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INSTITUTO DE MATEMATICA DE BAHIA BLANCA
INMABB (UNS - CONICET)

UNIVERSIDAD NACIONAL DEL SUR
Avda. ALEM 1253 - 8000 BAHIA BLANCA
República Argentina

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ON CERTAIN SPACES OF DIFFERENTIABLE FUNCTIONS VANISHING ON THE BOUNDARY

by

A. Benedek and R. Panzone

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1. INTRODUCTION. In [B] we introduced for Ω a bounded domain with C^∞ boundary, $1 \leqq p < \infty$, r and s positive integers, $r < s$, the space

(1) $$W^p_{r,s}(\Omega) := W^{r,p}_0(\Omega) \cap W^{s,p}(\Omega).$$

We proved that it coincides with the closure in $W^{s,p}(\Omega)$ of the family of functions

(2) $$D_r(\Omega) := \{\phi \in C^\infty(\overline{\Omega}) : D^\alpha\phi = 0 \text{ on } \partial\Omega,\ |\alpha| < r\}.$$

We owe to Professor A. P. Calderón a comment to our paper where he pointed out that the space $W^p_{r,s}(\Omega)$, $1 < p < \infty$, is the (closed) subspace of $W^{s,p}(\Omega)$ formed by the functions verifying

(3) $$D^\alpha f = 0 \quad \text{a.e. } \partial\Omega \text{ for } |\alpha| < r.$$

This result can be proved using [C], especially Theorem 11. The aim of the present note is to supply the details for the preceding characterization of the space $W^p_{r,s}(\Omega)$. Observe that the definition (1) can be given also for $r = s$ and in that case $W^p_{r,r}(\Omega) = W^{r,p}_0(\Omega)$. We prove

THEOREM. *Let $f \in W^{s,p}(\Omega)$, $1 \leqq r \leqq s$, $1 \leqq p < \infty$. $f \in W^p_{r,s}(\Omega)$ if and only if $tr\, D^\alpha f = 0$, $|\alpha| < r$.*

The trace operator will be defined in §3.

2. THE REGION Ω. We shall use E. Gagliardo's work [G]. In this section we show that our region satisfies his hypothesis.

In our case each $x^o \in \partial\Omega$ has an open neighborhood U, homeomorphic to the unit ball by an application $\Phi: U \to B = \{y \in R^n;\ |y| < 1\}$ such that $\Phi \in C^\infty(U)$, $\Psi := \Phi^{-1} \in C^\infty(B)$, and $\Phi(U \cap \Omega) = B^+ := \{y \in B;\ y_n > 0\}$. We have then for $x = (x_1, \dots, x_n) \in U$, $\Phi(x) = (\phi_1(x), \dots, \phi_n(x))$ and $x \in \Omega \cap U$ iff $\phi_n(x) > 0$; $x \in \partial\Omega \cap U$ iff $\phi_n(x) = 0$; $\nabla\phi_n(x) \neq 0\ \forall\, x \in U$. Therefore there exists h such that $\frac{\partial\phi_n}{\partial x_h}(x^o) \neq 0$. We may suppose without loss of generality that $x^o = 0$, $h = n$ and $\frac{\partial\phi_n}{\partial x_n}(0) > 0$. Let us consider the map

(4) $$\eta : x = (x', x_n) \to (x', \phi_n(x)) =: y$$

with $x' = (x_1, \dots, x_{n-1})$, $x \in P := \{|x_i| < d_i,\ i = 1, \dots, n\}$ and the parallelepiped P contained in U.

If the d_i's are adequately chosen, $\frac{\partial\phi_n}{\partial x_n}(x) > 0$ in P and η defines a C^∞ homeomorphism with a C^∞ inverse χ of the form

(5) $$\chi: y = (y', y_n) \to (y', \chi_n(y)) = x.$$

The equation of the boundary $\partial\Omega \cap P$ is then

(6) $$x_n = \gamma(x') := \chi_n(x', 0), \qquad x' \in P' := \{|x_i| < d_i;\ i < n\}.$$

Since $\gamma(0) = 0$, we may take, for fixed d_n, P' so small that $|\gamma(x')| < \frac{d_n}{2}$ in P'. Then the C^∞ homeomorphism

(7) $$\begin{cases} X_n = (x_n - \gamma(x'))\frac{2}{d_n}, & \\ X_j = x_j/d_j & , j < n \end{cases}$$

carries the neighborhood R_o of x^o defined by

$$R_o := \{\gamma(x') - (d_n/2) < x_n < \gamma(x') + (d_n/2), \quad x' \in P'\}$$

into the parallelepiped

$$R := \{|X_i| < 1,\ i = 1, \dots , n\}$$

and the images of $\Omega \cap R_o$ and $\partial\Omega \cap R_o$ are respectively

$$Q := \{|X_i| < 1 \text{ for } i < n,\ 0 < X_n < 1\}$$

and

$$S := \{|X_i| < 1 \text{ for } i < n,\ X_n = 0\}.$$

In consequence, a finite open covering R_j, $j = 1, \dots , N$, of $\partial\Omega$ can be obtained in such a way that each R_j maps homeomorphically onto R, while the images of $R_j \cap \Omega$ and $R_j \cap \partial\Omega$ are Q and S respectively. So Gagliardo's hypothesis is fulfilled, since these homeomorphisms are C^∞ in both ways. We shall denote them by Φ_j. A set $A \subset \partial\Omega$ is said to have **surface measure zero** if for every j, $\Phi_j(A \cap R_j)$ has measure zero in the $(n - 1)$-dimensional surface S.

3. THE TRACES. Using a partition of unity and the maps Φ_j one can verify that it is enough to define the restriction to S of a function in $W^{s,p}(Q)$ to give a meaning to the restriction of $u \in W^{s,p}(\Omega)$ to $\partial\Omega$. That is tr u.
Now if $f \in W^{s,p}(Q)$ then $D^\alpha f \in L^1(Q)$ for $|\alpha| \leq s$. So by Th. V pg. 57 [S], there is a representative of its class which is absolutely continuous on every segment $\{(x',t);\ 0 < t < 1\} \subset Q$. We shall call such an f a prototype of its class. For a prototype, $\frac{df}{dt}(x',t) = \frac{\partial f}{\partial x_n}(x',t) \in L^1(0,1)$ for almost all x'. In consequence, there exists the limit:

$$\lim_{t \to 0} f(x',t) =: (\text{tr } f)(x') =: f(x',0), \qquad \text{a.e. } x' \in S.$$

Besides, tr f does not depend of the prototype chosen. In [G] (footnote 7), p. 288) the following Lemma is proved.

LEMMA 1. *Let* $1 \leq p < \infty$ *and* $f \in W^{1,p}(Q)$. *Then* $tr\, f \in L^p(S)$ *and*

$$\|tr\, f;\ L^p(S)\| \leq C_p \|f; W^{1,p}(Q)\|. \tag{8}$$

Another way of defining the trace is the following. If $u \in C^\infty(\overline{Q}) \cap W^{s,p}(Q)$, we define $\mathrm{Tr}\, u(x',0) = u(x',0) = tr\, u(x',0)$. For $u \in W^{s,p}(Q)$ take a sequence $\{\phi_m\}$ such that $\phi_m \in C^\infty(\overline{Q}) \cap W^{s,p}(Q)$, $\phi_m \to u$ in $W^{s,p}(Q)$, and define

$$\mathrm{Tr}\, u := \lim_{m \to \infty} tr\, \phi_m.$$

Inequality (8) shows that this limit exists in $L^p(S)$ and coincides with tr u.

4. THE MAIN RESULT.

LEMMA 2. *Let* $f \in W^p_{r,s}(Q)$, $\overline{supp\, f} \subset R$, $1 \leq r \leq s$, $1 \leq p < \infty$. *Then* $tr\, D^\alpha f = 0$ *for* $|\alpha| < r$.

PROOF. There exists $\{\phi_j\} \subset D_r(Q)$ such that $D^\alpha \phi_j \to D^\alpha \phi$ in $L^p(Q)$, $|\alpha| \leq s$. For $|\alpha| < r$ we have $\|\, tr\, D^\alpha f - tr\, D^\alpha \phi_j\ ; L^p\| = \|tr\, D^\alpha f; L^p\|$, and from (8) we get the thesis, QED.

LEMMA 3. *Let* $f \in W^{s,p}(Q)$, $\overline{supp\, f} \subset R$, $1 \leq r \leq s$, $1 \leq p < \infty$, $tr\, D^\alpha f = 0$ *for* $|\alpha| < r$. *Then* $f \in W^p_{r,s}(Q)$.

PROOF. Let E be a strong s-extension for R^+_n. That is, a linear operator mapping functions defined in R^+_n into functions defined in R^n such that for every p, $1 \leq p < \infty$, and every k, $0 \leq k \leq s$, verifies

i) $Eu = u$ a.e. R^+_n, and

$$\|Eu; W^{s,p}(R_n)\| \leq K\|u; W^{s,p}(R^+_n)\|$$

ii) $D^{\alpha}Eu = E_{\alpha}D^{\alpha}u$, $|\alpha| \leq s$, where E_{α} is an operator similar to E but acting on $W^{s-|\alpha|,p}(R_n^+)$,

iii) if $u(x',t)$ is continuous in $t \in [0,\varepsilon)$ for fixed $x' \in R_{n-1}$ then $Eu(x',t)$ is continuous in $t \in (-\delta,\varepsilon)$, $\delta > 0$, (cf. [A], pp. 83-88).

A function $f \in W^{s,p}(Q)$ with $\overline{\text{supp } f} \subset R$ can be trivially extended to $W^{s,p}(R_n^+)$. Let $f_\varepsilon := Ef * \phi_\varepsilon$ where $\{\phi_\varepsilon\}$ is an approximation of δ. Then $f_\varepsilon \to Ef$ in $W^{s,p}(R_n)$. Let $\tilde{f}$ be the trivial extension of f to R_n. For $|\alpha| \leq r$, $\phi \in C_0^\infty(R_n)$, we have

$$<\tilde{f},D^\alpha\phi> = \lim_{\varepsilon \to 0} \int_{x_n>0} f_\varepsilon . D^\alpha\phi \, dx =$$

$$= \lim_{\varepsilon \to 0} [(-1)^{|\alpha|} \int_{x_n>0} D^\alpha f_\varepsilon . \phi \, dx + \sum_{|\gamma|+|\beta|<r} C_{\gamma\beta} \int_{x_n=0} D^\gamma f_\varepsilon . D^\beta \phi \, dx'] =$$

$$= (-1)^{|\alpha|} \int_{x_n>0} D^\alpha (Ef) . \phi \, dx + \sum_{|\gamma|+|\beta|<r} C_{\gamma\beta} \lim_{\varepsilon \to 0} \int_{x_n=0} (\mathrm{tr}\, D^\gamma f_\varepsilon) . D^\beta \phi \, dx' =$$

$$= (-1)^{|\alpha|} \int_{x_n>0} D^\alpha f . \phi \, dx + \sum_{|\gamma|+|\beta|<r} \int_{x_n=0} (\mathrm{tr}\, D^\gamma f) . D^\beta \phi \, dx' =$$

$$= (-1)^{|\alpha|} \int \widetilde{D^\alpha f} . \phi \, dx .$$

This means that $D^\alpha \tilde{f} = \widetilde{D^\alpha f}$ for $|\alpha| \leq r$ and so $\tilde{f} \in W^{r,p}(R^n)$. Since $f = \lim_{\varepsilon \to 0} \tilde{f}(x', x_n - \varepsilon)$ in $W^{r,p}(R_n^+)$, we get $f \in \overset{o}{W}{}^{r,p}(R_n^+)$, QED.

The theorem follows from Lemmas 2 and 3 and the localization described in §2. Observe that properties ii) and iii) are not used in the proof of Lemma 3.

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