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EXAMPLES OF BOUNDED DERIVATIONS ON SOME NON C^* ALGEBRAS

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ABSTRACT. We exhibit two examples: (1st) of a class of abelian - (in general non semi - simple) Banach algebras without non trivial bounded derivations; (2nd) of a Banach algebra with unbounded and non trivial bounded inner and outer derivations. These algebras do not admit a von Neumann nor a C^* algebra structure, frames where known results about automatic innerness, boundedness or triviality of derivations hold (cf. [3], [4])

1. Introduction

Let $\mathfrak{U}$ be a Banach algebra, δ a linear mapping in $\mathfrak{U}$. Then δ is said to be a *derivation* in $\mathfrak{U}$ if its domain is a dense subalgebra of $\mathfrak{U}$ within the usual *Leibnitz rule* holds. If δ is defined anywhere then it is simply called a derivation on $\mathfrak{U}$. If δ is bounded, it can uniquely be extended to a bounded derivation on $\mathfrak{U}$, i.e. it may be considered to be anywhere defined. Given $a \in \mathfrak{U}$ so that $\delta^2(a) = 0$ then $\delta(a)$ has null *spectral radius* (Kleinecke [2], Sirokov [9]). Therefore, if $\mathfrak{U}$ is an abelian Banach algebra and δ is a bounded derivation on $\mathfrak{U}$ its range is contained in the *radical* of $\mathfrak{U}$ (Singer & Wermer [8]). In particular, if $\mathfrak{U}$ is semi - simple every bounded derivation on $\mathfrak{U}$ is zero. In Section 2 we show a first example of abelian Banach algebras, that in general are not semi - simple, with no non zero bounded derivations. If $\mathfrak{U}$ is a C^* algebra every derivation on $\mathfrak{U}$ is bounded (cf. Sakai [5]), the same result being also true on any semi - simple Banach algebra (Johnson & Sinclair [1]). In case that $\mathfrak{U}$ be a von Neumann algebra every derivation δ on $\mathfrak{U}$ is inner, i.e. there is an element $a \in \mathfrak{U}$ such that $\delta(x) = [a, x]$ for all $x \in \mathfrak{U}$, where $[\cdot,\cdot]$ is the usual Lie bracket (Sakai [6], [7]). In Section 3 we give a second example of a non von Neumann Banach algebra plenty of either inner or outer bounded derivations. In this last case, a full description of the general structure of bounded derivations is given.

2. 1st example

Let α be a sequence of positive numbers so that $\alpha_{n+m} \leq \alpha_n \, \alpha_m$ if $n,\ m \in \mathbb{N}_0$. The weighted space $l^1(\alpha)$ becomes an abelian unitary Banach algebra if we define $a * b = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_{n-m} \, b_m\right) \mathbf{x}^n,\ a,\ b \in l^1(\alpha)$. A linear map Δ on $l^1(\alpha)$ will be

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a bounded derivation if and only if the extended number

$$\mathfrak{h}_\alpha(\Delta \mathbf{x}) = \sup_{n\in\mathbb{N}} (n/\alpha_n) \sum_{k=0}^{\infty} |\Delta \mathbf{x}(k)| \ \alpha_{k+n-1}$$

is finite. But, if Δ were a non zero bounded derivation the sequence $\{n/\alpha_n\}_{n\in\mathbb{N}_0}$ should be bounded. Then, let κ_1, κ_2 be positive constants such that $n/\kappa_1 \le \alpha_n$ for all $n \in \mathbb{N}_0$ and $n \ |\Delta\mathbf{x}(k)| \ \alpha_{k+n-1} \le \alpha_n \ \kappa_2$ if $n, \ k \in \mathbb{N}$. Let k be an integer greater than 2 such that $\Delta\mathbf{x}(k) \ne 0$. Since $\{\alpha_n\}_{n\in\mathbb{N}_0}$ becomes unbounded, there is a positive integer n_k so that $\kappa_1 |\Delta\mathbf{x}(k)| \ \alpha_{k+n-1}/\kappa_2 \ge 1$ if $n \ge n_k$.Then

$$\alpha_n \ge n \ \alpha_{k-1+n} \frac{|\Delta\mathbf{x}(k)|}{\kappa_2}$$

and therefore we obtain

$$\alpha_{n_k} \ge n_k \ \alpha_{k-1+n_k} \frac{|\Delta\mathbf{x}(k)|}{\kappa_2}, \ \alpha_{k+n_k-1} \ge (k+n_k-1) \ \alpha_{2(k-1)+n_k} \frac{|\Delta\mathbf{x}(k)|}{\kappa_2},$$

$$\alpha_{2k+n_k-2} \ge (2k+n_k-2) \ \alpha_{3(k-1)+n_k} \frac{|\Delta\mathbf{x}(k)|}{\kappa_2}, \ \ldots\ldots\ldots\ldots$$

So, if $j \in \mathbb{N}$ we have $\alpha_{n_k} \ge (j-1)!/\kappa_1 \ (|\Delta\mathbf{x}(k)| \, (k-1)/\kappa_2)^{j-1}$, which is impossible because j is any natural number. Thus $\Delta\mathbf{x}(k)$ must be zero for all k in contradiction with our initial assumption.

3. 2nd example

Let $l^2(\mathbb{N}^2, \mathbb{C})$ be the Hilbert space of infinite complex matrices $a = (a_{i,j})_{i,j\in\mathbb{N}}$ such that its Frobenius norm $\|a\|_2^2 = \sum_{i,j\in\mathbb{N}} |a_{i,j}|^2$ is finite. Moreover, it is a non abelian complex Banach algebra without unit if we define the product $a \cdot b$ of elements $a, \ b \in l^2(\mathbb{N}^2, \mathbb{C})$ as $a \cdot b = \{\sum_{k=1}^{\infty} a_{i,k} \ b_{k,j}\}_{i,j\in\mathbb{N}}$. Now, an element $\Delta \in \mathcal{B}[l^2(\mathbb{N}^2, \mathbb{C})]$ is a derivation if and only if there is a unique matrix $\alpha \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ so that the extended number $\kappa = \sup_{i,j\in\mathbb{N}} \sum_{h=1}^{\infty} (|\alpha_{h,i}|^2 + |\alpha_{j,h}|^2)$ is finite and

$$\Delta(z) = \sum_{k,l\in\mathbb{N}} e_{k,l} \sum_{j=1}^{\infty} (z_{j,l} \ \alpha_{k,j} - \alpha_{j,l} \ z_{k,j}), \ z \in l^2(\mathbb{N}^2, \mathbb{C}).$$

Indeed, if $\alpha \in l^2(\mathbb{N}^2, \mathbb{C})$ then Δ is inner and $\Delta = \Delta_\alpha$. It is readily seen the existence of unbounded derivations on $l^2(\mathbb{N}^2, \mathbb{C})$. For instance, let $\{e_{k,l}\}_{k,l\in\mathbb{N}}$ be the family of matrices $e_{k,l} = (\delta^{k,l}_{n,m})_{n,m\in\mathbb{N}}$, where $\delta^{k,l}_{n,m}$ denotes the usual Kronecker´ symbol. The operator $\Delta(z) = \sum_{k,l\in\mathbb{N}} (k-l) \ z_{k,l} \ e_{k,l}$ defined on the dense subalgebra

$$\mathcal{D}(\Delta) = \left\{ z \in l^2(\mathbb{N}^2) : \sum_{k,l\in\mathbb{N}} (k-l)^2 \ |z_{k,l}|^2 < \infty \right\}$$

is an unbounded derivation on $l^2(\mathbb{N}^2, \mathbb{C})$. An example of a bounded outer derivation on $l^2(\mathbb{N}^2, \mathbb{C})$ is

$$\Delta(z) = \sum_{i\in\mathbb{N}-\{1\},\ j\in\mathbb{N}} (z_{i-1,j} - z_{i,j+1})\ e_{i,j} - \sum_{j=1}^{\infty} z_{1,j+1}\ e_{1,j},\ z \in l^2(\mathbb{N}^2, \mathbb{C}).$$

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