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LAPLACIAN ON MEAN CURVATURE VECTOR FIELDS FOR SOME NON-LIGHTLIKE SURFACES IN THE 3-DIMENSIONAL LORENTZIAN SPACE

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ABSTRACT. Laplacian operator on functions is well known and quite different from the Laplacian operator on vector fields. Also let us recall that the mean curvature vector field is one of the most interesting vector fields in differential geometry.

In this work we study the Laplacian operator on vector fields and we apply it on the mean curvature vector fields on surfaces in the 3-dimensional Lorentzian space, L^3. We show the Laplacian operator on the mean curvature vector fields for the non-lightlike surfaces S_1^2, H_0^2, $S_1^1 \times \mathbb{R}$, $H_0^1 \times \mathbb{R}$, $L \times S^1$ and L^2.

1. INTRODUCTION

Laplacian operator on functions is well known and quite different from the Laplacian operator on vector fields. Also let us recall that the mean curvature vector field is one of the most interesting vector fields in differential geometry.

In this work we study the Laplacian operator on vector fields and we apply it on the mean curvature vector fields on surfaces in the 3-dimensional Lorentzian space, L^3. Since all surfaces of the same constant curvature K are isometric, we distinguish among the cases $K > 0$, $K < 0$ and $K = 0$. Hence, in section 4 we show the Laplacian operator on the mean curvature vector fields for the non-lightlike surfaces L^2, S_1^2, H_0^2, and for the cylinders $S_1^1 \times \mathbb{R}$, $H_0^1 \times \mathbb{R}$, $L \times S^1$. These results are summarized in a table in section 5.

This study was based, mainly, on the works of Chen, [3] , and O´Neill, [4].

In order to unify notation, we begin by giving to readers some basic notions in Lorentzian geometry.

2. The Lorentzian space L^3

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in the 3-dimensional vector space $\mathbb{R}^3$, the Lorentzian inner product of x and y is defined by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3.$$

Thus the square ds^2 of an element of arc-length is given by

$$ds^2 = -dx_1^2 + dx_2^2 + dx_3^2.$$

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The space $\mathbb{R}^3$ furnished with this metric is called a 3-dimensional Lorentzian space, or Lorentz-Minkowski 3-space. We write L^3 instead of $(\mathbb{R}^3,\ ds)$.

We say that a vector x in L^3 is timelike if $\langle x,x\rangle < 0$, spacelike if $\langle x,x\rangle > 0$, and null if $\langle x,x\rangle = 0$. The nulls vectors are also said to be lightlike.

The norm of the vector $x \in L^3$ is defined by $\|x\| = \sqrt{|\langle x,x\rangle|}$.

For any $x, y \in L^3$, the Lorentzian vector product of x and y is defined by

$$x \wedge y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

We say that x is orthogonal to y if $\langle x,y\rangle = 0$. Clearly, $x \wedge y$ is orthogonal to both of x and y.

We say that x is a unit vector if $\|x\| = 1$, that is, if $\langle x,x\rangle = 1$ or $\langle x,x\rangle = -1$.

We shall give a surface M in L^3 by expressing its coordinates x_i as functions of two parameters in a certain interval. We consider the functions x_i to be real functions of real variables.

We say that M is a non-lightlike surface if at every $p \in M$ its tangent plane T_pM is furnished with positive definite or Lorentzian metric.

Let x_1, x_2, x_3 be a coordinate system in L^3. We denote $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$, $\partial_3 = \frac{\partial}{\partial x_3}$, and, in classical way, $g_{ij} = \langle \partial_i, \partial_j\rangle$, $1 \le i,j \le 3$, and $(g^{ij}) = (g_{ij})^{-1}$. The Christoffel symbols are given by:

$$\Gamma_{ij}^k = \tfrac{1}{2}\sum_{m=1}^3 g^{km}\left\{\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\right\},\ \text{for all } 1 \le i,j,k \le 3.$$

In what follows, we consider the set $\{\partial_1, \partial_2, \partial_3\}$ to be an orthonormal basis or frame for L^3, that is, $\partial_1, \partial_2, \partial_3$ are 3-mutually orthogonal vectors such that $\langle\partial_1,\partial_1\rangle = -1$ and $\langle\partial_2,\partial_2\rangle = \langle\partial_3,\partial_3\rangle = 1$.

Relative to natural coordinates on L^3

(a) $g_{ij} = \begin{cases} -\delta_{ij} & \text{if } j = 1 \\ \delta_{ij} & \text{if } j \neq 1 \end{cases}$;

(b) $\Gamma_{ij}^k = 0$, for all $1 \le i,j,k \le 3$;

(c) if $V = \sum_{i=1}^3 v_i\partial_i$ and $W = \sum_{j=1}^3 w_j\partial_j$ are vector fields on L^3, the Levi-Civita connection $\overline{\nabla}$ of L^3 is given by: $\overline{\nabla}_V W = \sum_{i,j=1}^3 \{v_i\ \partial_i(w_j)\}\ \partial_j$.

3. Mean curvature vector field H and Laplacian operator Δ on vector fields

In what follows, we will write surface instead of non-lightlike surface.

Let M be a surface in L^3. For a coordinate system u, v in M, we denote the coefficients of the first fundamental form by:

$$E = g_{11} = \langle\partial_u,\partial_u\rangle,\quad F = g_{12} = g_{21} = \langle\partial_u,\partial_v\rangle,\quad G = g_{22} = \langle\partial_v,\partial_v\rangle. \tag{1}$$

Let u, v be an orthogonal coordinate system in M, that is, $F = 0$. In what follows, we consider this particular case in the theory of Lorentzian surfaces.

The Christoffel symbols are as follows:

$$EG\,\Gamma_{11}^1 = \tfrac{1}{2}\partial_u(E)\,G,\quad EG\,\Gamma_{12}^1 = \tfrac{1}{2}\partial_v(E)\,G,\quad EG\,\Gamma_{22}^1 = -\tfrac{1}{2}\partial_u(G)\,G,$$
$$EG\,\Gamma_{11}^2 = -\tfrac{1}{2}\partial_v(E)\,E,\quad EG\,\Gamma_{12}^2 = \tfrac{1}{2}\partial_u(G)\,E,\quad EG\,\Gamma_{22}^2 = \tfrac{1}{2}\partial_v(G)\,E.$$

The Gaussian curvature K of M is given by:

$$K = -\frac{1}{\sqrt{EG}}\left\{\text{sign }(E)\ \partial_u\left(\frac{\partial_u(\sqrt{G})}{\sqrt{E}}\right) + \text{sign }(G)\ \partial_v\left(\frac{\partial_v(\sqrt{E})}{\sqrt{G}}\right)\right\}.$$

We consider the unit normal vector to the surface M at (u,v) is

$$N = N(u,v) = \frac{\partial_v \wedge \partial_u}{\|\partial_u \wedge \partial_v\|}. \tag{2}$$

The Levi-Civita connection $\overline{\nabla}$ of L^3 gives rise in a natural way to a function $\overline{\nabla}:\Xi(M)\times\overline{\Xi}(M)\to\overline{\Xi}(M)$, called the induced connection on $M\subset L^3$. In [4], we shall find a definition of induced connection.

Let $V\in\Xi(M)$ and $X\in\overline{\Xi}(M)$. And for each $p\in M$ let $\overline{V}$ and $\overline{X}$ be smooth local extensions of V and X over a coordinate neighborhood $\mathcal{U}$ of p in L^3. Then the induced connection $\overline{\nabla}_V X$ is defined on each $\mathcal{U}\cap M$ to be restriction of $\overline{\nabla}_{\overline{V}}\overline{X}$ to $\mathcal{U}\cap M$.

Remark 1. *We use the same notation,* $\overline{\nabla}$, *for the induced connection on* M *and the Levi-Civita connection of* L^3 *because the induced connection on* M *is so closely related to the Levi-Civita connection of* L^3.

The induced connection $\overline{\nabla}$ on M satisfies the properties:
(1) $\overline{\nabla}_V X$ is $\mathcal{F}(M)$-linear in V;
(2) $\overline{\nabla}_V X$ is $\mathbb{R}$-linear in X;
(3) $\overline{\nabla}_V(fX) = VfX + f\,\overline{\nabla}_V X$;
(4) $[V,W] = \overline{\nabla}_V W - \overline{\nabla}_W V$;
(5) $V\langle X,Y\rangle = \langle\overline{\nabla}_V X, Y\rangle + \langle X, \overline{\nabla}_V Y\rangle$;
where $\mathcal{F}(M)$ denote the set of all smooth real-valued functions on M; $V,W\in\Xi(M)$, $X,Y\in\overline{\Xi}(M)$ and $f\in\mathcal{F}(M)$.

Two significative results about the connection induced for $M\subset L^3$ are the following (see §4 of [4]).

Lemma 2. *For* $M\subset L^3$, *if* $V,W\in\Xi(M)$ *and* ∇ *is the Levi-Civita connection of* M, *then*

$$\nabla_V W = tan\overline{\nabla}_V W. \tag{3}$$

Lemma 3. *The function* $\text{II}:\Xi(M)\times\Xi(M)\to\Xi(M)^{\perp}$ *such that*

$$\text{II}(V,W) = nor\overline{\nabla}_V W \tag{4}$$

is $\mathcal{F}(M)$*-bilinear and symmetric.*

The function II is called the shape tensor or second fundamental form tensor of M.

We summarize these lemmas by

$$\overline{\nabla}_V W = \nabla_V W + \text{II}(V,W). \tag{5}$$

In [4], O'Neill defines the shape operator for hypersurfaces of semi-riemannian manifolds. In particular, if N is the unit normal vector field on $M\subset L^3$, then the $(1,1)$ tensor field S on M such that, for all $V,W\in\Xi(M)$,

$$\langle S(V),W\rangle = \langle\text{II}(V,W),N\rangle \tag{6}$$

is called the shape operator of $M \subset L^3$ derived from N.

At each point $p \in M$, S determines a linear operator $S : T_pM \to T_pM$ defined by

$$S(\partial_u) = -\overline{\nabla}_{\partial_u} N, \quad S(\partial_v) = -\overline{\nabla}_{\partial_v} N. \tag{7}$$

where u, v is a coordinate system in M

In [3] , Chen makes very important contributions to the study of riemannian submanifolds. We adapt to semi-riemannian manifolds and submanifolds some results of §1 of [3]. We use the formalism of O´Neill to derive our results.

Hence, the mean curvature vector field H for M is given by

$$H = \frac{1}{2} \langle N, N \rangle \ \mathrm{tr}[S] \ N, \tag{8}$$

where $\begin{pmatrix} S(\partial_u) \\ S(\partial_v) \end{pmatrix} = [S] \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix}$.

We adapt to M in L^3 the definition of the Laplacian operator Δ on vector fields for riemannian submanifolds which was gived by Chen in [3] .

Hence, we define the Laplacian operator Δ on vector fields for surface M in L^3 by:

Definition 4. *Let M be a surface in L^3. For a coordinate system u, v in M, the Laplacian operator Δ on vector fields in $\Xi(M)$ is given by*

$$\Delta = g^{11} \overline{\nabla}_{\partial_u} \overline{\nabla}_{\partial_u} + g^{22} \overline{\nabla}_{\partial_v} \overline{\nabla}_{\partial_v} + g^{12} \left\{ \overline{\nabla}_{\partial_u} \overline{\nabla}_{\partial_v} + \overline{\nabla}_{\partial_v} \overline{\nabla}_{\partial_u} \right\}. \tag{9}$$

4. Laplacian on mean curvature vector fields for some surfaces in L^3

The trivial example of surface with curvature $K = 0$ is the plane L^2. In this case, the matrix $[S]$ is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. By (8), the mean curvature vector field H is zero. Hence, $\Delta H = 0$.

Now, we show the Laplacian operator Δ on mean curvature vector fields H for the surfaces: S_1^2, H_0^2, $S_1^1 \times \mathbb{R}$, $H_0^1 \times \mathbb{R}$, $L \times S^1$.

4.1. The pseudosphere S_1^2.

The pseudosphere S_1^2 in L^3 is the surface

$$S_1^2 = \{(x_1, x_2, x_3) \in L^3 : \ -x_1^2 + x_2^2 + x_3^2 = 1\}.$$

S_1^2 can be parametrized by:

$$\begin{cases} x_1 = & \sinh \omega \\ x_2 = & \cos\theta \cosh \omega \\ x_3 = & \sin\theta \cosh \omega \end{cases} \quad \text{, where } \omega \in \mathbb{R} \text{ and } 0 \leq \theta < 2\pi.$$

The tangent vectors are expressed as follows:

$$\partial_\omega = \tfrac{\partial}{\partial_\omega} = \cosh \omega \ \partial_1 + \sinh \omega \cos\theta \ \partial_2 + \sinh \omega \sin\theta \ \partial_3,$$
$$\partial_\theta = \tfrac{\partial}{\partial_\theta} = -\cosh \omega \ \sin\theta \ \partial_2 + \cosh \omega \cos\theta \ \partial_3.$$

Hence, the coefficients of the first fundamental form are:

$$E = \langle \partial_\omega, \partial_\omega \rangle = -1, \qquad F = \langle \partial_\omega, \partial_\theta \rangle = 0, \qquad G = \langle \partial_\theta, \partial_\theta \rangle = \cosh^2 \omega.$$

The Christoffel symbols are given by $\Gamma_{12}^2 = \Gamma_{21}^2 = \tanh\omega$, $\Gamma_{22}^2 = \cosh\omega\sinh\omega$, and $\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{22}^1 = 0$. Thus, a simple computation shows that $K = 1$.

By (2), we consider the unit normal vector to the surface S_1^2 at (ω,θ) is

$$N = (\sinh\omega, \cosh\omega\cos\theta, \cosh\omega\sin\theta).$$

Hence, $\langle N, N\rangle = 1$.

Since $[S] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,by (8), the mean curvature vector field H for S_1^2 is given by:

$$H = \tfrac{1}{2}(-2)\,N = -N.$$

By (9), the Laplacian operator on vector fields, Δ, for S_1^2 is given by

$$\Delta = -\overline{\nabla}_{\partial_\omega}\overline{\nabla}_{\partial_\omega} + \tfrac{1}{\cosh^2\omega}\overline{\nabla}_{\partial_\theta}\overline{\nabla}_{\partial_\theta}.$$

By applying to $H = -N$ the formula for the induced connection, we obtain

$$\overline{\nabla}_{\partial_\omega}\overline{\nabla}_{\partial_\omega}H = -\Gamma_{11}^1\partial_\omega - \Gamma_{11}^2\partial_\theta - \langle S(\partial_\omega), \partial_\omega\rangle N = -N$$

$$\overline{\nabla}_{\partial_\theta}\overline{\nabla}_{\partial_\theta}H = -\Gamma_{22}^1\partial_\omega - \Gamma_{22}^2\partial_\theta - \langle S(\partial_\theta), \partial_\theta\rangle N = -\cosh\omega\ \sinh\omega\ \partial_\omega + \cosh^2\omega\ N$$

Therefore,

$$\Delta H = 2\ N - \tanh\omega\ \partial_\omega.$$

4.2. **The pseudohyperbolic space H_0^2.** The pseudohyperbolic space H_0^2 in L^3 is the surface

$$H_0^2 = \{(x_1, x_2, x_3) \in L^3 :\ -x_1^2 + x_2^2 + x_3^2 = -1\}.$$

The two sheets of H_0^2 are characterized by $x_1 > 0$ and $x_1 < 0$. It follows that H_0^2 admits the parametrizations

$$\begin{cases} x_1 = \pm\cosh\omega \\ x_2 = \cos\theta\sinh\omega \\ x_3 = \sin\theta\sinh\omega \end{cases},\ \text{where } \omega\in\mathbb{R} \text{ and } 0\le\theta<2\pi.$$

Hence, the coefficients of the first fundamental form are

$$E = \langle\partial_\omega, \partial_\omega\rangle = 1, \qquad F = \langle\partial_\omega, \partial_\theta\rangle = 0, \qquad G = \langle\partial_\theta, \partial_\theta\rangle = \sinh^2\omega.$$

Therefore, $K = -1$.

By computing, we obtain that the Christoffel symbols different from zero are $\Gamma_{12}^2 = \Gamma_{21}^2 = \coth\omega$ and $\Gamma_{22}^1 = -\cosh\omega\sinh\omega$.

By (2), the unit normal vectors to the surface H_0^2 at (ω,θ) are given by

$$N = (\pm\cosh\omega, \sinh\omega\cos\theta, \sinh\omega\sin\theta),$$

respectively, which satisfy $\langle N, N\rangle = -1$.

We obtain the mean curvature vector field H for H_0^2 by computing analogous to S_1^2 :

$$H = -\tfrac{1}{2}\ (-2)\ N = N.$$

By (9),

$$\Delta H = -\coth\omega\ \partial_\theta + 2N.$$

4.3. The cylinders $S_1^1 \times \mathbb{R}$, $H_0^1 \times \mathbb{R}$ and $L \times S^1$. We now study the cylinders in L^3 defined by:

$$S_1^1 \times \mathbb{R} = \{(x_1, x_2, x_3) \in L^3 : \ -x_1^2 + x_2^2 = 1\}$$

$$H_0^1 \times \mathbb{R} = \{(x_1, x_2, x_3) \in L^3 : \ -x_1^2 + x_2^2 = -1\}$$

$$L \times S^1 = \{(x_1, x_2, x_3) \in L^3 : x_2^2 + x_3^2 = 1\}$$

They are given by

$$\begin{cases} x_1 = \sinh\omega \\ x_2 = \cosh\omega \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = \pm\cosh\omega \\ x_2 = \sinh\omega \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = t \\ x_2 = \cos\theta \\ x_3 = \sin\theta \end{cases}$$

respectively, where ω, $t \in \mathbb{R}$ and $0 < \theta < 2\pi$.

Hence, these cylinders are of constant curvature $K = 0$. All Christoffel symbols for these surfaces are zero.

In the case $S_1^1 \times \mathbb{R}$, we consider $\overline{\partial_\omega} = (r\ \cosh\omega, r\ \sinh\omega, 0)$, $\overline{\partial_t} = (0,0,1)$, $\overline{\partial_r} = (\sinh\omega, \cosh\omega, 0)$ to be smooth local extensions of ∂_ω, ∂_t and N, respectively. Thus, $S(\partial_\omega) = -\overline{\nabla}_{\overline{\partial_\omega}}\overline{\partial_r}/_{S_1^1\times\mathbb{R}} = -\partial_\omega$ and $S(\partial_t) = -\overline{\nabla}_{\overline{\partial_\omega}}\overline{\partial_r}/_{S_1^1\times\mathbb{R}} = 0$.

By (8), the mean curvature vector field H is given by

$$H = -\tfrac{1}{2}N,$$

where $N = (\sinh\omega, \cosh\omega, 0)$.

By (9),

$$\Delta H = \tfrac{1}{2}N.$$

We obtain in similar way the mean curvature vector fields H for $H_0^1 \times \mathbb{R}$ and $L \times S^1$, respectively, which are given by

$$H = \tfrac{1}{2}N \qquad \text{and} \qquad H = -\tfrac{1}{2}N.$$

In the case $H_0^1 \times \mathbb{R}$, by (9), we obtain:

$$\Delta H = \tfrac{1}{2}N,$$

where $N = (\cosh\omega, \sinh\omega, 0)$.

In the case $L \times S^1$, by (9), we obtain:

$$\Delta H = \tfrac{1}{2}N,$$

where $N = (0, \cos\theta, \sin\theta)$.

5. Table of Results

In this section, we summerize the results whose were obtained above, that is, the mean curvature vector fields, the Laplacian operator on vector fields and the Laplacian operator on the mean curvature vector fields for the non-lightlike surfaces S_1^2, H_0^2, $S_1^1 \times \mathbb{R}$, $H_0^1 \times \mathbb{R}$, $L \times S^1$ and L^2.

Surface	H	Δ	ΔH
S_1^2	$-N$	$-\overline{\nabla}_{\partial_\omega}\overline{\nabla}_{\partial_\omega} + \cosh^{-2}\omega\ \overline{\nabla}_{\partial_\theta}\overline{\nabla}_{\partial_\theta}$	$-\tanh\omega\ \partial_\omega - 2H$
H_0^2	N	$\overline{\nabla}_{\partial_\omega}\overline{\nabla}_{\partial_\omega} + \sinh^{-2}\omega\ \overline{\nabla}_{\partial_\theta}\overline{\nabla}_{\partial_\theta}$	$-\coth\omega\ \partial_\theta + 2H$
$S_1^1 \times \mathbb{R}$	$-\frac{1}{2}N$	$-\overline{\nabla}_{\partial_\omega}\overline{\nabla}_{\partial_\omega} + \overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}$	$-H$
$H_0^1 \times \mathbb{R}$	$\frac{1}{2}N$	$\overline{\nabla}_{\partial_\omega}\overline{\nabla}_{\partial_\omega} + \overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}$	H
$L \times S^1$	$-\frac{1}{2}N$	$-\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t} + \overline{\nabla}_{\partial_\theta}\overline{\nabla}_{\partial_\theta}$	$-H$
L^2	0	$-\overline{\nabla}_{\partial_1}\overline{\nabla}_{\partial_1} + \overline{\nabla}_{\partial_2}\overline{\nabla}_{\partial_2}$	0

Table of results.

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