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AN APPROXIMATION THEOREM FOR CERTAIN SUBSETS OF SOBOLEV SPACES

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SUMMARY. We show that a class of differentiable functions vanishing together with their derivatives of order less than r on the boundary of a smooth domain Ω is dense in the subset of $W^{m+r,p}(\Omega)$ defined by the functions already in $W_o^{r,p}(\Omega)$. We give a direct proof by introducing a particular extension operator and a related reflection operator.

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1. PRELIMINARIES AND NOTATION. Let Ω be a domain in R^n. By $(.,.)$ and $\|.\|$ we shall always denote the scalar product and norm in $L^2(\Omega)$. For r a nonnegative integer we denote by $H^r(\Omega)$ the Sobolev space $H^r(\Omega) := \{u \in D'(\Omega);\ D^\alpha u \in L^2(\Omega)$ for $|\alpha| \leq r\}$ with the norm $\|u;H^r(\Omega)\| = (\sum_{|\alpha|\leq r} \|D^\alpha u\|^2)^{1/2}$ and by $\overset{\circ}{H}{}^r(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^r(\Omega)$ (cfr. [A] where $H^r(\Omega) = W^{r,2}(\Omega)$ and $\overset{\circ}{H}{}^r(\Omega) = W_0^{r,2}(\Omega)$). We state some well known facts about these spaces that we shall need in what follows.

LEMMA 1. *If $u \in H^r(\Omega)$, $v \in \overset{\circ}{H}{}^r(\Omega)$ and $|\alpha| \leq r$, then*

$$(D^\alpha u, v) = (u, D^\alpha v).$$

PROOF. If $v_h \in C_0^\infty(\Omega)$ is a sequence such that $\|v_h - v;H^r(\Omega)\| \to 0$ then

$$(D^\alpha u,v) = \lim_{h\to\infty} (D^\alpha u,v_h) = \lim_{h\to\infty} (u,D^\alpha v_h) = (u,D^\alpha v), \qquad \text{Q.E.D.}$$

Let Ω be a bounded domain with C^∞ boundary (i.e. there exists a finite open covering of $\partial\Omega$, $\{U_j;\ j = 1, \ldots, N\}$, such that for each j there is a map ϕ_j from U_j onto $B = \{y \in R^n;\ |y| < 1\}$ with the properties: i) ϕ_j is one to one, ii) $\phi_j \in C^\infty(U_j)$, $\phi_j^{-1} \in C^\infty(B)$, iii) $\phi_j(U_j \cap \Omega) = B^+ = \{y \in B;\ y_n > 0\} = B \cap R_n^+$).

LEMMA 2. *If $u \in C^r(\overline{\Omega})$ and $D^\alpha u = 0$ on $\partial\Omega$ for $|\alpha| < r$, then $u \in \overset{\circ}{H}{}^r(\Omega)$.*

PROOF. Let U_0 be an open subset of Ω such that $\bigcup_{j=0}^{N} U_j \supset \overline{\Omega}$. Using a C^∞ partition of unity subordinate to this covering one sees that it is enough to prove that: if $u \in C^r(\overline{R_n^+})$, $D^\alpha u(x_1,\ldots,x_{n-1},0) = 0$ for $|\alpha| < r$ and supp u is bounded, then $u \in \overset{\circ}{H}{}^r(R_n^+)$, (cf. [A], T.3.35, particularly formula (15)). Now in that case let $\tilde{u}(x) := u(x)$ for $x \in R_n^+$ and 0 otherwise. Then Gauss' theorem yields for

$\phi \in C_0^\infty(R_n)$ and $|\alpha| \leq r$

$$\int_{R_n} (-1)^{|\alpha|}\tilde{u}D^\alpha\phi\, dx = (-1)^{|\alpha|}\int_{R_n^+} uD^\alpha\phi\, dx = \int_{R_n^+} D^\alpha u.\phi\, dx = \int_{R_n} \widetilde{D^\alpha u}.\phi\, dx$$

That is, $D^\alpha\tilde{u}$ is the function $\widetilde{D^\alpha u}$ for $|\alpha| \leq r$ and so $\tilde{u} \in H^r(R_n)$. But then $u = \lim_{\varepsilon \to 0} v_\varepsilon$ in $H^r(R_n^+)$ where $v_\varepsilon(x) = \tilde{u}(x_1,\ldots,x_{n-1},x_n - \varepsilon)$. Since supp v_ε is compact in R_n^+, $v_\varepsilon \in \overset{o}{H}{}^r(R_n^+)$ and the proof is complete, Q.E.D.

2. INTRODUCTION. For r, R positive integers, $r < R$, let us call $H_{r,R}(\Omega)$ the Hilbert space $H_{r,R}(\Omega) := \overset{o}{H}{}^r(\Omega) \cap H^R(\Omega)$ with the norm of $H^R(\Omega)$ and call $D_r(\Omega) := \{\phi \in C^\infty(\bar{\Omega});\ D^\alpha\phi = 0 \text{ on } \partial\Omega \text{ for } |\alpha| < r\}$. Now let Ω be a bounded domain with C^∞ boundary. By Lemma 2, $D_r(\Omega) \subset H_{r,R}(\Omega)$. (It also follows that this space contains **properly** the space $\overset{o}{H}{}^R(\Omega)$, cf. Th.5). In this paper we prove that $D_r(\Omega)$ is a dense subset of $H_{r,R}(\Omega)$. That is

THEOREM 1. *If $\mathcal{G}_{r,R}(\Omega)$:= closure of $D_r(\Omega)$ in $H^R(\Omega)$, then*

$$\mathcal{G}_{r,R}(\Omega) = \mathcal{H}_{r,R}(\Omega).$$

This theorem can be proved in the particular case $R = 2r$ using results of P.D.E. as follows. For $\lambda > 0$ the operator $(-\Delta)^r + \lambda$ maps $H_{r,2r}(\Omega)$ continuously into $L^2(\Omega)$. This map is also 1:1 since for $u \in H_{r,2r}(\Omega)$ using Lemma 1 we obtain

$$((-\Delta)^r u + \lambda u, u) = \sum_{|\alpha|=r} (r!/\alpha!)(D^{2\alpha}u,u) + \lambda\|u\|^2 =$$

$$= \sum_{|\alpha|=r} (r!/\alpha!)\|D^\alpha u\|^2 + \lambda\|u\|^2.$$

On the other hand for λ great enough $((-\Delta)^r + \lambda)G_{r,2r} = L^2(\Omega)$ (cfr. [S], Th. 9 - 27, pg. 219). In consequence $G_{r,2r}(\Omega) = H_{r,2r}(\Omega)$. We shall give a direct proof of this fact and moreover of Theorem 1. By using a partition of unity as in Lemma 2 it is enough to prove

THEOREM 2. *Let K be a compact set in B and $u \in H_{r,R}(R_n^+)$ with supp $u \subset K \cap R_n^+$. Then there exists a sequence $u_h \in D_r(R_n^+)$ such that supp $u_h \subset B^+$ and $\|u_h - u; H^R(R_n^+)\| \to 0$ for $h \to \infty$.*

Our proof relies on the following result.

3. AUXILIARY LEMMA. *Given R integers $K_1, K_2, \ldots, K_R$ there exists a polynomial $p(x)$ of degree $R - 1$ such that*

i) $p(2^j)$ *is an integer for* $j = 0, 1, \ldots$

ii) $p(2^{m-1}) = K_m \pmod 2$ *for* $1 \leqq m \leqq R$

iii) $p(2^{m-1}) = K_R \pmod 2$ *for* $R < m$.

PROOF. If $x_i = 2^{i-1}$, $i = 1, 2, \ldots, R$, define $p(x)$ by

$$p(x) := \sum_{j=1}^{R} h_j \prod_{k=1}^{j-1} ((x - x_k)/x_k) . \prod_{k=j+1}^{R} ((x - x_k)/x_j)$$

where $h_j = 0$ if k_j is even and $h_j = 1$ if K_j is odd.

Observe that $p(x)$ satisfies i) and ii) since $(x_j - x_k)/x_s$ is odd for $s = \min(j,k)$ and is even for $s < \min(j,k)$. By the same reasoning for $x = x_m$, $m > R$, the first product in the definition of p is odd and the last is even when not empty. So $p(x_m) - p(x_R)$ is even, Q.E.D.

COROLLARY. *Given R integers $K_1, \ldots, K_R$, there exists an entire function $f(z)$ without zeroes such that*

i) $f(2^{j-1}) = (-1)^{K_j}$ *for* $j = 1, \ldots, R$

ii) $f(2^{j-1}) = 1/f(2^{j-1})$ *for* $j \in \mathbf{N}$.

PROOF. Define

$$(1) \qquad f(z) := \exp(i\pi p(z))$$

where $p(z)$ is the polynomial in the preceding lemma. Then both $f(z)$ and $g(z) := 1/f(z)$ have the required properties, Q.E.D.

4. AN EXTENSION OPERATOR. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be an entire function. We associate to f the operator

$$(2) \qquad (T_f u)(x',t) := \sum_{k=0}^{\infty} c_k u(x',-2^k t)$$

where $x' = (x_1,\dots,x_{n-1}) \in R_{n-1}$, $t \in R_1$.

T_f is well defined if u vanishes outside a sphere.

THEOREM 3. *For* $u \in H^R(R_n^+)$, *supp* $u \subset B^+$, *we have:*

i) *supp* $T_f u \subset B^-$, $T_f u \in H^R(R_n^-)$

ii) $\|T_f u; H^R(R_n^-)\| \leq M(f)\|u; H^R(R_n^+)\|$ *and*

$$(3) \qquad D^\alpha T_f u = T_{f_h}(D^\alpha u) \quad \text{for } h = \alpha_n,\ |\alpha| \leq R$$

where f_h *is the entire function*

$$(4) \qquad f_h(z) := (-1)^h \sum_{k=0}^{\infty} c_k 2^{h.k} z^k = (-1)^h f(2^h z)\ .$$

PROOF. The first assertion of i) is immediate. The second follows from ii). Observe that if $x = (x',t) \in K$, a compact set in R_n^-, then the sum defining $T_f u(x',t)$ is finite. Therefore (3) is correct in $D'(R_n^-)$. To prove ii) it is therefore enough to prove

(5) $$\| T_{f_h} u;L^2(R_n^-)\| \leqq M(f_h)\| u;L^2(R_n^+)\| .$$

But $\|u(x',-2^k t)\| = 2^{-k/2} \|u\|$. Summing up, one gets

$$\|T_{f_h} u\| \leqq \left(\sum_{k=0}^{\infty} |c_k|.2^{k(2h-1)/2}\right) \|u\| \qquad \text{Q.E.D.}$$

Observe that the lemma remains true if the roles of R_n^+ and R_n^- are interchanged.

Now we define the **extension** operator E_f associated to $f(z) = \sum c_k z^k$ by

(6) $$E_f u(x',t) := \begin{cases} u(x',t) & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ T_f u(x',t) & \text{for } t < 0. \end{cases}$$

THEOREM 4. *Let $u \in H_{r,R}(R_n^+)$, closure in R_n of supp $u \subset B$. If the entire function $f(z)$ verifies*

(7) $$f(2^s) = (-1)^s \quad \text{for } s = r,\ r+1,\ \dots,\ R-1$$

then $E_f u \in H^R(R_n)$, supp $E_f u \subset B$ and

(8) $$\|E_f u;H^R(R_n)\| \leqq C(f)\|u;H^R(R_n^+)\| .$$

PROOF. We shall show that if $|\alpha| \leqq R$, $h = \alpha_n$ and f_h is defined by (4), then

(9) $$D^\alpha(E_f u) = E_{f_h}(D^\alpha u).$$

Therefore, the theorem will follow from Th.3. To prove (9) we consider two cases.

CASE 1: $\alpha = (0, \dots, 0, h)$. Let $\phi \in C_0^\infty(R_n)$. Then if we set $x = (x',t)$,

$$(10) \qquad \langle D^{\alpha}E_f u,\phi\rangle = (-1)^h\langle E_f u, D_t^h\phi\rangle =$$

$$= (-1)^h\int_{R_n^+} (uD_t^h\phi + (-1)^h \sum_{k=0}^{\infty} c_k u(x',2^k t).D_t^h\phi(x',-t))\, dx =$$

$$= (-1)^h\int_{R_n^+} uD_t^h\phi dx + \sum_{k=0}^{\infty} 2^{(h-1)k} c_k\int_{R_n^+} u(x)D_t^h(\phi(x',-2^{-k}t))\, dx =$$

$$= (-1)^h\int_{R_n^+} u(x)D_t^h\psi_h(x',t)\, dx'dt$$

with

$$(11) \qquad \psi_h(x',t) = \phi(x',t) - \sum_{k=0}^{\infty} (-2^k)^{h-1}\, c_k\phi(x',-2^{-k}t).$$

Since $\sum_{k=0}^{\infty} |c_k|.M^k < \infty$ for any $M > 0$, it is possible to interchange $\sum$ and $\int$ in (10). Also $\psi_h \in C_0^{\infty}(\overline{R_n^+}) \cap H^s(R_n^+)$ for any s.

Now we shall show that

$$(12) \qquad (-1)^h\int_{R_n^+} u(x)D_t^h\psi_h(x',t)\, dx = \int_{R_n^+} D_t^h u.\,\psi_h\, dx.$$

In fact, since $u \in \overset{o}{H}{}^r(R_n^+)$, by Lemma 1,

$$(13) \qquad (-1)^h\int_{R_n^+} uD_t^h\psi_h\, dx = (-1)^{h-j}\int_{R_n^+} D_t^j u.D_t^{h-j}\psi_h\, dx \quad \text{for } j = \min(h,r).$$

This proves (12) for $h \leqq r$. If $h > r$, then in view of (7), $\psi_h(x',0) = 0$ and also $D^{\gamma}\psi_h(x',0) = 0$ for $|\gamma| < h - r$. Then by Lemma 2, $\psi_h \in \overset{o}{H}{}^{h-r}(R_n^+)$, and we can apply again Lemma 1 to the right hand side of (13) ($j = r$ now!) thus obtaining (12).

The combination of (10) with (12) yields

$$\langle D^\alpha E_f u,\phi\rangle = \int D^\alpha u.\psi_h \, dx = \langle E_{f_h}(D^\alpha u),\phi\rangle.$$

CASE 2: $\alpha = (\alpha_1,\dots,\alpha_{n-1},0)$. Then (9) is true regardless condition (7) for $u \in H^q(R_n^+)$, $q \geq |\alpha|$. In fact, let $\eta(t) \in C^\infty(R_1)$, $\eta = 0$ for $|t| < 1/2$, $\eta = 1$ for $|t| > 1$, and call $\eta_\varepsilon(t) := \eta(t/\varepsilon)$. Then for $\phi \subset C_0^\infty(R_n)$ we have $\eta_\varepsilon\phi \in C_0^\infty(R_n^- \cup R_n^+)$ and so

$$\langle D^\alpha E_f u,\phi\rangle = (-1)^{|\alpha|}\langle E_f u, D^\alpha\phi\rangle = (-1)^{|\alpha|} \lim_{\varepsilon\to 0} \langle E_f u, \eta_\varepsilon D^\alpha\phi\rangle =$$

$$= \lim_{\varepsilon\to 0} (-1)^{|\alpha|}\langle E_f u, D^\alpha(\eta_\varepsilon\phi)\rangle = \lim_{\varepsilon\to 0} \langle E_f(D^\alpha u),\eta_\varepsilon\phi\rangle = \langle E_f(D^\alpha u),\phi\rangle.$$

To combine this two cases we write $\alpha = (\alpha_1,\dots,\alpha_{n-1},0) + (0,\dots,0,h) = \alpha' + \alpha''$ and obtain $D^\alpha(E_f u) = D^{\alpha'} E_{f_h}(D^{\alpha''}u) = E_{f_h}(D^\alpha u)$, Q.E.D.

5.A REFLECTION OPERATOR. Next we define an operator E which is a generalization of $\phi(x',t) \to -\phi(x',-t)$. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be the entire function constructed in the Corollary of section 3 for $K_i = i$ if $i \leq r$ and $K_i = i - 1$ for $r < i \leq R$. That is

(14) $f(2^{i-1}) = (-1)^i$ for $1 \leq i \leq r$; $f(2^{i-1}) = (-1)^{i-1}$ for $r < i \leq R$.

Further, let $g(z) := 1/f(z) = \sum_{k=0}^{\infty} d_k z^k$. For v a function with bounded support let us define

$$Ev(x',t) := \begin{cases} T_g v = \sum_{k=0}^{\infty} d_k v(x',-2^k t) & \text{for } t > 0, \\ T_f v = \sum_{k=0}^{\infty} c_k v(x',-2^k t), & t \leq 0. \end{cases}$$

LEMMA 3. *If* $\phi \in C_o^\infty(R_n)$, *then*

i) $\quad \mathcal{E}\phi \in C_o^\infty(R_n)$

ii) $\quad \mathcal{E}\phi = \phi$ *implies* $\phi \in D_r(R_n^+)$

iii) $\quad \mathcal{E}^2\phi = \phi$

iv) $\quad \|\mathcal{E}\phi; H^s(R_n)\| \leq M_s\|\phi; H^s(R_n)\| \qquad \forall\ s \in \mathbf{N}.$

v) $\quad$ *Let* $v \in H^s(R^n)$, *support of* $v \subset B$. *If the sequence* $\{\phi_m\} \subset C_o^\infty(B)$ *verifies* $\lim_{m\to\infty} \|\phi_m - v; H^s(R^n)\| = 0$, *then*

$$\lim_{m\to\infty} \|\mathcal{E}\phi_m - \mathcal{E}v; H^s(R^n)\| = 0.$$

PROOF. i) It is clear from the definition that supp $E\phi$ is bounded and that $E\phi \in C^\infty(R_n^- \cup R_n^+)$. Also

$$(15)\quad \begin{cases} D^\alpha E\phi(x',+0) = \left(\sum_{k=0}^{\infty} d_k(-2^k)^{\alpha_n}\right)D^\alpha\phi(x',0) = (-1)^{\alpha_n}g(2^{\alpha_n})D^\alpha\phi(x',0) \\[2ex] D^\alpha E\phi(x',-0) = \left(\sum_{k=0}^{\infty} c_k(-2^k)^{\alpha_n}\right)D^\alpha\phi(x',0) = (-1)^{\alpha_n}f(2^{\alpha_n})D^\alpha\phi(x',0). \end{cases}$$

i) then follows from

$$(16)\qquad f(2^h) = g(2^h) = \pm 1.$$

ii) Let $\alpha_n < r$. Using (15) and (14) it follows that

$$(17)\qquad D^\alpha E\phi(x',0) = (-1)^{\alpha_n}f(2^{\alpha_n})D^\alpha\phi(x',0) = -D^\alpha\phi(x',0).$$

But if $E\phi = \phi$ then

$$(18)\qquad : \qquad D^\alpha E\phi(x',0) = D^\alpha\phi(x',0).$$

Comparing (17) and (18) we get $D^\alpha\phi(x',0) = 0$ for $|\alpha| < r$, that is $\phi \in D_r(R_n^+)$.

iii) Observe that $T_g T_f \phi(x',t) = \sum_{k=0}^{\infty} d_k (\sum_{h=0}^{\infty} c_h \phi(x',2^{k+h}t)) =$

$= \sum_{j=0}^{\infty} \phi(x',2^j t)(\sum_{k=0}^{j} d_k c_{j-k}).$

Since $f(z).g(z) = 1$ we have $\sum_{k=0}^{j} d_k c_{j-k} = 1$ if $j = 0$ and 0 otherwise. Therefore, it holds pointwise that

(19) $$T_g T_f \phi(x',t) = \phi(x',t) = T_f T_g \phi(x',t).$$

iv) By i), $\|E\phi;H^s(R_n)\| \leq \|T_f\phi;H^s(R_n^-\| + \|T_g\phi;H^s(R_n^+)\|$. Now Theorem 3 yields iv).

v) By iv), $E\phi_m$ is a Cauchy sequence in $H^s(R^n)$. Therefore, there exists $U \in H^s(R^n)$ such that $\|E\phi_m - U;H^s(R^n)\|$ tends to zero.

But in virtue of Theorem 3, ii) both norms $\|E\phi_m - T_g v;H^s(R_n^+)\|$ and $\|E\phi_m - T_f v;H^s(R_n^-)\|$ tend to zero. So U restricted to R_n^+ is equal to $T_g v$ and U restricted to R_n^- is $T_f v$. Since the distribution U is a function of $L^2(R^n)$ it follows that $U = Ev$, Q.E.D.

Note that conditions (14) for $r < i \leq R$ are not really used in the proof of Lemma 3.

6. PROOF OF THEOREM 2. Let $u \in H_{r,R}(R_n^+)$, supp $u \subset K$ and call $u' := E_f u$ (cfr. (6)). Observe that by (14) the hypotheses of Theorem 4 are fullfilled. Thereby $u' \in H^R(R_n)$, supp $u' = K' =$ compact in B and $Eu' \in H^R(R_n)$. In consequence, from the definition of u' we have $Eu' = u'$ a.e. (cf. (19)). Now let $\phi_h' \in C_0^{\infty}(B)$ be a sequence converging to u' in $H^R(R_n)$. By Lemma 3, v), $E\phi_h'$ converges to $Eu' = u'$ in $H^R(R_n)$ and then

(20) $$\|u' - \phi_h;H^R(R_n)\| \to 0 \quad \text{for } h \to 0$$

if $\phi_h := (\phi_h' + E\phi_h')/2$.

Using Lemma 3, iii), we see that $E\phi_h = \phi_h$. Then by ii) of the same Lemma we obtain that $u_h := \phi_h$ restricted to R_n^+ belongs to $D_r(R_n^+)$. Since $\|u - u_h;H^R(R_n^+)\| =$

$= \|u' - \phi_h ; H^R(R_n^+)\| \leq \|u' - \phi_h ; H^R(R_n)\|$, we see by (20) that the sequence u_h satisfies all the requirements, Q.E.D.

7. FINAL REMARKS. a) The construction of our extension operator E_f is essentially the one used by Seeley in [Se] however corresponding to entire functions of different nature. In order that E_f extends $C^\infty(\overline{R_n^+})$ to $C^\infty(R_n)$, Seeley needs $f(2^h) = (-1)^h$ for $h = 0, 1, \dots$. This is never true for our f since we have $f(2^h) = (-1)^{h+1}$ for $h = 0, \dots, r - 1$ (and besides $f(2^h) = (-1)^{R-1}$ for $h \geq R$). On the other hand the coefficients a_k found by Seeley define an entire function of exponential type with zeroes and in that case $g = 1/f$ is not an entire function (cfr. [A], [Se]).

b) Our method can be applied to prove that $D_r(\Omega)$ is dense in other Banach spaces. For $1 \leq p < \infty$, $0 < r < R$, r, R integers, define

$$W^p_{r,R}(\Omega) := W_0^{r,p}(\Omega) \cap W^{R,p}(\Omega) \quad \text{with the norm } \|\,.\,;W^{R,p}\|.$$

THEOREM 1'. *If Ω is a bounded domain with C^∞ boundary, then $D_r(\Omega)$ is dense in $W^p_{r,R}(\Omega)$.*

This theorem reduces to prove

THEOREM 2'. *$\{u \in D_r(R_n^+)$: supp u bounded$\}$ is dense in $W^p_{r,R}(R_n^+)$.*

The proof follows the same lines as that of Theorem 2 noticing that the operator E_f defined by (6) is continuous from $W^p_{r,R}(R_n^+)$ into $W^{R,p}(R_n)$, and the operator E of Lemma 3 is continuous in $W^{R,p}(R_n)$. Lemma 2 should be replaced by

LEMMA 2'. *If $u \in C^r(\overline{\Omega})$ and $D^\alpha u = 0$ on $\partial\Omega$ for $|\alpha| < r$, then $u \in W_0^{r,p}(\Omega)$.*

THEOREM 5. *Let r be a positive integer and R a nonnegative one. The completion of $D_r(\Omega)$ in the norm $\|\,.\,;W^{R,p}(\Omega)\|$ is isomorphic to the space $W_0^{R,p}(\Omega)$ if $R \leq r$ and isomorphic to $W^p_{r,R}(\Omega) \supsetneq W_0^{R,p}(\Omega)$ if $R > r$.*

PROOF. In fact, for $R \leqq r$, because of Lemma 2', we have

$$C_o^\infty(\Omega) \subset D_r(\Omega) \subset D_R(\Omega) \subset W_o^{R,p}(\Omega).$$

If $R > r$, it follows from Theorem 1' that $W^p_{r,R} \supset W_o^{R,p}$. To prove that the inclusion is proper consider the function $k(x) = x_n^r \phi(x')\psi(x_n)$ restricted to R_n^+ where $\phi(x') \in C_o^\infty(R_{n-1})$, $\psi \in C_o^\infty(R_1)$, ϕ and ψ equal to one in a neighborhood of zero. Then, k is of bounded support and belongs to $W^{R,p}(R_n^+) \cap W_o^{r,p}(R_n^+)$. If k belonged to $W_o^{R,p}(R_n^+)$ then $\tilde{k}$ should belong to $W^{R,p}(R_n)$. However, $D^{r+1}_{x_n}\tilde{k}$ is not a function, Q.E.D.

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