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TOTAL MULTIGRAPHS. UNICITY OF THEIR SPECIAL SUBMULTIGRAPH

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ABSTRACT

In case of graphs without loops or parallel edges Behzad-Radjavi [1] show that if H is the total graph of a graph which is not a cycle or a complete graph then H contains one and only one subgraph G, such that H=T(G). In this work we show it is possible to extend this result to multigraphs and a constructive method for determining the special submultigraph is given.

1. INTRODUCTION

We consider finite, connected multigraphs with loops permitted. For simplicity we shall call these simply graph.

A graph is trivial if it consists of a single vertex.

Given a graph G, we denote by V(G) the set of its vertices and by U(G) the set of its edges.

With every graph G is associated a graph G°, called the LINE GRAPH of G , whose vertices are in a one to one correspondence with the edges of G in such a way that two vertices u', v' $(u' \neq v')$ of G° are joined by so many edges as common vertices have the corresponding edges u,v in G , and also, if u is a loop of G in the corresponding vertex u' there is only one loop.

In a similar way we can associate with G a graph $H = T(G)$ called the TOTAL GRAPH of G , such that $V(H) = V(G) \cup V(G^{\circ})$ and $U(H) = U(G) \cup U(G^{\circ}) \cup U(C(H))$, where C(H), called the MIDDLE GRAPH , has vertices $V(H)=V(G) \cup V(G^{\circ})$ and edges $[x,u']$ where $x \in V(G)$ is the end vertex of $u \in U(G)$.

H is a total graph if there is a graph G such that $H \cong T(G)$.

The precedent definitions have been considered in [2].

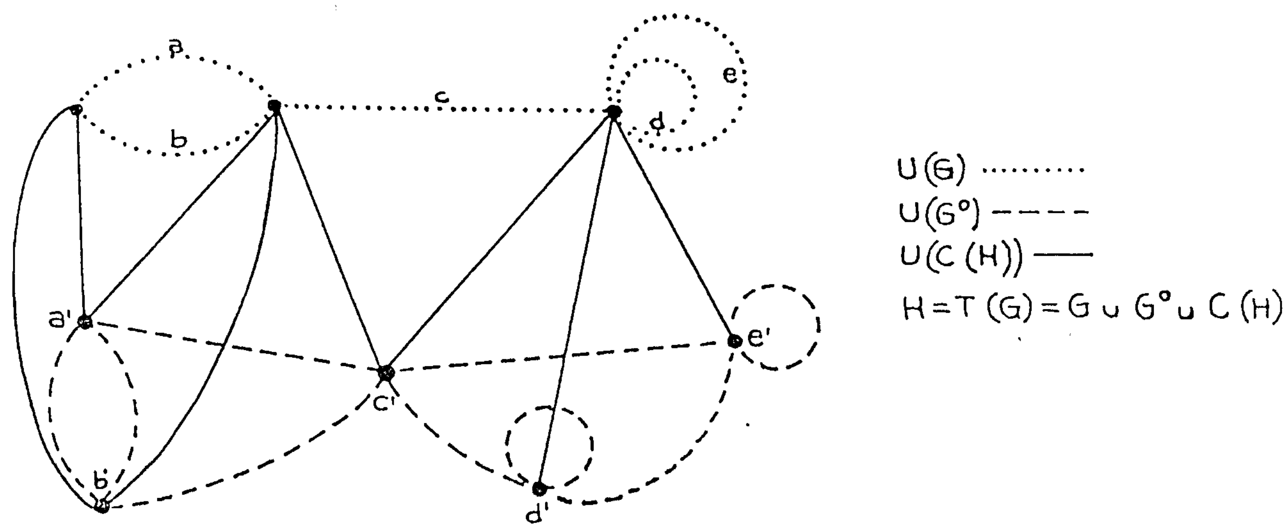

figura 1

REMARKS

If $H = T(G)$ then

1.1. G is connected if only if H is connected.

1.2. The number of loops of H is even.

1.3. If $u \in U(G)$, in the middle graph of H there are one or two edges with vertex u'.

1.4. If loops or parallel edges are not admitted , G is k-regular of orden n if only if H is 2k-regular of orden $n(1 + k/2)$.

1.5. If G is a cycle (in particular a loop or two parallel edges) or a complete graph, H contains at least two subgraphs G_i such that $H = T(G_i)$.

1.6. If H is the total graph of a cycle (in particular a loop or two parallel edges) or of a complete graph, H cannot be the total graph of another type of graph. (Follows from 1.4 and the definition of total graph applied to a loop or a pair of two parallel edges).

1.7. If H is the total graph of a path of order $n > 2$, H cannot be a total graph of another type of graph .

1.8. If $x \in V(H)$ is the end vertex of at least two loops, then $x \in V(G)$.

1.9. If there are three or more parallel edges $[x,y]$, then $x,y \in V(G)$.

1.10. If $x \in V(H)$ is the end vertex of parallel edges and at least one loop, $x \in V(G)$.

If $x \in V(G)$ we shall denote by d_x and D_x, respectively, the degree of x in G and in $T(G)$. For vertices $u' \in G^\circ$ we shall denote by $D_{u'}$ their degree in $H = T(G)$.

1.11. If G is not trivial, $D_x - d_x \geq 1$.

1.12. If $x \in V(G)$ and there are h_x loops, $h_x \geq 0$, with end vertex x, then $D_x = 2d_x - h_x$.

1.13. If $u = [x,x] \in U(G)$ and there are h_x loops, $h_x \geq 1$, with vertex x, $D_{u'} = d_x - h_x + 2 = D_x - d_x + 2$. If $d_x = 2$ then $D_{u'} = D_x$, otherwise, $D_{u'} < D_x$.

2. RESULTS

THEOREM 1

> Let H be a total graph. If A is a non trivial subgraph of H and A° is its corresponding line graph in H then there exists a unique graph G such that $A \subseteq G$ and $H = T(G)$.

proof

If $A \subset H$ is not trivial and A° is its corresponding line graph in H, we can construct $T(A) \subseteq H$. It is obvious that if $T(A) = H$ then $A = G$.

If $T(A) \neq H$, let $\hat{A} = \{u'_1, u'_2, \ldots, u'_k\}$ be the set of all vertices of $H = T(A)$ adyacent to some vertex of A°. Now we consider the edges of H with at least one vertex in A; the edges with the other vertex in $V(A^\circ) \cup \hat{A}$ are in $C(H)$. The remaining ones determine a well defined subgraph $A_1 \subset H$ such that $A \subset A_1$ and its corresponding line graph A_1° is the graph induced by $V(A^\circ) \cup \hat{A}$. We can repeat the same argument with A_1 and its corresponding line graph A_1° and in this way determine a sequence

$$A \subset A_1 \subset A_2 \subset \ldots\ldots \subset A_k$$

such that

$$H = T(A_k) \quad \text{and} \quad A_k = G. \blacksquare$$

Let $\mathbb{A}$ be the family of vertex subsets of a total graph that define cocycles without parallel edges. We say that a subgraph is an α- subgraph if it is a graph induced by the minimal elements of $\mathbb{A}$ that have loops and parallel edges.

REMARKS

Let S be a α-subgraph in $H = T(G)$

2.1. S is connected .

2.2. If S has no parallel edges, $|V(S)| = 1$ and its edges are loops.

2.3. If $|V(S)| = 2$, S has at least two parallel edges.

2.4. If $|V(S)| \geq 3$ and H has not three or more parallel edges

LEMMA 1

If $H = T(G)$ and $A \leq G$ is a α-subgraph, then it is possible to determine its corresponding $A^\circ \leq G^\circ$.

proof

If H has a single subgraph A° whose vertices are adyacent to the vertices of A just as the total graph definition implies, the lemma is trivial. Otherwise, the corresponding $A^\circ \subset G^\circ$ is the one whose vertices are not adyacent to the end vertex of a loop or parallel edges, except those of A.

LEMMA 2 (Behzad-Radjavi)

Let G be a connected graph without loops or parallel edges, which is not a path, a cycle or a complete graph and let a_o be a vertex of H = T(G) of maximal degree 2d.

Then $a_o \in V(G)$ if and only if the graph induced by the neighborhood of a_o has exactly one K_d as subgraph.

From the lemma it follows that the edges $[a_o, x_i]$, $i \in \{1,2,....,d\}$ such that $x_i \notin K_d$ determine a subgraph $A \leq G$ whose corresponding line graph A° is K_d.

THEOREM 2

Let H be a total graph, connected, with loops or parallel edges permitted.

a) If H is the total graph of a cycle or a complete graph, H contains two or more subgraphs G_i such that $H = T(G_i)$.

b) Otherwise, H has exactly one subgraph G whose total graph is H.

proof

a) This is a trivial consequence of 1.5 and 1.6.

b) We consider different cases and for each of them we shall show that there exists only one subgraph G such that $H = T(G)$.

Certainly, if we assume that we know a certain subgraph $A \subset H$ to which we can apply lemma 1 or 2 , following the constructive method used in the proof of theorem 1 we can determine the desired subgraph G.

That we can choose an adequate subgraph A in all cases is a consequence of the following considerations.

If H has no loops or parallel edges and is not the total graph of a cycle, a complete graph, or a path, to determine A we apply Lemma 2.

If H is the total graph of a path, we take A to be the smallest path which joins the unique vertices of degree 2.(see 1.7).

Otherwise H contains loops and parallel edges. If H contains α-subgraphs such that their vertices satisfy 1.8, 1.9, 1.10 we choose A among them. If in H there are no such α-subgraphs but there is a α-subgraph S such that $|V(S)| \geq 3$ then S can be taken as A.

If all α-subgraphs S consist of only a loop or two parallel edges, we take A to be one of the loops whose vertex has maximun degree in H (see 1.13) or one of the pairs of parallel edges with the minimun number of vertices adyacent to both end vertices.■

The following theorem is of interest in itself and generalizes the theorem of Behzad and Radjavi [1]. The proof is similar .

THEOREM 3

Let G_1 and G_2 be graphs. Then $T(G_1) \simeq T(G_2)$ if only if $G_1 \simeq G_2$

proof

It suffices to prove the result for connected graphs. If G_1 and G_2 are isomorphic so are their total graphs. The converse follows from 1.6, 1.7 and theorem 2.■

REFERENCES

[1] BEHZAD M. and RADJAVI H. "The total group of graph", Proc.Math.Soc.Vol. 19 (1968) n.1, 158-163.

[2] CHIAPPA R., MACCARI A. and ZILIANI A. " A Relation between Total Multigraphs and Total Multidigraphs", Rend. Mat. serie VII,vol. 12 (1992).