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INSTITUTO DE MATEMATICA DE BAHIA BLANCA

INMABB (UNS - CONICET)

UNIVERSIDAD NACIONAL DEL SUR

Avda. ALEM 1253 - 8000 BAHIA BLANCA

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A REMARK ON SELF-SIMILAR SETS

by A. Benedek and R. Panzone

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Summary. The Hausdorff dimension s of a self-similar set E as defined by J. E. Hutchinson ([H]) is a local property of the set. In this paper we prove that analogously its Hausdorff measure $H^s(E)$ can be determined looking at the construction process in an arbitrarily small neighborhood of any of its points.

1.Introduction. a) We consider in this paragraph a generalization of the Cantor construction in the real line as it is described in Falconer's book ([F],pp.14-19).

Let $m\geq 2$ be a fixed integer, r and d positive real numbers verifying $mr + (m-1)d = 1$. Given a closed interval $T=[a,b]$, $\mathfrak{D}(T)$ denotes the set of m equal and closed subintervals of T of length $r(b - a)$ with the same spacing $d(b - a)$ between consecutive intervals.

Starting with $T=[0,1]$, we call pieces (net intervals) of order one the intervals of $\mathfrak{D}(T)$, E_1 denotes its union. We define inductively the pieces of order j + 1 as the intervals belonging to $\mathfrak{D}(T)$ when T runs over all the pieces of order j. Let E_{j+1} be their union. Obviously $E_j=\overline{E}_j\supset E_{j+1}$. Let us define $E:=\cap E_j$ and s by $mr^s=1$. Since $r<1/m$, it follows that $0<s<1$. We want to show that E is an s-set, precisely, that $H^s(E)=1$. The family $\{I_h\}$ of the m^j pieces of order j covers E and each I_h has length r^j. Therefore, if we denote by $|I|$ the diameter of I we have

(1) $\sum_h |I_h|^s = m^j (r^s)^j = 1,$

and in consequence, $H^s(E) \leqslant 1$. Let us see that $H^s(E) \geqslant 1$. The concave function $\Psi(x) := (x(1 + 1/q) + 1)^s$, $q=r/d$, defined on $[0,m-1]$ verifies $\Psi(0)=1$, $\Psi(m - 1) = m$. In fact, the last relation follows from $mr^s=1=(mr + (m-1)d)^s$. Then,

(2) $x + 1 \leqslant \Psi(x) = (x + 1 + x/q)^s, \quad 0 \leqslant x \leqslant m-1.$

Let $\{C_j\}$ be a covering of E by open intervals. We must prove that $\sum |C_j|^s \geqslant 1$. We may assume that the covering is finite. There is a k such that any piece of order k is contained in some C_j. This follows from Lebesgue's lemma. Given C_j, we call O_j the least closed interval that contains all the pieces of order k that are entirely included in C_j. Then $\{O_j\}$ is a covering of E_k by closed intervals spanned by pieces of order k. Let us fix j and call $O=O_j$. If O is not a piece of order k then the pieces of order k in O are separated by "lacunas" that are open intervals of length d, dr, dr^2,..., or dr^{k-1}. Let $\{L_i\}$, $i=1,2,\dots,p \leqslant m-1$, the lacunas of equal, maximal length contained in O. They divide O in p + 1 closed intervals J_h such that

(3) $|J_h|/|L_1| \leq r/d = q.$

Then, $\Sigma|J_h| \leq (p+1)q\,|L_1|$. Since $|L_i|=|L_1|$, it follows that

(4) $|O|=\Sigma|J_h| + p|L_1| \geq (\Sigma|J_h|)\cdot(1 + p/(p+1)q).$

From this and Jensen's inequality we obtain

(5) $$\frac{\Sigma|J_h|^s}{p+1} \leq \left(\frac{\Sigma|J_h|}{p+1}\right)^s \leq \left(\frac{|O|}{p+1+p/q}\right)^s.$$

It follows from (2) and (5) that

(6) $\Sigma|J_h|^s \leq |O|^s$

and therefore,

(7) $\Sigma|C_j|^s \geq \Sigma|O_j|^s \geq \Sigma|J_{j,h}|^s.$

$\{J_{j,h}\}$ is again a covering of E_k by intervals with the properties that characterize the intervals O_j. We arrive after repeating this procedure a finite number of times to a covering of E_k made up by pieces of order k for which (7) holds. Thus, because of (1), $\Sigma|C_j|^s \geq 1$, qed.

b) Let $\gamma(O)$ denote the quotient of the number of pieces of order k contained in O divided by m^k= total number of such pieces. Then, $\gamma(O)=\Sigma\gamma(J_h)$ and it follows from (6) that

$$\frac{|O|^s}{\gamma(O)} \geq \inf_h \frac{|J_h|^s}{\gamma(J_h)}.$$

That is, if O contains *more than one* piece of order k *the quotient* $|O|^s/\gamma(O)$ decreases when O is replaced by a certain J. If this one coincides with a piece of order k, we get

$$(8) \quad \frac{|J|^s}{\gamma(J)} = \frac{(r^k)^s}{1/m^k} = 1.$$

In other words,

$$(9) \quad \inf |O|^s/\gamma(O) = H^s(E).$$

We shall show that (9) holds even for E an arbitrary self-similar set.

2. *Self-similar sets.* Assume that the family $\{\Psi_i(x): i=1,\ldots,m\}$ of similitudes in R^n, i.e., $\Psi_i(x)=c_i + r_i L_i(x)$, $0<r_i<1$, L_i an orthogonal map, satisfies the open set condition. This means that there exists a bounded open set $V\neq\phi$ such that $\Psi(V) := \bigcup_{i=1}^{m} \Psi_i(V)\subset V$, $\Psi_i(V)\cap\Psi_j(V)=\phi$ if $i\neq j$. Hutchinson's theorem ([H]) asserts that there exists a (unique) compact set E invariant under $\Psi:\Psi(E)=E$, such that $0<H^s(E)\leqslant|E|^s\leqslant|V|^s<\infty$ where s is determined by $\sum_{i=0}^{m} r_j^s=1$. Moreover, $E=\bigcap_{k=1}^{\infty} \Psi^k(\bar{V})$, $\Psi^k=\underbrace{\Psi\circ\cdots\circ\Psi}_{k}$. (Since $\Psi(\bar{V})\subset\bar{V}$, $\{\Psi^k(\bar{V})\}$ is a decreasing sequence of compact sets). Besides, it is self-similar, that is, $H^s(\Psi_i(E)\cap\Psi_j(E))=0$ for $i\neq j$, (cf. also [F] ch. 8). For such a family of similitudes and any set F we write $F_{j_1\ldots j_k}:= \Psi_{j_1}\circ\cdots\circ\Psi_{j_k}(F)$ and $(j):=j_1\ldots j_k$ when no confusion can arise. In this case we write $F_{(j)}$ for $F_{j_1\ldots j_k}$. Supose that A is a convex closed set of diameter $<\delta$ that covers $N>0$ pieces $\bar{V}_{j_1\ldots j_M}$ of the step M of the construction process: $\bar{V}_{j_1\ldots j_M}\subset\Psi^M(\bar{V})$.

Let $\tau=\tau(M,A)$ be the family of indices $(j)=j_1\ldots j_M$ such that $\bar{V}_{(j)}\subset A$.

Definition 1. $\rho_M(A) := \Sigma\{(r_{j_1}\ldots r_{j_M})^s : (j)\ \varepsilon\ \tau\}$.

Then $0 < R_M := 1-\rho_M < 1$.

Lemma 1. $H^s(E)\,.\,\rho_m(A)\leqslant|A|^s$.

Proof. Recall that $\bar{V}_{(i)\cdots(r)}\subset\bar{V}_{(i)}\subset A$ when $(i)\ \varepsilon\ \tau$.

Therefore, the union of the sets $A, \{A_{(i)} : (i)\overline{\varepsilon}\tau\}, \ldots,$ $\{A_{\underbrace{(i)\ldots(k)}_{v}} : (i)\overline{\varepsilon}\tau, \ldots, (k)\overline{\varepsilon}\tau\}$ cover all pieces of step $(v+1)M$ except those of the form $\overline{V}_{\underbrace{(i)\ldots(j)}_{v+1}}$ with $(i)\overline{\varepsilon}\tau, \ldots,$ $(j)\overline{\varepsilon}\tau$. These have also diameter less than δ for v large enough. Hence

$$H^s_\delta(E) \leqslant |A|^s + \Sigma\{|A_{(i)}|^s : (i)\overline{\varepsilon}\tau\} + \ldots +$$
$$+ \Sigma\{|A_{(i)\ldots(j)}|^s : \underbrace{(i), \ldots, (k)}_{v}\overline{\varepsilon}\tau\} +$$
$$+ \Sigma\{|\overline{V}_{(i)\ldots(\ell)}|^s : \underbrace{(i), \ldots, (\ell)}_{v+1}\overline{\varepsilon}\tau\}.$$

We write R, ρ for $R_M(A)$, $\rho_M(A)$ and obtain

$$H^s_\delta(E) \leqslant |A|^s(1+R+\ldots+R^v) + |\overline{V}|^s R^{v+1} \leqslant$$
$$\leqslant |A|^s(1 - R^{v+1})/\rho + |V|^s R^{v+1}.$$

Letting v tend to infinity, we arrive to the inequality

(10) $H^s_\delta(E) \leqslant |A|^s/\rho.$

If instead of A we begin with $A' = A_{j_1 \ldots j_Q}$ and instead of M we use $M' = M + Q$ we shall have $\delta' = \delta r_{j_1} \ldots r_{j_Q} < \varepsilon$ for Q great enough. Therefore, if $\rho' := \rho_{M+Q}(A')$, from (10) it follows that

(11) $H^s_\varepsilon(E) \leqslant |A'|^s/\rho' = (r_{j_1} \ldots r_{j_Q})^s |A|^s/\rho'.$

But τ' contains all the $M+Q$-tuples of the form $(j_1 \ldots j_Q)(i)$, $(i) \varepsilon \tau$. In consequence, $\rho' \geqslant (r_{j_1} \ldots r_{j_Q})^s \rho$. In view of (11) we finally get $|A'|^s/\rho' \leqslant |A|^s/\rho$, and the lemma follows, QED. Observe that if we replace A by a convex open set D, without changing other conditons, the lemma still holds with the same definitions for τ and ϱ.

Let $\mathfrak{F}(M)$ be the family of convex open sets of diameter $<\delta\leq 1$ that contain some piece $\overline{V}_{(i)}$ of a step M in the construction of E.

Definition 2. $f := \inf_M \inf\{|D|^s/\varrho_M(D) : D\in\mathfrak{F}(M)\}$.

Theorem 1. $H^s(E)=f$

Proof. It remains to prove that $H^s(E)\geq f$. Assume that, for a given $\varepsilon>0$ and $\delta>0$, $\{D_j\}$ is a countable covering of E by open convex sets such that $|D_j|<\delta$ and

(12) $\Sigma|D_j|^s< H^s(E) + \varepsilon$.

We may assume without loss of generality that the covering is finite. It follows from Lebesgue's lemma that there is an M such that any piece of step M is contained in some D_k.

Then

(13) $$1=\Sigma(r_{i_1}\cdots r_{i_M})^s\leq\sum_k \sum_{(j)\in\tau(M,D_k)} (r_{j_1}\cdots r_{j_M})^s= =\sum_k \varrho_M(D_k).$$

Since the definition of f implies $f\cdot\varrho_M(D_k)\leq|D_k|^s$, we obtain from (12) and (13): $f<H^s(E) + \varepsilon$, and the theorem follows, QED.

3. Final comments. a) Denoting with q(D,M) the quotient $q:=|D|^s/\varrho_M(D)$, we have already observed that

$q(D,M) \geqslant q(D_j, M+Q)$, where Q is the length of (j). Let υ be a neighborhood of $x \in E$. Then there exists $\overline{V}_{(j)}$ such that $x \in \overline{V}_{(j)} \subset \upsilon$. Assuming that the length of (j) is great enough, we also have $D_{(j)} \subset \upsilon$. This means that f can be calculated just looking at the open convex sets D contained in υ. Therefore, if we think of E as associated to a construction process, $H^s(E)$ becomes a kind of local property of the set.

b) We consider next the important particular case when $r_i = r$, $i=1,\ldots, m$. Then, $mr^s = 1$ and $\rho_M(D) = N(r^M)^s = N/m^M$. Thus $\rho_M(D)$ coincides with the quantity that was denoted by γ in the introduction, i.e., with the quotient of the number of M-pieces in D by the total number of M-pieces. In this situation, $\rho_M(D)$ increases with M for D fixed and if we call

$$L(D) := \inf_M |D|^s/\rho_M(D) = \lim_{M\to\infty} |D|^s/\rho_M(D),$$

we have

(13). $$H^s(E) = \inf_D \{L(D) : D \text{ an open convex set, } D \cap E \neq \phi\}.$$

c) Koch's curve K constructed from V, the open triangle determined by the points $A=(0,0)$, $B=(1,0)$, $C=(1/2, 1/2\sqrt{3})$, that has diameter equal to one and spans the interval

$[A,B]$, has dimension $s=\log 4/\log 3$ and verifies $H^s(E) \leqslant |V|^s=1$.

But the disc A of diameter $2/3^{3/2}$ and center

$(1/2, (\operatorname{sen}\pi/3)/9)$ contains two pieces of step 1. From lemma

1 and (13) it follows that

$$(14) \quad H^s(K) \leqslant \frac{(2/3^{3/2})^s}{2/4} = (2/3)^s < 1 .$$

d) Let E be the set of numbers between 0 and 1 that contain no even digit in their decimal expression. Then

$E = \bigcup_{i=1}^{5} \Psi_i(E)$ where $\Psi_i(x) = \frac{2i-1}{10} + \frac{x}{10}$.

Ψ_i satisfy the open set condition with $V=(1/9,1)$.

Hutchinson's theorem asserts that E is an s-set with Hausdorff dimension $s=\log 5/\log 10$. E can be obtained by the generalized Cantor construction process described in the introduction with $m=5$, $r=8/90$, starting with the interval $T=[1/9,1]$.

The set $E'=\varphi(E)$, $\varphi(x)=(9/8)x - 1/8$, is a generalized Cantor set fitting the mentioned construction process with $m=5$, $r=1/10$, $T'=[0,1]$. Therefore $H^s(E')=1$.

In consequence $H^s(E)=(8/9)^s \cdot H^s(E')=(8/9)^s$.

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Departamento e
Instituto de Matemática
Universidad Nac. del Sur
8000 Bahía Blanca