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Holonomy and the Theorem of Frobenius

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Abstract. By using the idea of holonomy we get a new proof of Frobenius Theorem on integrability of a given distribution on a given manifold. This proof also works for distributions given on Banach (infinite dimensional) manifolds. An interesting point is that Frobenius integrability condition, in terms of the Lie bracket of two vector fields, appears in a natural way being equivalent to "zero holonomy" for the given distribution. The paper begins with a description of tha ideas underlying previous proof of Frobenius Theorem to facilitate the comparison with ours.

§1. *INTRODUCTION*.

Let M be a C^{∞} manifold of dimension n. A distribution of p-planes on M is a p-dimensional subbundle of the tangent bundle TM . The distribution L is called *integrable* if for given C^{∞} vector fields on M, say X , $Y \in \mathcal{X}(M)$, the following condition is satisfied:

$$X(m), Y(m) \in L(m) \text{ for all } m \in M \Longrightarrow [X,Y](m) \in L(m) \qquad (1)$$

An *integral manifold* of L , is a C^{∞} p-submanifold $S \subseteq M$ such that $T_m S = L(m)$ for all $m \in S$.

A distribution of p-planes is said to *arise from a Regular Foliation* if for each $m \in M$ there is an integral submanifold S such that $m \in S$.

Theorem 1.(Frobenius). Let L be a given distribution on M. If is integrable then L arise from a regular foliation.

Frobeniu's Theorem has been proved by using several different arguments that we can found in the standard Differential Geometry literature . In this introduction we would like to sketch the basic geometric picture behind those arguments, for convenience of the interested reader, who may wish to compare them among themselves and, later ,also with our own appropach to the subjet. The latter will be discribed in § 2 . In §3 we discuss some ideas relating our main result in this paper to the notion of the curvature of a connection given on a principal bundle.

Before we continue a couple of remarks are in order. First, let us observe that infinite dimensional generalizations of Theorem 1 are also important and well known in the literature : for M a Banach manifold, L should be assumed to be a direct subbundle of TM. Then we get essentially the same result . Our proof in § 2 is also valid for infinite dimensional cases. Second, Theorem 1, as stated, is clearly local in nature, in the sense that, by choosing a local chart about any given $x_0 \in M$ we reduce the question to the case of M being an open ball . In the finite case this means that we can choose local coordinates $(x^1,\dots,x^n)$ about $x_0 = (0,\dots,0)$ such that the integral manifold is defined by equations $x^{p+1}=0,\dots x^n=0$, or , equivalently, L is spanned by $\frac{\partial}{\partial x^1},\dots\frac{\partial}{\partial x^p}$.

Some of the proofs sketched below are valid for infinite dimensions. Some others use induction on $p=\dim L(m)$, thus they are naturally adapted for M a finite dimensional manifold. The case $p=1$ is a trivial one, at least from our point of view, since it follows from ODE's theory (existence and uniqueness results) which we take for granted here , both for finite and infinite dimensional cases.

a) (See for instance Sternberg [7])

Let M be n dimensional and L p-dimensional . Choose $X_1,\dots X_p$ linearly independent vector fields on M spanning L on some neighborhood of a given $x_0 \in M$. By using ODE's existence and uniqueness theorems , we can inmediately show that there are coordinates $(x^1,\dots,x^n)$ about x_0 such that $X_1=\frac{\partial}{\partial x^1}$. Let W be defined by $x^1=0$. It is not difficult to show that, in this situation, if S is an integral manifold containing x_0 then

$S=(-\varepsilon,\varepsilon)\times S_1$, where $(-\varepsilon,\varepsilon)$ is some interval in the x^1 axis and $S_1=S\cap W$. Now, to apply the induction hypothesis, we only need to check that, L being integrable, the $(p-1)$- dimensional distribution $L\cap TW$ is also integrable . Then, S_1 will be simply the integral manifold of $L\cap TW$ containing x_0 . To establish integrability of $L\cap TW$ is perhaps the only part of the proof presenting some technical difficulty.

b) (See for instance Hermann [2])

The picture for this proof is similar to the previous one. First we must prove, that for any given p-dimensional integrable distribution on a given n-dimensional manifold M, we can find (locally) vectors fields $X_1,\ldots X_p$ spanning L and conmuting among themselves i.e. satisfying

$$[X_i,X_j]=0 \quad , \quad i,j=1,2,\ldots p \qquad (2)$$

Next we prove ,by induction on p, that (2) implies existence of local coordinates about any given point $x_0\in M$, such that $X_i=\frac{\partial}{\partial x^i}$ $i=1,2,\ldots p$ .Thus $x^{p+1}=0,\ldots x^n=0$ define the integral manifold containing x_0. To perform the induction step, we basically find coordinates $(y^1,\ldots,y^n)$ such that $X_1=\frac{\partial}{\partial y^1}$ and then prove that $L\cap TW$ is spanned by conmuting fields. (W defined by $y_1=0$)

c) (See Chern S.S, Wolfson J.S. [1])

This proof is technically simpler than the previous two ones.

However the underliying idea is the same.

d) (See for instance Lang [4] [5]).
This proof is valid also for infinite dimensional cases.
First , observe that any given distribution L on a (Banach) manifold M can be described ,by using a local chart about any given $x_0 \in M$,by a C^∞ map

$$f: U\times V \to L(E,F)$$

where E,F are Banach spaces and $U\subseteq E$, $V\subseteq F$ are open ball centered at 0.Then the plane $L(x_0,y_0)$ is defined by the equation

$$y-y_0 = f(x_0,y_0)\langle x-x_0\rangle$$

The integrability condition becomes

$$Df(x,y)\langle a,f(x,y)\langle a\rangle\rangle\langle b\rangle=Df(x,y)\langle b,f(x,y)\langle b\rangle\rangle\langle a\rangle \quad (3)$$
for all $(x,y) \in U\times V$ $a,b \in E$

Moreover we can always assume $x_0 = (0,0)$ and $f(0,0) = 0$.
Now let $X_t(x,y) = (0,f(tx,y)\langle x\rangle)$ be a time dependent vector field on $U\times V$ and let F_t be the flow of X_t.Then we can prove using (3) that there is ball U_1 centered at 0, $U_1 \subseteq U$, such that $F_t(x,0)$ is defined for all $(x,0) \in U_1\times\{0\}$, $t \in [0,1]$ and moreover,$F_1(U_1 \times \{0\})$ is an integral manifold of L containing $F_1(0,0)=(0,0)$,which is the graph of $S:U_1\to V$ defined by $S(x)=p_2 o F_1(x,0)$.Equivalently,S is the solution of the problem

$$DS(x) = f(x,S(x)) \quad (4)$$

$S(O) = O$

e)(See Penot [6]).This paper is indeed a rather comprehensive reference on Frobenius Theorem up to 1970.

Using the notation of d) we want to show existence of $S:U_1 \to V$ satisfying (4) . Roughly speaking,Penot's proof consists in using the Implicit Function Theorem.In fact some kind of equivalence between the later and Frobenius Theorem is established in Penot's article.

To start,set $G = \{ \gamma \in C^1([O,1],F) : \gamma(O)=O \}$. $H=C^O([O,1]),F)$. G and H are Banach spaces with norms $|\gamma| = sup(|\gamma(t)|+|\gamma'(t)|)$ and $|\eta| = sup|\eta(t)|$ where $t \in [O,1]$,respectively.Let $B_r(L)$ be the open ball of radius r centered at O in the Banach space L. The derivation map $\gamma \to \gamma'$ is a linear isomorphism from G onto H.Now define for some small $r>O$ $g:B_r(E)\times B_r(G) \to H$ as follows:

$$g(x,\gamma)(t)=\gamma'(t)-f(tx,\gamma(t))\langle x\rangle.$$

Since $D_2g(O,O)\langle\beta\rangle=\beta'$, the Implicit Function Theorem implies existence of $B_{r_1}(E) \subseteq B_r(E)$ and a map $\varphi : B_{r_1}(E) \to B_r(G)$ such that $\varphi(O)=O$ and $g(x,\varphi(x))=O$ for all $x \in B_{r_1}(E)$.

Define $S(x) = \varphi(x)(1)$. It remains to prove that S satisfies (4) of d). This calculation presents the same type of technical difficulty as it does the proof of (4) from (3) in d). Indeed we should remark the obvious similitude between $g(x,\gamma)(t)$ above and the vector field $X_t(x,y)$ in d).

There are some other approachs to Frobeniu's Theorem,using the language of differential forms,(See Sternberg [7]).However, they finally rely on a) or b) above.We claim that our proof,to be described in the next paragraph is based on an argument wich differs substantially from the previous ones.

§2. In this section we give a proof of Frobenius Theorem which, we believe, is new.

The situation being as in § 1.d) we want to prove that (3) implies (4).

First, let us introduce the following notation:

Let $X(t), t \in [t_0, t_1]$ be a C^{∞} curve on U having origin $x_0 = X(t_0)$

Let $P_0 = (x_0, y_0)$ be given. The *lifting* of X with origin P_0 (for a given distribution f) is the curve

$$l(P,X)(t) =: (X(t), Y(t))$$

on $U \times V$ defined as the (unique) solution of the problem

$$\frac{d}{dt} l(P_0,X)(t) = (X'(t), f(X(t),Y(t))\langle X'(t)\rangle)$$

$$l(P_0,X)(t_0) = (X_0, Y_0)$$

This means that the curve $l(P_0,X)$ is the (only) curve having origin P_0, such that

a) it is tangent to the linear manifold L defined by f at each point $l(P_0,X)(t)$, and

b) its projection on the first factor of U×V is the given curve X(t).

Obviously, we can also define $l(P_0,X)$ for a piecewise differentiable curve X.

In particular we will usually work with piecewise linear curves in this article, for convenience.

Let $Q(x_0)$ be the set of all piecewise linear curva $q(t)$

$t \in [t_0,t_1]$ in U such that $q(t_0)=x_0$. Given a curve $q \in Q(x_0)$, we will say that the endpoint of the *lifting* $l(P_0,q)(t_1)$ is *independent* of the path if for every $\bar{q} \in Q(x_0)$, say $\bar{q}(t)$, $t\in[\bar{t}_0,\bar{t}_1]$ such that $\bar{q}(\bar{t}_1)=q(t_1)$ we have $l(P_0,q)(t_1)=l(P_0,\bar{q})(\bar{t}_1)$.
Now we will describe the idea of the proof.

Suppose that, for every $q \in Q(x_0)$ the endpoint $l(P_0,q)(t_1)$ is independent of the path. Then define $S:U\to V$ as follows:
Given $x_1 \in U$ choosse $q \in Q(x_0)$ such that $q(t_1)=x_1$.
Let $l(P_0,q)(t_1) = (x_1,y_1)$. Then if $S(x_1) = y_1$ it is easy to check that S satisfies (4) of § 1.d). Therefore to conclude the proof it only remains to show that integrability condition (3) of § 1.d) implies the property of independence of the path which is the only point of the proof presenting some technical dificulty.

First, we will reduce the question of independence of the path to a very particular case of itself as follows.
Let $\boldsymbol{e_1},\boldsymbol{e_2}$ be any choise of two axis of the basis of E, and let R be a rectangle with vertices x_0,x_1,x_2,x_3 where the sides x_0x_1, x_2x_3 are paralell to $\boldsymbol{e_1}$ while the sides x_1x_2, x_0x_3 are paralell to $\boldsymbol{e_2}$.
The lenght of x_0x_1 is Δ_1, while the lenght of x_0x_3 is Δ.
Let $\alpha(t)$, $t \in [0,\Delta_1+\Delta_2]$ be the path $x_0x_1x_2$ (i.e, the union of the line segments x_0x_1 and x_1x_2). Similary, let $\beta(t)$, $t\in[0,\Delta_1+\Delta_2]$ be the path $x_0x_3x_2$. The property of independence of the path of endpoints $l(P_0,q)(t_1)$, for all $q \in Q(x_0)$, is equivalent to the following:

$$l(P_O,\alpha)(\Delta_1+\Delta_2)=l(P_O,\beta)(\Delta_1+\Delta_2) \qquad (5)$$

for all rectangles $R\subseteq U$ having sides paralell to a couple of axis $\boldsymbol{e_i},\boldsymbol{e_j}$ of E.

The argument to establish this equivalence can be described as follows. Let $q^i \in Q(x_O), i=1,2$ be two qiven paths having vertices $x_O = q_O^i$, q_1^i, ..., $q_r^i = x_1$. Two such paths q^1, q^2 are called "contiguous" if there is a number k, $O < k < r$, such that $q_O^1 = q_O^2, \dots, q_{k-1}^1 = q_{k-1}^2$, $q_{k+1}^1 = q_{k+1}^2, \dots, q_r^1 = q_r^2$, and q_{k-1}^1, q_k^1, q_{k+1}^1, q_k^2 are vertices of a rectangle. Now given any two paths q $\bar q \in Q(x_O)$, by (possibly) first introducing some new appropiate vertices in (the line segments of) q and $\bar q$ we can find a finite sequence $q= q^1, q^2, \dots, q^h=\bar q$ of paths $q^i \in Q(x_O)$ such that q^i is contiguous to q^{i+1} for $i=0,1,\dots,h-1$. We can then use (5) to show that the endpoint of the liftings $l(P_O,q^i)$ all coincide for $i=0,\dots,h-1$, thus in particular $l(P_O,q)(t_1) = l(P_O,\bar q)(\bar t_1)$. We can conclude that to finish the proof it is enough to prove the following:

Lema : (3) of § 1.d) implies (5).

Proof : Without loss of generality, we can assume from the beginning that $R \subseteq K$ where K is a compact ball centered at $O \in E$ and such that $K \subseteq U$. Let us also assume that the sides of R are paralell to $\boldsymbol{e_1},\boldsymbol{e_2}$.

A) Let ρ be the ratio $\rho=\frac{\Delta_2}{\Delta_1}$. Next we prove

$$|l(P_0,\alpha)(\Delta_1+\Delta_2)-l(P_0,\beta)(\Delta_1+\Delta_2)| \leqq Cd^3$$

where C is a constant provided that K,ρ are fixed.

To prove this , first observe that a given continuous function on K is bounded and that , ρ being fixed, we can write $O(\Delta x_i) = O(d)$, $O(\Delta_1,\Delta_2)=O(d^2)$, etc.

Now let us denote the vertices of the *liftings* of α and β as follows:

$$P_0 = (x_0,y_0) , \qquad P_1 = l(P_0,\alpha)(\Delta_1) = (x_1,y_1)$$

$$P_2 = l(P_0,\alpha)(\Delta_1+\Delta_2) = (x_2,y_2) , \qquad P_3 = l(P_0,\beta)(\Delta_2) = (x_3,y_3)$$

$$P_4 = l(P_0,\beta)(\Delta_1+\Delta_2) = (x_4,y_4).$$

Thus, what we want to show is that $|y_2-y_4| \leqq Cd^3$. From (3) of § 1.d) by using the Taylor development of $l(P_0,\alpha)$ at P_0 (up to order 2) we can show that

$$y_1-y_0=f_1(P_0)\Delta_1 + \frac{1}{2}\left[\frac{\partial f_1}{\partial x^1}(P_0)+\frac{\partial f_1}{\partial y^j}(P_0)f_1^j(P_0)\right]\Delta_1^2 + O(d^3) \qquad (\alpha)$$

Similary, by using the Taylor development of $l(P_0,\beta)$ at $P_3 = (x_3,y_3)$ we can show that

$$y_4-y_3=f_1(P_3)\Delta_1 + \frac{1}{2}\left[\frac{\partial f_1}{\partial x^1}(P_3)+\frac{\partial f_1}{\partial y^j}(P_3)f_1^j(P_3)\right]\Delta_1^2 + O(d^3) \qquad (\beta)$$

But we can approximate,again,by using Taylor development

$$f_1(P_3)=f_1(P_0) + \frac{\partial f_1}{\partial x^1}(P_0)\ \Delta_1 + \frac{\partial f_1}{\partial x^2}(P_0)\ \Delta_2 + \frac{\partial f_1}{\partial y^s}(P_0)f_2^s(P_0)\ \Delta_2 +$$
$$+ O(d^2)$$

$$\frac{\partial f_1}{\partial x^1}(P_3) = \frac{\partial f_1}{\partial x^1}(P_0) + O(d)$$

$$\frac{\partial f_1}{\partial y^s}(P_3) = \frac{\partial f}{\partial y^s}(P_0) + O(d)$$

$$f_1^s(P_3) = f_1^s(P_0) + O(d)$$

Thus y_4-y_3 becomes

$$y_4-y_3 = y_1-y_0 + \left[\frac{\partial f_1}{\partial x^2}(P_0)+\frac{\partial f_1}{\partial y^s}(P_0)f_2^s(P_0)\right]\Delta_1\Delta_2 + O(d^3) \qquad (\gamma)$$

Similarly we can prove

$$y_2-y_1 = y_3-y_0 + \left[\frac{\partial f_2}{\partial x^1}(P_0) + \frac{\partial f_2}{\partial y^s}(P_0)f_1^s(P_0)\right]\Delta_1\Delta_2 + O(d^3) \qquad (\delta)$$

By substracting (γ) from (δ),we get

$$|y_2-y_4| = O(d^3) \qquad \text{or} \qquad |y_2-y_4| \leq Cd^3$$

By going through the previous proof we can see that, for fixed ρ, the constant c comes from fixing bounds for certain Taylor remainders, which in turn, only depend on bounds for f^i_j and its derivatives up to order 4 at most, on a fixed compact ball K, contained in U, such that K contains all rectangles R' under consideration.

B) Now we will prove that the constant C that we have found in A) is, in fact 0. In other words, the endpoints $y^2=l(P_0,\alpha)(\Delta_1+\Delta_2)$ and $y^4 = l(P_0,\beta)(\Delta_1+\Delta_2)$ coincide, which establishes (5).

Let

$$x_0,\ x_0+\frac{1}{n}(x_1-x_0),\dots\dots,x_0+\frac{j}{n}(x_1-x_0),\dots,x_1$$

and

$$x_0,\ x_0+\frac{1}{n}(x_3-x_0),\dots\dots,x_0+\frac{j}{n}(x_3-x_0),\dots,x_3$$

be partitions of the segments x_0x_1 and x_0x_3 respectively. This gives rise to a partition of R into a n^2 rectangles, each having diagonal of lenght $\frac{d}{n}$. The ratio of the lenghts of the sides for each rectangle of the partition is the fixed number ρ. Now we can obviously find a finite sequence of elements $q^i\in Q(x_0)$ $i=0,\dots,n^2$, each of lenght $\Delta_1+\Delta_2$ such that

a) The vertices of each q^k are points of the vertices of the partition of R, i.e. of type

$$(x_0 + \frac{j}{n}(x_1-x_0) + \frac{i}{n}(x_3-x_0)).$$

In particular we have $q^i(0)=x_0$, $q^i(\Delta_1+\Delta_2)=x_2$ for $i=0,\dots n^2$

b) $q^0=\alpha$, $q^n = \beta$; and

c) Each q^i is contiguous to q^{i+1}, in the following sence:

For each i we can find $t_1^i < t_2^i \in [0,\Delta_1+\Delta_2]$ such that

$$q^i\big|[0,t_1^i] = q^{i+1}\big|[0,t_1^i] =: \gamma^i$$

$$q^i\big|[t_2^i,\Delta_1+\Delta_2] = q^{i+1}\big|[t_2^i,\Delta_1+\Delta_2] = \delta^i$$

$$q^i\big|[t_1^i,t_2^i] =: \alpha^i \quad , \quad q^{i+1}\big|[t_1^i,t_2^i] =: \beta^i$$

Where α^i, β^i both together form the border of a rectangle of the partition. Now, for each $i=0,\dots,n^2$, set:

$$P_0^i = l(P_0,q^i)(t_1^i)$$

$$P_1^i = l(P_0^i,\alpha^i)(t_2^i)$$

$$P_2^i = l(P_0^i,\beta^i)(t_2^i)$$

$$P_3^i = l(P_1^i,\delta^i)(\Delta_1+\Delta_2) = l(P_0,q^i)(\Delta_1+\Delta_2)$$

$$P_4^i = l(P_2^i,\delta^i)(\Delta_1+\Delta_2) = l(P_0,q^{i+1})(\Delta_1+\Delta_2)$$

Then we have

$$|y_4-y_2| = |(x_4,y_4)-(x_2,y_2)| = |P_4^{n^2} - P_3^0| = |l(P_0,q^0)-l(P_0,q^{n^2})| \leqq$$

$$\leqq \sum_{i=0}^{n^2-1} |l(P_0,q^i)(\Delta_1+\Delta_2)-l(P_0,q^{i+1})(\Delta_1+\Delta_2)|$$

Now for each i, we have for some $C_1>0$

$$|l(P_0,q^i)(\Delta_1+\Delta_2)-l(P_0,q^{i+1})(\Delta_1+\Delta_2)| =$$

$$|l(P_1^i,\delta^i)(\Delta_1+\Delta_2) - l(P_2^i,\delta^i)(\Delta_1+\Delta_2)| \leqq C_1 \; |P_1^i-P_2^i|$$

Since lenght of δ^i is bounded by $\Delta_1+\Delta_2$ by using the theorem on continuous dependence on the initial data for the O.D.E appearing in (3) of §1.d) we get the last inequality (for some $C_1>0$) . Now since the lenght of the diagonal of a rectangle of the partition is $\frac{d}{n}$ we can use part A) to show that

$$|P_1^i - P_2^i| \leqq C \left(d/n\right)^3$$

By collecting results we finally get

$$|y_4-y_2| \leqq C\,C_1 \sum_{i=0}^{n^2-1} (d/n)^3 = (C\,C_1\,d^3)/n$$

Since n is arbitrary,we can conclude that $y_4-y_2=0$.

This finishes the proof.

§3. Let $\Pi : P \to B$ be a principal bundle (See Kobayaschi S. and Nomizu K [1963]) with structure group G and let $W : TP \to \mathfrak{G}$ be a connection, $\mathfrak{G}$ being the Lie algebra of G. $H_p = Ker\ W_p$ is the corresponding distribution of horizontal planes, ($dim\ H_p = dim\ B$ for all $p \in P$) and let $h : T_pP \to H_p$ be the horizontal projection. Then the *curvature 2-form* Ω is defined by $\Omega = dW\ h := DW$.
Here W is viewed as a $\mathfrak{G}$-valued 1-form and d is exterior differentiation.

The holonomy theorem implies that the curvature Ω is 0 if and only if the holonomy groups $\Phi(u)$, $u \in P$ are all trivial, and the latter, in turn ,is equivalent to the distribution of horizontal planes being integrable.

Recall the definition of holonomy group. Let $\Pi(u) = x \in B$, and let $C(x)$ be the loopspace at x. Each $\tau \in C(x)$ determines and element, say a, of G such that $\tau(u) = ua$,where $\tau(u)$ is the endpoint of the horizontal lifting of τ having origin u.

We can easily see that the set of all such elements a form a subgroup $\Phi(u)$ of G.

By using a local trivialization we can represent P by a trivial bundle $U \times G \to U$ and in this representation the connection W becomes distribution $f : U \times G \to L(E,F)$ where E , F are vector spaces and $U \subseteq E$, $G \subseteq F$ are open as in §1.d). The concept of *horizontal lifting* coincides with the notion of *lifting* introduced in §2. The holonomy groups are all trivial if and only if the property of *independence of the path* ,considered in §2 holds.

Thus our proof of Frobenius Theorem in § 2 appears, at least at a local level, as a generalization, for any given distribution on a given manifold , of the known fact that , a given connection on a principal bundle is integrable iff its curvature form is identically 0.

According to the previous ideas, the curvature measures *how far* is a connection of being trivial.

In the same spirit we can define, at least at a local level, a sort of measure of *how far* is a given distribution on a given manifold of being integrable, as follows.

Let $f : U \times V \to L(E,F)$ be a local representative of the given distribution, as in § 1.d).

Let X, Y be given vectors fields on $U \times V$, say $X = (X_u, X_v)$, $Y = (Y_u, Y_v)$ and let $(u,v) \in U \times V$ be a given point.

For each $\lambda \in R^+$, form the paralelogram A_λ contained in U, whose vertices are: u, $u + \lambda X_u$, $u + \lambda X_u + \lambda X_v$, $u + \lambda X_v$.

Consider the border γ_λ of A_λ as a closed curve with both end-points coinciding with u, and parametrized by arc lenght.

Let $\tilde{\gamma}_\lambda$ be the *lifting* of γ_λ such that $\tilde{\gamma}_\lambda(0) = (u,v)$.

Thus $\tilde{\gamma}_\lambda(2\lambda(|X_u| + |X_v|)) = (u, v_\lambda)$ is the endpoint of $\tilde{\gamma}_\lambda$.

Now define the *F-valued 2-form* Ω on $U \times V$ by

$$\Omega\ (u,v)(X,Y) = \lim_{\lambda \to 0^+} \frac{1}{Area\ A_\lambda}\ (v_\lambda - v\ )$$

By using an argument similar to the one described in § 2 (but a little bit more involved, since, there, the paralelogram had sides of lenghts Δ_1, Δ_2, paralell to some axis)we can see that Ω is 0 iff the distribution is integrable. In this sense we can consider that Ω measures the *non integrability* of the distribution.

The similitude of this definition of Ω and the idea of holonomy become now obvious, at least, once a local chart has been chosen.

A more global version of all this with some possible applications to mechanics is being planned for a future work.

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