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THE AUSLANDER-REITEN QUIVER OF SOME QUOTIENTS OF TRIVIAL EXTENSIONS OF ARTIN ALGEBRAS

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The Auslander-Reiten Quiver Of Some Quotients Of Trivial Extensions Of Artin Algebras

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Introduction

Let $A = k(\vec{\Delta})/I$, where $k(\vec{\Delta})$ is the path algebra of the quiver $\vec{\Delta}$ over a field k, and I an admisible ideal. Let $\alpha_1, \cdots \alpha_t$ be arrows in $\vec{\Delta}$. In this paper we give a necessary and sufficient condition for an A-module M to be a module over the quotient algebra $A/ < \alpha_1, \cdots \alpha_t >$. This condition is more interesting when the algebra A is symmetric. If A is, moreover, the trivial extension $T(\Lambda)$ of a schurian algebra Λ then Λ is the quotient of $T(\Lambda)$ by an ideal generated by arrows, so the above applies. In this way we get information about the Auslander-Reiten quiver $\vec{\Gamma}_\Lambda$ of Λ from the Auslander-Reiten quiver $\vec{\Gamma}_{T(\Lambda)}$ of $T(\Lambda)$.

This is particularly useful where $T(\Lambda)$ is of finite representation type, case in which we obtain a complete description of $\vec{\Gamma}_\Lambda$ from $\vec{\Gamma}_{T(\Lambda)}$. This case is interesting because Λ is an iterated tilted algebra of Dynkin type if an only if $T(\Lambda)$ is of finite representation type. Moreover, this is the case if and only if $T(\Lambda) = T(\Lambda')$, with Λ' a tilted algebra (see[4] and [2]).

In many cases one can choose Λ' to be hereditary and such that $T(\Lambda) = T(\Lambda')$. The Auslander-Reiten quiver of trivial extensions of hereditary algebras can be constructed (see [7]), so we can construct $\vec{\Gamma}_\Lambda$ in these cases. We describe this procedure at the end of the section 2, giving also some examples to illustrate the techniques used. The fundamental tool in this work is a description, given by E.Fernández and M.I.Platzeck, of the quiver and relations of the trivial extension $T(\Lambda)$ of a schurian algebra $\Lambda = k(\vec{\Delta})/I$, for $\vec{\Delta}$ a quiver without oriented cycles, which we recall in the first section.

1 Preliminaries

Throughout the paper k denotes a field, $k\vec{\Delta}$ the path algebra associated to the finite quiver $\vec{\Delta} = ((\vec{\Delta})_0, (\vec{\Delta})_1)$, where $(\vec{\Delta})_0$ is the set of vertices and $(\vec{\Delta})_1$ the set of arrows of $\vec{\Delta}$.

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Also, we denote by Λ a finite dimensional k-algebra of the form $\Lambda = k(\vec{\Delta}_\Lambda)/I$, where $\vec{\Delta}_\Lambda$ is a quiver and I an admissible ideal. It is well known that when k is algebraically closed then any finite dimensional basic k-algebra is of this form.

Let $mod(\Lambda)$ denote the category of finite generated left Λ-modules and D_Λ the usual duality

$$Hom(-,k) : mod(\Lambda) \to mod(\Lambda^{op}).$$

We also assume that e_i denotes the trivial path corresponding to the vertex i, moreover S_i, P_i and I_i are the corresponding simple, projective and injetive indecomposable modules respectively.

The trivial extension $T(\Lambda)$ of Λ by the $\Lambda - \Lambda^{op}$ bimodule $D_\Lambda(\Lambda)$ is defined to be the k-vector space $\Lambda \amalg D_\Lambda(\Lambda)$ endowed with a multiplicative structure given by

$$(\lambda, \varphi)(\mu, \psi) = (\lambda\mu, \lambda\psi + \varphi\mu) \text{ for } \lambda, \mu \in \Lambda \text{ and } \varphi, \psi \in D_\Lambda(\Lambda).$$

The map $f : T(\Lambda) \to D_{T(\Lambda)}(T(\Lambda))$ defined by $[f(\lambda,\varphi)](\mu,\psi) = \varphi(\mu) + \psi(\lambda)$ is a $\Lambda - \Lambda^{op}$ bimodule isomorphism, so that $T(\Lambda)$ is a symmetric algebra and therefore selfinjective. On the other hand, the canonical epimorphism

$$\pi : T(\Lambda) \to \Lambda \ , \ \pi(\lambda, \varphi) = \lambda \text{ for } \lambda \in \Lambda \text{ and } \varphi \in D_\Lambda(\Lambda)$$

has kernel $D_\Lambda(\Lambda)$ an induces an embedding of $mod(\Lambda)$ in $mod(T(\Lambda))$ which identifies the category $mod(\Lambda)$ with the full subcategory of $mod(T(\Lambda))$ whose objects are the $T(\Lambda)$-modules M such that $D_\Lambda(\Lambda)\cdot M = 0$. In this way the vertices of the Auslander-Reiten quiver $\vec{\Gamma}_\Lambda$ of Λ can be identified with vertices of the Auslander-Reiten quiver of the trivial extension $T(\Lambda)$ of Λ.

We will describe next the quiver and relations of the trivial extension $T(\Lambda)$ of a schurian algebra $\Lambda = k(\vec{\Delta})/I$, for $\vec{\Delta}$ a quiver without oriented cycles. This description was given by E.Fernández and M.I.Platzeck, and will be an important tool in this work.

a) $(\vec{\Delta}_{T(\Lambda)})_0 = (\vec{\Delta})_0$.

b) $(\vec{\Delta}_{T(\Lambda)})_1 = (\vec{\Delta})_1 \cup \{\alpha_{\gamma_1}, \cdots \alpha_{\gamma_t}\}$ where $\{\gamma_1, \cdots \gamma_t\}$ is the set of maximal non zero paths in Λ, α_{γ_i} is an arrow starting at the endpoint of γ_i and ending at the origin of γ_i .

The relations in $\vec{\Delta}_{T(\Lambda)}$ are the following:

i) if δ_1 and δ_2 are directed cycles having the same origin, then $\delta_1 = \delta_2$ in $T(\Lambda)$.

ii) Let γ_i for $i = 1, 2$ be paths in $\vec{\Delta}_{T(\Lambda)}$ and Γ_i for $i = 1, 2$ be the set of paths δ in $\vec{\Delta}_{T(\Lambda)}$ such that $\gamma_i\delta$ or $\delta\gamma_i$ is a directed cycle.

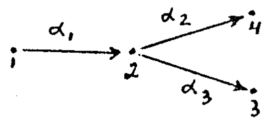
If $\Gamma_1 = \Gamma_2$ then $\gamma_1 = \gamma_2$ in $T(\Lambda)$.

iii) The composition of $n+1$ arrows in an oriented cycle of length n in $\vec{\Delta}_{T(\Lambda)}$ is zero in $T(\Lambda)$.

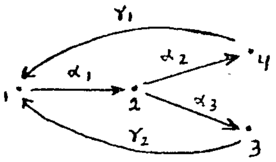
iv) The composition of arrows not belonging to a same cycle is zero in $T(\Lambda)$.

We illustrate this description with the following examples:

Example 1.1 *Let* $\Lambda_1 = k\vec{\Delta}_1$ *with* $\vec{\Delta}_1$:

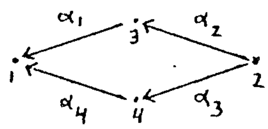


According to the preceding description, $\vec{\Delta}_{T(\Lambda_1)}$ *is given by the quiver with relations*



$$\alpha_1\gamma_1\alpha_2\alpha_1 = \alpha_2\alpha_1\gamma_1\alpha_2 = \gamma_1\alpha_2\alpha_1\gamma_1 = 0,$$
$$\alpha_1\gamma_2\alpha_3\alpha_1 = \alpha_3\alpha_1\gamma_2\alpha_3 = \gamma_2\alpha_3\alpha_1\gamma_2 = 0,$$
$$\alpha_3\alpha_1\gamma_1 = \alpha_2\alpha_1\gamma_2 = 0.$$

Example 1.2 *Let* $\Lambda_2 = k\vec{\Delta}/I_2$ *with* $\vec{\Delta}_2$:



and I_2 *generated by* $\alpha_1\alpha_2 - \alpha_4\alpha_3$. *Then* $T(\Lambda_1) \simeq T(\Lambda_2)$, *although* $\Lambda_1 \not\simeq \Lambda_2$.

The following result concerns algebras of finite global dimension with the same trivial extension, in the case one of them is hereditary.

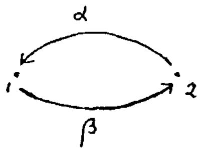
Theorem 1 *Let* $\vec{\Delta}$ *be a quiver without oriented cycles and* Λ *be a basic finite dimensional* k*-algebra. If* $T(\Lambda) = T(k\vec{\Delta})$ *and* Λ *has finite global dimension then* Λ *is tilted iterated of type* $\vec{\Delta}$.

Proof: The proof is based on known results about derived categories and repetitive algebras ([1] and [3]).
From Prop. 2.7 of [4] we get that the repetitive algebra $\hat{\Lambda}$ of Λ is isomorphic to the repetitive algebra $\widehat{k\vec{\Delta}}$ of $k\vec{\Delta}$. In particular, we obtain that the triangulated category $\underline{mod}(\hat{\Lambda})$ is triangle equivalent to $\underline{mod}(\widehat{k\vec{\Delta}})$. Since Λ and $k\vec{\Delta}$ have finite global dimension we have the diagram:

$$\begin{array}{ccc} \underline{mod}(\hat{\Lambda}) & \xrightarrow{\sim} & \underline{mod}(\widehat{k\Delta}) \\ \sim\uparrow & & \uparrow\sim \\ D^b(\Lambda) & & D^b(k\vec{\Delta}) \end{array}$$

Where $\xrightarrow{\sim}$ denotes a triangle equivalence. Thus $D^b(\Lambda)$ is triangle equivalent to $D^b(k\vec{\Delta})$, and therefore Λ is an iterated tilted algebra of type $\vec{\Delta}$ (see [1] or [3]). □
The following example, also considered in [3], shows that the above theorem does not hold if the global dimension of Λ is not finite.

Example 1.3 *Let* $\Lambda = k(\vec{\Delta})$ *with* $\vec{\Delta}$: 

Let $\Lambda' = k(\vec{\Delta}')/I$ *with* $\vec{\Delta}'$: 

$I = \langle \alpha\beta, \beta\alpha \rangle$.

In this case $T(\Lambda) \simeq T(\Lambda')$ *and* $gldim(\Lambda') = \infty$.

The following known results will be very useful for our purposes:

Theorem 2 *If* Λ *is and iterated tilted algebra of type* $\vec{\Delta}$, *then* $T(\Lambda)$ *is stable equivalent to* $T(k\vec{\Delta})$.

Proof: See [6]. □

Theorem 3 *If* Λ *is a basic finite dimensional k-algebra,the following are equivalent:*

a) $T(\Lambda)$ *is of finite representation type.*

b) There exists a tilted algebra B *of Dynkin type* $\vec{\Delta}$ *such that* $T(\Lambda) \simeq T(B)$.

c) Λ *is tilted iterated of Dynkin type* $\vec{\Delta}$.

Proof: See [1],[6] and [2]. □

2 Main results.

We start this section giving a characterization of modules M over the quotient algebra A modulo an ideal generated by arrows. Then we go on to study the case when A is the trivial extension of an artin algebra Λ. Finally, we give an application to the construction of the Auslander-Reiten quiver of some iterated tilted algebras.

It is well know that when a simple module S is a composition factor of M in $mod(\Lambda)$ then there are maps from the projective cover $P_0(S)$ of S to M, and from M to the injective envelope $I_0(S)$ of S.
The following Lemma shows that these maps can be choosen with non zero composition.

Lemma 4 *Let Λ be an artin ring and $h : P_0(S) \to M$ a non zero morphism in $mod(\Lambda)$, with S a simple module. Then there is a morphism $t : M \to I_0(S)$ such that $th \neq 0$.*

Proof: Since $h : P_0(S) \to h(P)$ is an essential epimorphism it follows that $h(P)/rad(h(P)) \simeq S$.
Then we have a commutative diagram

$$\begin{array}{ccc} h(P) & \xrightarrow{\ i\ } & M \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S \simeq h(P)/rad\, h(P) & \xrightarrow{\ i'\ } & M/rad\, h(P) \\ {\scriptstyle j}\downarrow & & \\ I_0(S) & & \end{array}$$

where i, j, i' and π, π' are the corresponding inclusions and canonical projections, respectively. Thus there is a morphism $t' : M/rad(h(P) \to I_0(S)$ such that $t'i' = j$. Then $t = t'\pi$ satisfies that $th = j\pi'h$ is non zero, proving the Lemma. □

Lemma 5 *Let $A = k\vec{\Delta}/I$ where I is an admissible ideal. Let $\alpha : i \to j$ be an arrow in $\vec{\Delta}$ and $M \in mod(\Lambda)$.*
The following conditions are equivalent:

a) $\overline{\alpha}M \neq 0$.

b) $Hom_A(r_{\overline{\alpha}}, M) : Hom_A(P_i, M) \to Hom_A(P_j, M)$ *is non zero, where* $r_{\overline{\alpha}} : P_j \to P_i$ *is the right multiplication by* $\overline{\alpha}$.

Proof: Assume that $\overline{\alpha}M \neq 0$ and let $m \in M$ such that $\overline{\alpha}m \neq 0$. Then $f : P_i \to M$ defined by $f(\lambda\overline{e_i}) = \lambda\overline{e_i}m$ for $\lambda \in A$, is an A-homomorphism and $f(\overline{\alpha}) = \overline{\alpha}m \neq 0$. So $fr_{\overline{\alpha}}(\overline{e_j}) = f(\overline{e_j\alpha}) = f(\overline{\alpha}) \neq 0$, thus $fr_{\overline{\alpha}} \neq 0$, proving that a) implies b).
Assume now that $Hom_A(r_{\overline{\alpha}}, M) \neq 0$, and let $f : P_i \to M$ such that $fr_{\overline{\alpha}} \neq 0$. Then $0 \neq fr_{\overline{\alpha}}(\overline{e_j}) = f(\overline{e_j\alpha}) = f(\overline{\alpha}) = \overline{\alpha}f(\overline{e_i}) \in \overline{\alpha}M$. So $\overline{\alpha}M \neq 0$, proving that b) implies a). □

Lemma 6 *Let $A = k\vec{\Delta}/I$ where I is an admissible ideal. Let $\alpha : i \to j$ be an arrow in $\vec{\Delta}$ and $M \in mod(\Lambda)$. Then:*

a) If $\overline{\alpha}M \neq 0$ *there are morphisms* $f : P_i \to M$, $g : M \to I_j$ *such that* $gf \neq 0$.

b) Assume that $Hom_A(r_{\overline{\alpha}}, I_j) : Hom_A(P_i, I_j) \to Hom_A(P_j, I_j)$ *is a monomorphism, where* $r_{\overline{\alpha}} : P_j \to P_i$ *is the right multiplication by* $\overline{\alpha}$.
If there are morphism $f : P_i \to M$, $g : M \to I_j$ *with* $gf \neq 0$, *then* $\overline{\alpha}M \neq 0$.

Proof: a) From Lemma 5 we know that there is a non zero morphism $f : P_i \to M$ such that $fr_{\overline{\alpha}} : P_j \to M$ is non zero. Then from Lemma 4 there is $g : M \to I_j$ such that $gfr_{\overline{\alpha}} \neq 0$, and consequently $gf \neq 0$.
b) We assume that $Hom_A(r_{\overline{\alpha}}, I_j)$ is a monomorphism and let $f : P_i \to M$, $g : M \to I_j$ such that $gf \neq 0$. Then $Hom_A(r_{\overline{\alpha}}, I_j)(gf) = (gf)r_{\overline{\alpha}} = g(fr_{\overline{\alpha}})$, proving that $fr_{\overline{\alpha}} \neq 0$. Thus $Hom_A(r_{\overline{\alpha}}, I_j)(f) \neq 0$ and by Lemma 5 we get that $\overline{\alpha}M \neq 0$. □
The preceding lemmas can be strengthened when A is the trivial extension of an algebra Λ, giving the following useful result.

Lemma 7 *Let $\Lambda = k\vec{\Delta}/I$ be a schurian algebra with I an admissible ideal, $\vec{\Delta}$ a quiver without oriented cycles and $\alpha : i \to j$ an arrow in $\vec{\Delta}_{T(\Lambda)}$. Then the following conditions are equivalents for a $T(\Lambda)$-module M.*

a) $\overline{\alpha}M \neq 0$.

b) There are morphisms $P_i \xrightarrow{f} M$, $M \xrightarrow{g} P_j$ with $gf \neq 0$.

Proof: Since $T(\Lambda)$ is a symmetric algebra then $P_j = I_j$ for any vertex j, so Lemma 6 a) implies that $a) \Rightarrow b)$.
To conclude that $b) \Rightarrow a)$ we only need to prove that the hypothesis of Lemma 6 b) are satisfied. This is, we need to prove that

$$Hom_{T(\Lambda)}(r_{\overline{\alpha}}, P_j) : Hom_{T(\Lambda)}(P_i, P_j) \to Hom_{T(\Lambda)}(P_j, P_j)$$

is a monomorphism. Since Λ is schurian and $\vec{\Delta}$ without oriented cycles we can use the description of $T(\Lambda)$ given by E.Fernández and M.I. Platzeck (see preliminaries) and conclude that $dim_k(Hom_{T(\Lambda)}(P_i, P_j)) = 1$. So the non zero morphism $Hom_{T(\Lambda)}(r_{\overline{\alpha}}, P_j)$ is a monomorphism. □
As a consequence of the preceding Lemma we obtain the following result.

Theorem 8 *Let $\Lambda = k\vec{\Delta}/I$ be a schurian algebra with I an admissible ideal, $\vec{\Delta}$ a quiver without oriented cycles. Let $\alpha_i : a_i \to b_i$ be arrows in $\vec{\Delta}_{T(\Lambda)}$ for $i = 1, 2, \cdots t$. Then the following conditions are equivalent for a $T(\Lambda)$-module M.*

a) M is a $T(\Lambda)/ < \overline{\alpha}_1, \cdots \overline{\alpha}_t >$ - module.

b) If $f : P_{a_i} \to M$, $g : M \to P_{b_i}$ are morphisms in $mod(\Lambda)$, then their comoposition gf is zero, for all $i = 1, 2, \cdots t$.

Proof: Follows from preceding lemma. □

The following corollary is important to describe the Auslander-Reiten quiver of iterated tilted algebras of Dynkin type.

Corollary 9 *Let $\Lambda = k\vec{\Delta}/I$ be a schurian algebra with I an admissible ideal, $\vec{\Delta}$ a quiver without oriented cycles.*
Let $\alpha_i : a_i \to b_i$ be arrows in $\vec{\Delta}_{T(\Lambda)}$ for $i = 1, 2, \cdots t$. If Λ is of finite representation type, then the following conditions are equivalent for a $T(\Lambda)$-module M.

a) M is a $T(\Lambda)/ < \overline{\alpha}_1, \cdots, \overline{\alpha}_t >$ - module.

b) Any chain of irreducible maps

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_j = M \xrightarrow{f_{j+1}} X_{j+1} \to \cdots \xrightarrow{f_r} X_r$$

with $X_0 = P_{a_i}, X_r = P_{b_i}$ has zero composition, for all $i = 1, 2, \cdots t$.

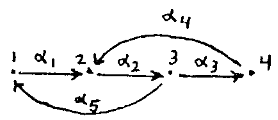
Proof: According to Theorem 8 we only need to prove that if $T(\Lambda)$ is of finite representation type then $\vec{\Delta}_\Lambda$ has no oriented cycles. This was proven by K. Yamagata in [8]. □

Assume now that Λ is an iterated tilted algebra of Dynkin type $\vec{\Delta}$. Then $T(\Lambda)$ is of finite representation type (see [1] and [2]), so the Corollary applies. It follows from [1] pag 176 that there is a tilted algebra Λ' such that $T(\Lambda) \simeq T(\Lambda')$. In many cases one can even choose such Λ' to be hereditary.

To describe the Auslander-Reiten quiver of Λ we proceed in the following way. We start by describing the quiver and relations of $T(\Lambda)$, using the description of Λ. To describe $\vec{\Gamma}_{T(\Lambda)}$ we look for an algebra Λ' such that $T(\Lambda')$ can be described, and $T(\Lambda) \simeq T(\Lambda')$. For example, we know ([7]) how to describe $T(\Lambda')$ if Λ' is hereditary. The algebras Λ' such that $T(\Lambda) \simeq T(\Lambda')$ are easily constructed ([5]): for each cycle c in $T(\Lambda)$ we choose exactly one arrow α_c. Then the quotient algebra $\Lambda' = T(\Lambda)/ < \{\alpha_c\}_c >$ satisfies $T(\Lambda) \simeq T(\Lambda')$. In the following examples we show how this can be done, and how to describe $\vec{\Gamma}_\Lambda$.

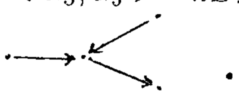
Example 2.1 *Let $\vec{Q}$ be a quiver* $\overset{1}{\cdot} \xrightarrow{\alpha_1} \overset{2}{\cdot} \xrightarrow{\alpha_2} \overset{3}{\cdot} \xrightarrow{\alpha_3} \overset{4}{\cdot}$ *and $\Lambda = k\vec{Q}/I$, where I is generated by $\alpha_3\alpha_2\alpha_1$.*

Then $\vec{\Delta}_{T(\Lambda)}$ is

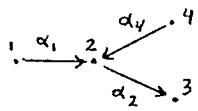


with the corresponding relations, as described in section 1.
We choose the arrows α_5 in the cycle $\alpha_5\alpha_2\alpha_1$, and α_3 in the cycle $\alpha_3\alpha_2\alpha_4$. Then $\Lambda' = T(\Lambda)/ < \alpha_3, \alpha_5 >= kD_4$ is the hereditary algebra given by the quiver

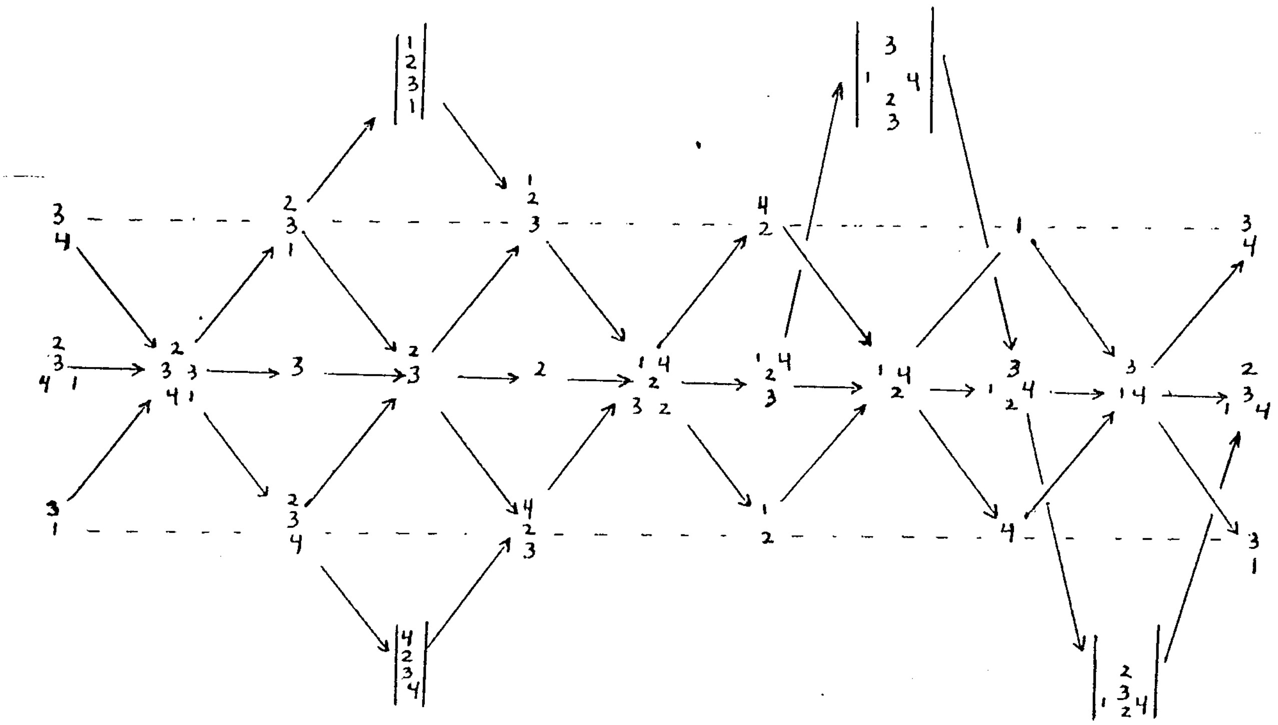
D_4 :



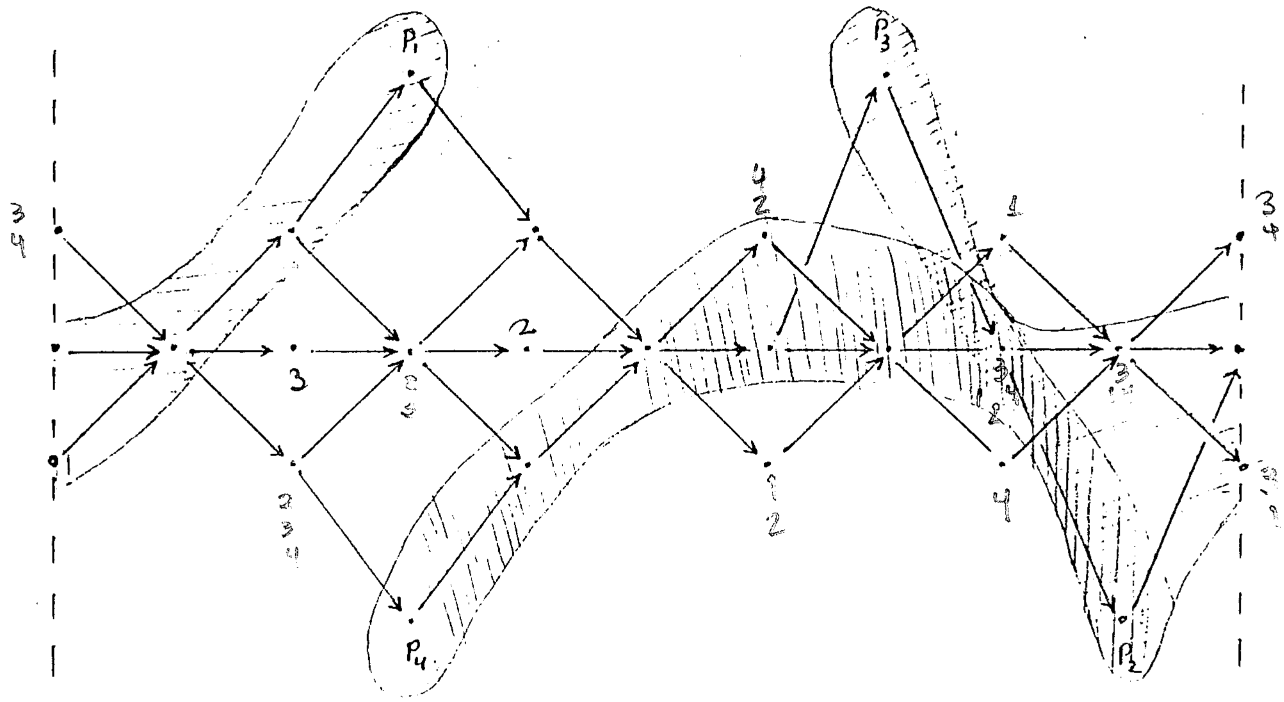
Since $T(\Lambda) \simeq T(kD_4)$ it follows that Λ in an iterated tilted algebra of tipe D_4 =



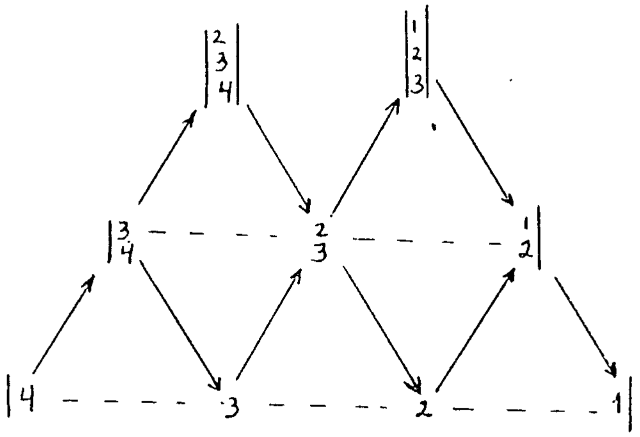
and $\vec{\Gamma}_{T(\Lambda)}$ *is given by:*

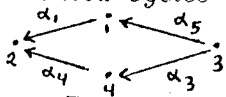


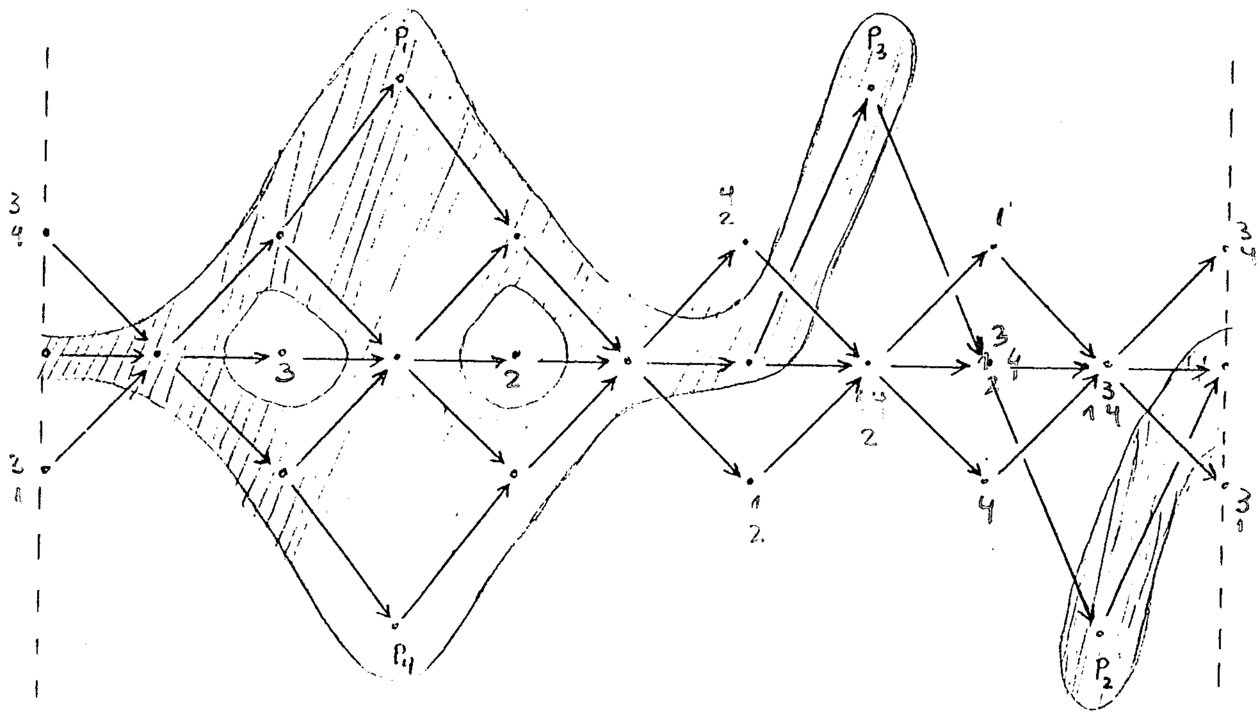
Now we use Corollary 9 to describe $\vec{\Gamma}_{\Lambda}$ *inside* $\vec{\Gamma}_{T(\Lambda)} = \vec{\Gamma}_{T(D_4)}$. *For this purpose we write* $\Lambda = k(T(D_4))/ < \alpha_4, \alpha_5 >$, *an look for the non zero paths in* $\vec{\Gamma}_{T(\Lambda)}$ *from* P_3 *to* P_1 *and from* P_4 *to* P_2. *Then we delete from the quiver the modules wich occurs in those paths.*



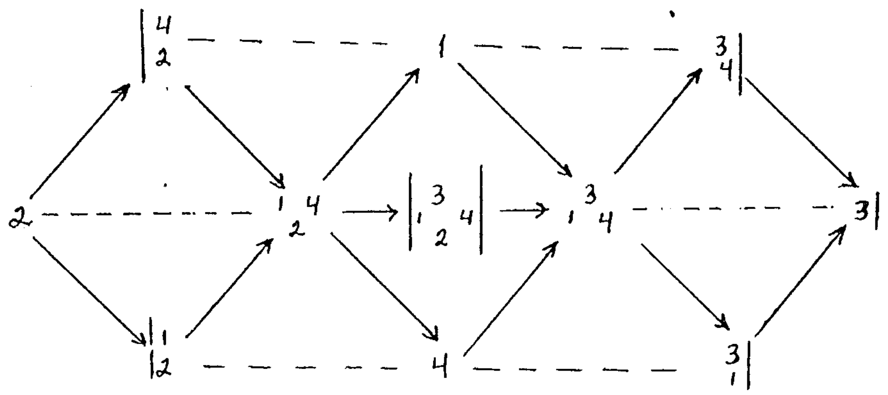
So $\vec{\Gamma}_\Lambda$ *is*



Example 2.2 *Let* Λ *as is Example 2.1 . There are other algebras* Λ" *such that* $T(\Lambda") = T(\Lambda)$. *We illustrate how to construt one of them, and its Auslander-Reiten quiver.*
We choose the arrow α_2 , *which belongs to both oriented cycles in* $T(\Lambda)$. *Then* $\Lambda" = T(\Lambda)/ < \alpha_2 >$ *is the path algebra of the quiver*  *with conmutativity relation.*
With  *we indicate the non zero paths from* P_2 *to* P_3 .



Then $\vec{\Gamma}_{\Lambda''}$ *is:*



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