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Proofs of Some Propositions of the Semi-Intuitionistic Logic with Strong Negation

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Abstract

We offer the proofs that complete our article introducing the propositional calculus called semi-intuitionistic logic with strong negation.

1 Introduction

The proofs of Lemmas 3.1 and 3.3 (Lemmas 2.1 and 2.3 below) were left out of our article [CV17]. We detail them here, together with the necessary axioms for the semi-intuitionistic logic with strong negation. Please refer to that article for motivation and more results on this calculus.

2 Semi-intuitionistic logic with strong negation

A *logical language* $\mathbf{L}$, as defined in [FJP03], is a set of connectives, each with a fixed arity $n \geq 0$. For a countably infinite set Var of propositional variables, the *formulas* of the logical language $\mathbf{L}$ are inductively defined as usual.

A *logic*, in the language $\mathbf{L}$, is a pair $\mathcal{L} = \langle Fm_{\mathbf{L}}, \vdash_{\mathcal{L}} \rangle$ where $Fm_{\mathbf{L}}$ is the set of formulas and $\vdash_{\mathcal{L}}$ is a substitution-invariant consequence relation on $Fm_{\mathbf{L}}$. As usual, the set $Fm_{\mathbf{L}}$ may also be endowed with an algebraic structure, just by regarding the connectives of the language as operation symbols. The resulting algebra is the *algebra of formulas*, denoted by $\mathsf{Fm}_{\mathbf{L}}$. The finitary logic is presented by means of their "Hilbert style" sets of axioms and inferences rules.

We define *semi-intuitionistic logic with strong negation* $\mathcal{SN}$ over the language $\mathbf{L} = \{\top, \sim, \wedge, \vee, \rightarrow\}$ in terms of the following set of axiom schemata, in which we use the following definitions:

- $\alpha \rightarrow_N \beta := \alpha \rightarrow (\alpha \wedge \beta)$,
- $\alpha \Rightarrow \beta := (\alpha \rightarrow_N \beta) \wedge (\sim \beta \rightarrow_N \sim \alpha)$.

$(A1)$ $(\alpha \rightarrow_N \beta) \rightarrow_N ((\beta \rightarrow_N \gamma) \rightarrow_N (\alpha \rightarrow_N \gamma))$,

$(A2)$ $(\alpha \rightarrow_N \beta) \rightarrow_N ((\alpha \rightarrow_N \gamma) \rightarrow_N (\alpha \rightarrow_N (\beta \wedge \gamma)))$,

$(A3)$ $(\alpha \wedge \beta) \rightarrow_N \alpha$,

$(A4)$ $(\alpha \wedge \beta) \rightarrow_N \beta$, (A5) $\alpha \to_N (\alpha \vee \beta)$,

(A6) $\beta \to_N (\alpha \vee \beta)$,

(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,

(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,

(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,

(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,

(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,

(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,

(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,

(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,

(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,

(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,

(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,

(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,

(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,

(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,

(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,

(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,

(A23) $\top$.

The only inference rule is Modus Ponens for the implication $\to_N$, which we denominate $\mathcal{N}$-*Modus Ponens* ($\mathcal{N}$-MP): $\Gamma \vdash_{\mathcal{SN}} \phi$ and $\Gamma \vdash_{\mathcal{SN}} \phi \to_N \gamma$ yield $\Gamma \vdash_{\mathcal{SN}} \gamma$.

Lemma 2.1. *Let $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. In $\mathcal{SN}$ the following properties hold:*

(a) *If $\Gamma \vdash \alpha$ then $\Gamma \vdash \beta \to_N \alpha$,*

(b) $\Gamma \vdash \alpha \to_N \alpha$,

(c) *If $\Gamma \vdash \alpha \Rightarrow \beta$ then $\Gamma \vdash \alpha \to_N \beta$ and $\Gamma \vdash \sim \beta \to_N \sim \alpha$,*

(d) $\Gamma \vdash \sim \alpha \to_N \sim (\alpha \wedge \beta)$,

(e) $\Gamma \vdash \sim \beta \to_N \sim (\alpha \wedge \beta)$,

(f) $\Gamma, \alpha, \alpha \Rightarrow \beta \vdash \beta$,

(g) *If $\Gamma \vdash \alpha \Rightarrow \beta$ and $\Gamma \vdash \alpha$ then $\Gamma \vdash \beta$,*

(h) *If $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$ then $\Gamma \vdash \alpha \wedge \beta$,*

(i) $\Gamma \vdash \alpha \wedge \beta \Rightarrow \alpha$ *and* $\Gamma \vdash \alpha \wedge \beta \Rightarrow \beta$,

(j) $\Gamma \vdash \alpha \Rightarrow \alpha \vee \beta$ *and* $\Gamma \vdash \alpha \Rightarrow \beta \vee \alpha$,

(k) $\Gamma \vdash \alpha \Rightarrow \alpha$,

(l) *If* $\Gamma \vdash \alpha \Rightarrow \beta$ *and* $\Gamma \vdash \beta \Rightarrow \gamma$ *then* $\Gamma \vdash \alpha \Rightarrow \gamma$,

(m) $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash \alpha \Rightarrow \gamma$,

(n) $\Gamma \vdash \alpha \rightarrow_N \beta$ *then* $\Gamma \vdash (\gamma \wedge \alpha) \rightarrow_N (\gamma \wedge \beta)$ *and* $\Gamma \vdash (\alpha \wedge \gamma) \rightarrow_N (\beta \wedge \gamma)$,

(o) $\Gamma \vdash \alpha \rightarrow_N \beta$ *then* $\Gamma \vdash (\gamma \vee \alpha) \rightarrow_N (\gamma \vee \beta)$ *and* $\Gamma \vdash (\alpha \vee \gamma) \rightarrow_N (\beta \vee \gamma)$,

(p) $\Gamma \vdash (\alpha \vee \beta) \rightarrow_N (\beta \vee \alpha)$,

(q) $\Gamma \vdash (\alpha \wedge \beta) \rightarrow_N (\beta \wedge \alpha)$,

(r) $\Gamma, \alpha \Rightarrow \beta \vdash (\alpha \vee \gamma) \Rightarrow (\beta \vee \gamma)$,

(s) $\Gamma, \alpha \Rightarrow \beta \vdash (\gamma \vee \alpha) \Rightarrow (\gamma \vee \beta)$,

(t) $\Gamma, \alpha \Rightarrow \beta, \gamma \Rightarrow t \vdash (\alpha \vee \gamma) \Rightarrow (\beta \vee t)$,

(u) $\Gamma, \beta \Rightarrow \alpha \vdash (\sim \alpha) \Rightarrow (\sim \beta)$,

(v) $\Gamma \vdash (\sim (\alpha \rightarrow \beta)) \rightarrow_N (\sim (\alpha \rightarrow_N \beta))$.

Proof. (a)

1. $\Gamma \vdash [(\alpha \wedge \beta) \rightarrow_N \alpha] \Rightarrow [\alpha \rightarrow_N (\beta \rightarrow_N \alpha)]$ by axiom $(A15)$.
2. $\Gamma \vdash [[(\alpha \wedge \beta) \rightarrow_N \alpha] \Rightarrow [\alpha \rightarrow_N (\beta \rightarrow_N \alpha)]] \rightarrow_N [[(\alpha \wedge \beta) \rightarrow_N \alpha] \rightarrow_N [\alpha \rightarrow_N (\beta \rightarrow_N \alpha)]]$ by axiom $(A3)$.
3. $\Gamma \vdash [[(\alpha \wedge \beta) \rightarrow_N \alpha] \rightarrow_N [\alpha \rightarrow_N (\beta \rightarrow_N \alpha)]$ by ($\mathcal{N}$-MP) applied to 1 and 2.
4. $\Gamma \vdash (\alpha \wedge \beta) \rightarrow_N \alpha$ by axiom $(A3)$.
5. $\Gamma \vdash \alpha \rightarrow_N (\beta \rightarrow_N \alpha)$ by ($\mathcal{N}$-MP) applied to 3 and 4.
6. $\Gamma \vdash \alpha$ by hypothesis.
7. $\Gamma \vdash \beta \rightarrow_N \alpha$ by ($\mathcal{N}$-MP) applied to 5 and 6.

(b) Let ϕ be any axiom of $\mathcal{SN}$.

1. $\Gamma \vdash \phi$.
2. $\Gamma \vdash [(\phi \wedge \alpha) \rightarrow_N \alpha] \Rightarrow [\phi \rightarrow_N (\alpha \rightarrow_N \alpha)]$ by axiom $(A15)$.
3. $\Gamma \vdash \{[(\phi \wedge \alpha) \rightarrow_N \alpha] \Rightarrow [\phi \rightarrow_N (\alpha \rightarrow_N \alpha)]\} \rightarrow_N \{[(\phi \wedge \alpha) \rightarrow_N \alpha] \rightarrow_N [\phi \rightarrow_N (\alpha \rightarrow_N \alpha)]\}$ by axiom $(A3)$.
4. $\Gamma \vdash [(\phi \wedge \alpha) \rightarrow_N \alpha] \rightarrow_N [\phi \rightarrow_N (\alpha \rightarrow_N \alpha)]$ by ($\mathcal{N}$-MP) applied to 2 and 3.
5. $\Gamma \vdash (\phi \wedge \alpha) \rightarrow_N \alpha$ by axiom $(A4)$.
6. $\Gamma \vdash \phi \rightarrow_N (\alpha \rightarrow_N \alpha)$ by ($\mathcal{N}$-MP) applied to 4 and 5.
7. $\Gamma \vdash \alpha \rightarrow_N \alpha$ by ($\mathcal{N}$-MP) applied to 1 and 6.

(c)

1. $\Gamma \vdash \alpha \Rightarrow \beta$ by hypothesis.
2. $\Gamma \vdash (\alpha \rightarrow_N \beta) \wedge (\sim \beta \rightarrow_N \sim \alpha)$ by the definition of $\Rightarrow$.

3. $\Gamma \vdash [(\alpha \to_N \beta) \wedge (\sim \beta \to_N \sim \alpha)] \to_N (\alpha \to_N \beta)$ by axiom $(A3)$.
4. $\Gamma \vdash \alpha \to_N \beta$ by ($\mathcal{N}$-MP).
5. $\Gamma \vdash [(\alpha \to_N \beta) \wedge (\sim \beta \to_N \sim \alpha)] \to_N (\sim \beta \to_N \sim \alpha)$ by axiom $(A4)$.
6. $\Gamma \vdash \sim \beta \to_N \sim \alpha$ by ($\mathcal{N}$-MP).

(d)
1. $\Gamma \vdash (\sim \alpha) \to_N (\sim \alpha \vee \sim \beta)$ by axiom $(A5)$.
2. $\Gamma \vdash (\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$ by axiom $(A17)$.
3. $\Gamma \vdash (\sim \alpha \vee \sim \beta) \to_N (\sim (\alpha \wedge \beta))$ by part (c) applied to 2.
4. $\Gamma \vdash (\sim \alpha) \to_N (\sim (\alpha \wedge \beta))$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 1 and 3.

(e)
1. $\Gamma \vdash (\sim \beta) \to_N (\sim \alpha \vee \sim \beta)$ by axiom $(A6)$.
2. $\Gamma \vdash (\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$ by axiom $(A17)$.
3. $\Gamma \vdash (\sim \alpha \vee \sim \beta) \to_N (\sim (\alpha \wedge \beta))$ by part (c) applied to 2.
4. $\Gamma \vdash (\sim \beta) \to_N (\sim (\alpha \wedge \beta))$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 1 and 3.

(f)
1. $\Gamma, \alpha, \alpha \Rightarrow \beta \vdash \alpha \Rightarrow \beta$.
2. $\Gamma, \alpha, \alpha \Rightarrow \beta \vdash \alpha \to_N \beta$ by part (c).
3. $\Gamma, \alpha, \alpha \Rightarrow \beta \vdash \alpha$
4. $\Gamma, \alpha, \alpha \Rightarrow \beta \vdash \beta$ by ($\mathcal{N}$-MP) applied to 2 and 3.

(g)
1. $\Gamma \vdash \alpha \Rightarrow \beta$ by hypothesis.
2. $\Gamma \vdash \alpha \to_N \beta$ by part (c).
3. $\Gamma \vdash \alpha$ by hypothesis.
4. $\Gamma \vdash \beta$ by ($\mathcal{N}$-MP) applied to 2 and 3.

(h)
1. $\Gamma \vdash [(\beta \wedge \alpha) \to_N \beta] \Rightarrow [\beta \to_N (\alpha \to_N \beta)]$ by axiom $(A15)$.
2. $\Gamma \vdash (\beta \wedge \alpha) \to_N \beta$ by axiom $(A3)$.
3. $\Gamma \vdash \beta \to_N (\alpha \to_N \beta)$ by (g) applied to 1 and 2.
4. $\Gamma \vdash \beta$ by hypothesis.
5. $\Gamma \vdash \alpha \to_N \beta$ by ($\mathcal{N}$-MP) applied to 3 and 4.
6. $\Gamma \vdash (\alpha \to_N \alpha) \to_N ((\alpha \to_N \beta) \to_N (\alpha \to_N (\alpha \wedge \beta)))$ by axiom $(A2)$.
7. $\Gamma \vdash \alpha \to_N \alpha$ by part (b).
8. $\Gamma \vdash (\alpha \to_N \beta) \to_N (\alpha \to_N (\alpha \wedge \beta))$ by ($\mathcal{N}$-MP) applied to 6 and 7.
9. $\Gamma \vdash \alpha \to_N (\alpha \wedge \beta)$ by ($\mathcal{N}$-MP) applied to 5 and 8.
10. $\Gamma \vdash \alpha$ by hypothesis.
11. $\Gamma \vdash \alpha \wedge \beta$ by ($\mathcal{N}$-MP) applied to 9 and 10.

(i) Follows immediately from items (h) and (d), axioms $(A3)$, $(A4)$ and item (e).

(j) Is a direct consequence of item (h) and axioms $(A5)$, $(A6)$, $(A7)$, and $(A8)$.

(k)
1. $\Gamma \vdash \alpha \to_N \alpha$ by part (b).
2. $\Gamma \vdash (\sim \alpha) \to_N (\sim \alpha)$ by part (b).

3. $\Gamma \vdash (\alpha \to_N \alpha) \wedge ((\sim \alpha) \to_N (\sim \alpha))$ by part (h).

(l)
1. $\Gamma \vdash \alpha \Rightarrow \beta$ by hypothesis.
2. $\Gamma \vdash \alpha \to_N \beta$ by 1 and (c).
3. $\Gamma \vdash \beta \Rightarrow \gamma$ by hypothesis.
4. $\Gamma \vdash \beta \to_N \gamma$ by 3 and (c).
5. $\Gamma \vdash (\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$ by axiom $(A1)$.
6. $\Gamma \vdash (\beta \to_N \gamma) \to_N (\alpha \to_N \gamma)$ by ($\mathcal{N}$-MP) applied to 2 and 5.
7. $\Gamma \vdash \alpha \to_N \gamma$ by ($\mathcal{N}$-MP) applied to 4 and 6.
8. $\Gamma \vdash (\alpha \Rightarrow \beta) \to_N (\sim \beta \to_N \sim \alpha)$ by axiom $(A4)$.
9. $\Gamma \vdash \sim \beta \to_N \sim \alpha$ by ($\mathcal{N}$-MP) applied to 1 and 8.
10. $\Gamma \vdash (\beta \Rightarrow \gamma) \to_N (\sim \gamma \to_N \sim \beta)$ by axiom $(A4)$.
11. $\Gamma \vdash \sim \gamma \to_N \sim \beta$ by ($\mathcal{N}$-MP) applied to 3 and 10.
12. $\Gamma \vdash (\sim \gamma \to_N \sim \beta) \to_N ((\sim \beta \to_N \sim \alpha) \to_N (\sim \gamma \to_N \sim \alpha))$ by axiom $(A1)$.
13. $\Gamma \vdash (\sim \beta \to_N \sim \alpha) \to_N (\sim \gamma \to_N \sim \alpha)$ by ($\mathcal{N}$-MP) applied to 11 and 12.
14. $\Gamma \vdash \sim \gamma \to_N \sim \alpha$ by ($\mathcal{N}$-MP) applied to 9 and 13.
15. $\Gamma \vdash \alpha \Rightarrow \gamma$ by part (h) applied to 7 and 14.

(m)
1. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash \alpha \Rightarrow \beta$
2. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash \beta \Rightarrow \gamma$
3. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash \alpha \Rightarrow \gamma$ by part (l).

(n)
1. $\Gamma \vdash ((\gamma \wedge \alpha) \to_N \gamma) \to_N [((\gamma \wedge \alpha) \to_N \beta) \to_N [(\gamma \wedge \alpha) \to_N (\gamma \wedge \beta)]]$ by axiom $(A2)$.
2. $\Gamma \vdash (\gamma \wedge \alpha) \to_N \gamma$ by axiom $(A3)$.
3. $\Gamma \vdash ((\gamma \wedge \alpha) \to_N \beta) \to_N [(\gamma \wedge \alpha) \to_N (\gamma \wedge \beta)]$ by ($\mathcal{N}$-MP) applied to 1 and 2.
4. $\Gamma \vdash (\gamma \wedge \alpha) \to_N \alpha$ by axiom $(A4)$
5. $\Gamma \vdash \alpha \to_N \beta$ by hypothesis.
6. $\Gamma \vdash (\gamma \wedge \alpha) \to_N \beta$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 4 and 5.
7. $\Gamma \vdash (\gamma \wedge \alpha) \to_N (\gamma \wedge \beta)$ by ($\mathcal{N}$-MP) applied to 3 and 6.
8. $\Gamma \vdash ((\alpha \wedge \gamma) \to_N \beta) \to_N [((\alpha \wedge \gamma) \to_N \gamma) \to_N [(\alpha \wedge \gamma) \to_N (\beta \wedge \gamma)]]$ by axiom $(A2)$.
9. $\Gamma \vdash (\alpha \wedge \gamma) \to_N \alpha$ by axiom $(A3)$.
10. $\Gamma \vdash (\alpha \wedge \gamma) \to_N \beta$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 9 and 5.
11. $\Gamma \vdash ((\alpha \wedge \gamma) \to_N \gamma) \to_N [(\alpha \wedge \gamma) \to_N (\beta \wedge \gamma)]$ by ($\mathcal{N}$-MP) applied to 8 and 10.
12. $\Gamma \vdash (\alpha \wedge \gamma) \to_N \gamma$ by axiom $(A4)$.
13. $\Gamma \vdash (\alpha \wedge \gamma) \to_N (\beta \wedge \gamma)$ by ($\mathcal{N}$-MP) applied to 11 and 12.

(o)
1. $\Gamma \vdash (\gamma \to_N (\gamma \vee \beta)) \to_N [(\alpha \to_N (\gamma \vee \beta)) \to_N [(\gamma \vee \alpha) \to_N (\gamma \vee \beta)]]$ by axiom $(A9)$.
2. $\Gamma \vdash \gamma \to_N (\gamma \vee \beta)$ by axiom $(A5)$.

3. $\Gamma \vdash (\alpha \rightarrow_N (\gamma \vee \beta)) \rightarrow_N [(\gamma \vee \alpha) \rightarrow_N (\gamma \vee \beta)]$ by ($\mathcal{N}$-MP) applied to 1 and 2.
4. $\Gamma \vdash \alpha \rightarrow_N \beta$ by hypothesis.
5. $\Gamma \vdash \beta \rightarrow_N (\gamma \vee \beta)$ by axiom ($A6$).
6. $\Gamma \vdash \alpha \rightarrow_N (\gamma \vee \beta)$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 4 and 5.
7. $\Gamma \vdash (\gamma \vee \alpha) \rightarrow_N (\gamma \vee \beta)$ by ($\mathcal{N}$-MP) applied to 3 and 6.
8. $\Gamma \vdash (\alpha \rightarrow_N (\beta \vee \gamma)) \rightarrow_N [(\gamma \rightarrow_N (\beta \vee \gamma)) \rightarrow_N [(\alpha \vee \gamma) \rightarrow_N (\beta \vee \gamma)]]$ by axiom ($A9$).
9. $\Gamma \vdash \beta \rightarrow_N (\beta \vee \gamma)$ by axiom ($A5$).
10. $\Gamma \vdash \alpha \rightarrow_N (\beta \vee \gamma)$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 4 and 9.
11. $\Gamma \vdash (\gamma \rightarrow_N (\beta \vee \gamma)) \rightarrow_N [(\alpha \vee \gamma) \rightarrow_N (\beta \vee \gamma)]$ by ($\mathcal{N}$-MP) applied to 8 and 10.
12. $\Gamma \vdash \gamma \rightarrow_N (\beta \vee \gamma)$ by axiom ($A6$).
13. $\Gamma \vdash (\alpha \vee \gamma) \rightarrow_N (\beta \vee \gamma)$ by ($\mathcal{N}$-MP) applied to 11 and 12.

(p)
1. $\Gamma \vdash \alpha \rightarrow_N (\beta \vee \alpha)$ by axiom ($A6$).
2. $\Gamma \vdash \beta \rightarrow_N (\beta \vee \alpha)$ by axiom ($A5$).
3. $\Gamma \vdash (\alpha \rightarrow_N (\beta \vee \alpha)) \rightarrow_N [(\beta \rightarrow_N (\beta \vee \alpha)) \rightarrow_N ((\alpha \vee \beta) \rightarrow_N (\beta \vee \alpha))]$ by axiom ($A9$).
4. $\Gamma \vdash (\beta \rightarrow_N (\beta \vee \alpha)) \rightarrow_N ((\alpha \vee \beta) \rightarrow_N (\beta \vee \alpha))$ by ($\mathcal{N}$-MP) applied to 1 and 3.
5. $\Gamma \vdash (\alpha \vee \beta) \rightarrow_N (\beta \vee \alpha)$ by ($\mathcal{N}$-MP) applied to 2 and 4.

(q)
1. $\Gamma \vdash (\alpha \wedge \beta) \rightarrow_N \alpha$ by axiom ($A3$).
2. $\Gamma \vdash (\alpha \wedge \beta) \rightarrow_N \beta$ by axiom ($A4$).
3. $\Gamma \vdash ((\alpha \wedge \beta) \rightarrow_N \beta) \rightarrow_N [((\alpha \wedge \beta) \rightarrow_N \alpha) \rightarrow_N ((\alpha \wedge \beta) \rightarrow_N (\beta \wedge \alpha))]$ by axiom ($A2$).
4. $\Gamma \vdash ((\alpha \wedge \beta) \rightarrow_N \alpha) \rightarrow_N ((\alpha \wedge \beta) \rightarrow_N (\beta \wedge \alpha))$ by ($\mathcal{N}$-MP) applied to 2 and 3.
5. $\Gamma \vdash (\alpha \wedge \beta) \rightarrow_N (\beta \wedge \alpha)$ by ($\mathcal{N}$-MP) applied to 1 and 4.

(r)
1. $\Gamma, \alpha \Rightarrow \beta \vdash \gamma \rightarrow_N (\beta \vee \gamma)$ by axiom ($A6$).
2. $\Gamma, \alpha \Rightarrow \beta \vdash \beta \Rightarrow (\beta \vee \gamma)$ by part (j).
3. $\Gamma, \alpha \Rightarrow \beta \vdash \alpha \Rightarrow \beta$
4. $\Gamma, \alpha \Rightarrow \beta \vdash \alpha \rightarrow_N \beta$ by part (c) applied to 3.
5. $\Gamma, \alpha \Rightarrow \beta \vdash \alpha \Rightarrow (\beta \vee \gamma)$ by part (l) applied to 3 and 2.
6. $\Gamma, \alpha \Rightarrow \beta \vdash (\alpha \vee \gamma) \rightarrow_N (\beta \vee \gamma)$ by part (o) applied to 4.
7. $\Gamma, \alpha \Rightarrow \beta \vdash \sim (\beta \vee \gamma) \rightarrow_N \sim \gamma$ by axiom ($A8$).
8. $\Gamma, \alpha \Rightarrow \beta \vdash \sim (\beta \vee \gamma) \rightarrow_N \sim \alpha$ by part (c) applied to 5.
9. $\Gamma, \alpha \Rightarrow \beta \vdash (\sim (\beta \vee \gamma) \rightarrow_N \sim \alpha) \rightarrow_N [[\sim (\beta \vee \gamma) \rightarrow_N \sim \gamma] \rightarrow_N [\sim (\beta \vee \gamma) \rightarrow_N \sim (\alpha \vee \gamma)]]$ by axiom ($A10$).
10. $\Gamma, \alpha \Rightarrow \beta \vdash [\sim (\beta \vee \gamma) \rightarrow_N \sim \gamma] \rightarrow_N [\sim (\beta \vee \gamma) \rightarrow_N \sim (\alpha \vee \gamma)]$ by ($\mathcal{N}$-MP) applied to 8 and 9.
11. $\Gamma, \alpha \Rightarrow \beta \vdash \sim (\beta \vee \gamma) \rightarrow_N \sim (\alpha \vee \gamma)$ by ($\mathcal{N}$-MP) applied to 7 and 10.
12. $\Gamma, \alpha \Rightarrow \beta \vdash (\alpha \vee \gamma) \Rightarrow (\beta \vee \gamma)$ by part (h) applied to 6 and 11.

(s) 1. $\Gamma, \alpha \Rightarrow \beta \vdash \beta \Rightarrow (\gamma \vee \beta)$ by part (j).

2. $\Gamma, \alpha \Rightarrow \beta \vdash \alpha \Rightarrow \beta$.

3. $\Gamma, \alpha \Rightarrow \beta \vdash \alpha \to_N \beta$ by part (c).

4. $\Gamma, \alpha \Rightarrow \beta \vdash \alpha \Rightarrow (\gamma \vee \beta)$ by part (l) applied to 2 and 1.

5. $\Gamma, \alpha \Rightarrow \beta \vdash (\gamma \vee \alpha) \to_N (\gamma \vee \beta)$ by part (o) and 3.

6. $\Gamma, \alpha \Rightarrow \beta \vdash \sim (\gamma \vee \beta) \to_N \sim \gamma$ by axiom $(A7)$.

7. $\Gamma, \alpha \Rightarrow \beta \vdash \sim (\gamma \vee \beta) \to_N \sim \alpha$ by part (c) applied to 4.

8. $\Gamma, \alpha \Rightarrow \beta \vdash (\sim (\gamma \vee \beta) \to_N \sim \gamma) \to_N [[\sim (\gamma \vee \beta) \to_N \sim \alpha] \to_N [\sim (\gamma \vee \beta) \to_N \sim (\gamma \vee \alpha)]]$ by axiom $(A10)$.

9. $\Gamma, \alpha \Rightarrow \beta \vdash [\sim (\gamma \vee \beta) \to_N \sim \alpha] \to_N [\sim (\gamma \vee \beta) \to_N \sim (\gamma \vee \alpha)]$ by ($\mathcal{N}$-MP) applied to 6 and 8.

10. $\Gamma, \alpha \Rightarrow \beta \vdash \sim (\gamma \vee \beta) \to_N \sim (\gamma \vee \alpha)$ by ($\mathcal{N}$-MP) applied to 7 and 9.

11. $\Gamma, \alpha \Rightarrow \beta \vdash (\gamma \vee \alpha) \Rightarrow (\gamma \vee \beta)$ by part (h) applied to 5 and 10.

(t) 1. $\Gamma, \alpha \Rightarrow \beta, \gamma \Rightarrow t \vdash (\alpha \vee \gamma) \Rightarrow (\beta \vee \gamma)$ by part (r).

2. $\Gamma, \alpha \Rightarrow \beta, \gamma \Rightarrow t \vdash (\beta \vee \gamma) \Rightarrow (\beta \vee t)$ by part (s).

3. $\Gamma, \alpha \Rightarrow \beta, \gamma \Rightarrow t \vdash (\alpha \vee \gamma) \Rightarrow (\beta \vee t)$ by part (l).

(u) 1. $\Gamma, \beta \Rightarrow \alpha \vdash \alpha \Rightarrow \sim\sim \alpha$ by axiom $(A11)$.

2. $\Gamma, \beta \Rightarrow \alpha \vdash \beta \Rightarrow \alpha$

3. $\Gamma, \beta \Rightarrow \alpha \vdash \beta \Rightarrow \sim\sim \alpha$ by part (l) applied to 2 and 1.

4. $\Gamma, \beta \Rightarrow \alpha \vdash \sim\sim \beta \Rightarrow \beta$ by axiom $(A12)$.

5. $\Gamma, \beta \Rightarrow \alpha \vdash \sim\sim \beta \Rightarrow \sim\sim \alpha$ by part (l) applied to 4 and 3.

6. $\Gamma, \beta \Rightarrow \alpha \vdash \sim\sim \beta \to_N \sim\sim \alpha$ by part (c).

7. $\Gamma, \beta \Rightarrow \alpha \vdash \sim \alpha \to_N \sim \beta$ by part (c) applied to 2.

8. $\Gamma, \beta \Rightarrow \alpha \vdash (\sim \alpha \to_N \sim \beta) \wedge (\sim\sim \beta \to_N \sim\sim \alpha)$ by part (h) applied to 6 and 7.

9. $\Gamma, \beta \Rightarrow \alpha \vdash \sim \alpha \Rightarrow \sim \beta$ by the definition of $\Rightarrow$ applied to 8.

(v) 1. $\Gamma \vdash (\sim (\alpha \to \beta)) \to_N (\alpha \wedge (\sim \beta))$ by axiom $(A20)$.

2. $\Gamma \vdash \sim \beta \to_N (\sim (\alpha \wedge \beta))$ by part (e).

3. $\Gamma \vdash (\alpha \wedge \sim \beta) \to_N [\alpha \wedge (\sim (\alpha \wedge \beta))]$ by 2 and part (n).

4. $\Gamma \vdash (\sim (\alpha \to \beta)) \to_N [\alpha \wedge (\sim (\alpha \wedge \beta))]$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 1 and 3.

5. $\Gamma \vdash [\alpha \wedge (\sim (\alpha \wedge \beta))] \to_N (\sim (\alpha \to_N \beta))$ by axiom $(A21)$.

6. $\Gamma \vdash (\sim (\alpha \to \beta)) \to_N (\sim (\alpha \to_N \beta))$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 4 and 5. □

Theorem 2.2. *(Deduction Theorem) Let* $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. *Then*

$$\Gamma \vdash \alpha \to_N \beta \text{ if and only if } \Gamma, \alpha \vdash \beta$$

Proof. For one implication we have:

1. $\Gamma \vdash \alpha \to_N \beta$ by hypothesis.

2. $\Gamma, \alpha \vdash \alpha \to_N \beta$.

3. $\Gamma, \alpha \vdash \alpha$.

4. $\Gamma, \alpha \vdash \beta$ by ($\mathcal{N}$-MP) applied to 2 and 3.

For the other one, assume that $\Gamma, \alpha \vdash \beta$. We prove the result by induction on the lenght of the proof of $\Gamma, \alpha \vdash \beta$.

- If $\vdash \beta$ or $\beta \in \Gamma$ then $\Gamma \vdash \beta$. By Lemma 2.1 (a) we have that $\vdash \alpha \to_N \beta$. Consequently, $\Gamma \vdash \alpha \to_N \beta$.

- If $\beta = \alpha$, using Lemma 2.1 (b), $\Gamma \vdash \alpha \to_N \beta$.

- If β comes from applying the inference rule then there exist $\gamma \in Fm_{\mathbf{L}}$ such that $\Gamma, \alpha \vdash \gamma$ and $\Gamma, \alpha \vdash \gamma \to_N \beta$. Then

  1. $\Gamma \vdash \alpha \to_N (\gamma \to_N \beta)$ by inductive hypothesis.
  2. $\Gamma \vdash \alpha \to_N \gamma$ by inductive hypothesis.
  3. $\Gamma \vdash [\alpha \to_N (\gamma \to_N \beta)] \Rightarrow [(\alpha \wedge \gamma) \to_N \beta]$ by axiom ($A19$).
  4. $\Gamma \vdash [[\alpha \to_N (\gamma \to_N \beta)] \Rightarrow [(\alpha \wedge \gamma) \to_N \beta]] \to_N [[\alpha \to_N (\gamma \to_N \beta)] \to_N [(\alpha \wedge \gamma) \to_N \beta]]$ by axiom ($A3$).
  5. $\Gamma \vdash [\alpha \to_N (\gamma \to_N \beta)] \to_N [(\alpha \wedge \gamma) \to_N \beta]$ by ($\mathcal{N}$-MP) applied to 3 and 4.
  6. $\Gamma \vdash (\alpha \wedge \gamma) \to_N \beta$ by ($\mathcal{N}$-MP) applied to 1 and 5.
  7. $\Gamma \vdash \alpha \to_N \alpha$ by Lemma 2.1 (b).
  8. $\Gamma \vdash (\alpha \to_N \alpha) \to_N [(\alpha \to_N \gamma) \to_N (\alpha \to_N (\alpha \wedge \gamma))]$ by axiom ($A2$).
  9. $\Gamma \vdash (\alpha \to_N \gamma) \to_N (\alpha \to_N (\alpha \wedge \gamma))$ by ($\mathcal{N}$-MP) applied to 7 and 8.
  10. $\Gamma \vdash \alpha \to_N (\alpha \wedge \gamma)$ by ($\mathcal{N}$-MP) applied to 2 and 9.
  11. $\Gamma \vdash (\alpha \to_N (\alpha \wedge \gamma)) \to_N [[(\alpha \wedge \gamma) \to_N \beta] \to_N [\alpha \to_N \beta]]$ by axiom ($A1$).
  12. $\Gamma \vdash [(\alpha \wedge \gamma) \to_N \beta] \to_N [\alpha \to_N \beta]$ by ($\mathcal{N}$-MP) applied to 10 and 11.
  13. $\Gamma \vdash \alpha \to_N \beta$ by ($\mathcal{N}$-MP) applied to 6 and 12. □

We use $\alpha \leftrightarrow_N \beta$ as an abbreviation for the formula $(\alpha \to_N \beta) \wedge (\beta \to_N \alpha)$.

Lemma 2.3. *Let $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. In $\mathcal{SN}$ the following properties hold:*

(a) $\Gamma \vdash (\sim \alpha \wedge \sim \beta) \leftrightarrow_N \sim (\alpha \vee \beta)$,

(b) $\Gamma \vdash (\alpha \wedge (\alpha \to_N \beta)) \Rightarrow (\alpha \wedge (\sim \alpha \vee \beta))$,

(c) *If* $\Gamma \vdash \alpha \leftrightarrow_N \beta$ *then* $\Gamma \vdash \beta \leftrightarrow_N \alpha$,

(d) $\Gamma \vdash \alpha \Rightarrow (\alpha \wedge (\alpha \vee \beta))$,

(e) $\Gamma \vdash [\alpha \wedge [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \Rightarrow [\alpha \wedge (\beta \vee \gamma)]$,

(f) $\Gamma \vdash (\alpha \wedge (\beta \vee \gamma)) \Rightarrow [\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha))]$,

(g) $\Gamma \vdash (\sim\beta \to_N \sim\alpha) \to_N ((\sim\gamma \to_N \sim\alpha) \to_N (\sim(\beta\wedge\gamma) \to_N \sim\alpha))$,

(h) *If* $\Gamma \vdash \alpha \Rightarrow \beta$ *and* $\Gamma \vdash \alpha \Rightarrow \gamma$ *then* $\Gamma \vdash \alpha \Rightarrow \beta\wedge\gamma$ *and* $\Gamma \vdash \alpha \Rightarrow \gamma\wedge\beta$,

(i) *If* $\Gamma \vdash \alpha \Rightarrow \beta$ *then* $\Gamma \vdash \alpha \Rightarrow (\beta\vee\gamma)$ *and* $\Gamma \vdash \alpha \Rightarrow (\gamma\vee\beta)$,

(j) *If* $\Gamma \vdash \alpha \Rightarrow \beta$ *then* $\Gamma \vdash \alpha\wedge\gamma \Rightarrow \beta$ *and* $\Gamma \vdash \gamma\wedge\alpha \Rightarrow \beta$,

(k) $\Gamma, \alpha \Rightarrow \beta \vdash (\alpha\wedge\gamma) \Rightarrow (\beta\wedge\gamma)$,

(l) $\Gamma, \alpha \Rightarrow \beta \vdash (\gamma\wedge\alpha) \Rightarrow (\gamma\wedge\beta)$,

(m) $\Gamma, \alpha \Rightarrow \beta, \gamma \Rightarrow t \vdash (\alpha\wedge\gamma) \Rightarrow (\beta\wedge t)$,

(n) $\Gamma \vdash [\sim(\alpha \to_N \beta)] \to_N (\alpha\wedge\sim\beta)$,

(o) $\Gamma \vdash (\alpha\wedge\sim\alpha) \to_N \beta$,

(p) $\Gamma \vdash (\alpha\wedge\sim\alpha) \Rightarrow (\beta\vee\sim\beta)$,

(q) $\Gamma \vdash (\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\sim(\beta\to\gamma)) \to_N (\sim(\alpha\to\gamma))]]$,

(r) $\Gamma \vdash (\sim\alpha \to_N \sim\beta) \to_N [(\sim\beta \to_N \sim\alpha) \to_N [(\sim(\gamma\to\alpha)) \to_N (\sim(\gamma\to\beta))]]$,

(s) $\Gamma \vdash \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\alpha\to\gamma) \Rightarrow (\beta\to t)$,

(t) $\Gamma, \alpha \vdash \beta \Rightarrow \alpha$,

(u) $\Gamma \vdash (\alpha\to\beta) \to_N (\alpha \to_N \beta)$.

Proof. (a)

1. $\Gamma \vdash (\sim\sim\alpha\vee\sim\sim\beta) \Rightarrow \sim(\sim\alpha\wedge\sim\beta)$ by axiom ($A17$).
2. $\Gamma \vdash \alpha \Rightarrow (\sim\sim\alpha)$ by axiom ($A11$).
3. $\Gamma, \alpha \Rightarrow (\sim\sim\alpha) \vdash (\alpha\vee\beta) \Rightarrow (\sim\sim\alpha\vee\beta)$ by Lemma 2.1 (r).
4. $\Gamma \vdash (\alpha \Rightarrow (\sim\sim\alpha)) \to_N [(\alpha\vee\beta) \Rightarrow (\sim\sim\alpha\vee\beta)]$ by Theorem 2.2.
5. $\Gamma \vdash (\alpha\vee\beta) \Rightarrow (\sim\sim\alpha\vee\beta)$ by ($\mathcal{N}$-MP) applied to 2 and 4.
6. $\Gamma \vdash \beta \Rightarrow (\sim\sim\beta)$ by axiom ($A11$).
7. $\Gamma, \beta \Rightarrow (\sim\sim\beta) \vdash (\sim\sim\alpha\vee\beta) \Rightarrow (\sim\sim\alpha\vee\sim\sim\beta)$ by Lemma 2.1 (s).
8. $\Gamma \vdash (\beta \Rightarrow (\sim\sim\beta)) \to_N [(\sim\sim\alpha\vee\beta) \Rightarrow (\sim\sim\alpha\vee\sim\sim\beta)]$ by Theorem 2.2.
9. $\Gamma \vdash (\sim\sim\alpha\vee\beta) \Rightarrow (\sim\sim\alpha\vee\sim\sim\beta)$ by ($\mathcal{N}$-MP) applied to 6 and 8.
10. $\Gamma \vdash (\alpha\vee\beta) \Rightarrow (\sim\sim\alpha\vee\sim\sim\beta)$ by Lemma 2.1 (l) applied to 5 and 9.
11. $\Gamma \vdash (\alpha\vee\beta) \Rightarrow \sim(\sim\alpha\wedge\sim\beta)$ by Lemma 2.1 (l) applied to 1 and 10.
12. $\Gamma \vdash [\sim\sim(\sim\alpha\wedge\sim\beta)] \to_N (\sim(\alpha\vee\beta))$ by Lemma 2.1 (c).
13. $\Gamma \vdash (\sim\alpha\wedge\sim\beta) \to_N (\sim\sim(\sim\alpha\wedge\sim\beta))$ by axiom ($A11$) and Lemma 2.1 (c).
14. $\Gamma \vdash (\sim\alpha\wedge\sim\beta) \to_N (\sim(\alpha\vee\beta))$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 12 and 13.
15. $\Gamma \vdash (\sim(\alpha\vee\beta)) \to_N (\sim\alpha)$ by axiom ($A7$).
16. $\Gamma \vdash (\sim(\alpha\vee\beta)) \to_N (\sim\beta)$ by axiom ($A8$).
17. $\Gamma \vdash (\sim(\alpha\vee\beta)) \to_N (\sim\alpha\wedge\sim\beta)$ by axiom ($A2$) and ($\mathcal{N}$-MP) applied to 15 and 16.

18. $\Gamma \vdash (\sim\alpha \wedge \sim\beta) \leftrightarrow_N (\sim(\alpha \vee \beta))$ by Lemma 2.1 (h) applied to 14 and 17.

(b)

1. $\Gamma \vdash (\alpha \wedge \sim\beta) \to_N \sim\beta$ by axiom ($A4$).
2. $\Gamma \vdash \sim\beta \to_N \sim(\alpha \wedge \beta)$ by 2.1 (e).
3. $\Gamma \vdash (\alpha \wedge \sim\beta) \to_N (\alpha \wedge \sim(\alpha \wedge \beta))$ by Lemma 2.1 (n).
4. $\Gamma \vdash (\alpha \wedge \sim(\alpha \wedge \beta)) \to_N (\sim(\alpha \to (\alpha \wedge \beta)))$ by axiom ($A21$).
5. $\Gamma \vdash (\alpha \wedge \sim(\alpha \wedge \beta)) \to_N (\sim(\alpha \to_N \beta))$ by definition of $\to_N$ applied to 4.
6. $\Gamma \vdash (\alpha \wedge \sim\beta) \to_N (\sim(\alpha \to_N \beta))$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 3 and 5.
7. $\Gamma \vdash (\sim\alpha \vee (\alpha \wedge \sim\beta)) \to_N (\sim\alpha \vee (\sim(\alpha \to_N \beta)))$ by Lemma 2.1 (o).
8. $\Gamma \vdash \sim(\alpha \wedge (\sim\alpha \vee \beta)) \to_N (\sim\alpha \vee \sim(\sim\alpha \vee \beta))$ by axiom ($A16$) and by Lemma 2.1 (c).
9. $\Gamma \vdash (\sim\sim\alpha \wedge \sim\beta) \leftrightarrow_N \sim(\sim\alpha \vee \beta))$ by Lemma 2.3 (a).
10. $\Gamma \vdash [(\sim\sim\alpha \wedge \sim\beta) \leftrightarrow_N \sim(\sim\alpha \vee \beta))] \to_N [(\sim(\sim\alpha \vee \beta)) \to_N (\sim\sim\alpha \wedge \sim\beta)]$ by axiom ($A4$).
11. $\Gamma \vdash (\sim(\sim\alpha \vee \beta)) \to_N (\sim\sim\alpha \wedge \sim\beta)$ by ($\mathcal{N}$-MP) applied to 9 and 10.
12. $\Gamma \vdash (\sim\alpha \vee (\sim(\sim\alpha \vee \beta))) \to_N (\sim\alpha \vee (\sim\sim\alpha \wedge \sim\beta))$ by Lemma 2.1 (o) applied to 11.
13. $\Gamma \vdash \sim(\alpha \wedge (\sim\alpha \vee \beta)) \to_N (\sim\alpha \vee (\sim\sim\alpha \wedge \sim\beta))$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 8 and 12.
14. $\Gamma \vdash \sim\sim\alpha \to_N \alpha$ by axiom ($A12$) and by Lemma 2.1 (c).
15. $\Gamma \vdash (\sim\sim\alpha \wedge \sim\beta) \to_N (\alpha \wedge \sim\beta)$ by Lemma 2.1 (n) applied to 14.
16. $\Gamma \vdash (\sim\alpha \vee (\sim\sim\alpha \wedge \sim\beta)) \to_N (\sim\alpha \vee (\alpha \wedge \sim\beta))$ by Lemma 2.1 (o) applied to 15.
17. $\Gamma \vdash (\sim\alpha \vee (\sim(\alpha \to_N \beta))) \to_N (\sim(\alpha \wedge (\alpha \to_N \beta)))$ by axiom ($A17$) and by Lemma 2.1 (c).
18. $\Gamma \vdash \sim(\alpha \wedge (\sim\alpha \vee \beta)) \to_N (\sim\alpha \vee (\alpha \wedge \sim\beta))$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 13 and 16.
19. $\Gamma \vdash \sim(\alpha \wedge (\sim\alpha \vee \beta)) \to_N (\sim\alpha \vee (\sim(\alpha \to_N \beta)))$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 18 and 7.
20. $\Gamma \vdash \sim(\alpha \wedge (\sim\alpha \vee \beta)) \to_N (\sim(\alpha \wedge (\alpha \to_N \beta)))$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 19 and 17.
21. $\Gamma, \alpha, \alpha \to_N \beta \vdash \alpha$.
22. $\Gamma, \alpha, \alpha \to_N \beta \vdash \alpha \to_N \beta$.
23. $\Gamma, \alpha, \alpha \to_N \beta \vdash \beta$ by ($\mathcal{N}$-MP) applied to 21 and 22.
24. $\Gamma, \alpha, \alpha \to_N \beta \vdash \beta \to_N (\sim\alpha \vee \beta)$ by axiom ($A6$).
25. $\Gamma, \alpha, \alpha \to_N \beta \vdash \sim\alpha \vee \beta$ by ($\mathcal{N}$-MP) applied to 23 and 24.
26. $\Gamma, \alpha, \alpha \to_N \beta \vdash \alpha \wedge (\sim\alpha \vee \beta)$ by Lemma 2.1 (h) applied to 21 and 25.
27. $\Gamma, \alpha \vdash (\alpha \to_N \beta) \to_N (\alpha \wedge (\sim\alpha \vee \beta))$ by Theorem 2.2.
28. $\Gamma \vdash \alpha \to_N [(\alpha \to_N \beta) \to_N (\alpha \wedge (\sim\alpha \vee \beta))]$ by Theorem 2.2.

29. $\Gamma \vdash [\alpha \to_N [(\alpha \to_N \beta) \to_N (\alpha \wedge (\sim \alpha \vee \beta))]] \Rightarrow [(\alpha \wedge (\alpha \to_N \beta)) \to_N (\alpha \wedge (\sim \alpha \vee \beta))]$ by axiom $(A19)$.
30. $\Gamma \vdash [\alpha \to_N [(\alpha \to_N \beta) \to_N (\alpha \wedge (\sim \alpha \vee \beta))]] \to_N [(\alpha \wedge (\alpha \to_N \beta)) \to_N (\alpha \wedge (\sim \alpha \vee \beta))]$ by Lemma 2.1 (c).
31. $\Gamma \vdash (\alpha \wedge (\alpha \to_N \beta)) \to_N (\alpha \wedge (\sim \alpha \vee \beta))$ by ($\mathcal{N}$-MP) applied to 28 and 30.
32. $\Gamma \vdash (\alpha \wedge (\alpha \to_N \beta)) \Rightarrow (\alpha \wedge (\sim \alpha \vee \beta))$ by Lemma 2.1 (h) applied to 20 and 31.

(c)
1. $\Gamma \vdash (\alpha \to_N \beta) \wedge (\beta \to_N \alpha)$ by hypothesis.
2. $\Gamma \vdash [(\alpha \to_N \beta) \wedge (\beta \to_N \alpha)] \to_N [(\beta \to_N \alpha) \wedge (\alpha \to_N \beta)]$ by Lemma 2.1 (q).
3. $\Gamma \vdash (\beta \to_N \alpha) \wedge (\alpha \to_N \beta)$ by ($\mathcal{N}$-MP).
4. $\Gamma \vdash \beta \leftrightarrow_N \alpha$ by the definition of $\leftrightarrow_N$.

(d)
1. $\Gamma \vdash \alpha \to_N \alpha$ by Lemma 2.1 (b).
2. $\Gamma \vdash \alpha \to_N (\alpha \vee \beta)$ by axiom $(A5)$.
3. $\Gamma \vdash \alpha \to_N (\alpha \wedge (\alpha \vee \beta))$ by axiom $(A2)$ and ($\mathcal{N}$-MP) applied to 1 and 2.
4. $\Gamma \vdash [\sim (\alpha \wedge (\alpha \vee \beta))] \to_N [\sim \alpha \vee \sim (\alpha \vee \beta)]$ by axiom $(A16)$ and Lemma 2.1 (c).
5. $\Gamma \vdash \sim (\alpha \vee \beta) \leftrightarrow_N (\sim \alpha \wedge \sim \beta)$ by Lemma 2.3 (a) and (c).
6. $\Gamma \vdash [\sim (\alpha \vee \beta) \leftrightarrow_N (\sim \alpha \wedge \sim \beta)] \to_N [\sim (\alpha \vee \beta) \to_N (\sim \alpha \wedge \sim \beta)]$ by axiom $(A3)$.
7. $\Gamma \vdash \sim (\alpha \vee \beta) \to_N (\sim \alpha \wedge \sim \beta)$ by ($\mathcal{N}$-MP) applied to 5 and 6.
8. $\Gamma \vdash (\sim \alpha \vee \sim (\alpha \vee \beta)) \to_N (\sim \alpha \vee (\sim \alpha \wedge \sim \beta))$ by Lemma 2.1 (o).
9. $\Gamma \vdash (\sim \alpha \wedge \sim \beta) \to_N \sim \alpha$ by axiom $(A3)$.
10. $\Gamma \vdash \sim \alpha \to_N \sim \alpha$ by Lemma 2.1 (b).
11. $\Gamma \vdash [\sim \alpha \to_N \sim \alpha] \to_N [[(\sim \alpha \wedge \sim \beta) \to_N \sim \alpha] \to_N [(\sim \alpha \vee (\sim \alpha \wedge \sim \beta)) \to_N \sim \alpha]]$ by axiom $(A9)$.
12. $\Gamma \vdash [(\sim \alpha \wedge \sim \beta) \to_N \sim \alpha] \to_N [(\sim \alpha \vee (\sim \alpha \wedge \sim \beta)) \to_N \sim \alpha]$ by ($\mathcal{N}$-MP) applied to 10 and 11.
13. $\Gamma \vdash (\sim \alpha \vee (\sim \alpha \wedge \sim \beta)) \to_N \sim \alpha$ by ($\mathcal{N}$-MP) applied to 9 and 12.
14. $\Gamma \vdash [\sim (\alpha \wedge (\alpha \vee \beta))] \to_N (\sim \alpha \vee (\sim \alpha \wedge \sim \beta))$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 4 and 8.
15. $\Gamma \vdash [\sim (\alpha \wedge (\alpha \vee \beta))] \to_N (\sim \alpha)$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 13 and 14.
16. $\Gamma \vdash \alpha \Rightarrow (\alpha \wedge (\alpha \vee \beta))$ by Lemma 2.1 (h) applied to 3 and 15.

(e)
1. $\Gamma \vdash (\gamma \wedge \alpha) \to_N \alpha$ by axiom $(A4)$.
2. $\Gamma \vdash (\beta \wedge \alpha) \to_N \alpha$ by axiom $(A4)$.
3. $\Gamma \vdash [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)] \to_N \alpha$ by axiom $(A9)$ and ($\mathcal{N}$-MP) applied to 1 and 2.
4. $\Gamma \vdash (\gamma \wedge \alpha) \to_N \gamma$ by axiom $(A3)$.
5. $\Gamma \vdash \gamma \to_N (\beta \vee \gamma)$ by axiom $(A6)$.
6. $\Gamma \vdash (\gamma \wedge \alpha) \to_N (\beta \vee \gamma)$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 4 and 5.
7. $\Gamma \vdash (\beta \wedge \alpha) \to_N \beta$ by axiom $(A3)$.

8. $\Gamma \vdash \beta \to_N (\beta \vee \gamma)$ by axiom ($A5$).
9. $\Gamma \vdash (\beta \wedge \alpha) \to_N (\beta \vee \gamma)$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 7 and 8.
10. $\Gamma \vdash [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)] \to_N (\beta \vee \gamma)$ by axiom ($A9$) and ($\mathcal{N}$-MP) applied to 6 and 9.
11. $\Gamma \vdash [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)] \to_N [\alpha \wedge (\beta \vee \gamma)]$ by axiom ($A2$) and ($\mathcal{N}$-MP) applied to 3 and 10.
12. $\Gamma \vdash [\alpha \wedge [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]$ by axiom ($A4$).
13. $\Gamma \vdash [\alpha \wedge [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \to_N [\alpha \wedge (\beta \vee \gamma)]$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 11 and 12.
14. $\Gamma \vdash \sim (\alpha \wedge (\beta \vee \gamma)) \to_N [\sim \alpha \vee \sim (\beta \vee \gamma)]$ by axiom ($A16$) and Lemma 2.1 (c).
15. $\Gamma \vdash \sim (\beta \vee \gamma) \to_N \sim \gamma$ by axiom ($A8$).
16. $\Gamma \vdash (\sim \alpha \vee \sim (\beta \vee \gamma)) \to_N (\sim \alpha \vee \sim \gamma)$ by Lemma 2.1 (o) applied to 15.
17. $\Gamma \vdash (\sim \alpha \vee \sim \gamma) \to_N (\sim \gamma \vee \sim \alpha)$ by Lemma 2.1 (p).
18. $\Gamma \vdash (\sim \alpha \vee \sim (\beta \vee \gamma)) \to_N (\sim \gamma \vee \sim \alpha)$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 16 and 17.
19. $\Gamma \vdash \sim (\beta \vee \gamma) \to_N \sim \beta$ by axiom ($A7$).
    In a similar manner,
20. $\Gamma \vdash (\sim \alpha \vee \sim (\beta \vee \gamma)) \to_N (\sim \beta \vee \sim \alpha)$.
21. $\Gamma \vdash (\sim \alpha \vee \sim (\beta \vee \gamma)) \to_N [(\sim \gamma \vee \sim \alpha) \wedge (\sim \beta \vee \sim \alpha)]$ by axiom ($A2$) and ($\mathcal{N}$-MP) applied to 18 and 20.
22. $\Gamma \vdash [(\sim \gamma \vee \sim \alpha) \wedge (\sim \beta \vee \sim \alpha)] \to_N (\sim \gamma \vee \sim \alpha)$ by axiom ($A3$).
23. $\Gamma \vdash (\sim \gamma \vee \sim \alpha) \to_N [\sim (\gamma \wedge \alpha)]$ by axiom ($A17$) and Lemma 2.1 (c).
24. $\Gamma \vdash [(\sim \gamma \vee \sim \alpha) \wedge (\sim \beta \vee \sim \alpha)] \to_N [\sim (\gamma \wedge \alpha)]$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 22 and 23.
25. $\Gamma \vdash [(\sim \gamma \vee \sim \alpha) \wedge (\sim \beta \vee \sim \alpha)] \to_N (\sim \beta \vee \sim \alpha)$ by axiom ($A4$).
26. $\Gamma \vdash (\sim \beta \vee \sim \alpha) \to_N [\sim (\beta \wedge \alpha)]$ by axiom ($A17$).
27. $\Gamma \vdash [(\sim \gamma \vee \sim \alpha) \wedge (\sim \beta \vee \sim \alpha)] \to_N [\sim (\beta \wedge \alpha)]$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 25 and 26.
28. $\Gamma \vdash [(\sim \gamma \vee \sim \alpha) \wedge (\sim \beta \vee \sim \alpha)] \to_N [[\sim (\gamma \wedge \alpha)] \wedge [\sim (\beta \wedge \alpha)]]$ by axiom ($A2$) and ($\mathcal{N}$-MP) applied to 24 and 27.
29. $\Gamma \vdash [\sim [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \leftrightarrow_N [(\sim (\gamma \wedge \alpha)) \wedge (\sim (\beta \wedge \alpha))]$ by Lemma 2.3 (a).
30. $\Gamma \vdash \{[\sim [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \leftrightarrow_N [(\sim (\gamma \wedge \alpha)) \wedge (\sim (\beta \wedge \alpha))]\} \to_N \{[(\sim (\gamma \wedge \alpha)) \wedge (\sim (\beta \wedge \alpha))] \to_N [\sim [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]]\}$ by axiom ($A4$).
31. $\Gamma \vdash [(\sim (\gamma \wedge \alpha)) \wedge (\sim (\beta \wedge \alpha))] \to_N [\sim [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]]$ by ($\mathcal{N}$-MP) applied to 29 and 30.
32. $\Gamma \vdash [\sim [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \to_N [\sim [\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha))]]$ by Lemma 2.1 (e).
33. $\Gamma \vdash \sim (\alpha \wedge (\beta \vee \gamma)) \to_N [(\sim \gamma \vee \sim \alpha) \wedge (\sim \beta \vee \sim \alpha)]$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 14 and 21.
34. $\Gamma \vdash \sim (\alpha \wedge (\beta \vee \gamma)) \to_N [[\sim (\gamma \wedge \alpha)] \wedge [\sim (\beta \wedge \alpha)]]$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 28 and 33.

35. $\Gamma \vdash\sim (\alpha \wedge (\beta \vee \gamma)) \to_N [\sim [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]]$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 31 and 34.
36. $\Gamma \vdash\sim (\alpha \wedge (\beta \vee \gamma)) \to_N [\sim [\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha))]]$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 32 and 35.
37. $\Gamma \vdash [\alpha \wedge [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \Rightarrow [\alpha \wedge (\beta \vee \gamma)]$ by Lemma 2.1 (h) applied to 13 and 36.

(f)

1. $\Gamma, \alpha, \beta \vdash \beta$
2. $\Gamma, \beta \vdash \alpha \to_N \beta$ by Theorem 2.2 applied to 1.
3. $\Gamma, \beta \vdash \alpha \to_N \alpha$ by Lemma 2.1 (b).
4. $\Gamma, \beta \vdash (\alpha \to_N \beta) \to_N [(\alpha \to_N \alpha) \to_N (\alpha \to_N (\beta \wedge \alpha))]$ by axiom ($A2$).
5. $\Gamma, \beta \vdash (\alpha \to_N \alpha) \to_N (\alpha \to_N (\beta \wedge \alpha))$ by ($\mathcal{N}$-MP) applied to 2 and 4.
6. $\Gamma, \beta \vdash \alpha \to_N (\beta \wedge \alpha)$ by ($\mathcal{N}$-MP) applied to 3 and 5.
7. $\Gamma, \alpha, \beta \vdash \beta \wedge \alpha$ by Theorem 2.2.
8. $\Gamma, \alpha, \gamma \vdash \gamma \wedge \alpha$ similar to 7.
9. $\Gamma, \alpha, \beta \vdash (\beta \wedge \alpha) \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]$ by axiom ($A6$).
10. $\Gamma, \alpha, \beta \vdash (\gamma \wedge \alpha) \vee (\beta \wedge \alpha)$ by ($\mathcal{N}$-MP) applied to 7 and 9.
11. $\Gamma, \alpha \vdash \beta \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]$ by Theorem 2.2 applied to 10.
12. $\Gamma, \alpha, \gamma \vdash (\gamma \wedge \alpha) \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]$ by axiom ($A5$).
13. $\Gamma, \alpha, \gamma \vdash (\gamma \wedge \alpha) \vee (\beta \wedge \alpha)$ by ($\mathcal{N}$-MP) applied to 8 and 12.
14. $\Gamma, \alpha \vdash \gamma \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]$ by Theorem 2.2 applied to 13.
15. $\Gamma, \alpha \vdash [\beta \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \to_N [[\gamma \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \to_N [(\beta \vee \gamma) \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]]]$ by axiom ($A9$).
16. $\Gamma, \alpha \vdash [\gamma \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \to_N [(\beta \vee \gamma) \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]]$ by ($\mathcal{N}$-MP) applied to 11 and 15.
17. $\Gamma, \alpha \vdash (\beta \vee \gamma) \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]$ by ($\mathcal{N}$-MP) applied to 14 and 16.
18. $\Gamma \vdash \alpha \to_N [(\beta \vee \gamma) \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]]$ by Theorem 2.2 applied to 17.
19. $\Gamma \vdash [\alpha \to_N [(\beta \vee \gamma) \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]]] \Rightarrow [(\alpha \wedge (\beta \vee \gamma)) \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]]$ by axiom ($A19$).
20. $\Gamma \vdash (\alpha \wedge (\beta \vee \gamma)) \to_N [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]$ by Lemma 2.1 (g) applied to 18.
21. $\Gamma \vdash (\alpha \wedge (\beta \vee \gamma)) \to_N \alpha$ by axiom ($A3$).
22. $\Gamma \vdash (\alpha \wedge (\beta \vee \gamma)) \to_N [\alpha \wedge [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]]$ by axiom ($A2$) and ($\mathcal{N}$-MP) applied to 20 and 21.
23. $\Gamma \vdash [\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$ by axiom ($A22$).
24. $\Gamma \vdash (\alpha \wedge (\beta \vee \gamma)) \Rightarrow [\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha))]$ by Lemma 2.1 (h) applied to 22 and 23.

(g)

1. $\Gamma, \sim \beta \to_N \sim \alpha, \sim \gamma \to_N \sim \alpha, \sim (\beta \wedge \gamma) \vdash (\sim \beta \to_N \sim \alpha) \to_N [(\sim \gamma \to_N \sim \alpha) \to_N ((\sim \beta \vee \sim \gamma) \to_N \sim \alpha)]$ by axiom ($A9$).
2. $\Gamma, \sim \beta \to_N \sim \alpha, \sim \gamma \to_N \sim \alpha, \sim (\beta \wedge \gamma) \vdash\sim \beta \to_N \sim \alpha$.
3. $\Gamma, \sim \beta \to_N \sim \alpha, \sim \gamma \to_N \sim \alpha, \sim (\beta \wedge \gamma) \vdash (\sim \gamma \to_N \sim \alpha) \to_N ((\sim \beta \vee \sim \gamma) \to_N \sim \alpha)$ by ($\mathcal{N}$-MP) applied to 1 and 2.

4. $\Gamma, \sim\beta \to_N \sim\alpha, \sim\gamma \to_N \sim\alpha, \sim(\beta \wedge \gamma) \vdash \sim\gamma \to_N \sim\alpha$.
5. $\Gamma, \sim\beta \to_N \sim\alpha, \sim\gamma \to_N \sim\alpha, \sim(\beta \wedge \gamma) \vdash (\sim\beta \vee \sim\gamma) \to_N \sim\alpha$ by ($\mathcal{N}$-MP) applied to 3 and 4.
6. $\Gamma, \sim\beta \to_N \sim\alpha, \sim\gamma \to_N \sim\alpha, \sim(\beta \wedge \gamma) \vdash \sim(\beta \wedge \gamma)$.
7. $\Gamma, \sim\beta \to_N \sim\alpha, \sim\gamma \to_N \sim\alpha, \sim(\beta \wedge \gamma) \vdash [\sim(\beta \wedge \gamma)] \Rightarrow (\sim\beta \vee \sim\gamma)$ by axiom ($A16$).
8. $\Gamma, \sim\beta \to_N \sim\alpha, \sim\gamma \to_N \sim\alpha, \sim(\beta \wedge \gamma) \vdash [\sim(\beta \wedge \gamma)] \to_N (\sim\beta \vee \sim\gamma)$ by Lemma 2.1 (c).
9. $\Gamma, \sim\beta \to_N \sim\alpha, \sim\gamma \to_N \sim\alpha, \sim(\beta \wedge \gamma) \vdash \sim\beta \vee \sim\gamma$ by ($\mathcal{N}$-MP) applied to 6 and 8.
10. $\Gamma, \sim\beta \to_N \sim\alpha, \sim\gamma \to_N \sim\alpha, \sim(\beta \wedge \gamma) \vdash \sim\alpha$ by ($\mathcal{N}$-MP) applied to 5 and 9.
11. $\Gamma, \sim\beta \to_N \sim\alpha, \sim\gamma \to_N \sim\alpha \vdash (\sim(\beta \wedge \gamma)) \to_N (\sim\alpha)$ by Theorem 2.2 applied to 10.
12. $\Gamma, \sim\beta \to_N \sim\alpha \vdash (\sim\gamma \to_N \sim\alpha) \to_N [(\sim(\beta \wedge \gamma)) \to_N (\sim\alpha)]$ by Theorem 2.2 applied to 11.
13. $\Gamma \vdash (\sim\beta \to_N \sim\alpha) \to_N [(\sim\gamma \to_N \sim\alpha) \to_N [(\sim(\beta \wedge \gamma)) \to_N (\sim\alpha)]]$ by Theorem 2.2 applied to 12.

(h)
1. $\Gamma \vdash \alpha \Rightarrow \beta$ by hypothesis.
2. $\Gamma \vdash \alpha \to_N \beta$ by Lemma 2.1 (c).
3. $\Gamma \vdash \sim\beta \to_N \sim\alpha$ by Lemma 2.1 (c).
4. $\Gamma \vdash \alpha \Rightarrow \gamma$ by hypothesis.
5. $\Gamma \vdash \alpha \to_N \gamma$ by Lemma 2.1 (c).
6. $\Gamma \vdash \sim\gamma \to_N \sim\alpha$ by Lemma 2.1 (c).
7. $\Gamma \vdash (\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$ by axiom ($A2$).
8. $\Gamma \vdash (\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma))$ by ($\mathcal{N}$-MP) applied to 2 and 7.
9. $\Gamma \vdash \alpha \to_N (\beta \wedge \gamma)$ by ($\mathcal{N}$-MP) applied to 5 and 8.
10. $\Gamma \vdash (\sim\beta \to_N \sim\alpha) \to_N ((\sim\gamma \to_N \sim\alpha) \to_N (\sim(\beta \wedge \gamma) \to_N \sim\alpha))$ by (g).
11. $\Gamma \vdash (\sim\gamma \to_N \sim\alpha) \to_N (\sim(\beta \wedge \gamma) \to_N \sim\alpha)$ by ($\mathcal{N}$-MP) applied to 3 and 10.
12. $\Gamma \vdash \sim(\beta \wedge \gamma) \to_N \sim\alpha$ by ($\mathcal{N}$-MP) applied to 6 and 11.
13. $\Gamma \vdash \alpha \Rightarrow \beta \wedge \gamma$ by Lemma 2.1 (h) applied to 9 and 12.
In a similar manner,
14. $\Gamma \vdash \alpha \Rightarrow \gamma \wedge \beta$.

(i)
1. $\Gamma \vdash \alpha \Rightarrow \beta$ by hypothesis.
2. $\Gamma \vdash \alpha \to_N \beta$ by 2.1 (c).
3. $\Gamma \vdash \beta \to_N (\beta \vee \gamma)$ ($A5$).
4. $\Gamma \vdash \alpha \to_N (\beta \vee \gamma)$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 2 and 3.
5. $\Gamma \vdash \sim\beta \to_N \sim\alpha$ by 2.1 (c) applied to 1.
6. $\Gamma \vdash \sim(\beta \vee \gamma) \to_N \sim\beta$ by axiom ($A7$).
7. $\Gamma \vdash \sim(\beta \vee \gamma) \to_N \sim\alpha$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 5 and 6.

8. $\Gamma \vdash \alpha \Rightarrow (\beta \vee \gamma)$ by 2.1 (h) applied to 4 and 7.
   In a similar manner, using axioms ($A6$) and ($A8$),
9. $\Gamma \vdash \alpha \Rightarrow (\gamma \vee \beta)$

(j) Follows from 2.1 (l) and 2.1 (i).

(k)
1. $\Gamma, \alpha \Rightarrow \beta \vdash \alpha \Rightarrow \beta$.
2. $\Gamma, \alpha \Rightarrow \beta \vdash \alpha \rightarrow_N \beta$ by 2.1 (c).
3. $\Gamma, \alpha \Rightarrow \beta \vdash (\alpha \wedge \gamma) \Rightarrow \alpha$ by 2.1 (i).
4. $\Gamma, \alpha \Rightarrow \beta \vdash (\alpha \wedge \gamma) \Rightarrow \beta$ by 2.1 (l) applied to 1 and 3.
5. $\Gamma, \alpha \Rightarrow \beta \vdash (\alpha \wedge \gamma) \rightarrow_N (\beta \wedge \gamma)$ by Lemma 2.1 (c) and (n).
6. $\Gamma, \alpha \Rightarrow \beta \vdash \sim \beta \rightarrow_N \sim (\alpha \wedge \gamma)$ by 2.1 (c) applied to 4.
7. $\Gamma, \alpha \Rightarrow \beta \vdash \sim \gamma \rightarrow_N \sim (\alpha \wedge \gamma)$ by 2.1 (e).
8. $\Gamma, \alpha \Rightarrow \beta \vdash (\sim \beta \rightarrow_N \sim (\alpha \wedge \gamma)) \rightarrow_N ((\sim \gamma \rightarrow_N \sim (\alpha \wedge \gamma)) \rightarrow_N (\sim (\beta \wedge \gamma) \rightarrow_N \sim (\alpha \wedge \gamma)))$ by Lema 2.3 (g).
9. $\Gamma, \alpha \Rightarrow \beta \vdash (\sim \gamma \rightarrow_N \sim (\alpha \wedge \gamma)) \rightarrow_N (\sim (\beta \wedge \gamma) \rightarrow_N \sim (\alpha \wedge \gamma))$ by ($\mathcal{N}$-MP) applied to 6 and 8.
10. $\Gamma, \alpha \Rightarrow \beta \vdash \sim (\beta \wedge \gamma) \rightarrow_N \sim (\alpha \wedge \gamma)$ by ($\mathcal{N}$-MP) applied to 7 and 9.
11. $\Gamma, \alpha \Rightarrow \beta \vdash (\alpha \wedge \gamma) \Rightarrow (\beta \wedge \gamma)$ by 2.1 (h) applied to 5 and 10.

(l)
1. $\Gamma, \alpha \Rightarrow \beta \vdash (\gamma \wedge \alpha) \Rightarrow \alpha$ by 2.1 (i).
2. $\Gamma, \alpha \Rightarrow \beta \vdash \alpha \Rightarrow \beta$.
3. $\Gamma, \alpha \Rightarrow \beta \vdash \alpha \rightarrow_N \beta$ by 2.1 (c).
4. $\Gamma, \alpha \Rightarrow \beta \vdash (\gamma \wedge \alpha) \Rightarrow \beta$ by 2.1 (l) applied to 1 and 2.
5. $\Gamma, \alpha \Rightarrow \beta \vdash (\gamma \wedge \alpha) \rightarrow_N (\gamma \wedge \beta)$ by 3 and Lemma 2.1 (n).
6. $\Gamma, \alpha \Rightarrow \beta \vdash \sim \beta \rightarrow_N \sim (\gamma \wedge \alpha)$ by Lemma 2.1 (c) applied to 4.
7. $\Gamma, \alpha \Rightarrow \beta \vdash \sim \gamma \rightarrow_N \sim (\gamma \wedge \alpha)$ by Lemma 2.1 (d).
8. $\Gamma, \alpha \Rightarrow \beta \vdash (\sim \gamma \rightarrow_N \sim (\gamma \wedge \alpha)) \rightarrow_N [(\sim \beta \rightarrow_N \sim (\gamma \wedge \alpha)) \rightarrow_N (\sim (\gamma \wedge \beta) \rightarrow_N \sim (\gamma \wedge \alpha))]$ by Lemma 2.3 (g).
9. $\Gamma, \alpha \Rightarrow \beta \vdash (\sim \beta \rightarrow_N \sim (\gamma \wedge \alpha)) \rightarrow_N (\sim (\gamma \wedge \beta) \rightarrow_N \sim (\gamma \wedge \alpha))$ by ($\mathcal{N}$-MP) applied to 7 and 8.
10. $\Gamma, \alpha \Rightarrow \beta \vdash \sim (\gamma \wedge \beta) \rightarrow_N \sim (\gamma \wedge \alpha)$ by ($\mathcal{N}$-MP) applied to 6 and 9.
11. $\Gamma, \alpha \Rightarrow \beta \vdash (\gamma \wedge \alpha) \Rightarrow (\gamma \wedge \beta)$ by Lemma 2.1 (h) applied to 5 and 10.

(m)
1. $\Gamma, \alpha \Rightarrow \beta, \gamma \Rightarrow t \vdash (\alpha \wedge \gamma) \Rightarrow (\beta \wedge \gamma)$ by part (k).
2. $\Gamma, \alpha \Rightarrow \beta, \gamma \Rightarrow t \vdash (\beta \wedge \gamma) \Rightarrow (\beta \wedge t)$ by part (l).
3. $\Gamma, \alpha \Rightarrow \beta, \gamma \Rightarrow t \vdash (\alpha \wedge \gamma) \Rightarrow (\beta \wedge t)$ by Lemma 2.1 (l).

(n)
1. $\Gamma \vdash [\sim (\alpha \rightarrow (\alpha \wedge \beta))] \rightarrow_N [\alpha \wedge (\sim (\alpha \wedge \beta))]$ by axiom ($A20$)
2. $\Gamma \vdash [\sim (\alpha \rightarrow_N \beta)] \rightarrow_N [\alpha \wedge (\sim (\alpha \wedge \beta))]$ by definition of $\rightarrow_N$ applied to 1.
3. $\Gamma, (\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta) \vdash [\alpha \wedge (\sim (\alpha \wedge \beta))] \Rightarrow [\alpha \wedge (\sim \alpha \vee \sim \beta)]$ by part (l).

4. $\Gamma \vdash [(\sim(\alpha \wedge \beta)) \Rightarrow (\sim\alpha \vee \sim\beta)] \rightarrow_N [[\alpha \wedge (\sim(\alpha \wedge \beta))] \Rightarrow [\alpha \wedge (\sim\alpha \vee \sim\beta)]]$ by Theorem 2.2 applied to 3.
5. $\Gamma \vdash (\sim(\alpha \wedge \beta)) \Rightarrow (\sim\alpha \vee \sim\beta)$ by axiom ($A16$).
6. $\Gamma \vdash [\alpha \wedge (\sim(\alpha \wedge \beta))] \Rightarrow [\alpha \wedge (\sim\alpha \vee \sim\beta)]$ by ($\mathcal{N}$-MP) applied to 4 and 5.
7. $\Gamma \vdash [\alpha \wedge (\sim(\alpha \wedge \beta))] \rightarrow_N [\alpha \wedge (\sim\alpha \vee \sim\beta)]$ by Lemma 2.1 (c).
8. $\Gamma \vdash [\sim(\alpha \rightarrow_N \beta)] \rightarrow_N [\alpha \wedge (\sim\alpha \vee \sim\beta)]$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 2 and 7.
9. $\Gamma \vdash [\alpha \wedge (\sim\alpha \vee \sim\beta)] \rightarrow_N [\alpha \wedge (\alpha \rightarrow_N (\sim\beta))]$ by axiom ($A18$) and Lemma 2.1 (c).
10. $\Gamma \vdash [\sim(\alpha \rightarrow_N \beta)] \rightarrow_N [\alpha \wedge (\alpha \rightarrow_N (\sim\beta))]$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 8 and 9.
11. $\Gamma \vdash [\alpha \wedge (\alpha \rightarrow_N (\sim\beta))] \rightarrow_N \alpha$ by axiom ($A3$).
12. $\Gamma \vdash [\sim(\alpha \rightarrow_N \beta)] \rightarrow_N \alpha$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 10 and 11.
13. $\Gamma, \sim(\alpha \rightarrow_N \beta) \vdash \alpha$ by Theorem 2.2.
14. $\Gamma \vdash [\alpha \wedge (\alpha \rightarrow_N (\sim\beta))] \rightarrow_N (\alpha \rightarrow_N (\sim\beta))$ by axiom ($A4$).
15. $\Gamma \vdash [\sim(\alpha \rightarrow_N \beta)] \rightarrow_N (\alpha \rightarrow_N (\sim\beta))$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 10 and 14.
16. $\Gamma, \sim(\alpha \rightarrow_N \beta) \vdash \alpha \rightarrow_N (\sim\beta)$ by Theorem 2.2 applied to 15.
17. $\Gamma, \sim(\alpha \rightarrow_N \beta) \vdash \sim\beta$ by ($\mathcal{N}$-MP) applied to 13 and 16.
18. $\Gamma, \sim(\alpha \rightarrow_N \beta) \vdash (\alpha \wedge \sim\beta)$ by Lemma 2.1 (h) applied to 13 and 17.
19. $\Gamma \vdash [\sim(\alpha \rightarrow_N \beta)] \rightarrow_N (\alpha \wedge \sim\beta)$ by Theorem 2.2.

(o)
1. $\Gamma, \alpha, \sim\alpha \vdash \sim\alpha$.
2. $\Gamma, \alpha, \sim\alpha \vdash \sim\alpha \rightarrow_N (\sim\alpha \vee \beta)$ by axiom ($A5$).
3. $\Gamma, \alpha, \sim\alpha \vdash \sim\alpha \vee \beta$ by ($\mathcal{N}$-MP) applied to 1 and 2.
4. $\Gamma, \alpha, \sim\alpha \vdash \alpha$.
5. $\Gamma, \alpha, \sim\alpha \vdash \alpha \wedge (\sim\alpha \vee \beta)$ by Lemma 2.1 (h) applied to 3 and 4.
6. $\Gamma, \alpha, \sim\alpha \vdash [\alpha \wedge (\sim\alpha \vee \beta)] \rightarrow_N [\alpha \wedge (\alpha \rightarrow_N \beta)]$ by axiom ($A18$) and Lemma 2.1 (c).
7. $\Gamma, \alpha, \sim\alpha \vdash \alpha \wedge (\alpha \rightarrow_N \beta)$ by ($\mathcal{N}$-MP) applied to 5 and 6.
8. $\Gamma, \alpha, \sim\alpha \vdash (\alpha \wedge (\alpha \rightarrow_N \beta)) \rightarrow_N (\alpha \rightarrow_N \beta)$ by axiom ($A4$).
9. $\Gamma, \alpha, \sim\alpha \vdash \alpha \rightarrow_N \beta$ by ($\mathcal{N}$-MP) applied to 7 and 8.
10. $\Gamma, \alpha, \sim\alpha \vdash \beta$ by ($\mathcal{N}$-MP) applied to 4 and 9.
11. $\Gamma, \alpha \vdash \sim\alpha \rightarrow_N \beta$ by Theorem 2.2 applied to 10.
12. $\Gamma \vdash \alpha \rightarrow_N (\sim\alpha \rightarrow_N \beta)$ by Theorem 2.2 applied to 11.
13. $\Gamma \vdash [\alpha \rightarrow_N (\sim\alpha \rightarrow_N \beta)] \rightarrow_N [(\alpha \wedge \sim\alpha) \rightarrow_N \beta]$ by axiom ($A19$) and Lemma 2.1 (c).
14. $\Gamma \vdash (\alpha \wedge \sim\alpha) \rightarrow_N \beta$ by ($\mathcal{N}$-MP) applied to 12 and 13.

(p)
1. $\Gamma \vdash (\sim\beta \wedge \sim\sim\beta) \leftrightarrow_N \sim(\beta \vee \sim\beta)$ by 2.3 (a).

2. $\Gamma \vdash [(\sim \beta \wedge \sim\sim \beta) \leftrightarrow_N \sim (\beta \vee \sim \beta)] \to_N [\sim (\beta \vee \sim \beta) \to_N (\sim \beta \wedge \sim\sim \beta)]$ by axiom ($A4$).
3. $\Gamma \vdash \sim (\beta \vee \sim \beta) \to_N (\sim \beta \wedge \sim\sim \beta)$ by ($\mathcal{N}$-MP) applied to 1 and 2.
4. $\Gamma \vdash (\sim \beta \wedge \sim\sim \beta) \to_N \sim (\alpha \wedge \sim \alpha)$ by 2.3 (o).
5. $\Gamma \vdash \sim (\beta \vee \sim \beta) \to_N \sim (\alpha \wedge \sim \alpha)$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 3 and 4.
6. $\Gamma \vdash (\alpha \wedge \sim \alpha) \to_N (\beta \vee \sim \beta)$ by 2.3 (o).
7. $\Gamma \vdash (\alpha \wedge \sim \alpha) \Rightarrow (\beta \vee \sim \beta)$ by Lemma 2.1 (h) applied to 6 and 5.

(q)
1. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha, \beta \vdash \beta \to_N \alpha$.
2. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha, \beta \vdash \beta$.
3. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha, \beta \vdash \alpha$ by ($\mathcal{N}$-MP) applied to 1 and 2.
4. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha, \beta \vdash \sim \gamma \to_N \alpha$ by Lemma 2.1 (a) applied to 3.
5. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha \vdash \beta \to_N (\sim \gamma \to_N \alpha)$ by Theorem 2.2 applied to 4.
6. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha \vdash [\beta \to_N (\sim \gamma \to_N \alpha)] \to_N [(\beta \wedge \sim \gamma) \to_N \alpha]$ by axiom ($A19$) and by Lemma 2.1 (c).
7. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha \vdash (\beta \wedge \sim \gamma) \to_N \alpha$ by ($\mathcal{N}$-MP) applied to 5 and 6.
8. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha \vdash (\beta \wedge \sim \gamma) \to_N \sim \gamma$ by axiom ($A4$).
9. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha \vdash (\beta \wedge \sim \gamma) \to_N (\alpha \wedge \sim \gamma)$ by axiom ($A2$) and ($\mathcal{N}$-MP) applied to 7 and 8.
10. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha \vdash \sim (\beta \to \gamma) \to_N (\beta \wedge \sim \gamma)$ by axiom ($A20$).
11. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha \vdash \sim (\beta \to \gamma) \to_N (\alpha \wedge \sim \gamma)$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 10 and 9.
12. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha \vdash (\alpha \wedge \sim \gamma) \to_N \sim (\alpha \to \gamma)$ by axiom ($A21$).
13. $\Gamma, \alpha \to_N \beta, \beta \to_N \alpha \vdash \sim (\beta \to \gamma) \to_N \sim (\alpha \to \gamma)$ by axiom ($A1$) and ($\mathcal{N}$-MP) applied to 11 and 12.
14. $\Gamma, \alpha \to_N \beta \vdash (\beta \to_N \alpha) \to_N [\sim (\beta \to \gamma) \to_N \sim (\alpha \to \gamma)]$ by Theorem 2.2 applied to 13.
15. $\Gamma \vdash (\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [\sim (\beta \to \gamma) \to_N \sim (\alpha \to \gamma)]]$ by Theorem 2.2 applied to 14.

(r)
1. $\Gamma, \sim \alpha \to_N \sim \beta, \sim \beta \to_N \sim \alpha \vdash \sim \alpha \to_N \sim \beta$.
2. $\Gamma, \sim \alpha \to_N \sim \beta, \sim \beta \to_N \sim \alpha \vdash \gamma \to_N (\sim \alpha \to_N \sim \beta)$ by Lemma 2.1 (a).
3. $\Gamma, \sim \alpha \to_N \sim \beta, \sim \beta \to_N \sim \alpha \vdash [\gamma \to_N (\sim \alpha \to_N \sim \beta)] \to_N [(\gamma \wedge \sim \alpha) \to_N \sim \beta]$ by axiom ($A19$) and by Lemma 2.1 (c).
4. $\Gamma, \sim \alpha \to_N \sim \beta, \sim \beta \to_N \sim \alpha \vdash (\gamma \wedge \sim \alpha) \to_N \sim \beta$ by ($\mathcal{N}$-MP) applied to 2 and 3.
5. $\Gamma, \sim \alpha \to_N \sim \beta, \sim \beta \to_N \sim \alpha \vdash (\gamma \wedge \sim \alpha) \to_N \gamma$ by axiom ($A3$).
6. $\Gamma, \sim \alpha \to_N \sim \beta, \sim \beta \to_N \sim \alpha \vdash (\gamma \wedge \sim \alpha) \to_N (\gamma \wedge \sim \beta)$ by axiom ($A2$) and ($\mathcal{N}$-MP) applied to 5 and 4.
7. $\Gamma, \sim \alpha \to_N \sim \beta, \sim \beta \to_N \sim \alpha \vdash \sim (\gamma \to \alpha) \to_N (\gamma \wedge \sim \alpha)$ by axiom ($A20$).

8. $\Gamma, \sim\alpha \to_N \sim\beta, \sim\beta \to_N \sim\alpha \vdash \sim(\gamma \to \alpha) \to_N (\gamma \wedge \sim\beta)$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 6 and 7.
9. $\Gamma, \sim\alpha \to_N \sim\beta, \sim\beta \to_N \sim\alpha \vdash (\gamma \wedge \sim\beta) \to_N \sim(\gamma \to \beta)$ by axiom $(A21)$.
10. $\Gamma, \sim\alpha \to_N \sim\beta, \sim\beta \to_N \sim\alpha \vdash \sim(\gamma \to \alpha) \to_N \sim(\gamma \to \beta)$ by axiom $(A1)$ and ($\mathcal{N}$-MP) applied to 8 and 9.
11. $\Gamma, \sim\alpha \to_N \sim\beta \vdash (\sim\beta \to_N \sim\alpha) \to_N [\sim(\gamma \to \alpha) \to_N \sim(\gamma \to \beta)]$ by Theorem 2.2.
12. $\Gamma \vdash (\sim\alpha \to_N \sim\beta) \to_N [(\sim\beta \to_N \sim\alpha) \to_N [\sim(\gamma \to \alpha) \to_N \sim(\gamma \to \beta)]]$ by Theorem 2.2.

(s)
1. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash \alpha \Rightarrow \beta$.
2. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash \alpha \to_N \beta$ by 2.1 (c).
3. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash \sim\beta \to_N \sim\alpha$ by 2.1 (c).
4. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash \beta \Rightarrow \alpha$.
5. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash \beta \to_N \alpha$ by 2.1 (c).
6. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash \sim\alpha \to_N \sim\beta$ by 2.1 (c).
7. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$ by axiom $(A13)$.
8. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]$ by ($\mathcal{N}$-MP) applied to 2 and 7.
9. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\alpha \to \gamma) \to_N (\beta \to \gamma)$ by ($\mathcal{N}$-MP) applied to 5 and 8.
10. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\sim(\beta \to \gamma)) \to_N (\sim(\alpha \to \gamma))]]$ by Lemma 2.3 (q).
11. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\beta \to_N \alpha) \to_N [(\sim(\beta \to \gamma)) \to_N (\sim(\alpha \to \gamma))]$ by ($\mathcal{N}$-MP) applied to 2 and 10.
12. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\sim(\beta \to \gamma)) \to_N (\sim(\alpha \to \gamma))$ by ($\mathcal{N}$-MP) applied to 5 and 11.
13. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\alpha \to \gamma) \Rightarrow (\beta \to \gamma)$ by 2.1 (h) applied to 9 and 12.
14. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash \gamma \Rightarrow t$.
15. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash \gamma \to_N t$ by 2.1 (c).
16. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash \sim t \to_N \sim\gamma$ by 2.1 (c).
17. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash t \Rightarrow \gamma$.
18. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash t \to_N \gamma$ by 2.1 (c).
19. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash \sim\gamma \to_N \sim t$ by 2.1 (c).
20. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\gamma \to_N t) \to_N [(t \to_N \gamma) \to_N [(\beta \to \gamma) \to_N (\beta \to t)]]$ by axiom $(A14)$.
21. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (t \to_N \gamma) \to_N [(\beta \to \gamma) \to_N (\beta \to t)]$ by ($\mathcal{N}$-MP) applied to 15 and 20.
22. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\beta \to \gamma) \to_N (\beta \to t)$ by ($\mathcal{N}$-MP) applied to 18 and 21.

23. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\sim t \rightarrow_N \sim \gamma) \rightarrow_N [(\sim \gamma \rightarrow_N \sim t) \rightarrow_N [(\sim (\beta \rightarrow t)) \rightarrow_N (\sim (\beta \rightarrow \gamma))]]$ by 2.3 (r).
24. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\sim \gamma \rightarrow_N \sim t) \rightarrow_N [(\sim (\beta \rightarrow t)) \rightarrow_N (\sim (\beta \rightarrow \gamma))]$ by ($\mathcal{N}$-MP) applied to 16 and 23.
25. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\sim (\beta \rightarrow t)) \rightarrow_N (\sim (\beta \rightarrow \gamma))$ by ($\mathcal{N}$-MP) applied to 19 and 24.
26. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\beta \rightarrow \gamma) \Rightarrow (\beta \rightarrow t)$ by 2.1 (h) applied to 22 and 25.
27. $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\alpha \rightarrow \gamma) \Rightarrow (\beta \rightarrow t)$ by 2.1 (l) applied to 13 and 26.

(t)
1. $\Gamma, \alpha \vdash [(\alpha \wedge \beta) \rightarrow_N \alpha] \Rightarrow [\alpha \rightarrow_N (\beta \rightarrow_N \alpha)]$ by axiom ($A15$).
2. $\Gamma, \alpha \vdash [(\alpha \wedge \beta) \rightarrow_N \alpha] \rightarrow_N [\alpha \rightarrow_N (\beta \rightarrow_N \alpha)]$ by Lemma 2.1 (c) applied to 1.
3. $\Gamma, \alpha \vdash (\alpha \wedge \beta) \rightarrow_N \alpha$ by axiom ($A3$).
4. $\Gamma, \alpha \vdash \alpha \rightarrow_N (\beta \rightarrow_N \alpha)$ by ($\mathcal{N}$-MP) applied to 2 and 3.
5. $\Gamma, \alpha \vdash \alpha$.
6. $\Gamma, \alpha \vdash \beta \rightarrow_N \alpha$ by ($\mathcal{N}$-MP) applied to 4 and 5.
7. $\Gamma, \alpha \vdash [(\alpha \wedge \sim \alpha) \rightarrow_N \sim \beta] \Rightarrow [\alpha \rightarrow_N (\sim \alpha \rightarrow_N \sim \beta)]$ by axiom ($A15$).
8. $\Gamma, \alpha \vdash [(\alpha \wedge \sim \alpha) \rightarrow_N \sim \beta] \rightarrow_N [\alpha \rightarrow_N (\sim \alpha \rightarrow_N \sim \beta)]$ by Lemma 2.1 (c) applied to 7.
9. $\Gamma, \alpha \vdash (\alpha \wedge \sim \alpha) \rightarrow_N \sim \beta$ by 2.3 (o).
10. $\Gamma, \alpha \vdash \alpha \rightarrow_N (\sim \alpha \rightarrow_N \sim \beta)$ by ($\mathcal{N}$-MP) applied to 8 and 9.
11. $\Gamma, \alpha \vdash \sim \alpha \rightarrow_N \sim \beta$ by ($\mathcal{N}$-MP) applied to 5 and 10.
12. $\Gamma, \alpha \vdash \beta \Rightarrow \alpha$ by Lemma 2.1 (h) applied to 6 and 11.

(u)
1. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash \alpha$.
2. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash \beta \rightarrow_N \alpha$ by 2.1 (a).
3. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash \beta \rightarrow_N \beta$ by 2.1 (b).
4. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash \beta \rightarrow_N (\alpha \wedge \beta)$ by axiom ($A2$) and ($\mathcal{N}$-MP) applied to 2 and 3.
5. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash (\alpha \wedge \beta) \rightarrow_N \beta$ by axiom ($A3$).
6. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash (\alpha \rightarrow \beta) \rightarrow_N (\alpha \rightarrow (\alpha \wedge \beta))$ by axiom ($A14$) and ($\mathcal{N}$-MP) applied to 4 and 5.
7. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash \alpha \rightarrow \beta$.
8. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash \alpha \rightarrow (\alpha \wedge \beta)$ by ($\mathcal{N}$-MP) applied to 6 and 7.
9. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash \alpha \rightarrow_N \beta$.
10. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash \alpha$.
11. $\Gamma, \alpha \rightarrow \beta, \alpha \vdash \beta$ by ($\mathcal{N}$-MP) applied to 9 and 10.
12. $\Gamma, \alpha \rightarrow \beta \vdash \alpha \rightarrow_N \beta$ by Theorem 2.2.
13. $\Gamma \vdash (\alpha \rightarrow \beta) \rightarrow_N (\alpha \rightarrow_N \beta)$ by Theorem 2.2. □

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