{m,0}^{\eta,\alpha_0}}{\alpha-\alpha_0} + \psi(\alpha_0)\,H_{m,0}^{\eta,\alpha_0} - K_{m,\mu}^{\eta,\alpha_0}\right|(t,1)\,t^{-1/p} = \tag{14}$$

$$\begin{aligned} &= m\,t^{\mathrm{Re}(m\eta+\mu)+m-1-1/p}\left|\frac{\frac{[1-t^m]^{\alpha-1}}{\Gamma(\alpha)} - \frac{[1-t^m]^{\alpha_0-1}}{\Gamma(\alpha_0)}}{\alpha-\alpha_0} + \frac{1}{\Gamma(\alpha_0)}\frac{\psi(\alpha_0)-\log(1-t^m)}{[1-t^m]^{1-\alpha_0}}\right| \\ &\le m\,t^{\mathrm{Re}(m\eta+\mu)+m-1-1/p}\frac{(1-t^m)^{\mathrm{Re}(\alpha_0)-1}}{\Gamma(\alpha_0)}[C_2+|\psi(\alpha_0)|-\log(1-t^m)].\end{aligned}$$

In general, for $a,\ b \in \mathbb{I}$ the integral $\int_0^\infty t^{a-1}(1+t)^{-a-b}\ \log(1+t)\ dt$ converges. For , if $0 < \nu < a$ there exists $T = T(\nu) > 0$ such that $(1+t)^{-\nu}\log(1+t) \leq 1$ if $t \geq T$. Thus $t^{a-1}(1+t)^{-a-b}\ \log(1+t) \leq t^{a-1}(1+t)^{-(a-\nu)-b}$ and

$$\int_T^\infty t^{a-1}(1+t)^{-a-b}\ \log(1+t)\ dt \leq \int_0^\infty t^{a-1}(1+t)^{-(a-\nu)-b} < +\infty.$$

On the other hand, there exists $0 < \delta < T$ such that $\log(1+t) \leq 1$ if $0 < t < \delta$ and so

$$\int_0^\delta t^{a-1}(1+t)^{-a-b}\ \log(1+t)\ dt \leq \int_0^\infty t^{a-1}(1+t)^{-a-b} dt < +\infty.$$

Therefore $\int_0^1 t^{a-1}(1-t)^{b-1}\ \log(1-t)\ dt$ converges absolutely. In particular, if $a = \mathrm{Re}(\alpha_0)$, $b = \mathrm{Re}[\eta + (\mu - 1/p)/m] + 1$ in (14) we have an integrable majorant and letting $\alpha \to \alpha_0$ in (13) we have

$$\gamma_n^{p,0}\left(\frac{I_m^{\eta,\alpha}\varphi(z) - I_m^{\eta,\alpha_0}\varphi(z)}{\alpha - \alpha_0} + \psi(\alpha_0)\ I_m^{\eta,\alpha_0}\varphi(z) - J_m^{\eta,\alpha_0}\varphi(z)\right) \to 0,$$

i. e. (9) is the Fréchet derivative with respect to α for $\alpha = \alpha_0$ on $\mathbb{F}_{p,0}(\mathbb{I},\ \mathbb{E})$ of the operator $I_m^{\eta,\alpha}$. Now, by (9) and Corollary 2.8 it will suffice to show that $J_m^{\eta,\alpha}$ is a bouded operator on $\mathbb{F}_{p,0}(\mathbb{I},\ \mathbb{E})$. For, let $K_{m,\mu}^{\eta,\alpha} : \mathbb{I} \times \mathbb{I} \to \mathbb{C}$ be the function

$$K_{m,\mu}^{\eta,\alpha}(t,\ z) = m\ z^{-m(\eta+\alpha)-\mu}\ \frac{(z^m - t^m)^{\alpha-1}}{\Gamma(\alpha)}\log\left[1 - \left(\frac{t}{z}\right)^m\right]\ t^{m\eta+m+\mu-1}\chi_{(0,z)}(t).$$

So $J_m^{\eta,\alpha} = T_{K_{m,0}^{\eta,\alpha}}$, evidently $K_{m,0}^{\eta,\alpha}$ is a homogeneous function of degree -1, the integral

$$\int_0^1 s^{\eta-1/(pm)}\ \frac{(1-s)^{\mathrm{Re}(\alpha)-1}}{|\Gamma(\alpha)|}\ |\log(1-s)|\ ds = \int_0^\infty \left|K_{m,0}^{\eta,\alpha}(t,1)\right| t^{-1/p}\ dt$$

converges and we must only apply Corollary 2.5. □

Corollary 10 *If* α *has a positive real part and* $\mathrm{Re}(m\eta + \mu) + m > 1/p$ *then* $I_m^{\eta,\alpha}$ *hcs a linear continuous Fréchet derivative on* $\mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E})$.

Proof.
It suffices to note that $J_m^{\eta,\alpha} = z^\mu\ T_{K_{m,\mu}^{\eta,\alpha}}\ z^{-\mu}$. □

The proofs of the following vectorial propositions are analogous to the scalar case (c. f . [7]).

Proposition 11 *If* $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(m\eta + \mu) + m > 1/p$, $\varphi \in \mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E})$ *then*

$$I_m^{\eta,\alpha}\varphi = I_m^{\eta,\alpha+1}(\eta + \alpha + 1 + \delta/m)\varphi, \tag{15}$$

where $\delta = z\ d/dz$.

The right member in (15) is defined if $\mathrm{Re}(\alpha) > -1$ and $\mathrm{Re}(m\eta+\mu)+m > 1/p$. By an analytic continuation argument we may extend $I_m^{\eta,\alpha}$ for all complex α such that $\mathrm{Re}(\alpha) > -1$. Then by an itereted process $I_m^{\eta,\alpha}$ is defined for $\alpha \in \mathbb{C}$.

Proposition 12 *Let* $\mathrm{Re}(m\eta + \mu) + m > 1/p$. *Then*

(i) $I_m^{\eta,\alpha}$ is a linear continuous operator on $\mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E})$.
(ii) $I_m^{\eta,\alpha}$ has a linear continuous Fréchet derivative with respect to any complex α. In particular, given $\varphi \in \mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E}),\ z > 0,\ \alpha \to I_m^{\eta,\alpha}\varphi(z)$ defines an entire function.
(iii) The operator $I_m^{\eta,\alpha}$ has Frechét derivative with respect to η in the halfplane $\mathrm{Re}(m\eta + \mu) + m > 1/p$ and

$$\frac{\partial I_m^{\eta,\alpha}}{\partial\eta}\varphi(z) = z^{-m(\eta+\alpha)} \int_0^z \frac{(z^m - t^m)^{\alpha-1}}{\Gamma(\alpha)}\ t^{m\eta}\ \log\left(\frac{t}{z}\right)^m\ \varphi(t)\ dt^m$$

for $\varphi \in \mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E}),\ z > 0$. In particular, the operator $\varphi \to \partial I_m^{\eta,\alpha}\varphi/\partial\eta$ is bounded on $\mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E})$.[1]
(iv) $I_m^{\eta,0}$ is the identity operator on $\mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E})$.

Proposition 13 *If* $\mathrm{Re}(m\eta+\mu)+m > 1/p$, $\mathrm{Re}(m\eta+m\alpha+\mu)+m > 1/p$, $\alpha,\ \beta$ *are complex numbers and* $\varphi \in \mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E})$ *then* $I_m^{\eta+\alpha,\beta}\ I_m^{\eta,\alpha}\varphi = I_m^{\eta,\alpha+\beta}\varphi$.

Corollary 14 $I_m^{\eta,\alpha}$ *is linear continuous on* $\mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E})$ *if* $\mathrm{Re}(m\eta + \mu) + m > 1/p$, *it being an homeomorphism with* $(I_m^{\eta,\alpha})^{-1} = I_m^{\eta+\alpha,-\alpha}$ *if* $\mathrm{Re}(m\eta + m\alpha + \mu) + m > 1/p$. *So,* I_m^{α} *is a linear continuous operator between* $\mathbb{F}_{p,\mu}(\mathbb{I},\ \mathbb{E})$ *and* $\mathbb{F}_{p,\mu+m\alpha}(\mathbb{I},\ \mathbb{E})$ *if* $\mathrm{Re}(\mu)+m > 1/p$, *becoming an homeomorfism if* $\mathrm{Re}(\mu+m\alpha)+m > 1/p$.

[1] Observe that $\partial I_m^{\eta,\alpha}/\partial\eta = T_L$, where L is the Schur's kernel

$$L(t,\ z) = m\ \frac{t^{m-1}}{z^m}\ \frac{(1-t^m/z^m)^{\alpha-1}}{\Gamma(\alpha)}\ \left(\frac{t}{z}\right)^{m\eta}\ \log\left(\frac{t}{z}\right)^m.$$

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