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A NEW PROOF OF FROBENIUS THEOREM AND APPLICATIONS

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ABSTRACT . A proof of Frobenius theorem on local integrability of a given distribution on a finite or infinite dimensional manifold under weak differentiability conditions is given. The Inverse Problem in the Calculus of Variations appears as a particular case. A local two-form which measures the non-integrability of a given distribution is also studied, with applications.

Key Words : integrability (of) distributions; Inverse Problem (in the) Calculus of Variations; holonomy.

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INTRODUCTION. Frobenius' Theorem establishes, roughly speaking, that a given sub-bundle of the tangent bundle is integrable if and only if for every pair of vector fields tangent to the sub-bundle, their Lie bracket is also tangent to the sub-bundle. This basic statement has been proved in different situations for finite or infinite dimensional manifolds.Each case assumes some kind of regularity condition on the sub-bundle (distribution) and the manifold (See [1],[5],[8],[9],[10],[12],[15])).There are various methods of proof (See [6] ,[13] and previous references) of the existence of a local integral manifold. Then existence of global integrals is usually established by some extension procedure.

In this paper we concentrate on a case of the local existence theory where (some of) the involved spaces are normed spaces, not necessarily complete and the regularity conditions on the distribution are expressed in terms of its restrictions to 2- dimensional subspaces. This makes our result easily applicable to some cases where the distribution is only densely defined on a Banach manifold. As an interesting example we study the Inverse Problem in the Calculus of Variations. The integrability condition of Frobenius in terms of the Lie bracket,becomes in this example the usual self- adjointness condition ([2],[14],[16]). Also as a consequence of our methods we find some kind of (local) "obstruction" to the integrability of a given distribution consisting in a vector-valued "two form" ω wich vanishes if and only if the integrability condition holds. We give some physical interpretation of ω in the context of constrained Lagrangian sistems.

Our proof is entirely different from those appearing in the literature and is inspired in the idea of holonomy of a connection on a principal bundle ([11]), the two form ω being related to the curvature tensor.See also [4] for a related idea.

A similar obstruction appears as the curvature tensor of connections in the almost tangent structure of a tangent bundle [7]. We thank J.Marsden and T.Ratiu for their valuable suggestions.

1.- Let H be a normed vector space and let G be a Banach space. Norms will be denoted "$|\ |$" in this paper. Let $U \subseteq H$ be an open ball centered at $x_0 \in H$. For each choice of a couple of unit linearly independent vectors $a, b \in H$, we define

$$H' = \{\lambda a + \mu b : \lambda, \mu \in \mathbf{R}\} \qquad , \qquad U' = x_0 + H' \cap U$$

For given $\alpha, \beta > 0$ we write

$$R = \{ x_0 + \lambda a + \mu b : (\lambda, \mu) \in [0,\alpha] \times [0,\beta] \}$$

We always assume $R \subseteq U$. Thus shrinking U implies shrinking R. Let $E : U \times G \times H \to G$ be given. For each choice of U, x_0, a, b as before, we define conditios (I) and (L) as follows

(I) $E(x,y).h$ continuous for $(x,y,h) \in U' \times G \times H'$ and linear in h. $E(x,y).h$ is k-Lipschitz in y in the following sense. There exists $k = k(U')$ such that

$$|E(x,y).h - E(x,y').h| \leq k|y-y'| \quad \text{for all } x \in U', |h| \leq 1;\ y, y' \in G$$

Now we define $DE : U \times G \times H \times G \times H \to G$ as follows

$$DE(x,y).(h,k).l = \frac{d}{ds}\Big|_{s=0} E(x+sh, y+sk).l$$

It is important to notice that this notion of derivative is weaker than that of *Frechet derivative*. It is usually called *Gateaux derivative*.

(L) $DE(x,y).(h,E(x,y).h).l$ is continuous for $(x,y,h,l) \in U'\times G\times H'\times H'$ and

$$DE(x,y)(h,E(x,y).h).l = DE(x,y)(l,E(x,y).l).h$$

for $(x,y,h,l) \in U'\times G\times H'\times H'$

Finally, fix $y_0 \in G$. Then, under the previous conditions we have

Theorem 1 There exists one and only one $S:U \to G$ such that

$$DS(x).h = E(x,S(x)).h \qquad \text{for all } (x,h) \in U\times H$$
$$S(x_0) = y_0$$

Here $DS(x).h = \frac{d}{d\lambda}S(x+\lambda h)\Big|_{\lambda=0}$,by definition

Remark: a) Using Zorn's lemma, we can always define an inner product norm on a given vector space H . In examples H is given to us and we have some freedom to choose the norm so as to satisfy (I),(L).

b) Replace the Lipschitz condition in (I) by the following: there exist balls $U\subseteq H$, $V\subseteq G$ about x_0,y_0 and a constant k such that $|E(x,y).h - E(x,y').h| \leq k|y - y'|$ for all $x\in U$, $y,y'\in V$, $h\in H$, $|h|\leq 1$. Then the conclusions of theorems ***1*** and ***2*** subsist, with a possibly smaller U.

c) Replace the Lipschitz condition in (I) by the following. Let $x_0 \in H$, $y_0 \in G$ be given. Let $U \subseteq H$ be a ball centered at x_0. Let H', $U' = (x_0 + H') \cap U$ as before. Assume that foe each ball $B \subseteq G$ centered at y_0, there is a constant $k = k(U', B)$ such that

$$|E(x,y).h - E(x,y').h| \leq k|y - y'|$$

for all $x \in U'$, $|h| \leqq 1$, $y, y' \in B$

Then the conclusion of ***Theorem 1*** subsists in a weaker form,namely the domain of S is a star-shaped subset W of H about x_0. Moreover W has the following property: for each H' as before, $(x_0 + H') \cap W$ is a bidimensional ball centered at x_0.

The conclusion of ***Theorem 2*** subsists.

To prove this theorem we need some previous lemmas.

For each $n = 1,2,3...$ we have an n-partition of R into n^2 smaller rectangles whose sides have lenght $\frac{\alpha}{n}$, $\frac{\beta}{n}$. We will work with piecewise-linear maps $q : [0, \alpha + \beta] \to R$ parametrized by arc-lenght, each linear piece being parallel to either a or b and having length $\frac{\alpha}{n}$ or $\frac{\beta}{n}$ respectively and coinciding with some side of some rectangle of the n-partition of R. We also assume that the length of q is $\alpha + \beta$ and $q(0) = x_0$, $q(\alpha + \beta) = x_0 + \alpha a + \beta b$. We denote by Q_n the set of such q's. Any $q \in Q_n$ (and more generally any piecewise linear curve) will be represented by the sequence of its vertices as follows

$$q \equiv \langle q_0, q_1, ... q_{2n} \rangle, \quad \text{where } q_0 \equiv x_0, \; q_{2n} = x_0 + \alpha a + \beta b$$

We set $Q=\bigcup_{n=1}^{\infty} Q_n$. Given $q \in Q_n$, we can write the differential equation and initial condition problem

$$\begin{aligned} \dot{y}(t) &= E(q(t),y(t)).\dot{q}(t) \\ y(0) &= y_0 \end{aligned} \qquad (1)$$

On each linear piece the differential equation has unique solution (as a consequence of the Lipschitz condition in y) and by glueing pieces together we get a unique continuous $y(t)$, $t\in[0,\alpha+\beta]$ called the "*lifting of the curve q with origin* y_0" denoted $y_q(t,y_0)$ or sometimes simply $y_q(t)$. Given two curves $C_1:[a_1, b_1] \to Z$, $C_2:[a_2, b_2] \to Z$ where Z is a given space, such that $C_1(b_1) = C_2(a_2)$ we can form the sum C_1+C_2: $[a_1, b_2] \to Z$ as follows : $(C_1+C_2)(t) = C_1(t)$ if $t\in [a_1, b_1]$ and $(C_1+C_2)(t) = C_2(t)$ if $t \in [a_2, b_2]$. We can also define $-C_1(t)=C_1(a_1+b_1-t)$. Thus a piecewise linear curve say $q = \{q_1,q_2,q_3,\ldots\}$ equals the sum of its linear pieces, namely $q = [q_1, q_2]+[q_2, q_3]+\ldots$ For liftings, we obviously have $y_{C_1+C_2}(\cdot,y_0)=y_{C_1}(\cdot,y_0)+y_{C_2}(\cdot,y_{C_1}(b_1,y_0))$

Let $L > 0$ be a given number, and choose U' as before. Then the Lipschitz condition implies that there is a constant say $T > 0$ such that, for any given piecewise-C^1 curve $C : [t_1, t_3] \to U'$ parametrized by arc lenght, and having lenght $t_2 - t_1 < L$ we have $|y_C(t_2,y_0) - y_0| < T(t_2 - t_1)$ for any $y_0 \in G$

Thus in particular, given $r>0$ we can shrink $R \subseteq U'$ so as to satisfy the following

$$|y_q(t,y_0) - y_0| < r \qquad \text{for all } t\in[0\ ,\ \alpha+\beta],\ q\in Q\ ,\ y_0\in G$$

By definition (See [5] pag 116) an *ε-approximate* solution to a differential equation $y'=f(t,y)$ where $f : U \to F$, U open in $\mathbf{R}\times F$ F a Banach space, is a differentiable map $\varphi : I \to F$, where $I \subseteq R$ is an open interval such that for $t\in I$ we have

$$\text{(i)} \qquad (t,\varphi(t)) \in U$$

$$\text{(ii)} \qquad |\varphi'(t) - f(t,\varphi(t))| \leqq \varepsilon$$

Lemma 1. Let $\varphi_i : I \to F$ be ε_i-approximate solutions of the equation $y' = f(t,y)$ for $i=1,2$. Let $x_i = \varphi_i(t_0)$ the initial values for φ_i, i=1,2. Then if f is k-Lipschitz in y and continuous in $(t,y)\in U$ we have for $t \in I$ the following

$$|\varphi_1(t) - \varphi_2(t)| \leqq |x_1 - x_2|e^{k|t-t_0|} + (\varepsilon_1+\varepsilon_2)\ \frac{e^{k|t-t_0|} - 1}{k}$$

Proof: See [5] pag 116.

In the following lemmas, x_0, y_0, H', will remain fixed while U, R, V, will be eventually shrinked so as to satisfy certain conditions.

Let $V \subseteq G$ be an open ball centered at $y_0 \in G$. Using our continuity assumptions (I),(L) and shrinking U,V if necesary, we can assume that $E(x,y).h$ and $DE(x,y).(h,E(x,y).h).1$ are bounded for

$(x,y) \in U' \times V$ and $h, l \in H'$ bounded. Besides we shall assume that the clousure $\bar{K}$ of the set $K = \{ y_q(t,y_0) : q \in Q,\ t \in [0\ ,\ \alpha+\beta]\}$ satisfies $\bar{K} \subseteq V$. This can be achieved using the continuity and Lipschitz conditions (I), and the observation before ***Lemma 1.***
A number of bounds, either constants like C, D, etc., or functions like $\varepsilon, \varepsilon_1$, etc., will appear along the line in the following lemmas. It is understood that they may depend on the choice of $U, V, R, y_0,\ x_0$, but apart of this, they are fixed.

Lemma 2 : Let $u, v \in R,\ v=u+h,\ h \neq 0,\ w \in V$. Set $\frac{h}{|h|} = X$ and let $y(t,w)$ be the solution of

$$\dot{y} = E(u+tX\ ,\ y(t)).X$$

$$y(0) = w$$

Let

$$\varphi(t,w) = y_0 + tE(u,w).X + \frac{t^2}{2} DE(u,w)(X, E(u,w).X).X$$

a) Then there is a continuous function $\varepsilon(t,X,u,w) > 0$ such that $|y(t,w) - \varphi(t,w)| < \varepsilon(t,X,u,w)t^2$ for all $0 \leq t \leq |h|,\ u \in R, w \in V$, provided that U, V are small. In particular ε is bounded provided that U, V are small.

b) Moreover if w varies on a compact set $\bar{K} \subseteq V$, then there is a continuous function $\bar{\varepsilon}(t)$ such that $\bar{\varepsilon}(t) \to 0$ as $t \to 0$ and $\varepsilon(t,X,u,w) <$

$\bar{\varepsilon}(t)$ for all $t \in [0\ ,\ |h|]$, $u \in R$, $w \in V$.

Proof: a) The Taylor's expansion of the solution $y(t) \equiv y(t,X,u,v)$ is $y(t) = \varphi(t) + t^2 \varepsilon(t,X,u,w)$. Let $A(u,v,X) = DE(u,v)(X,E(u,v).X).X$ Then the integral form of the remainder gives the following

$$\varepsilon(t,X,u,w) = \int_0^1 (1-s)[A(u+tsX\ ,y(ts),X) - A(u,w,X)]ds$$

Our continuity assumptions imply that ε is continuous, and therefore it is bounded provided that U,V are small.

b) It follows from the previous formula that $\varepsilon(t,X,u,w) \to 0$ as $t \to 0$ for each (X,u,w) . Since $(X,u,w) \in S^1 \times R \times \bar{K}$ which is compact we can find $\bar{\varepsilon}(t)$ having the required properties. ∎

Let $u \in R$, $w \in V$ and for each positive integer n let $B = u + \frac{\alpha}{n}a$, $C = u + \frac{\beta}{n}b$, $D = u + \frac{\alpha}{n}a + \frac{\beta}{n}b$ such that u,B,C,D are vertices of a rectangle contained in R. Set $\gamma = \{A,B,D\}$, $\gamma' = \{A,C,D\}$, piecewise linear maps parametrized by arc lenght. Set $y_B = y_\gamma(\frac{\alpha}{n}\ ,\ w)$, $y_D = y_\gamma(\frac{\alpha}{n}+\frac{\beta}{n}\ ,w)$, $y_C = y_{\gamma'}(\frac{\beta}{n}\ ,\ w)$, $y'_D = y_{\gamma'}(\frac{\alpha}{n}+\frac{\beta}{n}\ ,\ w)$. We want to estimate the difference $|y_D - y'_D|$.

Lemma 3: a) There is a constant C such that $|y_D - y'_D| \leq C\frac{1}{n^2}$

b) Assume that w happens to vary on a compact set $\bar{K} \subseteq G$ and conditions (L) is satisfied. Then for some $D(\frac{1}{n})$, $D(\frac{1}{n}) \to 0$ as $n \to \infty$, we have $|y_D - y'_D| \leq D(\frac{1}{n}) \frac{1}{n^2}$.

***Proof*:** a) Using the previous lemma with $v \equiv B$, $h = \frac{\alpha}{n} a$, $t \equiv \frac{\alpha}{n}$, $x = a$ we obtain,

$$y_B - w = \frac{\alpha}{n} E(u,w).a + (\frac{\alpha}{n})^2 \frac{1}{2} DE(u,w)(a,E(u,w).a).a + \varepsilon(\frac{\alpha}{n},a,u,w)(\frac{\alpha}{n})^2 \quad (1)$$

Using the previous lemma again with $u \equiv B$, $v \equiv D$, $w \equiv y_B$, $h \equiv \frac{\beta}{n} b$, $x \equiv b$ we obtain,

$$y_D - y_B = \frac{\beta}{n} E(B,y_B).b + (\frac{\beta}{n})^2 \frac{1}{2} DE(B,y_B)(b,E(B,y_B).b).b + \varepsilon(\frac{\beta}{n},b,B,y_B)(\frac{\beta}{n})^2 \quad (2)$$

We replace B, y_B from (1) on the second side of (2) and using the continuity of E and DE we obtain, after some rearrangements,

$$y_D - y_B = \frac{\beta}{n} E(u,w).b + (\frac{\beta}{n})^2 \frac{1}{2} DE(u,w)(b,E(u,w).b).b + \varepsilon_1(\frac{1}{n},u,w) \frac{1}{n^2} \quad (3)$$

Where ε_1 is continuous and therefore bounded if U, V are small. In a similar manner we can prove the following

$$y_C - w = \frac{\beta}{n} E(u,w).b + \left(\frac{\beta}{n}\right)^2 \frac{1}{2} DE(u,w)(b,E(u,w).b).b + \epsilon\left(\frac{\beta}{n},b,u,w\right) \left(\frac{\beta}{n}\right)^2 \quad (4)$$

and

$$y_{D'} - y_C = \frac{\alpha}{n} E(u,w).a + \left(\frac{\alpha}{n}\right)^2 \frac{1}{2} DE(u,w)(a,E(u,w).a).a + \epsilon_2\left(\frac{1}{n},u,w\right) \frac{1}{n^2} \quad (5)$$

Using (1),(3),(4) and (5) we can calculate $y_D - y_{D'}$ as follows

$$y_D - y_{D'} = \frac{\alpha\beta}{n^2} \frac{1}{2}\Big[DE(u,w)(b,E(u,w).b - DE(u,w)(a,E(u,w).a).a\Big] + \\ + \mu\left(\frac{1}{n},u,w\right) \frac{1}{n^2} \quad (6)$$

where μ is continuous and therefore bounded if U,V are small. From this the proof of part a) follows easily

b) Note that, for each $(u,w) \in R \times V$ we have $\mu(\frac{1}{n},u,w) \to 0$ as $n \to \infty$. Since $(u,w) \in R \times \bar{K}$ compact, we can find $D(\frac{1}{n}) > \mu(\frac{1}{n},u,w)$ and $D(\frac{1}{n}) \to 0$ as $n \to \infty$, satisfying the required condition.

∎

Let $q,q' \in Q_n$. They are *contiguous* if and only if they differ by one vertex only namely $q=p+\gamma+s$, $q'=p+\gamma'+s$ where $p=\{q_1,...q_{k-1}\}$, $\gamma=\{q_{k-1},q_k,q_{k+1}\}$, $\gamma'=\{q_{k-1},q'_k,q_{k+1}\}$, $s=\{q_{k+1},...q_{2n}\}$.

Let t_1=lenght p , $t_2-t_1 = \frac{\alpha}{n}+\frac{\beta}{n}$ =lenght γ = lenght γ'.Choose $t_3 \geqq t_2$

$t_3-t_2 \leqq$ lenght s.

Set $w_1 = y_p(t_1,y_0)$, $w_2 = y_\gamma(t_2-t_1,w_1)$, $w'_2 = y_{\gamma'}(t_2-t_1,w_1)$

$w_3 = y_s(t_3-t_2,w_2) = y_q(t_3,y_0)$, $w'_3 = y_s(t_3-t_2,w'_2) = y_{q'}(t_3,y_0)$

Lemma 4: There is a constant C_1 such that $|w_3 - w'_3| < C_1 \frac{1}{n^2}$

Proof : Using ***Lemma 3***,a) we have $|w_2 - w'_2| < C \frac{1}{n^2}$. Then applying ***Lemma 1*** to the *exact* solutions $y_s(t,w_2)$, $y_s(t,w'_2)$ with *different* initial data w_2,w'_2 we get

$$|w_3 - w'_3| \leqq |w_2 - w'_2| \, e^{k(t_3-t_2)}$$

Since $t_3 - t_2 \leqq \alpha+\beta$, we obtain

$$|w_3 - w'_3| \leqq C \, e^{k(\alpha+\beta)} \frac{1}{n^2} \equiv C_1 \frac{1}{n^2}$$

∎

Let $C_1,C_2 : [a , b] \to Z$, C_1,C_2 continuous , Z a metric space. We define as usual

$d(C_1,C_2) = \sup d(C_1(t) , C_2(t))$

Lemma 5: There is a constant C_2 such if $q, q' \in Q_n$ *satisfy* $d(q,q') < \delta$ and $q(t) = q'(t)$ for some $t = \bar{t}$ then

$$|y_q(\bar{t},y_0) - y_{q'}(\bar{t},y_0)| < C_2\ \delta$$

Proof: An elementary reasoning will show that there is a constant say F, such that the number of rectangles of the n-partition of R which lie "in between" the restrictions $q|[0\ ,\bar{t}]\equiv\bar{q}\ ,q'|[0,\bar{t}]\equiv\bar{q}'$ is less or equal than $F\delta n^2$. This means that there is a sequence of at most $N < F\delta n^2$ elements $q^1,q^2,\ldots,q^N \in Q_n$ such that q^i,q^{i+1} are contiguous for $i=1,\ldots,N-1$, $q^1\equiv q$, $q^N\equiv q'$ and $q^i(\bar{t}) = q(\bar{t})$ for $i=1,\ldots,N$. Using ***Lemma 4*** repeatedly with $\bar{t}\equiv t_3$ we can easily show that $C_2 = C_1F$ satisfies the requirement of the lemma.

∎

Lemma 6: There is a constant C_3 such that for $q,q' \in Q$ we have

$$d(q,q') < \delta \quad \text{implies} \quad d(y_q,y_{q'}) < C_3\delta$$

Proof: Let $t_0 \in [0\ ,\ \alpha+\beta]$. An elementary reasoning shows that we can find piecewise linear maps $\xi,\eta : [0\ ,\ \Delta] \rightarrow R$ satisfying the following conditions, where F_1 is an appropriate constant.

i) $\xi(\Delta) = \eta(\Delta)$, $\xi(0) = q(t_0)$, $\eta(0) = q'(t_0)$
$\Delta = lenght\ \xi = lenght\ \eta < F_1\delta$

ii) Each linear piece of ξ and η is contained in some side of some rectangle of the n-partition of R.

Let $p=q|[0,t_0] + \xi$, $\Gamma = q'|[0,t_0] + \eta$ .Then $p(t_0+\Delta) = \Gamma(t_0+\Delta)$. Then we can apply the previous lemma with $\bar{t} = t_0+\Delta$ and we obtain

$$|y_q(\bar{t},y_0) - y_\Gamma(\bar{t},y_0)| < C_2\delta$$

Using ***Lemma 1*** (or more directly, the uniform Lipschitz condition on y) we can find a constant, say F_2, such that $|\xi(\Delta) - \xi(0)|$, $|\eta(\Delta) - \eta(0)| < F_2\delta$. From this and the previous inequality we finally obtain

$$|y_q(t_0,y_0) - y_{q'}(t_0,y_0)| < (2F_2 + C_2)\delta \equiv C_3\delta$$

∎

Lemma 7: The set $K = \left\{ y_q(t,y_0) \ : \ q\in Q \ , \ t \in [0 \ , \ \alpha+\beta] \right\}$ is relatively compact as a subset of G

Proof: The family (of continuous functions from $[0 \ , \ \alpha+\beta]$ into R) Q is totally bounded as a consequence of Arzelá's theorem. Then using the previous lemma, we can show that $\{y_q\}_{q \in Q}$ is totally bounded as well. Finally, the evaluation map

$$\{y_q\}_{q\in Q} \times [0 \ , \ \alpha+\beta] \to G$$
$$(y_q,t) \to y_q(t)$$

can be continuously extended to the closure of its domain which is compact. Since K is contained in the image of this extended map, it is relatively compact.

∎

Lemma 8: Let the same situation as in ***Lemma 4***, and assume that condition (L) is satisfied. Then there is a function $\nu(\frac{1}{n})$ such that $\nu(\frac{1}{n}) \to 0$ as $n \to \infty$ and $|w_3 - w_3'| < \nu(\frac{1}{n}) . \frac{1}{n^2}$

Proof: Using ***Lemma 7***, we can show that $w_1, w_2, w_2', w_3, w_3'$ all vary on a compact set $\bar{K}$. Therefore using ***Lemma 3 b)*** we see that $|w_2 - w_2'| < D(\frac{1}{n}) \frac{1}{n^2}$ .Argueing like in the last part of the proof of ***Lemma 4*** we can show that

$$|w_3 - w_3'| \leq |w_2 - w_2'| \, e^{k(\alpha+\beta)} \leq e^{k(\alpha+\beta)} D(\tfrac{1}{n}) \frac{1}{n^2} \equiv \nu(\tfrac{1}{n^2}) \frac{1}{n^2}$$

∎

Proof of the theorem : Let $q \equiv q^1, q^2, \ldots, q^N \equiv q'$, where $N = n^2$ is a sequence of elements of Q_n such that q^i, q^{i+1} are contiguous for $i=1,\ldots N-1$ and q lies in the sum of the two edges $[x_0 , x_0+\alpha a]+ [x_0+\alpha a , x_0+\alpha a+\beta b]$ of R while q' lies in the sum of the remaining edges $[x_0 , x_0+\beta b] + [x_0+\beta b , x_0+\alpha a+\beta b]$. Using the previous lemma we have

$$|y_q(\alpha+\beta) - y_{q'}(\alpha+\beta)| \leq |y_{q'}(\alpha+\beta) - y_{q^2}(\alpha+\beta)| + |y_{q^2}(\alpha+\beta) - y_{q^3}(\alpha+\beta)| + \ldots + |y_{q^{N-1}}(\alpha+\beta) - y_{q'}(\alpha+\beta)| \leq \nu(\tfrac{1}{n})$$

Since $\nu(\frac{1}{n}) \to 0$ as $n \to \infty$ we obtain $\quad y_q(\alpha+\beta\ ,\ y_0) = y_{q'}(\alpha+\beta\ ,\ y_0)$

Now we define the map S appearing in the statement of the theorem.For this,given R as before,define $S_R : R \to G$ as follows. For any $x_0+\lambda a+\mu b=x \in R$, let $q = \{x_0, x_0+\lambda a, x\}$, $q' = \{x_0, x_0+\mu b, x\}$ be piecewise linear maps (having two linear pieces each) contained in R. So far we have proved the following. Let $x_0 \in U'$, $U'=x_0+H'\cap U$ $y_0 \in G$ and assume that (I),(L) are satisfied.Then there exist $\alpha,\beta>0$ and $R\equiv R(x_0, a, b, \alpha, \beta) \subseteq U'$ as before such that, for each $x=x_0+\lambda a+\mu b$ $\in R$ we have

$$y_q(\lambda+\mu\ ,\ y_0) = y_{q'}(\lambda+\mu, y_0)$$

where $q=\{x_0\ ,\ x_0+\lambda a\ ,\ x\}$, $q'=\{x_0\ ,\ x_0+\mu b,\ x\}$

Define $S_R : R \to G$ by $S_R(x) = y_q(\lambda+\mu\ ,\ y_0) = y_{q'}(\lambda+\mu, y_0)$. It follows that

$$\frac{\partial S_R(x)}{\partial \lambda} = E(x, S(x)).a \quad , \quad \frac{\partial S_R(x)}{\partial \mu} = E(x, S(x)).b$$

Consequently S_R is of differentiability class C^1 and therefore for every C^1 curve $x(s) = x_0 + \lambda(s)a + \mu(s)b \in R$ we have $\frac{d}{ds}S_R(x(s)) = E(x(s)\ ,\ S_R(x(s))).\dot{x}(s)$. In particular for $x(s)=x+sh$ we obtain

$$\frac{d}{ds}\Big|_{s=0} S_R(x+sh) = E(x\ ,\ S_R(x)).h$$

This implies that for any piecewise-C^1 curve $\gamma(s)$ on R such that $\gamma(o) = x_0$ we have

$$y_\gamma(s\ ,\ y_0) = S_R(\gamma(s))$$

We can obviously find a finite family of rectangles

$R_I \equiv R(x_O, a_I, b_I, \alpha_I, \varepsilon_I)$ $a_I, b_I \in H'$ such that its union covers a ball centered at x_O, say $D(x_O) \subseteq U'$. By glueing the S_{R_I} togheter we can find $S_{D(x_O)}: D(x_O) \to G$ satisfying the Frobenius differential equation and initial condition

$$DS_{D(x_O)}(x).h = E(x, S_{D(x_O)}(x)).h$$
$$S_{D(x_O)}(x_O) = y_O$$

for all $x \in D(x_O)$, $h \in H'$

Next we shall extend $S_{D(x_O)}$ to a map $S_{U'}: U' \to G$ satisfying the same differential equation for $x \in U'$ and initial condition. Choose x_O coinciding with the center of the ball $U \equiv U_{r_O}$ of radius r_O. Let $U'_r = U_r \cap H'$ be the ball having the center x_O and radius $r < r_O$ in the space H', and let $\bar{U}'_r$ be its closure. Assume that $S_{D(x_O)}$ has an extension $S_{U'_r}: U'_r \to G$ satisfying the required conditions for some r. Using the Lipschitz condition we can show that $S_{U'_r}$ has a continuous extension $S_{\bar{U}'_r}$ to the closure $\bar{U}'_r$. For each $x'_O \in \partial\bar{U}'_r$ find $D(x'_O)$ and $S_{D(x'_O)}: D(x'_O) \to G$ satifying the Frobenius differential equation and the initial condition: $S_{D(x'_O)}(x'_O) = S_{\bar{U}'_r}(x'_O)$. By compactness of $\partial\bar{U}'_r$ we can find a finite number of such $D(x'_0)$, covering $\partial\bar{U}'_r$ and glueing all the corresponding $S_{D(x'_0)}$ and $S_{U'_r}$ together , we find an extension $S_{U'_{r+\delta}}$ for some $\delta>0$. From this we can easily conclude that there is an extension $S_{U'}$ where $U'=U\cap x_0+H'$ with the originally given U.

Finally let $H',H''\subseteq H$ be given bidimensional subespaces and let $U'=(x_0+H')\cap U$, $U''=(x_0+H'')\cap U$. Let $x\in U'\cap U''$ and $C(s)=x_0+s(x-x_0)$ Then $y_C(s,y_0) = S_{U'}(C(s)) = S_{U''}(C(s))$. This shows that we can coherently define $S:U\to G$ using its restrictions to each $U' = (x_0+H')\cap U$ where H' is an arbitrary bidimensional subespace of H, namely $S|U' = S_{U'}$.

∎

With the notation introduced in the previous theorem,we have as a corollary the following result:

Theorem 2.For each n,let $\gamma(t)$ be the boundary of the rectangle R parametrized with arc length $t\in[0\ ,\ 2\ \frac{\alpha+\beta}{n}\ ]$,such that

$\gamma(0) = \gamma(2\ \frac{\alpha+\beta}{n}) = x_0$, $\gamma(\frac{\alpha}{n}) = x_0+\frac{\alpha}{n}a$, $\gamma(\frac{\alpha+\beta}{n})= x_0+\frac{\alpha}{n}a+\frac{\beta}{n}b$

$\gamma(2\ \frac{\alpha}{n} + \frac{\beta}{n}) = x_0+\frac{\beta}{n}b$

Define

$$\omega(x_0,y_0)(a,b) = \frac{1}{2} \lim_{n\to\infty} \frac{y_\gamma(2(\frac{\alpha}{n} + \frac{\beta}{n}) \ ,y_0) - y_\gamma(0,y_0)}{\frac{\alpha\beta}{n^2}}$$

Set $y_0=y_\gamma(0)$. Then we have

$\omega(x_0,y_0)(a,b)$ =

$$\frac{1}{2}\Big[DE(x_0,y_0)(b,E(x_0,y_0).b).a - DE(x_0,y_0)(a,E(x_0,y_0).a).b\Big]$$

***Proof*:** The result follows directly from of formula (6).

2.- EXAMPLES

A. Inverse Problem in the Calculus of Variations

Let $H = \{ q \in C^{\infty}([t_1, t_2], \mathbf{R}^n) : q(t_i)=q_i, \ i=1,2 \}$ where q_i are fixed elements of $\mathbf{R}^n$ for $i=1,2$. Let $O \in H$, $O(t)=q_1 + \frac{t-t_1}{t_2-t_1}(q_2 - q_1)$

By replacing $q \in H$ by $q - O$ we can assume without any loss of generality that $q_i = O$, $i=1,2$. Thus the affine space H becomes a vector space where the neutral element is O. This minor change makes the situation exactly as in ***Theorem 1***.

Let $L : T\mathbf{R}^n \to \mathbf{R}$ be a given Lagrangian of differentiability class C^2. The Euler-Lagrange operator is by definition

$$E(q).h = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}(q,\dot{q}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q,\dot{q}) \right].h \, dt \qquad \text{where } q,h \in H$$

This is an example of the situation described in §1, where $U \equiv H$ $G \equiv \mathbf{R}$, $x \equiv q$ and $E(q,y).h \equiv E(q).h$ is *independent of* y. The Inverse Problem in the Calculus of Variations consists in finding necessary and sufficient conditions to ensure existence of L, for a given $E(q).h$. It can be shown that this is in turn equivalent to existence of a potential $S: U \to \mathbf{R}$ such that $DS(q).h = E(q).h$

It is interesting to notice that in this example, the integrability condition (L) becomes the usual *self-adjointness condition* (See[9], [14], [16]) namely

$$DE(q).h.k = DE(q).k.h$$

This is known to be the necessary and sufficient condition for existence of a potential. Checking our hypothesis (I),(L) for a given operator $E(q).h \in \mathbf{R}$, $(q,h) \in H \times H$ becomes a routine task . Let $x_0, a, b \in H$ and write $q = x_0 + \lambda a(t) + \mu b(t)$, $t \in [t_1 , t_2]$. Similarly $h = \alpha a + \beta b$ (or $h(t) = \alpha a(t) + \beta b(t)$, $t \in [t_1 , t_2]$) and $k = \gamma a + \delta b$.

This way $E(q).h$, $DE(q).h.k$, $DE(q).k.h$ become continuous functions on $\lambda, \mu, \alpha, \beta, \gamma, \delta \in \mathbf{R}$ and the integrability condition is also checked very easily.

The conclusion is that standard theorems on existence of potential operators ([16]) under a self-adjointness condition are partiticular cases of a version of Frobenius Theorem. This also opens the door for some kind of generalization of the Inverse Problem.

B.Constrained Lagrangian Systems

Let us assume for simplicity $dim\ H = n$, $dim\ G = m$, E of class C^1. Let $L : T(U \times G) \to R$ be a given Lagrangian and let us interpret E as a (time independent) constraint (See [4]). A system is called *holonomic* or *non-holonomic* according to whether the imposed constraints are integrable or not.

A curve $P(t) = (q(t), y(t)) \in U \times G$ is *compatible with the constraint* E if $\dot{y}(t) = E(q(t), y(t)).\dot{q}(t)$ (i.e. if $y(t)$ is the lifting of $q(t)$ with $y(t_0) = y_0$, for some y_0). Choose variations $P(t,\lambda) = (q(t,\lambda) , y(t,\lambda))$ such that for each fixed t, $y(t,\lambda)$ is the lifting of $q(t,\lambda)$ with initial condition $y(t)$, and $q(t_i,\lambda) = q(t_i)$, $i=0,1$, $y(t_0,\lambda) = y_0$, $y(t_1,\lambda) = y_1$ are fixed.

Remark. A different kind of variations is also interesting. Namely $(q(t,\lambda),\tilde{y}(t,\lambda))$, where for each λ , $\tilde{y}(t,\lambda)$ is the lifting of $q(t,\lambda)$ with fixed origin y_0 . Thus in general $\tilde{y}(t_1,\lambda) \neq \tilde{y}(t_1,0) = y(t,a)$.

Each vector $\frac{\partial P}{\partial\lambda} \equiv \delta P \in T_{P(t)}U{\times}G$ compatible with the constraint is called a *virtual velocity*, and D'Alambert-Lagrange Principle establishes that $P(t)$ is a motion if and only if it is a critical point of

$$\mathcal{L}(P) = \int_{t_0}^{t_1} L(P(t),\dot{P}(t))\; dt$$

with respect to variations δP of P compatible with the constraints (*virtual displacements*). This is equivalent to

$$\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\,\frac{\partial L}{\partial\dot{q}}\right]\delta q + \left[\frac{\partial L}{\partial y} - \frac{d}{dt}\,\frac{\partial L}{\partial\dot{y}}\right]E.\delta q = 0$$

$$\dot{y} = E(q\ ,\ y).\dot{q}$$

for all δq such that $\delta q(t_i) = 0, i=0,1.$

It is well known that for holonomic constraints we can simply restrict L to the integral manifold of E,obtaining an equivalent restricted unconstrained system . This is in turn equivalent to finding curves $(q(t),y(t))$ such that

$$0 = \frac{\partial}{\partial\lambda}\int_{t_0}^{t_1} L(q(t,\lambda),y(t,\lambda),\frac{\partial q}{\partial t}(t,\lambda),E(q(t,\lambda),y(t,\lambda)).\frac{\partial q}{\partial t}(t,\lambda))dt$$

The later is no longer true for non-holonomic constraints.However by using our two-form ω and expanding the previous equality we get the following general formula

$$\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right]\delta q + \left[\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}\right]E.\delta q = -2\frac{\partial L}{\partial \dot{y}}\omega(\dot{q},\delta q)$$

Thus restriction of the Lagrangian to a nonholonomic constraint is equivalent to adding an external force.If $n=2,m=1$ then $2\omega(\dot{q},.)$ looks like a Coriolis force.

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