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On the measure of self-similar sets

por

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ON THE MEASURE OF SELF-SIMILAR SETS

by Pablo A. Panzone

Abstract. We exhibit a method by which we can approximate the Hausdorff measure of a certain class of self-similar sets .

0. Introduction. In 1. we show a procedure for approximating the measure of a certain kind of self-similar sets. In 2. we use these methods to show that if K is the Koch curve then $0.26 \le \mathcal{H}^s(K) \le 0.598$ (example 2). We calculate in an exact way the measure of certain self-similar sets in R^2 (example 1).

Despite the fact that we repeate arguments and use ideas borrowed from the works of Hutchinson [Hut] and Marion [Mar 1], on the whole the method shown seems to be new.

1. The Hausdorff metric is defined on the collection of all non empty compact subsets of R^n by

$$d_H(E,F)=\inf\{t: \quad F \subseteq [E]_t \text{ and } E \subseteq [F]_t \}$$

where $[E]_t = \{x \in R^n : \inf_{y \in E} \|x-y\| = d(x,y) \le t\}$ and $\|\cdot\|$ ($d(\cdot,\cdot)$) is the usual norm (distance). We shall write $F_j \xrightarrow{H} K$ instead of $d_H(F_j,K) \xrightarrow[j\to\infty]{} 0$.

We state here the well-known selection theorem due to Blaschke:

Let $\mathcal{F}$ be an infinite collection of non empty compact sets all lying in a bounded portion B of R^n . Then there exists a sequence $\{F_j\}$ of distinct sets of $\mathcal{F}$ convergent in the Hausdorff metric to a non-empty compact set K.

For a proof see page 37 of [Fal].

$|A|$ denotes the diameter of a set $A \subseteq R^n$ and $\mathcal{H}^s(\cdot)$ its s-Hausdorff measure.

A convex body is a compact convex set with non-empty interior.

We use ∎ to indicate the end of a proof.

The following is a corollary of Blaschke theorem.

Lemma 1: Let F_i be a sequence of compact convex non-empty sets of R^n such that

a) $\lim_{i\to\infty} |F_i| = \alpha > 0$

b) There exists a compact convex set F such that

$F_i \subseteq F$ for all i

Then there exists a subsequence F_{i_j} such that

i) $F_{i_j} \xrightarrow[H]{} K$, K compact and convex

ii) $|K| = \alpha$

iii) $K \subseteq F$

Proof: By the mentioned Blaschke selection theorem we know that there is a subsequence F_{i_j} such that $F_{i_j} \xrightarrow[H]{} K$ where K is a non-empty compact set. Obviously $K \subseteq F$. As $F_{i_j} \xrightarrow[H]{} K$, we have $d_H(F_{i_j}, K) < \epsilon_j$ with $\epsilon_j \to 0$. But then $K \subseteq [F_{i_j}]_{\epsilon_j}$ for all j (notice that $[F_{i_j}]_{\epsilon_j}$ are compact convex sets) and

$$|[F_{i_j}]_{\epsilon_j}| \to \alpha$$

Thus $|K| \leq \alpha$. Suppose that $|K| < \alpha$. Since $F_{i_j} \subseteq [K]_{\epsilon_j}$ we have $|F_{i_j}| \leq |[K]_{\epsilon_j}|$ which is absurd taking limits. This proves ii) and iii).

We now prove that K is convex. Observe that $[F_{i_j}]_{\epsilon_j}$ tends in the Hausdorff metric to K because

$$d_H(K,[F_{i_j}]_{\epsilon_j}) \leq d_H(K,F_{i_j}) + d_H(F_{i_j},[F_{i_j}]_{\epsilon_j})$$

Thus given $\epsilon > 0$ we have j_o such that

$$[F_{i_j}]_{\epsilon_j} \subseteq [K]_\epsilon \quad \text{if } j \geq j_o$$

Then $\bigcap [F_{i_j}]_{\epsilon_j} \subseteq K$. The inclusion $K \subseteq \bigcap [F_{i_j}]_{\epsilon_j}$ was already established. This finishes the proof of the lemma ∎

Let K be a compact set in R^n such that $\mathcal{H}^s(K) < \infty$ $(s > 0)$. Define for $\delta > 0$:

$$\mu(\delta) := \sup_{\substack{C_\delta \text{ compact convex} \\ \text{set of diameter } \delta}} \mathcal{H}^s(K \cap C_\delta)$$

This function is the key to measure self-similar sets.

Theorem 1: $\mu(\delta)$ is continuous from the right and non-decreasing and for any $\delta > 0$, $\mu(\delta) = \mathcal{H}^s(K \cap C^o_\delta)$ where C^o_δ is a particular compact convex set of diameter δ .

Morover if for any compact convex set C we have

$$\mathcal{H}^s(K \cap \partial C) = 0$$

then $\mu(\delta)$ is continuous.

Proof : From the definition of $\mu(\delta)$ we know that there exists a sequence C^i_δ of compact convex sets of diameter δ, all lying in a bounded portion of R^n, such that

$$\mu(\delta) = \lim_{i \to \infty} \mathcal{H}^s(K \cap C^i_\delta)$$

By lemma 1 there exists C^o_δ a compact convex set of diameter δ and a subsequence $C^{i_j}_\delta$ of C^i_δ such that

$$C^{i_j}_\delta \xrightarrow[H]{} C^o_\delta$$

But $\mathcal{H}^s(K \cap \overset{\circ}{C}_\delta) = \lim_{k \to \infty} \mathcal{H}^s(K \cap [\overset{\circ}{C}_\delta]_{1/2^k})$

and $C_\delta^{i_j} \subseteq [\overset{\circ}{C}_\delta]_{1/2^k}$ if i_j is large enough with k fixed.

Then $\mu(\delta) = \mathcal{H}^s(K \cap \overset{\circ}{C}_\delta)$.

From this one easily gets that $\mu(\delta)$ is non-decreasing.

Let $\delta_o > 0$ and $\delta_i > 0$; $i = 1,2,3,\ldots$, $\delta_i \dashrightarrow \delta_o$

Then

$$\mu(\delta_j) = \mathcal{H}^s(K \cap \overset{\circ}{C}_{\delta_j}) \qquad \text{if } j = 0,1,2,3,\ldots$$

with $\overset{\circ}{C}_{\delta_j}$ a compact convex set of diameter δ_j lying in a bounded portion of R^n.

From the sequence $\overset{\circ}{C}_{\delta_j}$ we can extract by lemma 1 a subsequence (we call this subsequence in the same way) such that

$$\overset{\circ}{C}_{\delta_j} \xrightarrow[H]{} \overset{\circ}{C}, \quad \text{where } \overset{\circ}{C} \text{ is a compact convex set of diameter } \delta_o$$

But $\mathcal{H}^s(K \cap \overset{\circ}{C}) = \lim_{i \to \infty} \mathcal{H}^s(K \cap [\overset{\circ}{C}]_{1/2^i})$ and

$\mathcal{H}^s(K \cap [\overset{\circ}{C}]_{1/2^i}) \geq \mathcal{H}^s(K \cap \overset{\circ}{C}_{\delta_j}) = \mu(\delta_j)$ if i is fixed and $j \geq j(i)$. Thus

$$\overline{\lim_{j \to \infty}}\, \mu(\delta_j) \leq \mathcal{H}^s(K \cap \overset{\circ}{C}) \leq \mu(\delta_o)$$

This proves that $\mu(\delta)$ is continuous from the right.

We show now that if C is any compact convex set and

$$\mathcal{H}^s(K \cap \partial C) = 0$$

then $\mu(\delta)$ must be continuous.

Recall $\mu(\delta_o) = \mathcal{H}^s(K \cap \overset{\circ}{C}_{\delta_o})$, $\overset{\circ}{C}_{\delta_o}$ a compact convex set of diameter δ_o .

If $\overset{\circ}{C}_{\delta_o}$ is not a convex body then it is easy to see that

$$\mu(\delta) = 0 \qquad \text{if } \delta \leq \delta_o$$

and therefore $\mu(\delta)$ is continuous from the left at δ_o.

Assume $\overset{\circ}{C}_{\delta_o}$ is a convex body.

Let $]\overset{\circ}{C}_{\delta_o}[_\epsilon = \{ x : d(x, R^n - \overset{\circ}{C}_{\delta_o}) > \epsilon \}$. Thus from the hypothesis we get

$$\mu(\delta_o) = \mathcal{H}^s(K \cap \overset{\circ}{C}_{\delta_o}) = \mathcal{H}^s(K \cap \partial \overset{\circ}{C}_{\delta_o}) + \mathcal{H}^s(K \cap \operatorname{int}(\overset{\circ}{C}_{\delta_o}))$$

$$= \mathcal{H}^s(K \cap \operatorname{int}(\overset{\circ}{C}_{\delta_o})) = \lim_{i \to \infty} \mathcal{H}^s(K \cap]\overset{\circ}{C}_{\delta_o}[_{1/2^i}).$$

This implies the continuity of $\mu(\delta)$ at δ_o ∎

A mapping $Y : R^n \to R^n$ is called a contraction if $\|Y(x)-Y(y)\| \leq k.\|x-y\|$ for all $x,y \in R^n$, where $0<k<1$. Clearly a contraction is a continuous function. A contraction that transforms every subset of R^n to a geometrically similar set is called a similitude. Thus a similitude is a composition of a dilatation, a rotation and a translation.

Let Y_i $i = 1,\ldots,m$ be a set of similitudes with contraction ratios k_i . We know that there exists a unique non-void compact set K such that

$$K = \bigcup_{i=1}^{m} Y_i(K)$$

(see [Fal]). We assume also the following (s is the Hausdorff dimension of K):

I) $0 < \mathcal{H}^s(K) < \infty \qquad (s > 0)$

II) $\mathcal{H}^s(Y_i(K) \cap Y_j(K)) = 0 \qquad \text{if } i \neq j$

Such a K will be called a self-similar set.

Notice that if K is a self-similar set then the following equality holds:

$$\sum_{i=1}^{m} k_i^s = 1.$$

By $\mathcal{C}(A)$ we denote the convex hull of a set A.

Let K be a self-similar set. It is clear that $Y_i(\mathcal{C}(K)) \subseteq \mathcal{C}(K)$ for all i. We rename the sets $Y_{i_1} \circ \dots \circ Y_{i_q}(\mathcal{C}(K))$ in the following way: $\mathcal{C}(K)$ is called T, $Y_i(\mathcal{C}(K))$ is called T_i, $Y_i \circ Y_j(\mathcal{C}(K)) = Y_i(Y_j(\mathcal{C}(K))) = T_{ij}$, etc. Fix $r \geq 1$. Take all $T_{i_1 \dots i_r}$ possible. This family has m^r elements and we call it G_r. Notice $Y_{i_1} \circ \dots \circ Y_{i_r} \circ Y_{i_{r+1}}(K) \subseteq T_{i_1 \dots i_r i_{r+1}} \subseteq T_{i_1 \dots i_r}$.

Property Z: Let K be self-similar. We say that K has property Z if there exists an index $i_1 \dots i_{r_0}$ such that

$$T_{i_1 \dots i_{r_0}} \subseteq \operatorname{int} \mathcal{C}(K)$$

Corollary 1: Let K be a self-similar set and let K have property Z

Then for any compact convex set C we have

$$\mathcal{H}^s(K \cap \partial C) = 0$$

and $\mu(\delta)$ is continuous.

Proof: If the hypotheses of the corollary hold for K then $\mathcal{C}(K)$ is a convex body and the following property is true: there exist $\epsilon_0 > 0$ and a natural number r_1 ($\geq r_0$, r_0 of property Z) such that for all convex compact sets C and all $t \leq \epsilon_0$ the set

$$[\partial C]_t = \{ p : d(p, \partial C) \leq t \}$$

does not intersect all elements of G_{r_1}.

The proof of this fact is as follows. Let r_1 be such that $r_1 \geq r_0$ and

(1) Max diameter of elements of $G_{r_1} = (\max k_i)^{r_1} \cdot |K| <$

$$< \quad d(\partial\mathcal{E}(K), T_{i_1 \ldots i_{r_0}})/2$$

Let $\epsilon_0 = (\max k_i)^{r_1} \cdot |K|/2$. Take all elements Γ of G_{r_1} such that $\Gamma \cap \partial\mathcal{E}(K) \neq \{\emptyset\}$. Call this set G'_{r_1} (notice $G'_{r_1} \neq \{\emptyset\}$)and observe that

(2) $$\mathcal{E}(\bigcup_{\Gamma \in G'_{r_1}} \Gamma) = \mathcal{E}(K)$$

Let C be a compact convex set and assume that $[\partial C]_{\epsilon_0}$ intersects all elements of G_{r_1} . For each set $\Gamma \in G'_{r_1}$ take a point $q_j \in \Gamma \cap [\partial C]_{\epsilon_0}$ Thus $\mathcal{E}(\bigcup_j q_j) \subseteq [C]_{\epsilon_0}$ and $\mathcal{E}(\bigcup_j q_j) \subseteq \mathcal{E}(K)$. But by **(1)** and **(2)** $\mathcal{E}(K) \subseteq [\mathcal{E}(\bigcup_j q_j)]_{2\epsilon_0}$. Thus we have that if $p \in \mathcal{E}(K)$, $p \notin \mathcal{E}(\bigcup_j q_j)$ then

(3) $$d(p, \partial\mathcal{E}(K)) \leq 2\epsilon_0 \quad *)$$

***)** Let C_1, C_2 be two compact convex sets such that $C_2 \subseteq [C_1]_\epsilon$ for some $\epsilon > 0$. If $p \in C_2$, $p \notin C_1$ then $d(p, \partial C_2) \leq \epsilon$

Proof: Let $q \in C_1$ be such that

(4) $$d(p,q) = d(p,C_1)$$

Obviously $d(p,q) \leq \epsilon$.Let $v_1 = (p-q)/d(p,q)$, v_2, ... , v_n be an orthonormal vector family. Notice that the ray $L = \{ v_1 t + q : t \in [0,\infty)\}$ intersects ∂C_2 . Therefore to prove *) it is only necessary to prove that $\partial C_2 \cap \{ v_1 t + q : t \in (\epsilon, \infty) \} = \{\emptyset\}$

Suppose this is not true ie. $v_1 t_0 + q \in \partial C_2$ with $t_0 > \epsilon$ From the hypothesis there exists $q' \in C_1$ such that $d(v_1 t_0 + q, q') \leq \epsilon$ But the function $g(t) := d(tq' + (1-t)q, p) = d(t(v_1 t_1 + \sum_{i=2}^{n} t_i v_i) + q, p)$ is continuously differentiable in a neighbourhood of zero.

Also observe that $t_1 > 0$ so $g'(0) < 0$. Thus there exists t', $0 < t' < 1$ such that

$$g(t') = d(t'q' + (1-t')q,p) < d(p,q)$$

and this contradicts **(4)**.

Therefore $T_{i_1 \dots i_{r_0}} \subseteq \operatorname{int} \mathcal{E}(\bigcup_j q_j)$ and by **(1)** and **(3)**

$$d(T_{i_1 \dots i_{r_0}}, \partial \mathcal{E}(\bigcup_j q_j)) > 2\epsilon_0$$

Using ***)** again one obtains $d(T_{i_1 \dots i_{r_0}}, \partial C) > \epsilon_0$ and therefore $[\partial C]_{\epsilon_0}$ cannot intersect $T_{i_1 \dots i_{r_0}}$.

The proof is completed if we notice that there are elements of G_{r_1} contained in $T_{i_1 \dots i_{r_0}}$.

We return to the proof of the corollary.

Let C be a convex compact set and $t > 0$. We define

$$W(t,C) := \mathcal{H}^s(K \cap [\partial C]_t)$$

Suppose $t < \epsilon_0$ then

$$\textbf{(5)} \qquad W(t,C) \leq \sum_{\substack{\text{all index } i_1 \dots i_{r_1} \\ \text{such that} \\ T_{i_1 \dots i_{r_1}} \cap [\partial C]_t \neq \{\emptyset\}}} \mathcal{H}^s(Y_{i_1}(\dots(Y_{i_{r_1}}(K))\dots) \cap [\partial C]_t)$$

But

$$\textbf{(6)} \qquad \mathcal{H}^s(Y_{i_1}(\dots(Y_{i_{r_1}}(K))\dots) \cap [\partial C]_t) =$$

$$= k_{i_1}^s \cdot \dots \cdot k_{i_{r_1}}^s \cdot \mathcal{H}^s(K \cap [\partial C^{i_1 \dots i_{r_1}}]_{t/k_{i_1} \dots k_{i_{r_1}}})$$

where $C^{i_1 \dots i_{r_1}}$ is a convex compact set. More precisely $C^{i_1 \dots i_{r_1}} = Y_{i_{r_1}}^{-1}(\dots(Y_{i_1}^{-1}(C))\dots)$.

Using (5), (6) and the proposition just proved we have

$$\left(\text{recall } 1 = \left(\sum_{i=1}^{m} k_i^s\right)^n = \sum_{\substack{\text{all} \\ n\text{-tuples}}} \underbrace{k_i^s \ldots k_j^s}_{n}\right)$$

$$W(t,C) \le \Big(\sum_{\substack{\text{all index} \\ i_1 \ldots i_{r_1} \text{ such that} \\ T_{i_1 \ldots i_{r_1}} \cap [\partial C]_t \neq \{\emptyset\}}} k_{i_1}^s \ldots k_{i_{r_1}}^s \Big) \cdot \mathcal{H}^s(K \cap [\partial C']_{t/(\min k_i)^{r_1}}) \le$$

$$\le \left(1 - (\min k_i)^{r_1 s} \right) . W(t/(\min k_i)^{r_1}, C')$$

where C′is some convex set, ie. we have proved that there exist $\epsilon_0 > 0$, r_1 a natural number and a fixed α, $0<\alpha<1$, such that for any C compact convex set and any $t < \epsilon_0$ there exists C′ a compact convex set such that

(7) $$W(t,C) \le \alpha . W(t/(\min k_i)^{r_1 s}, C')$$

Thus using (7) and the fact that $W(t,C) \le \mathcal{H}^s(K)$ for any C and $t>0$ we get

$$\lim_{t \to 0} W(t,C) = 0$$

∎

Now we define functions u,U and U which 'approximate' in some sense the function μ. For defining these functions we need other auxiliary functions.

Recall that G_r is the set of all $T_{i_1 \ldots i_r}$ possible with r (≥1) fixed.

Let $P(G_r)$ be all the subsets of the set G_r ({∅} is not included) . Define $J_r : P(G_r) \to R$ in the following way:

if $\{ T_{i_1 \ldots i_r} , \ldots , T_{j_1 \ldots j_r} \}$ is an element of $P(G_r)$ then

$$J_r (\{ T_{i_1 \ldots i_r} , \ldots , T_{j_1 \ldots j_r} \}) :=$$

$$=(k^s_{i_1} \circ \dots \circ k^s_{i_r}) + \dots + (k^s_{j_1} \circ \dots \circ k^s_{j_r})$$

It is not difficult to check that $J_r(P(G_r))$ is a finite set of points of R such that if $\alpha \in J_r(P(G_r))$ then $0<\alpha\le 1$, and $1 \in J_r(P(G_r))$. Also $J_r(P(G_r)) \subseteq J_{r+1}(P(G_{r+1}))$ for all $r \ge 1$. Besides, for all $\epsilon > 0$ there exists $r_o \ge 1$ such that for all $r \ge r_o$, if $x \in [0,1]$ then there exists $\alpha \in J_r(P(G_r))$ such that $|x-\alpha| < \epsilon$.

We will define functions H_r, h_r on the set $J_r(P(G_r))$, ie. $H_r, h_r : J_r(P(G_r)) \longrightarrow R$.

Let $\alpha \in J_r(P(G_r))$, we define

$$G^\alpha_r := J^{-1}_r(\alpha)$$

and

$$H_r(\alpha) := \min_{\beta \in G^\alpha_r} \left(\max_{\Gamma,\Gamma' \in \beta} |\Gamma \cup \Gamma'| \right) =$$

$$= \min_{\beta \in G^\alpha_r} (\text{ diameter of } \beta) \quad ;$$

$$h_r(\alpha) := \min_{\beta \in G^\alpha_r} \left(\max_{\Gamma,\Gamma' \in \beta} d(\Gamma, \Gamma') \right)$$

where $d(\cdot,\cdot)$ is the distance between sets. Remember that Γ,Γ' are elements of the form $T_{i_1 \dots i_r}$.

From the definitions of H_r and h_r it is clear that $h_r(\alpha) \le H_r(\alpha) \le |K|$ and $H_r(1) = |K|$. It is not difficult to see that $H_r(\alpha) - h_r(\alpha) < \epsilon$ for all $\alpha \in J_r(P(G_r))$ if r is big enough. Also $H_{r+1}(\alpha) \le H_r(\alpha)$.

Let $0 < \epsilon_1 < \epsilon_2$. We define functions U_r, $\tilde{U}_r$ and u_r which 'approximate' $\mu(\delta)$ on $[\epsilon_1,\epsilon_2]$.

Let

$$U_r(\delta) := \max \{ \alpha : h_r(\alpha) \le \delta \} ,$$

$$u_r(\delta) := \max \{ \alpha : H_r(\alpha) \le \delta \}$$

Thus $U_r(\delta)$ is defined for $\delta \geq \min_{\alpha \in J_r(P(G_r))} h_r(\alpha)$ and $u_r(\delta)$ is defined for $\delta \geq \min_{\alpha \in J_r(P(G_r))} H_r(\alpha)$. It is easy to see that there exists r_o and $\alpha \in J_{r_o}(P(G_{r_o}))$ such that $H_{r_o}(\alpha) < \epsilon_1$. Thus U_r and u_r are defined on $[\epsilon_1, \infty)$ if $r \geq r_o$.

Let $\tilde{h}_r(\alpha) := H_r(\alpha) - ((\max k_i)^r . |K| . 2)$ and $\tilde{U}_r(\delta) := \max \{ \alpha : \tilde{h}_r(\alpha) \leq \delta \}$. Thus $\tilde{U}_r(\delta)$ is defined for $\delta \geq \min_{\alpha \in J_r(P(G_r))} H_r(\alpha) - ((\max k_i)^r . |K| . 2)$.

Morover $u_r(\delta + ((\max k_i)^r . |K| . 2)) = \tilde{U}_r(\delta)$ and therefore $\tilde{U}_r$ is defined on $[\epsilon_1, \infty)$ if $r \geq r_o$.

All functions $u_r(\delta)$, $U_r(\delta)$ and $\tilde{U}_r(\delta)$ are jump functions with a finite number of jumps, continuous from the right non-decreasing and positive.

The following theorem shows that the above functions are approximations of $\mu(\delta)$.

Theorem 2 : Let K be a self-similar set. Let $u_r(\delta)$, $U_r(\delta)$, $\tilde{U}_r(\delta)$ as above then

a) $u_r(\delta)/\delta^s \leq \mu(\delta)/(\delta^s . \mathcal{H}^s(K)) \leq U_r(\delta)/\delta^s \leq \tilde{U}_r(\delta)/\delta^s$

if $\delta \geq \min_{\alpha \in J_r(P(G_r))} H_r(\alpha)$.

b) if $\mu(\delta)$ is continuous on $(0,\infty)$ then

$|\tilde{U}_r(\delta) - u_r(\delta)| \longrightarrow 0$ uniformly on $[\epsilon_1, \epsilon_2]$ as $r \longrightarrow \infty$.

c) $\lim_{r \to \infty} (\sup_{\delta \in [\epsilon_1, \epsilon_2]} u_r(\delta)/\delta^s) = \lim_{r \to \infty} (\sup_{\delta \in [\epsilon_1, \epsilon_2]} \tilde{U}_r(\delta)/\delta^s) =$

$= (\sup_{\delta \in [\epsilon_1, \epsilon_2]} \mu(\delta)/\delta^s)/\mathcal{H}^s(K)$ if $\mu(\delta)$ is continuous at ϵ_2.

d) If we replace $\tilde{U}_r$ by U_r, b) and c) hold.

Proof : We show first that

$$u_r(\delta) \le \mu(\delta)/\mathcal{H}^s(K) \le U_r(\delta) \text{ if } \delta \ge \min_{\alpha\in J_r(P(G_r))} H_r(\alpha)$$

From theorem 1 we know that $\mu(\delta) = \mathcal{H}^s(\overset{\circ}{C}_\delta \cap K)$ where $\overset{\circ}{C}_\delta$ is a compact convex set of diameter δ. But $\overset{\circ}{C}_\delta$ interesects l elements of G_r. Let $T_{i_1 \ldots i_r}, \ldots, T_{j_1 \ldots j_r}$ be such elements

Then

$$\mu(\delta) = \mathcal{H}^s(\overset{\circ}{C}_\delta \cap K) \le \underbrace{[(k_{i_1} \ldots k_{i_r})^s + \ldots + (k_{j_1} \ldots k_{j_r})^s]}_{=\alpha} . \mathcal{H}^s(K)$$

(this last inequality due to the fact that $K = \bigcup_{\substack{\text{all} \\ r\text{-index}}} Y_{i_1} \circ \ldots \circ Y_{i_r}(K)$)

Also $h_r(\alpha) \le |\overset{\circ}{C}_\delta| = \delta$. Then $\mu(\delta) \le U_r(\delta) . \mathcal{H}^s(K)$

The other inequality is proved as follows: let $u_r(\delta)=\alpha$, ie. $H_r(\alpha) \le \delta$. Thus there exists l elements of G_r, say $T_{i_1 \ldots i_r}, \ldots, T_{j_1 \ldots j_r}$, such that

i) $J_r(\{T_{i_1 \ldots i_r}, \ldots, T_{j_1 \ldots j_r}\}) =$

$$=(k_{i_1} \ldots k_{i_r})^s + \ldots + (k_{j_1} \ldots k_{j_r})^s = \alpha$$

ii) $H_r(\alpha) = | T_{i_1 \ldots i_r} \cup \ldots \cup T_{j_1 \ldots j_r} |$

Using $\mathcal{H}^s(Y_i(K) \cap Y_j(K))=0$ if $i \ne j$ it follows that $u_r(\delta) \le \mu(\delta)/\mathcal{H}^s(K)$.

Now we prove that $U_r(\delta) \le \tilde{U}_r(\delta)$ if $\delta \ge \min_{\alpha\in J_r(P(G_r))} H_r(\alpha)$.

For this we only have to prove that $\tilde{h}_r(\alpha) \le h_r(\alpha)$ if $\alpha \in J_r(P(G_r))$

Fix α. From the definition of $h_r(\alpha)$ we then have l elements of G_r, say $T_{i_1 \ldots i_r}, \ldots, T_{j_1 \ldots j_r}$, such that

i) $J_r(\{T_{i_1 \ldots i_r}, \ldots, T_{j_1 \ldots j_r}\}) = \alpha$

ii) $h_r(\alpha) = \max\limits_{\Gamma,\Gamma' \in \{T_{i_1 \ldots i_r}, \ldots, T_{j_1 \ldots j_r}\}} (d(\Gamma,\Gamma'))$

where $d(\cdot,\cdot)$ is the distance between sets.

But any element of $\{T_{i_1 \ldots i_r}, \ldots, T_{j_1 \ldots j_r}\}$ has diameter less than or equal to $(\max k_i)^r \cdot |K|$. Thus

$$|T_{i_1 \ldots i_r} \cup \ldots \cup T_{j_1 \ldots j_r}| \le h_r(\alpha) + (\max k_i)^r \cdot |K| \cdot 2$$

and therefore $H_r(\alpha) \le h_r(\alpha) + (\max k_i)^r \cdot |K| \cdot 2$ ie. $\tilde{h}_r(\alpha) \le h_r(\alpha)$ and we are done. This proves a).

To prove c) we need the following: suppose $\mu(\delta)$ is continuous at ϵ_2, then $\lim\limits_{r\to\infty} u_r(\epsilon_2) = \mu(\epsilon_2)/\mathcal{H}^s(K)$. Suppose this is not true, then we would have $\epsilon > 0$ and a subsequence r_j such that

for all j $\qquad u_{r_j}(\epsilon_2) < (\mu(\epsilon_2)/\mathcal{H}^s(K)) - \epsilon$

But then

$$\mu(\epsilon_2-((\max k_i)^{r_j} \cdot |K| \cdot 2))/\mathcal{H}^s(K) \le \tilde{U}_{r_j}(\epsilon_2-((\max k_i)^{r_j} \cdot |K| \cdot 2)) =$$
$$= u_{r_j}(\epsilon_2) < \mu(\epsilon_2)/\mathcal{H}^s(K) - \epsilon$$

which is an absurd taking $j \to \infty$.

Let $\epsilon > 0$. Let r_1 be such that $\mu(\epsilon_2 + ((\max k_i)^{r_1} \cdot |K| \cdot 2)) - \mu(\epsilon_2) < \epsilon \cdot \mathcal{H}^s(K)$, $\mu(\epsilon_2)/\mathcal{H}^s(K) - u_r(\epsilon_2) < \epsilon$ if $r \ge r_1$ and

$|1/x^s - 1/y^s| < \epsilon$ if $|x-y| \le ((\max k_i)^{r_1} \cdot |K| \cdot 2)$ and $x,y \in [\epsilon_1, \infty)$. Let $\tau = \sup\limits_{\delta \in (0,\epsilon_2]} \mu(\delta)/\mathcal{H}^s(K)$.

Let's prove c): due to the fact that $\tilde{U}_r$ is non-decreasing and continuous from the right we have that

$\sup\limits_{\delta \in [\epsilon_1,\epsilon_2]} \tilde{U}_r(\delta)/\delta^s$ is taken on a particular point δ_o of $[\epsilon_1, \epsilon_2]$.

Thus if we assume $r \ge r_1$ we have

$$\sup_{\delta\in[\epsilon_1,\epsilon_2]} \tilde{U}_r(\delta)/\delta^s = \tilde{U}_r(\delta_o)/\delta_o^s = u_r(\delta_o + ((\max k_i)^r.|K|.2))/\delta_o^s$$

and we therefore have two choices: $(\delta_o + ((\max k_i)^r.|K|.2))=\delta_o'$ belongs to $[\epsilon_1,\epsilon_2]$ or not.

Suppose that it belongs. Then

$$u_r(\delta_o')/\delta_o^s = u_r(\delta_o').(1/\delta_o^s-1/\delta_o'^s) + u_r(\delta_o')/\delta_o'^s \le$$

$$\tau.\epsilon + \sup_{\delta\in[\epsilon_1,\epsilon_2]} u_r(\delta)/\delta^s$$

If δ_o' does not belong to $[\epsilon_1,\epsilon_2]$ then

$$u_r(\delta_o')/\delta_o^s = ((u_r(\delta_o')-u_r(\epsilon_2))/\delta_o^s) + u_r(\epsilon_2).(1/\delta_o^s-1/(\epsilon_2)^s) +$$

$$+ u_r(\epsilon_2)/(\epsilon_2)^s \le 2.\epsilon/(\epsilon_1)^s + \tau.\epsilon + \sup_{\delta\in[\epsilon_1,\epsilon_2]} u_r(\delta)/\delta^s$$

Thus c) is proved.

We end the proof of theorem 2 proving that b) holds.

From previous discussions of the definitions of u_r and $\tilde{U}_r$ we can assume that these functions are defined on $[\epsilon_1-\epsilon,\epsilon_2+\epsilon] \subseteq (0,\infty)$ if $r \ge R$ for some R, where $\epsilon > 0$ and $((\max k_i)^r.|K|.2)<\epsilon$ if $r\ge R$.

Suppose that $\tilde{U}_r(\delta)-u_r(\delta)$ does not tend to zero uniformly on $[\epsilon_1,\epsilon_2]$. Then we would have a sequence of points $\delta_j \in [\epsilon_1,\epsilon_2]$ such that $(r_j\ge R)$

$$0 < \theta \le \tilde{U}_{r_j}(\delta_j) - u_{r_j}(\delta_j) = u_{r_j}(\delta_j+ q_j) - u_{r_j}(\delta_j)$$

where $q_j := (\max k_i)^{r_j}.|K|.2$. Then

$$\mu(\delta_j+q_j)/\mathcal{H}^s(K) - \mu(\delta_j-q_j)/\mathcal{H}^s(K)=\mu(\delta_j+q_j)/\mathcal{H}^s(K) \pm u_{r_j}(\delta_j+q_j) \pm$$

$$\pm \tilde{U}_{r_j}(\delta_j-q_j) - \mu(\delta_j-q_j)/\mathcal{H}^s(K) \ge \theta \quad \text{for all } j$$

and this contradicts the uniform continuity of $\mu(\delta)$ on $[\epsilon_1-\epsilon,\epsilon_2+\epsilon]$.■

Set $f(\delta) := \mu(\delta)/\delta^s$.

Theorem 3 : Let K be a self-similar set. Then

$$f(\delta) \leq 1 \text{ for all } \delta \in (0,\infty)$$

Proof: Suppose the theorem is not true. Then there exists C_δ a compact convex set of diameter δ such that

$$\mathcal{H}^s(K \cap C_\delta)/|K \cap C_\delta|^s \geq \mathcal{H}^s(K \cap C_\delta)/|C_\delta|^s \geq \beta > 1$$

From property II) (see page 5) of a self-similar set we obtain

$$\mathcal{H}^s(Y_i(K \cap C_\delta) \cap Y_j(K \cap C_\delta)) = 0 \quad \text{if} \quad i \neq j$$

Also $\mathcal{H}^s(Y_i(K \cap C_\delta)) = k_i^s . \mathcal{H}^s(K \cap C_\delta)$.

Thus for all i we have

$$\mathcal{H}^s(Y_i(K \cap C_\delta))/|Y_i(K \cap C_\delta)|^s = \mathcal{H}^s(K \cap C_\delta)/|K \cap C_\delta|^s \geq \beta > 1$$

and therefore by induction, for any l=1,2,.., the following holds:

a) $\mathcal{H}^s(\underbrace{Y_i \circ \ldots \circ Y_j}_{=l}(K \cap C_\delta) \cap \underbrace{Y_{i'} \circ \ldots \circ Y_{j'}}_{=l}(K \cap C_\delta)) = 0$

if the l-tuples i ... j and i′ ... j′ are different.

b) $\mathcal{H}^s(Y_i \circ \ldots \circ Y_j(K \cap C_\delta))/|Y_i \circ \ldots \circ Y_j(K \cap C_\delta)|^s \geq \beta > 1$

for all l-tuples.

Set $A_n := \bigcup_{\substack{\text{all the} \\ \text{l-tuples with } l \geq n}} \underbrace{Y_i \circ \ldots \circ Y_j}_{=l} (K \cap C_\delta)$

Then $A_{n+1} \subseteq A_n$.

Set $B_n := \bigcup_{\substack{\text{all the} \\ \text{n-tuples}}} \underbrace{Y_i \circ \ldots \circ Y_j}_{=n} (K \cap C_\delta)$

Clearly $B_n \subseteq A_n$. Also from a) and b) we have

$$\mathcal{H}^s(B_n) = \sum_{\substack{\text{all the} \\ \text{n-tuples}}} \mathcal{H}^s(Y_i \circ \ldots \circ Y_j(K \cap C_\delta)) \geq \beta . \sum_{\substack{\text{all the} \\ \text{n-tuples}}} |Y_i \circ \ldots \circ Y_j(K \cap C_\delta)|^s =$$

$$= \beta . \sum_{\substack{\text{all the} \\ \text{n-tuples}}} k_i^s . \ldots . k_j^s \, |K \cap C_\delta|^s = \beta . |K \cap C_\delta|^s$$

(the last inequality because $(\sum_{i=1}^{m} k_i^s)^n = \sum_{\text{all the n-tuples}} k_i^s \cdot \ldots \cdot k_j^s = 1$)

But $\mathcal{H}^s(A_n) \le \mathcal{H}^s(K)$. Thus $\lim_{n\to\infty} \mathcal{H}^s(A_n) = \mathcal{H}^s(\bigcap_1^\infty A_n = A) \ge$

$\ge \beta . |K \cap C_\delta|^s > 0$.

Clearly $\underbrace{Y_i o \ldots o Y_j}_{=l} (K \cap C_\delta)$ for all the l-tuples

$l \ge n$, form a Vitali family V_n for A, ie.they are compact sets and for any $\epsilon > 0$ and any $x \in A$ there exists $Y_i o \ldots o Y_j (K \cap C_\delta)$ of (+) diameter less than ϵ such that $x \in Y_i o \ldots o Y_j (K \cap C_\delta)$.

Let n_0 and $\epsilon>0$ be such that $\mathcal{H}^s(A_{n_0}) + \epsilon < \beta . \mathcal{H}^s(A)$. Then there exists a disjoint subfamily V'_{n_0} of V_{n_0} such that([Fal]pg.11)

(8) $$\mathcal{H}^s(A) \le \left(\sum_{Y_i o \ldots o Y_j (K \cap C_\delta) \in V'_{n_0}} |Y_i o \ldots o Y_j (K \cap C_\delta)|^s \right) + \epsilon/\beta = W + \epsilon/\beta$$

and (or $W=\infty$ or ($W<\infty$ and $\mathcal{H}^s(A - \bigcup_{Y_i o \ldots o Y_j (K \cap C_\delta) \in V'_{n_0}} Y_i o \ldots o Y_j (K \cap C_\delta)) = 0$))

But if $W=\infty$ by **(8)** and b) it follows

$$\beta . \mathcal{H}^s(A) \le \sum_{Y_i o \ldots o Y_j (K \cap C_\delta) \in V'_{n_0}} \beta . |Y_i o \ldots o Y_j (K \cap C_\delta)|^s + \epsilon \le$$

$$\le \sum_{Y_i o \ldots o Y_j (K \cap C_\delta) \in V'_{n_0}} \mathcal{H}^s(Y_i o \ldots o Y_j (K \cap C_\delta)) + \epsilon \le \mathcal{H}^s(K) + \epsilon$$

and then $\mathcal{H}^s(K) = \infty$.

If $W < \infty$ then by **(8)** and b) we get

$$\beta . \mathcal{H}^s(A) \le \sum_{Y_i o \ldots o Y_j (K \cap C_\delta) \in V'_{n_0}} \mathcal{H}^s(Y_i o \ldots o Y_j (K \cap C_\delta)) + \epsilon \le \mathcal{H}^s(A_{n_0}) + \epsilon < \beta . \mathcal{H}^s(A)$$

∎

Property A : Let K be a self-similar set. We say that property A holds for K if there exists $\Delta > 0$ such that for any $x \in K$ and any $B_{x,r}$ (ball centered at x and radius r) with $r \le \Delta$ we have $y \in K$ and Y, a similitude with contraction ratio $k = 1$, $Y: R^n \rightarrow R^n$, such that

a) $\gamma (B_{y,r} \cap K) = B_{x,r} \cap K$

b) $(B_{y,r} \cap K) - \gamma_{i_o} (K) = \{\emptyset\}$ for some i_o $1 \le i_o \le m$.

Lemma 2 : Let K be a self-similar set having property A. Then for any δ, $0 < \delta \le \Delta$ there exists j , j=1,..,m , such that

$$f(\delta) = f(\delta/k_j)$$

Proof: Suppose $0 < \delta \le \Delta$. By theorem 1 we know that $\mu(\delta) = \mathcal{H}^s(K \cap C_\delta)$, where C_δ is a convex compact set of diameter δ. By property A there exists C'_δ a convex compact set of diameter δ such that $\mathcal{H}^s(K \cap C'_\delta) = \mathcal{H}^s(K \cap C_\delta)$ and $(K \cap C'_\delta) - \gamma_{i_o}(K) = \{\emptyset\}$ with $1 \le i_o \le m$.

Then $\gamma^{-1}_{i_o}(C'_\delta) = C_{\delta/k_{i_o}}$ is a compact convex set of diameter δ/k_{i_o}. It is easy to check that $\mathcal{H}^s(K \cap C_{\delta/k_{i_o}}) = 1/k^s_{i_o} \cdot \mathcal{H}^s(K \cap C'_\delta)$ Clearly $\mu(\delta/k_{i_o}) \ge \mathcal{H}^s(K \cap C_{\delta/k_{i_o}})$.

Also by theorem 1

$$\mu(\delta/k_{i_o}) = \mathcal{H}^s(K \cap C'_{\delta/k_{i_o}})$$

where $C'_{\delta/k_{i_o}}$ is a convex compact set of diameter δ/k_{i_o}

But

$$\mu(\delta/k_{i_o}) = \mathcal{H}^s(K \cap C'_{\delta/k_{i_o}}) \le 1/k^s_{i_o} \cdot \mathcal{H}^s(K \cap \gamma_{i_o}(C'_{\delta/k_{i_o}})) \le$$

$$\le 1/k^s_{i_o} \cdot \mathcal{H}^s(K \cap C'_\delta) = \mathcal{H}^s(K \cap C_{\delta/k_{i_o}})$$

Then $\mu(\delta/k_{i_o}) = \mathcal{H}^s(K \cap C_{\delta/k_{i_o}}) = 1/k^s_{i_o} \cdot \mathcal{H}^s (K \cap C'_\delta) = \mu(\delta)/k^s_{i_o}$ ∎

Theorem 4 : Let K be a self-similar set. Then

i) $\overline{\lim_{\delta \to 0}} f(\delta) = 1$

ii) Let also K have property A. Let $0 < \epsilon_1 < \epsilon_2$ be such that

a) $\epsilon_1 \le \Delta$ with Δ of property A.

b) $\epsilon_1 . (\max 1/k_i) \le \epsilon_2$

Then $f(\delta) = 1$ for some $\delta \in [\epsilon_1, \epsilon_2]$

Proof : We prove first that $\overline{\lim}_{\delta \to 0} f(\delta) = 1$. Suppose this is not true, then there exists $a_1 > 0$, $a_2 > 0$ such that $f(\delta) \le 1 - a_1$ if $\delta \in (0, a_2)$. From the definition of Hausdorff measure of K we have that for any $\epsilon > 0$ there exists a numerable family E_i of compact convex sets of diameter less than ϵ such that $\mathcal{H}^s(K \cap E_i) \neq 0$ for all i, $\sum_i \mathcal{H}^s(K \cap E_i) \ge \mathcal{H}^s(K)$ and

$$\text{(9)} \qquad \mathcal{H}^s(K) + \epsilon \ge \sum_i |E_i|^s$$

But let $\epsilon < a_2$, then

$$\sum_i |E_i|^s = \sum_i \mathcal{H}^s(K \cap E_i) \,.\, |E_i|^s / \mathcal{H}^s(K \cap E_i) \ge$$

$$\ge \sum_i \mathcal{H}^s(K \cap E_i)/f(|E_i|) \ge \sum_i \mathcal{H}^s(K \cap E_i)/(1-a_1) \ge \mathcal{H}^s(K)/(1-a_1)$$

which is a contradiction with **(9)**.

We prove now ii). Suppose K has property A. For proving ii) it is only necessary to show that $\sup_{\delta \in [\epsilon_1, \epsilon_2]} f(\delta) = 1$ because $\mu(\delta)$ is continuous from the right and non-decreasing. Also to prove that $\sup_{\delta \in [\epsilon_1, \epsilon_2]} f(\delta) = 1$ it is enough by i to prove that if $0 < \delta < \epsilon_1$ then there exists $\delta' \in [\epsilon_1, \epsilon_2]$ such that $f(\delta') = f(\delta)$. But this follows easily from lemma 2.∎

A combination of theorems 2,3 and 4 gives us a method by which we can calculate in a practical way the measure of a self similar set K if property A holds and $T = \mathcal{C}(K)$ is known.

The method is as follows: we observe first that the function $J_r : P(G_r) \to R$, defined on page 10 is a function whose values we can calculate. Thus H_r and h_r are functions which we can also calculate because this involves taking the distance (or the diameter) between sets of the form

$Y_{i_1}(\dots Y_{i_r}(\mathcal{E}(K)) \dots) = T_{i_1 \dots i_r}$ (recall $T = \mathcal{E}(K)$ is known!).

Thus, the functions $\tilde{U}_r$, U_r and u_r are known.

But these functions are of the form

$$\sum_{i=1}^{l} q_i \cdot S(x-\xi_i)$$

where $\xi_i \in (0,\infty)$, $q_i > 0$ (ξ_i and q_i are known !) and

$$S(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Let ϵ_1 , ϵ_2 be as in theorem 4. Then

$$\sup_{\delta\in[\epsilon_1,\epsilon_2]} U_r(\delta)/\delta^s = \max_{\delta\in\{\epsilon_1,\text{all points } \delta'\in[\epsilon_1,\epsilon_2] \text{ such that } U_r \text{ has a jump at } \delta'\}} U_r(\delta)/\delta^s$$

and a similar expression holds for $\tilde{U}_r$ and u_r.

Thus $B_r = \sup_{\delta\in[\epsilon_1,\epsilon_2]} U_r(\delta)/\delta^s$, $\tilde{B}_r = \sup_{\delta\in[\epsilon_1,\epsilon_2]} \tilde{U}_r(\delta)/\delta^s$ and

$\beta_r = \sup_{\delta\in[\epsilon_1,\epsilon_2]} u_r(\delta)/\delta^s$ are all numbers which we can calculate. Then by theorem 2,3,4 we have

$$\beta_r \leq 1/\mathcal{H}^s(K) \leq B_r \leq \tilde{B}_r$$

and $\beta_r \leq \beta_{r+1}$ (this because $H_r \geq H_{r+1}$). From theorem 2 we know that $\tilde{B}_r - \beta_r \text{ ---> } 0$ if $r \text{--> } \infty$ ie.

$$1/\tilde{B}_r \leq 1/B_r \leq \mathcal{H}^s(K) \leq 1/\beta_r$$

and $1/\beta_r - 1/\tilde{B}_r \text{ --> } 0$ if $r \text{-->} \infty$.

In the next section we compute exact and approximate measures of some self-similar sets.

Observation 1: Let K be a self-similar set. Suppose that property Z does not hold, then it easy to see that

$$(\text{int } \mathcal{E}(K)) \cap K = \{\emptyset\} \quad \text{ie. } K \subseteq \mathcal{E}(K).$$

2. Example 1. The sets K_n will be self-similar sets in R^2 for each

$n \geq 3$ and they are defined as follows. Let P_n be a regular poligon of n sides and $|P_n| = 1$. Thus, for example, P_3 is an equilateral triangle whose base has lenght 1, P_4 is a cube of side equal to $1/\sqrt{2}$, P_5 is a pentagon, etc.

We define Y_i^n i=1,...,n a similitude in the following way: for each vertex V_i^n , $1 \leq i \leq n$ of the regular poligon P_n, Y_i^n is a contraction of ratio 1/n and a translation (ie. there is no rotation) and $Y_i^n(V_i^n) = V_i^n$. K_n is defined to be the unique compact set such that $\bigcup_{i=1}^{n} Y_i^n(K_n) = K_n$.

From the definitions of Y_i^n one easily gets an 'open set condition':

$$\bigcup_{i=1}^{n} Y_i^n(\text{int}\, \mathfrak{C}(P_n)) \subseteq \text{int}\, \mathfrak{C}(P_n)$$

and $Y_i^n(\text{int}\mathfrak{C}(P_n))$ are disjoint (see begining of proof of lemma 4). Thus, by Hutchinson's theorem (see [Fal]pg. 119) we get that

a) $0 < \mathcal{H}^{s_n}(K_n) < \infty$

b) $\mathcal{H}^{s_n}(Y_i^n(K_n) \cap Y_j^n(K_n)) = 0 \quad$ if $i \neq j$

where s_n is the Hausdorff dimension of K_n. Thus $s_n = 1$ for all n.

Observing that V_i^n must belong to K_n it is not difficult to show that $\mathfrak{C}(K_n) = \mathfrak{C}(P_n)$. Recall that $\mathfrak{C}(K_n) = T^n$, $Y_i^n(\mathfrak{C}(K_n)) = T_i^n$, etc.

Notice that property Z holds for K_n.

For the sets K_n we can compute their measure in an exact way:

Theorem 5 : $\mathcal{H}^1(K_n) = 1$ for all $n \geq 3$.

The proof of this theorem will come later. We will need some lemmas first. To motivate the reading of these auxiliary lemmas the reader may go directly to the proof of theorem 5 on page 30 . Figures 7 and 8 show how K_3 and K_5 look like.

We write $\mu(\delta,n)$ for the function $\mu(\delta)$ of K_n .

Lemma 3 : Let n,j be natural numbers. Then

a) $\frac{1/n}{(1-1/n).\sin(\pi/n)-1/n} \leq 1$ if $n \geq 5$

b) $\frac{2/n}{(1-1/n).\sin(\pi/n)} \leq 1$ if $n \geq 5$

c) $\frac{(j+1)/n}{\sin(j\pi/n) - 2/n} \leq 1$ if $n \geq 7$ and [(n is even $2 \leq j \leq n/2$) or (n is odd and $2 \leq j \leq (n-1)/2$)]

d) $(1-1/n).\sin(\pi/n) < \sin(2\pi/n) - 2/n$ if $n \geq 6$

e) $(1 - 1/n).\sin^2(\pi/n) \leq 2/n$ if $n \geq 6$

f) $\sqrt{2/(1 + \cos(\pi/n))}.(1 - 1/n).\sin(\pi/n).\sin(\pi/2n) \leq$

$\leq 2/n$ if $n \geq 7$

Proof: From Taylor's series of sin x we obtain

(1) $\sin x - x \geq - x^3/3!$ if $x \in [0,\pi/2]$.

In the following x denotes real values and n (or j) denote natural values.

a) Let $f(x) := (\pi - 2) - \pi/x - \pi^3.(x-1)/(x^3.3!)$.Then $f(x) \geq 0$ if $x \in [5,\infty)$ because f(x) is non-decreasing if $x \in [5,\infty)$ and $f(5) > 0$. But using **(1)** we get for $n \geq 5$ that

$$1 \leq 1 + f(n) \leq n.[(1-1/n).\sin(\pi/n) - 1/n]$$

and a) follows.

b) Follows from a) immediately.

c) Let $g(n,j) = (n/j)^2. ((\pi - 1) - 3/j)$. Then $g(n,j) \geq \pi^3/3!$ if $n \geq 8$ and [(n is even and $4 \leq j \leq n/2$) or (n is odd and $4 \leq j \leq (n-1)/2$)] because

$g(n,j) \geq 4.(\pi - 7/4) \geq \pi^3/3!$ for the above values of n and j

If $j = 2$ and $n \geq 7$ we get $g(n,2) \geq (7/2)^2.(\pi - 5/2) \geq$ $\geq \pi^3/3!$. If $j = 3$ and $n \geq 7$ we get $g(n,3) \geq (7/3)^2.(\pi - 2) \geq \pi^3/3!$.

Thus

(2) $g(n,j) \geq \pi^3/3!$ if $n \geq 7$ and [(n is even $2 \leq j \leq n/2$) or (n is odd and $2 \leq j \leq (n-1)/2$)]

Thus using **(1)** and **(2)** we get

$$0 \leq (g(n,j) - \pi^3/3!).(j/n)^3 \leq \sin(j\pi/n) - j/n - 3/n$$

and c) follows.

d) Let $h(x) := (\pi.(\sqrt{3} - 1) - 2).x^2 - \pi^3 .(\sqrt{3} - 1)/3!$.

Then $h(6) > 0$ and therefore $h(x) > 0$ if $x \geq 6$. But using **(1)** we get if $n \geq 6$ that

$$0 < h(n)/n^3 \leq (\sqrt{3} - 1).\sin(\pi/n) - 2/n \leq$$
$$\leq (2.\cos(\pi/n) - 1 + 1/n).\sin(\pi/n) - 2/n$$

and d) follows.

e) and f) Let $f(x) := \sin^2(\pi x) - \sqrt{(1+\cos(\pi/7))/2}\ .2x$. It is not difficult to prove that $f(x) \leq 0$ if $x \in (0, \infty)$. Using this inequality e) and f) follow.

Lemma 4 : Let n be a natural number. Then

a) $\mu((1 - 1/n).\sin(\pi/n),\ n) \leq \mathcal{H}^1(K_n)/n$ if $n \geq 6$ and n is even

b) $\mu(\ \sin(j\pi/n) - 2/n,\ n) \leq \mathcal{H}^1(K_n).j/n$ if $n \geq 6$, n is even and $2 \leq j \leq n/2$

c) $\mu(\ \sqrt{2/(1 + \cos(\pi/n))}.(1-1/n).\sin(\pi/n),\ n) \leq \mathcal{H}^1(K_n)/n$ if $n \geq 5$ and n is odd.

d) $\mu(\ \sqrt{2/(1 + \cos(\pi/n))}\ .\sin(j\pi/n) - 2/n,\ n) \leq \mathcal{H}^1(K_n).j/n$ if $n \geq 5$, n is odd and $2 \leq j \leq (n-1)/2$

Proof: Let $n \geq 5$. Recall that $\mathfrak{C}(K_n) = \mathfrak{C}(P_n) = T^n$, $Y_i^n(\mathfrak{C}(K_n)) = T_i^n$ $Y_j^n(Y_i^n(\mathfrak{C}(K_n))) = T_{ji}^n$, etc. We call C_e^n the center of P_n ie.

$C_e^n = \sum_i V_i^n / n$. Thus it is easy to check that (recall $|P_n| = 1$)

$$d(V_i^n, C_e^n) = \begin{cases} 1/2 & \text{if } n \text{ is even} \\ 1/\sqrt{2.(1+\cos(\pi/n))} & \text{if } n \text{ is odd} \end{cases}$$

As T_i^n contains V_i^n , $|T_i^n| = 1/n$ and

$$d(V_1^n, V_{j+1}^n) = d(V_1^n, V_{n-j+1}^n) = \begin{cases} \sin(j\pi/n) & \text{if } n \text{ is even, } 1 \leq j \leq (n/2) \\ \sqrt{2/(1+\cos(\pi/n))}.\sin(j\pi/n) & \text{if } n \text{ is odd, } 1 \leq j \leq (n-1)/2 \end{cases}$$

we get

(3)

$$d(T_1^n, T_{j+1}^n) = d(T_1^n, T_{n-j+1}^n) \geq \begin{cases} \sin(j\pi/n) - (2/n) & \text{if } n \text{ is even, } 1 \leq j \leq n/2 \\ \sqrt{2/(1+\cos(\pi/n))}.\sin(j\pi/n) - (2/n) & \text{if } n \text{ is odd, } 1 \leq j \leq (n-1)/2 \end{cases}$$

Set center $T_i^n = Y_i^n(C_e^n)$. Then

(4)

$$d(\text{center } T_1^n, \text{center } T_2^n) = d(\text{center } T_1^n, \text{ center } T_n^n) =$$

$$= \begin{cases} (1-1/n).\sin(\pi/n) & \text{if } n \text{ is even} \\ \sqrt{2/(1+\cos(\pi/n))}.(1-1/n).\sin(\pi/n) & \text{if } n \text{ is odd} \end{cases}$$

The above formulae could be sharpened to get the following (**(3')**)which will be used directly in the proof of theorem 5:

(3')

$$\min_{i \neq j} d(T_i^n, T_j^n) \geq \begin{cases} (1-1/n).\sin(\pi/n) - 1/n & \text{if } n \text{ is even } n \geq 6 \\ \sqrt{2/(1+\cos(\pi/n))}.[(1-1/n).\sin(\pi/n) - 1/n] & \text{if } n \text{ is odd } n \geq 5 \end{cases}$$

Also using d) of lemma 3 we get, for n even, $n \geq 6$,

(5) $$(1-1/n).\sin(\pi/n) < \sin(2\pi/n) - (2/n) <$$
$$< \sin(3\pi/n) - (2/n) < \ldots < \sin(\pi/2) - (2/n)$$

and for n odd, $n \geq 5$

(6) $$\sqrt{2/(1+\cos(\pi/n))}.(1-1/n).\sin(\pi/n) <$$
$$< \sqrt{2/(1+\cos(\pi/n))}.\sin(2\pi/n) - (2/n) <$$
$$< \sqrt{2/(1+\cos(\pi/n))}.\sin(3\pi/n) - (2/n) < \ldots <$$
$$< \sqrt{2/(1+\cos(\pi/n))}.\sin((n-1)\pi/2n) - (2/n)$$

(the reader can check the first inequality of **(6)** for $n = 5$)

a) Let C be a compact convex set of diameter $(1-1/n).\sin(\pi/n)$. Suppose $T_1^n \cap C \neq \{\emptyset\}$. From **(3)** and **(5)** we have that $C \cap T_j^n = \{\emptyset\}$ if $j \neq 1,2,n$. Thus from simmetry C can only intersect two elements of $\{T_1^n, T_2^n, T_n^n\}$. We assume C intersects T_1^n and T_2^n. Observe that $\mathcal{H}^1(T_i^n \cap K_n) = \mathcal{H}^1(K_n)/n$ By corollary 1 we get that if L is any line in R^2 then

(7) $$\mathcal{H}^1(L \cap K_n) = 0$$

Let L_1, L_2 be two parallel lines at a distance $(1-1/n).\sin(\pi/n)$, perpendicular to the segment joining the centers of T_1^n, T_2^n;

such that $C \subseteq W$ where W is (see figure 1) the 'strip' obtained with L_1 and L_2, ie. $W = L_1 \cup L_2 \cup$ all points 'between' L_1 and L_2. Recall that d(center T_1^n, center T_2^n) = $=(1-1/n).\sin(\pi/n)$ and observe that $K_n \cap T_1^n$ and $K_n \cap T_2^n$ are translations one from the other. Then from simmetry and **(7)** follows that $\mathcal{H}^1((K_n \cap (T_1^n \cup T_2^n))-W) \geq \mathcal{H}^1(K_n)/n$. Thus a) follows. This last argument will be used quite often. Case c) is proved in an analogous way using **(3)**, **(4)** and **(6)**.

(b) Let n and j be as in b). Let C be a compact convex set of diameter $\sin(j\pi/n) -(2/n)$. Assume $C \cap T_1^n \neq \{\emptyset\}$. Then from **(3)** and **(7)** we get

$$0 = \mathcal{H}^1(K_n \cap T_{j+1}^n \cap C) = \mathcal{H}^1(K_n \cap T_{j+2}^n \cap C) = \dots =$$
$$= \mathcal{H}^1(K_n \cap T_{n-j+1}^n \cap C)$$

Thus we could assume that C intersects in a non-trivial way only T_{n-j+2}^n, T_{n-j+3}^n, ..., T_n^n, T_1^n, T_2^n, ... , T_{j-1}^n, T_j^n . By simmetry and using this last argument repeatedly we verify that C interesects in a non-trivial way at most j elements of $\{T_i\}$ and b) follows.

Case d) is proved in a similar way using **(3)** and **(7)**.

Lemma 5 : Let n and i be integers. Then

a) $\mu(1-(1/n^i),n) \le (1-(3/n^i)).\mathcal{H}^1(K_n)$ if $n \ge 6$, $i \ge 1$

b) $\mu(1-(3/n^i),n) \le (1-(1/n^{i-1})).\mathcal{H}^1(K_n)$ if $n \ge 6$, $i \ge 2$

c) $\mu(1-(1/5^i),5) \le (1-(2/5^i)).\mathcal{H}^1(K_5)$ if $i \ge 1$

d) $\mu(1-(2/5^i),5) \le (1-(1/5^{i-1})).\mathcal{H}^1(K_5)$ if $i \ge 2$

e) $\mu(1-(3/n),n) \le \mathcal{H}^1(K_n)/2$ if $n \ge 6$

f) $\mu(1-(2/5),5) \le \mathcal{H}^1(K_5).2/5$

Proof: Let $n\ge 5$. It is clear that $\mathcal{H}^1(K_n \cap T^n_{j_1..j_i}) = \mathcal{H}^1(K_n)/n^i$ and $|T^n_{j_1..j_i}| = 1/n^i$. Also $V^n_1 \in T^n_{\underbrace{1..1}_{i}}$, $V^n_{n/2+1} \in T^n_{\underbrace{n/2+1,..,n/2+1}_{i}}$

$d(V^n_1,V^n_{n/2+1}) = 1$ if n is even or $V^n_1 \in T^n_{\underbrace{1..1}_{i}}$, $V^n_{(n-1)/2+1} \in T^n_{\underbrace{(n-1)/2+1,..,(n-1)/2+1}_{i}}$, $d(V^n_1,V^n_{(n-1)/2+1}) = 1$ if n is odd.

Let C be a compact convex set of diameter $1-(1/n^i)$. Assume $C \cap T^n_{\underbrace{1..1}_{i}} \ne \{\emptyset\}$. Let L_1, L_2 be two lines perpendicular to the line that joints V^n_1 and $V^n_{n/2+1}$ if n is even (replace n/2 + 1 by (n-1)/2 + 1 if n is odd in this and all the following expressions) and such that $d(L_1,L_2) = 1 - (1/n^i)$,

$C \subseteq W$ where W is the strip enclosed by L_1 and L_2.

Then as $K_n \cap T^n_{\underbrace{1..1}_{i}}$ and $K_n \cap T^n_{\underbrace{n/2+1,..,n/2+1}_{i}}$ are translations one of the other we have using **(7)** that

$$\textbf{(8)} \quad \mathcal{H}^1(K_n \cap (T^n_{\underbrace{1..1}_{i}} \cup T^n_{\underbrace{n/2+1,..,n/2+1}_{i}}) \cap C) \le \mathcal{H}^1(K_n)/n^i$$

But a similar expression holds for pairs ($T^n_{\underbrace{j..j}_{i}}, T^j_{\underbrace{n/2+j,..,n/2+j}_{i}}$) $j=2,..,n/2$ if n is even(or for pairs ($T^n_{\underbrace{j..j}_{i}}, T^n_{\underbrace{(n-1)/2+j..(n-1)/2+j}_{i}}$) $j=2,...,(n-1)/2$ if n is odd with $n/2 + 1$ replaced by $(n-1)/2 + 1$ in **(8)**). If we assume $n \geq 6$ then there are at least 3 such pairs and a) is proved. If $n=5$ there are 2 such pairs and c) is proved.

e) Let $n \geq 6$. Observe that $d(T^n_j, T^n_{n/2+j}) \geq 1-(2/n) > 1-(3/n)$ if n is even, $1 \leq j \leq n/2$ (replace $n/2$ by $(n-1)/2$ if n is odd). Thus if C is a convex compact set of diameter $1-(3/n)$ and $C \cap T^n_j \neq \{\emptyset\}$ then $C \cap T^n_{n/2+j} = \{\emptyset\}$ (replace $n/2$ by $(n-1)/2$ if n is odd) and e) follows easily.

f) It is easy to check that if C is a convex compact set of diameter $1-(2/5)$ and $C \cap T^5_1 \neq \{\emptyset\}$ then $\mathcal{H}^1(C \cap (T^5_3 \cup T^5_4) \cap K_5) = 0$. Using simmetry f) follows.

b and d) Let $i \geq 2$, $n \geq 5$ and let Q^n_i be (see fig.2) the point intersection of the line L joining V^n_1 and $V^n_{n/2+1}$ if n is even (or $V^n_{(n-1)/2+1}$ if n is odd) and the line L' perpendicular to L such that L' contains the point $\underbrace{Y^n_1(Y^n_1 ... Y^n_1}_{i}(Y^n_n(V^n_1))...) \in T^n_{\underbrace{11..1n}_{i}}$.

It is easy to check that

$$d(V^n_1, Q^n_i) = \begin{cases} 1/n^{i-1}.(1-1/n).\sin^2(\pi/n) & \text{if } n \text{ is even} \\ \sqrt{2/(1+\cos(\pi/n))}.1/n^{i-1}.(1-1/n).\sin(\pi/n).\sin(\pi/2n) & \text{if } n \text{ is odd} \end{cases}$$

Let be $n \geq 6$ and C be a compact convex set of diameter $1-(3/n^i)$. Assume $C \cap T^n_{\underbrace{n/2+1..n/2+1}_{i}} \neq \{\emptyset\}$ if n is even (replace $n/2 +1$ by $(n-1)/2 +1$ if n is odd in this and the following formulas). Then from the fact that

$$d(T^n_{\underbrace{1..1}_{i}}, T^n_{\underbrace{n/2+1..n/2+1}_{i}}) = 1-(2/n^i) > 1-(3/n^i)$$

we get $C \cap T^n_{\underbrace{1..1}_{i}} = \{\emptyset\}$. Also by e (or f) if n is odd) of lemma 3

$d(V^n_1, Q^n_i) \leq 2/n^i$.

Then as $T^n_{\underbrace{11..1n}_{i}}$ and $T^n_{\underbrace{n/2+1..n/2+1}_{i}}$ are translations one of the other we can use an argument similar to the one used in a) and get

$$\mathcal{H}^1(K_n \cap (T^n_{\underbrace{11..1n}_{i}} \cup T^n_{\underbrace{n/2+1..n/2+1}_{i}}) \cap C) \leq \mathcal{H}^1(K_n)/n^i$$

which combined with the fact that $C \cap T^n_{\underbrace{1..1}_{i}} = \{\emptyset\}$ gives

(9)

$$\mathcal{H}^1(K_n \cap (T^n_{\underbrace{1..1}_{i}} \cup T^n_{\underbrace{1..1n}_{i}} \cup T^n_{\underbrace{n/2+1..n/2+1}_{i}}) \cap C) \leq \mathcal{H}^1(K_n)/n^i$$

Note that if C does not intersect $T^n_{\underbrace{1..1}_{i}}$ nor $T^n_{\underbrace{n/2+1 .. n/2+1}_{i}}$ then (9) holds. But b) follows from (9) and the fact that the same argument may be done with all the triples $(T^n_{\underbrace{j..j}_{i}}, T^n_{\underbrace{j..j(j-1)}_{i}}, T^n_{\underbrace{n/2+j..n/2+j}_{i}})$ $2 \leq j \leq n/2$ (for n odd observe that C can only intersect (n−1)/2 elements of the form $\{T^n_{\underbrace{j..j}_{i}}\}$ $j=1,...,n$).

Case d) (n=5) is proved in an almost similar way using $d(V^5_1, Q^5_i) \leq 2/5^i$.

Lemma 6: Let i be an integer. Then

a) $\mu(1-(2/3^{i+1}),3) \leq (1-(1/3^i)).\mathcal{H}^1(K_3)$ if $i \geq 1$

b) $\mu(1-(1/3^i),3) \leq (1-(5/3^{i+1})).\mathcal{H}^1(K_3)$ if $i \geq 1$

c) $\mu(1-(5/3^{i+1}),3) \leq (1-(2/3^i)).\mathcal{H}^1(K_3)$ if $i \geq 1$

Proof: Recall that $T^3_{j_1..j_i}$ is an equilateral triangle of base equal to $1/3^i$.

a) Let $i \geq 1$ and let C be a convex compact set of diameter $1-(2/3^{i+1})$. Then if $C \cap (T^3_{\underbrace{1..1}_{i+1}} \cup T^3_{\underbrace{2..2}_{i+1}} \cup T^3_{\underbrace{3..3}_{i+1}})=\{\emptyset\}$ we have

$$\mathcal{H}^1(K_3 \cap C) \leq (1-(1/3^i)). \mathcal{H}^1(K_3)$$

Therefore we assume $C \cap T^3_{\underbrace{1..1}_{i+1}} \neq \{\emptyset\}$.

Since $d(T^3_{\underbrace{1..1}_{i+1}}, T^3_{\underbrace{2..2}_{i+1}}) = d(T^3_{\underbrace{1..1}_{i+1}}, T^3_{\underbrace{3..3}_{i+1}}) = 1-(2/3^{i+1})$ (see fig. 3) we have

$$\mathcal{H}^1(K_3 \cap (T^3_{\underbrace{2..2}_{i+1}} \cup T^3_{\underbrace{3..3}_{i+1}}) \cap C) = 0$$

It is not difficult to check that the segment $[P_{i+1}, Q_{i+1}]$ is perpendicular to $[V^3_1, V^3_2]$. Thus $d(P_{i+1}, V^3_1) \geq 1-(2/3^{i+1})$ and by an argument similar to that given in lemma 5 a) we have that

$$\mathcal{H}^1(K_3 \cap (T^3_{\underbrace{2..23}_{i+1}} \cup T^3_{\underbrace{11..1}_{i+1}}) \cap C) \leq \mathcal{H}^1(K_3)/3^{i+1}$$

and a) follows.

b) Let $i \geq 1$ and C be a convex compact set of diameter $1-(1/3^i)$. Then if $C \cap (T^3_{\underbrace{1..1}_{i}} \cup T^3_{\underbrace{2..2}_{i}} \cup T^3_{\underbrace{3..3}_{i}}) = \{\emptyset\}$ we have

$$\mathcal{H}^1(K_3 \cap C) \leq (1- 1/3^{i-1}). \mathcal{H}^1(K_3)$$

Thus we assume $C \cap T^3_{\underbrace{1..1}_{i}} \neq \{\emptyset\}$. From simmetry we have only three subcases:

b1) $C \cap T^3_{\underbrace{1..1}_{i+1}} \neq \{\emptyset\}$

b2) $C \cap T^3_{\underbrace{j..j}_{i+1}} = \{\emptyset\}$, $j=1,2,3$; $C \cap T^3_{\underbrace{1..12}_{i+1}} = \{\emptyset\}$,

$C \cap T^3_{\underbrace{1..13}_{i+1}} \neq \{\emptyset\}$

b3) $C \cap T^3_{\underbrace{j..j}_{i+1}} = \{\emptyset\}, j=1,2,3;\ C \cap T^3_{\underbrace{1..12}_{i+1}} \neq \{\emptyset\}$,

$C \cap T^3_{\underbrace{1..13}_{i+1}} \neq \{\emptyset\}$,

b1) It is easy to see that (see fig. 3)

$$d(T^3_{\underbrace{1..1}_{i+1}}, T^3_{\underbrace{2..2}_{i+1}}) \text{ and } d(T^3_{\underbrace{1..1}_{i+1}}, T^3_{\underbrace{2..23}_{i+1}}) \geq d(T_{i+1}, Q_{i+1}) = 1-(1/3^i)$$

Thus $\mathcal{H}^1(K_3 \cap (T^3_{\underbrace{2..2}_{i+1}} \cup T^3_{\underbrace{2..23}_{i+1}}) \cap C) = 0$ and by simmetry

$\mathcal{H}^1(K_3 \cap (T^3_{\underbrace{3..3}_{i+1}} \cup T^3_{\underbrace{3..32}_{i+1}}) \cap C) = 0$. Also as $d(V^3_1, R^{i+1}) = 1-(1/3^i)$ we have that $\mathcal{H}^1(K_3 \cap (T^3_{\underbrace{1..1}_{i+1}} \cup T^3_{\underbrace{2..21}_{i+1}}) \cap C) \leq \mathcal{H}^1(K_3)/3^{i+1}$.

b2) As $d(S_{i+1}, P_{i+1}) = 1-(1/3^i)$ one gets

$$\textbf{(10)}\ \mathcal{H}^1(K_3 \cap (T^3_{\underbrace{1..13}_{i+1}} \cup T^3_{\underbrace{2..23}_{i+1}}) \cap C) \leq \mathcal{H}^1(K_3)/3^{i+1}$$

b3) From b2) one gets **(10)** again and by simmetry

$$\mathcal{H}^1(K_3 \cap (T^3_{\underbrace{1..12}_{i+1}} \cup T^3_{\underbrace{3..32}_{i+1}}) \cap C) \leq \mathcal{H}^1(K_3)/3^{i+1}$$

c) Let $i \geq 1$ and let C be a compact convex set of diameter $1-(5/3^{i+1})$. We assume $C \cap T^3_{\underbrace{1..1}_{i}} \neq \{\emptyset\}$ (if $C \cap (T^3_{\underbrace{1..1}_{i}} \cup T^3_{\underbrace{2..2}_{i}} \cup T^3_{\underbrace{3..3}_{i}}) = \{\emptyset\}$ then $\mathcal{H}^1(K_3 \cap C) \leq (1-(1/3^{i-1})) \cdot \mathcal{H}^1(K_3)$).

Then by simmetry, only two choices are possible:

c1) $C \cap T^3_{\underbrace{1..1}_{i+1}} \neq \{\emptyset\}$ and therefore

$C \cap (T^3_{\underbrace{2..2}_{i}} \cup T^3_{\underbrace{3..3}_{i}}) = \{\emptyset\}$.

c2) $C \cap \underbrace{T^3_{j..j}}_{i+1} = \{\emptyset\}, j=1,2,3; C \cap \underbrace{T^3_{1..12}}_{i+1} \neq \{\emptyset\}$ and therefore

$$\mathcal{H}^1(K_3 \cap (\underbrace{T^3_{3..3}}_{i} \cup \underbrace{T^3_{2..2}}_{i+1} \cup \underbrace{T^3_{2..23}}_{i+1} \cup \underbrace{T^3_{1...1}}_{i+1}) \cap C) = 0. \quad \blacksquare$$

Proof (of theorem 5) : Recall that property Z holds for K_n $n \geq 3$. Thus $\mu(\delta,n)$ is continuous on $(0,\infty)$. Let $f(\delta,n)=\mu(\delta,n)/\delta$. Then, if $1<\delta$, $f(\delta,n) = \mathcal{H}^1(K_n)/\delta < f(1,n) = \mathcal{H}^1(K_n) \leq 1$ (th.3). Therefore to prove the theorem we must show $\mathcal{H}^1(K\ ) \geq 1$. Observe that any number $0 < \Delta_n < \min_{i \neq j} d(T_i, T_j)$ maybe used in property A as Δ. Therefore from theorems 3 and 4 we get

i′) $f(\delta,n) \leq 1$ on $[\Delta_n ,1]$

ii′) $f(\delta_0,n) = 1$ for some $\delta_0 \in [\Delta_n,1]$

From the continuity of $\mu(\delta,n)$ one gets i′) and ii′) for $\Delta_n = \min_{i \neq j} d(T^n_i,T^n_j)$ ie.

i) $f(\delta,n) \leq 1$ on $[\min_{i \neq j} d(T^n_i,T^n_j) \ ,1]$

ii) $f(\delta_0,n) = 1$ for some $\delta_0 \in [\min_{i \neq j} d(T^n_i,T^n_j) \ ,1]$

We recall formulae **(3′)** of lemma 4

(3′)

$$\min_{i \neq j} d(T^n_i,T^n_j) \geq \begin{cases} (1-1/n).\sin(\pi/n) - 1/n & \text{if } n \text{ is even } n \geq 6 \\ \sqrt{2/(1+\cos(\pi/n))}.[(1-1/n).\sin(\pi/n) - 1/n] & \text{if } n \text{ is odd } n \geq 5 \end{cases}$$

and $\min_{i \neq j} d(T^3_i, T^3_j) = 1/3$. Let n be even, $n \geq 8$. Define functions $g(\delta,n)$ and $h(\delta,n)$ as follows:

$$g(\delta,n)= \begin{cases} 1/n & \text{if } \delta \in [(1-1/n).\sin(\pi/n)-(1/n),(1-1/n).\sin(\pi/n)) \\ 2/n & \text{if } \delta \in [(1-1/n).\sin(\pi/n),\sin(2\pi/n)-(2/n)) \\ (j+1)/n & \text{if } \delta \in [\sin(j\pi/n)-(2/n),\sin((j+1)\pi/n)-(2/n)) \\ & \text{and } 2 \leq j \leq (n/2)-1 \end{cases}$$

(11)

$$h(\delta,n)= \begin{cases} 1-1/n^i & \text{if } \delta \in [1-1/n^i,1-3/n^{i+1}) \ , \ i=1,2,\ldots \\ 1-3/n^{i+1} & \text{if } \delta \in [1-3/n^{i+1},1-1/n^{i+1}) \ , \ i=0,1,2,\ldots \\ 1/2 & \text{if } \delta \in [1/2,1-3/n) \end{cases}$$

Then $h(\delta,n)$ is defined on $[1/2,1)$ and $g(\delta,n)$ on

$[(1-1/n).\sin(\pi/n)-(1/n),1-2/n)$. Also $h(\delta,n)/\delta \le 1$ and by lemma 3 a,b,c) we get $g(\delta,n)/\delta \le 1$. By lemmas 5 a,b,e) 4 a,b) and from the fact that $\mu(\delta,n)$ is non decreasing we get

(12) $$f(\delta,n)/\mathcal{H}^1(K_n) \le h(\delta,n)/\delta \le 1 \text{ if } \delta\in[1/2,1)$$

and

$$f(\delta,n)/\mathcal{H}^1(K_n) \le g(\delta,n)/\delta \le 1 \text{ if }$$
$$\delta\in[(1-1/n).\sin(\pi/n)-(1/n),1-2/n)$$

and using the continuity of $\mu(\delta,n)$

(13) $$f(\delta,n)/\ \mathcal{H}^1(K_n) \le 1 \quad \text{if} \quad \delta\in[\min_{i\neq j} d(T_i^n,T_j^n),1]$$

Using property ii) above we get $\mathcal{H}^1(K_n) \ge 1$.

Thus $\mathcal{H}^1(K_n) = 1$ if n is even $n\ge 8$.

The proof of the other cases are similar.

Let n be odd, $n\ge 7$. Define $h(\delta,n)$ as in **(11)** and

$$g(\delta,n)=\begin{cases} 1/n \text{ if } \delta\in[\sqrt{2/(1+\cos(\pi/n))}.[(1-1/n).\sin(\pi/n)-(1/n)], \\ \qquad\qquad \sqrt{2/(1+\cos(\pi/n))}.(1-1/n).\sin(\pi/n)) \\ 2/n \text{ if } \delta\in[\sqrt{2/(1+\cos(\pi/n))}.(1-1/n).\sin(\pi/n), \\ \qquad\qquad \sqrt{2/(1+\cos(\pi/n))}.\sin(2\pi/n)-(2/n)) \\ (j+1)/n \quad \text{if } \delta\in[\sqrt{2/(1+\cos(\pi/n))}.\sin(j\pi/n)-(2/n), \\ \qquad\qquad \sqrt{2/(1+\cos(\pi/n))}.\sin((j+1)\pi/n)-(2/n)) \\ \qquad\qquad 2 \le j \le (n-1)/2 - 1 \end{cases}$$

$g(\delta,n)$ is defined on $[\sqrt{2/(1+\cos(\pi/n))}.[(1-1/n).\sin(\pi/n)-(1/n)]$, $\sqrt{2/(1+\cos(\pi/n))}.\sin((n-1)\pi/2n)-(2/n))$.

Using lemma 3 a,b,c) we get $g(\delta,n)/\delta \le 1$. By lemma 4 c,d follows that $f(\delta,n)/\mathcal{H}^1(K_n) \le g(\delta,n)/\delta \le 1$. Using lemma 5 a,b,e) **(12)** holds for $h(\delta,n)$. Thus **(13)** holds and the proofs ends as the other case.

Case n=4 may be found in [Fal] . The proof given there is different.

For n=6, $h(\delta,6)$ is defined as in **(11)** and

$$g(\delta,6)=\begin{cases} 1/6 & \text{if } \delta\in[(1-1/6).\sin(\pi/6) - 1/6,(1-1/6).\sin(\pi/6)\ ) \\ 1/3 & \text{if } \delta\in[(1-1/6).\sin(\pi/6),\sin(\pi/3) - 1/3] \end{cases}$$

and the proof runs in a similar way using lemma 3 a,b), lemma 5 a,b,e), lemma 4 a,b).

For n=5 let

$$h(\delta,5) = \begin{cases} 1 - 2/5^{i} & \text{if } \delta\in[1 - 2/5^{i},1 - 1/5^{i}) & i=1,2,\dots \\ 1 - 1/5^{i-1} & \text{if } \delta\in[1 - 1/5^{i-1},1 - 2/5^{i}\ ) & i= 2,3,\dots \\ 2/5 & \text{if } \delta\in[2/5,\ 1 - 2/5) \end{cases}$$

$g(\delta,5) = 1/5$, if $\delta\in[\sqrt{2/(1+\cos(\pi/5)}.[(1-1/5).\sin(\pi/5)-1/5]$, $\sqrt{2/(1+\cos(\pi/5)}.(1-1/5).\sin(\pi/5)]$ and use lemma 5 c,d,f),4c),3a).

For n=3 we define only one function $g(\delta,3)$ in the following way

$$g(\delta,3) = \begin{cases} 1-2/3^{i} & \text{if } \delta\in[1 - 2/3^{i},1 - 5/3^{i+1}) & i \geq 1 \\ 1 - 5/3^{i+1} & \text{if } \delta\in[1 - 5/3^{i+1},\ 1 - 1/3^{i}) & i \geq 1 \\ 1 - 1/3^{i} & \text{if } \delta\in[1 - 1/3^{i},\ 1 - 2/3^{i+1}) & i \geq 1 \end{cases}$$

Thus $g(\delta,3)$ is defined on [1/3,1) and this case follows from lemma 6.

Example 2. The Koch curve is the unique compact set K such that

$$K = \bigcup_{i=1}^{4} Y_i(K)$$

where Y_i are similitudes of the complex plane defined by $Y_1(z)=z/3$; $Y_2(z)= z.(1/2 + i\sqrt{3}/2)/3 + 1/3$; $Y_3(z) = z.(1/2-i\sqrt{3}/2)/3 + (1/2 + i/2\sqrt{3})$; $Y_4(z) = z/3 + 2/3$.

It is not difficult to see that $\mathfrak{C}(K)=\mathfrak{C}(\{0,1,1/2+i/2\sqrt{3}\})$ and therefore using int $\mathfrak{C}(K)$ one can prove that an 'open set condition' holds for K. Therefore K is self similar (see [Fal]). Morover s=log 4/log 3.

K can indeed be defined with only two similitudes ie.

$$K = \bigcup_{i=1}^{2} Y'_i((K)$$

where $Y'_1(z)= z.(-\sqrt{3}/2 - i/2)/\sqrt{3} + (1/2 + i/2\sqrt{3})$; $Y'_2(z)= z.(-\sqrt{3}/2 + i/2)/\sqrt{3} + 1$ (primes will be used to describe elements that arise from this definition).

Property Z holds for K and therefore $\mu(\delta)$ is continuous.

Figure 4 shows how K looks like.

Let C be a compact set of diameter $\delta < 1/\ 3.\sqrt{3}$ such that (by theorem 1) $\mu(\delta) = \mathcal{H}^s(C \cap K)$. If C intersects T_1' or T_2' but not both then using ${Y_1'}^{-1}$ (or ${Y_2'}^{-1}$) one can prove that

(1) $$\mu(\delta.\sqrt{3}) = (\sqrt{3})^s.\ \mathcal{H}^s(C \cap K)$$

If C intersects both T_1' and T_2' then C must intersect at most the sets $\{T_{23}, T_{24}, T_{31}, T_{32}\}$ (fig 5) . But $Y(K \cap (T_{23} \cup T_{24} \cup T_{31} \cup T_{32})) = K \cap \{T_{11} \cup T_{12} \cup T_{13} \cup T_{14}\}$ where Y is a similitude with contraction ratio 1. Therefore one could assume that C only intersects T_1' and **(1)** holds. Thus we have proved that if $\delta < 1/3.\sqrt{3}$ then $f(\delta) = f(\sqrt{3}.\ \delta)$. Therefore theorem 4 holds with $\epsilon_1 = \Delta$, Δ any number less than $1/3.\sqrt{3}$ and $\epsilon_2 = \Delta\ .\sqrt{3}$ because in its proof we have not used property A but the thesis of lemma 2.

Also using the continuity of μ we have that

i) $f(\delta) \leq 1 \quad \delta \in [1/3\ .\sqrt{3}\ ,\ 1/3]$

ii) $f(\delta_o) = 1$ for some $\delta_o \in [1/3\ .\sqrt{3}\ ,\ 1/3\ ]$

We note that property A holds for K for some $\Delta << 1/3.\sqrt{3}$.

Upper and lower bounds for K had been given in [BePa]

$$0.026 \simeq 2^{-s-4} \leq \mathcal{H}^s(K) \leq 2^{s-2} \simeq 0.5995$$

In [Mar 2] was given an alternative proof of the upper bound and it was conjectured that $\mathcal{H}^s(K) = 2^{s-2}$. But we shall see that indeed $\mathcal{H}^s(K) < 2^{s-2}$.

Now to get a lower bound for $\mathcal{H}^s(K)$ we need to compute h_r. The following is a table of a function $\tilde{h}_2$ which is an

---

$\tilde{h}_2(6/16) = 1/3.\ \sqrt{3}$

$\tilde{h}_2(7/16) = 2/9$

$\tilde{h}_2(8/16) = 2/9$

$\tilde{h}_2(9/16) = 0.29397$

$\tilde{h}_2(10/16) = 1/3$

$\tilde{h}_2(11/16)=4/9$

---

approximation of h_2 . We recall the definition of h_2 :

$$h_2(\alpha) = \min_{\beta \in G_2^\alpha} (\max_{\Gamma,\Gamma' \in\beta} d(\Gamma ,\Gamma'))$$

where d(.,.) is the usual distance between sets.

Let $p_1=0, p_2=1, p_3=1/2 + i/2.\sqrt{3}$, $p_4=1/3$, $p_5= 2/3$, $p_6= 1/6 + i/6.\sqrt{3}$, $p_7= 1/3 + i/ 3.\sqrt{3}$, $p_8= 2/3 + i/3.\sqrt{3}$, $p_9= 5/6 + i/6.\sqrt{3}$ and let $\Gamma= T_{i_1 i_2}$, $\Gamma'= T_{j_1 j_2}$. Then set

$$\tilde{d} (\Gamma, \Gamma') = \min_{1\le k,l\le 9} d(Y_{i_1} \circ Y_{i_2} (p_k), Y_{j_1} \circ Y_{j_2} (p_l))$$

and define $\tilde{h}_2$ in the folllowing way:

$$\tilde{h}_2(\alpha) = \min_{\beta \in G_2^\alpha} (\max_{\Gamma,\Gamma' \in\beta} \tilde{d}(\Gamma ,\Gamma'))$$

Notice that $\tilde{d}(\Gamma, \Gamma') - 1/54 \le d(\Gamma, \Gamma') \le \tilde{d}(\Gamma, \Gamma')$. Therefore

$$\tilde{\tilde{h}}_2 := \tilde{h}_2 -1/54 \le h_2 \le \tilde{h}_2$$

and if we define $\tilde{\tilde{U}}_2 (\delta):= \max\{ \alpha : \tilde{\tilde{h}}_2(\alpha) \le \delta \}$ then $U_2 \le \tilde{\tilde{U}}_2$.

Since h_2 (and $\tilde{h}_2$) are non decreasing, to compute the supremum of $\tilde{\tilde{U}}_2$ on $[1/3.\sqrt{3},1/3]$ we do not need all the values of $\tilde{h}_2$ but those stated in the above table.

This is a general fact ie. H_r and h_r are non decreasing functions if $k_i= k_j$ for all i ,j the contraction ratios of K a self similar set. We left this proof to the reader. Hint: the set $G_r^{\alpha=1/k_i^r}$ is the set of all subsets of l-elements of G_r.

From i,ii) above and theorem 2 a) we get

$$1/\tilde{\tilde{B}}_2 \le 1/B_2 \le \mathcal{H}^s(K)$$

where $\tilde{\tilde{B}}_2= \sup_{\delta\in[1/3.\sqrt{3},1/3]} \tilde{\tilde{U}}_2(\delta)/\delta^s$; $B_2= \sup_{\delta\in[1/3.\sqrt{3},1/3]} U_2(\delta)/\delta^s$. Using the above table is easy to compute that $\tilde{\tilde{B}}_2 = 3.723$. Then

$$0.26 \leq \mathcal{H}^s(K)$$

We compute now an upper bound. Observe that $Q=\{T_{212} \cup T_{213} \cup T_{214} \cup T_{221} \cup T_{222} \cup T_{223} \cup T_{224} \cup T_{231} \cup T_{232} \cup T_{233} \cup T_{234} \cup T_{241} \cup T_{242} \cup T_{243} \cup T_{244} \cup T_{311} \cup T_{312} \cup T_{313} \cup T_{314} \cup T_{321} \cup T_{322} \cup T_{323} \cup T_{324} \cup T_{331} \cup T_{332} \cup T_{333} \cup T_{334} \cup T_{341} \cup T_{342} \cup T_{343}\}$ has diameter $\delta' = \sqrt{(292/243)}\ /3 \cong 0.36539$

and that $\mathcal{H}^s(Q \cap K) = 30/64 \cdot \mathcal{H}^s(K)$ (see fig. 6).

Therefore

$$30 \cdot \mathcal{H}^s(K) \ /\ (64 \cdot \delta'^s) \leq \mu(\delta')/\delta'^s \leq 1$$

ie $\mathcal{H}^s(K) \leq 0.5988 < 0.5995 \cong 2^{s-2}$. This disproves the mentioned conjecture. The numbers stated in example 2 are all exact up to the last digit .

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[Hut] Hutchinson J.E., 'Fractals and Self-Similarity', Indiana University Mathematics Journal, 30, 713-47 (1981).

[Mar 1] Marion Jacques, 'Mesure de Hausdorff d'un Fractal a Similitude Interne', Ann. sc. math. Quebec, 1986, Vol.10 No 1, pp 51-84.

[Mar 2] Marion J., 'Mesures de Hausdorff d'ensembles Fractals', Ann. sc. math. Quebec, 1987, vol. 11, No 1, pp. 111-132.

[BePa] Benedek A. y Panzone R.,'La Isla de von Koch', Rev. de la Academia Nacional de Ciencias Exactas, Físicas y Naturales, 1991.

Note:

Figures 4,5,6,7,8 are computer-made. This is the work of Pedro Panzone who kindly also made the software to compute the $\tilde{h}_2$- table.

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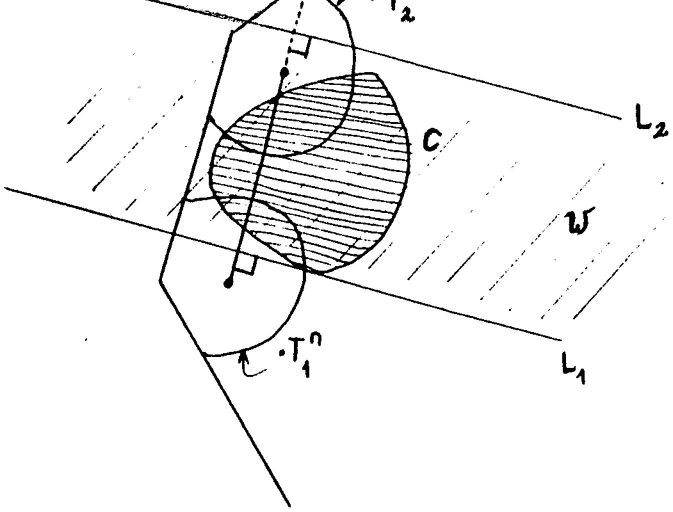

Fig. 1

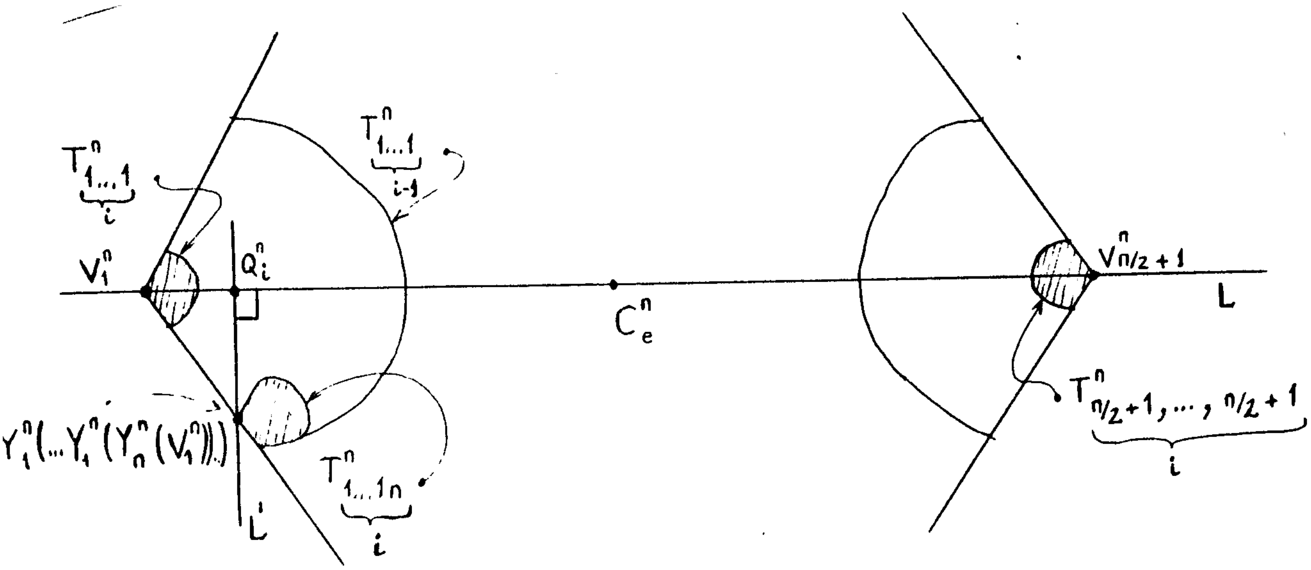

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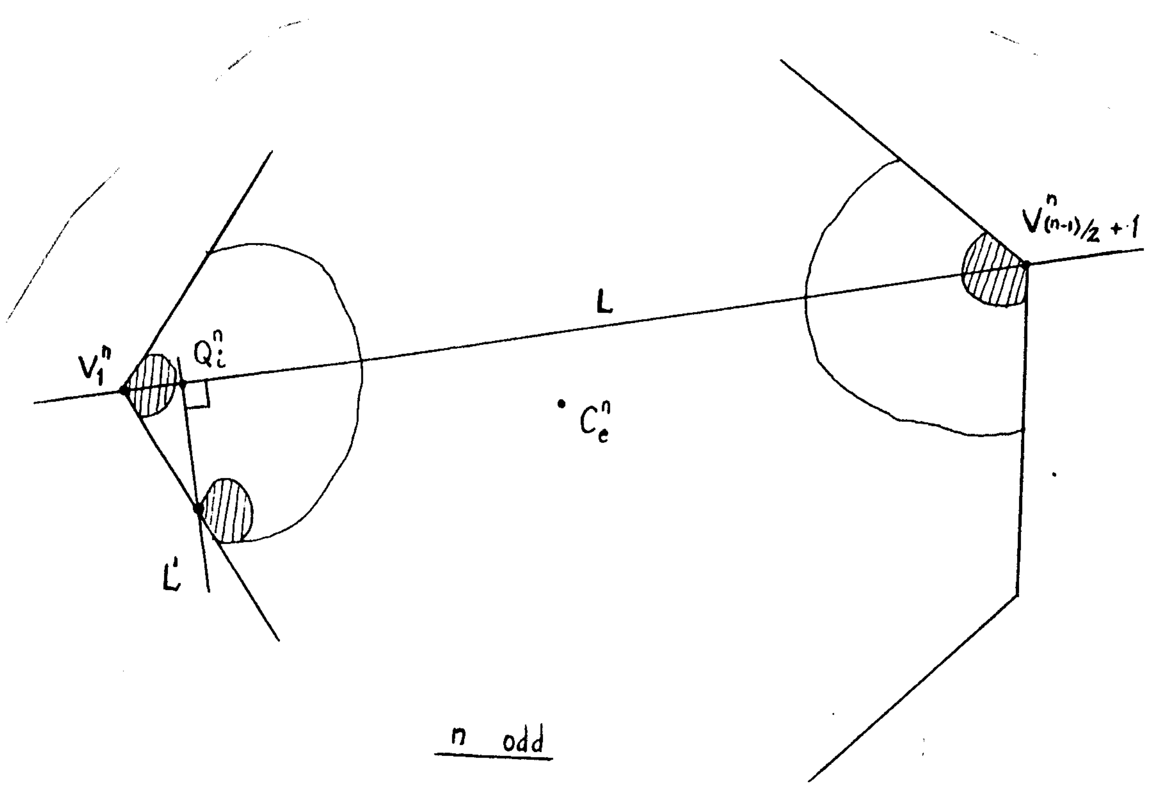
Fig. 2



n odd

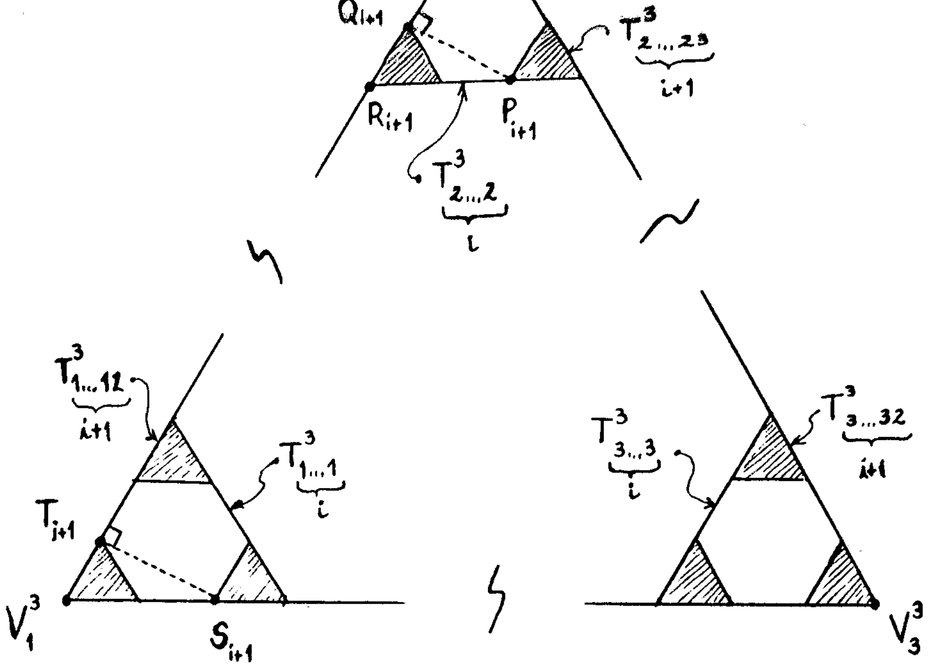


Fig. 3

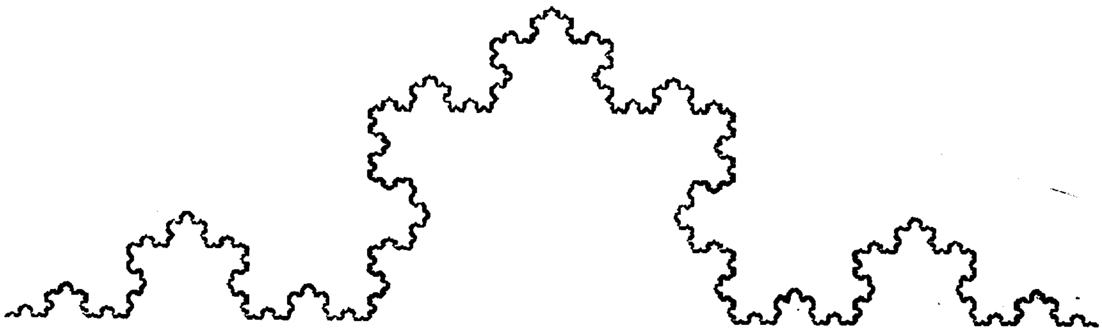


Fig. 4

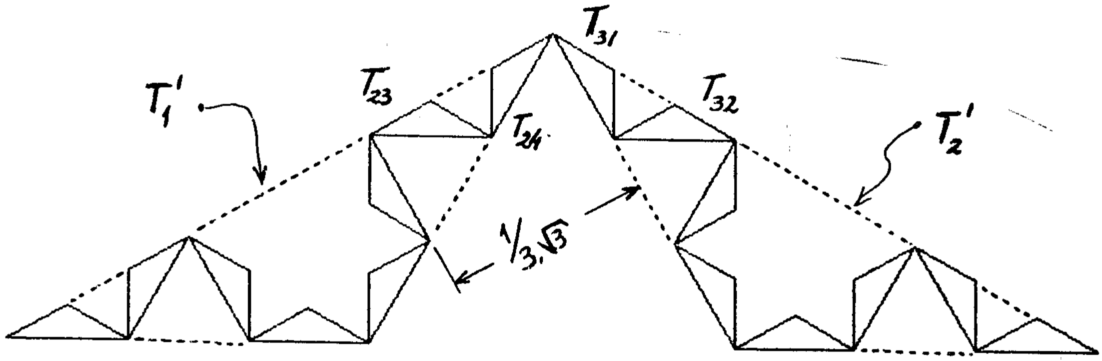


Fig. 5

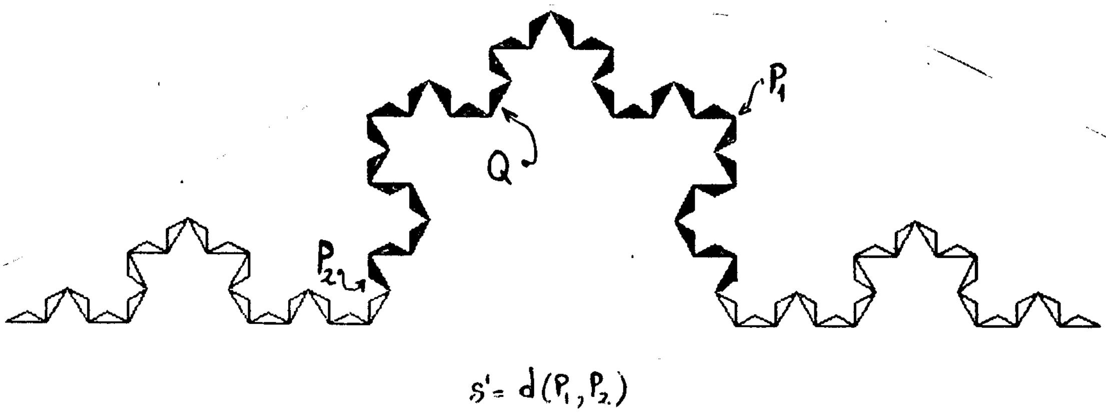
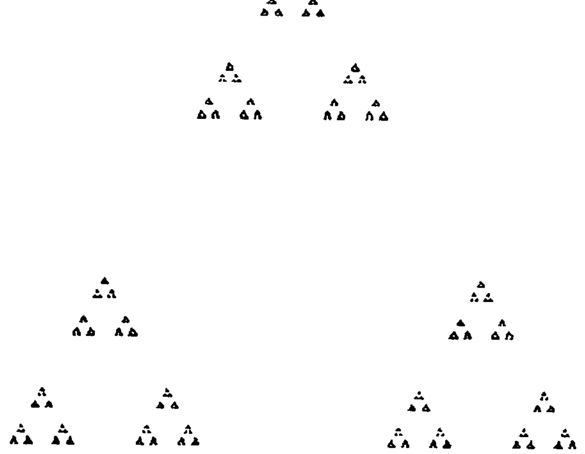


Fig. 6



K_3 **Fig. 7**

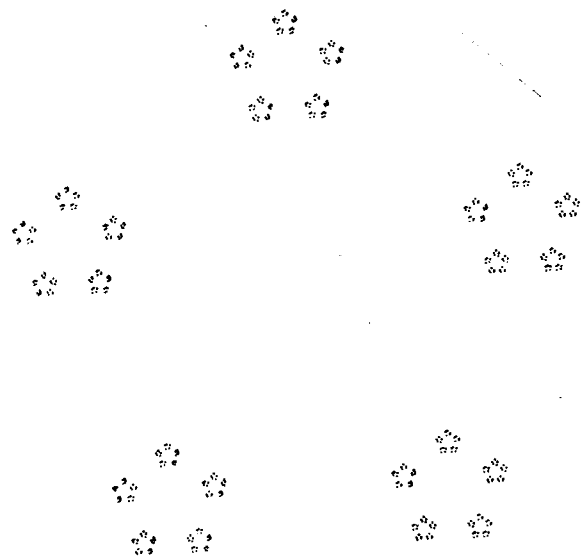


Fig. 8 K_5