NOTAS DE ALGEBRA Y ANALISIS

10

AGNES ILONA BENEDEK AND RAFAEL PANZONE

ON STURM-LIOUVILLE PROBLEMS WITH THE SQUARE-ROOT OF THE EIGENVALUE PARAMETER CONTAINED IN THE BOUNDARY CONDITIONS

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BAHIA BLANCA - ARGENTINA

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RÉSUMÉ

À ce travail-ci nous considérons en $a \leqslant x \leqslant b$ l'équation différentielle de 2ème ordre $y'' + (s^2 - q(x))y = 0$ avec de conditions de contour de la forme:

$$P(s)y(a) + Q(s)y'(a) = 0 = \tilde{P}(s)y(b) + \tilde{Q}(s)y'(b) \ ,$$

où $P, Q, \tilde{P}$ et $\tilde{Q}$ ce sont des polynômes dans la racine carrée du valeur caractéristique $\lambda = s^2$.

Sauf par peu de restrictions, pratiquement imposées par la nature du problème, ces polynômes sont arbitraires.

Traitons ici, au cas où les polynomes sont réels, le comportement asymptotique des valeurs caractéristiques, le développement en fonctions caractéristiques des fonctions de carré sommable, la liberté de choissir des coefficients dans le développement d'une fonction donnée et d'autres problèmes qui surgent naturellement dans l'étude de cette sorte de systèmes différentiellles de Sturm-Liouville.

Au cas où les polynômes sont linéaires en λ, on analyse les zéros des fonctions caractéristiques.

Quelques détails especiaux sont traités dans le cas:

$$P \equiv 1 \quad , \quad Q \equiv 0 \quad , \quad \tilde{P} \equiv s \quad , \quad Q \equiv a \ \text{réel}.$$

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1. INTRODUCTION.

Boundary value problems like

$$
(\mathrm{I}) \qquad \begin{cases} y''+(\lambda-q)y = 0 \quad , \quad -\infty < a \leqslant x \leqslant b < +\infty \ , \ q \in L^1(a,b) \ , \\ P(\lambda)y(a) + Q(\lambda)y'(a) = 0, \\ \tilde{P}(\lambda)y(b) + \tilde{Q}(\lambda)y'(b) = 0, \end{cases}
$$

were studied by many authors, among others by R.E.Langer, C.Miranda, R.L.Peak, E.Hille, R.Davies, R.V.Churchill, W.F.Bauer, W.Dück, W.D.Evans, J.Walter, Ch.Fulton, E.M.Russakovskii, E.N.Güichal, and the authors of this monograph. Most of the applications involve only linear dependence on the parameter λ and were considered by almost all the authors mentioned. Some appear already in books, e.g., B.Friedman "Principles and Techniques of Applied Mathematics", A.Tijonov and A.Samarsky "Ecuaciones de la Física matemática", S.Timoshenko and D.H.Young "Vibration Problems in Engineering", etc..

Some characteristic features that distinguish problem (I) from an ordinary boundary value problem, i.e., one with $P,Q,\tilde{P}$ and $\tilde{Q}$ constants, are: a) the eigenfunctions are not orthogonal with respect to Lebesgue measure on $[a,b]$, b) null series are present, i.e., non-trivial series that converge to zero in the mean. (It seems that one of the first authors that considered these series was J.Tamarkin).

Well-known results on the eigenvalues or the eigenfunctions that hold in the ordinary case are not valid any more; for example, the eigenvalues are not necessarily real for $q,P,Q,\tilde{P}$ and $\tilde{Q}$ real, and the number of zeroes of the eigenfunctions in (a,b) do not increase necessarily with the eigenvalues, even in the most simple cases where the eigenvalues are real and simple. This is shown in Appendix I. In Appendix II we prove that if the spectral parameter enters via polynomials in the boundary conditions these do not fall into the class of "boundary values for a differential operators" considered in [DS] , where it is shown that such a boundary value can be reduced to a linear relation between the values of the function and its first derivative at the end points of the interval [a,b]. But, as is shown in [BP], p.162, for $Q \not\equiv 0$

$$P(\lambda)y(a) + Q(\lambda)y'(a) = \sum_{j=0}^{m} c_j \, y^{(j)}(a) \quad , \quad m = (2p)\vee(2q+1),$$

with c_j constants independent of y, and $p = \deg P$, $q = \deg Q$.

The boundary problems that interest us in this paper are those ones where boundary conditions in (I) are replaced by

$$\text{(II)} \quad \begin{cases} P(\sqrt{\lambda})y(a) + Q(\sqrt{\lambda})y'(a) = 0, \\ \tilde{P}(\sqrt{\lambda})y(b) + \tilde{Q}(\sqrt{\lambda})y'(b) = 0, \quad P,Q,\tilde{P} \text{ and } \tilde{Q} \text{ polynomials.} \end{cases}$$

Problems of this kind were studied by H.Hochstadt, Y.Li and others for $P,Q,\tilde{P}$ and $\tilde{Q}$ linear. The methods of proof that we follow in many places are the same that were used in [BGP], and the details are much the same as those in [G].

A particular case of (II) is

$$\text{(III)} \quad \begin{cases} y'' + (s^2-q)y = 0, \quad q \in L^1 \quad (0 \leqslant x \leqslant \pi) \text{ real}, \\ y(0) = 0, \quad ay'(\pi) + s.y(\pi) = 0, \quad a \neq 0 \text{ and real.} \end{cases}$$

Here, $P \equiv 1$, $Q \equiv 0$, $\tilde{P} \equiv s$, $\tilde{Q} \equiv a$. Most of the propositions in sections 3-7, where problem II is considered in detail, apply to this particular case. At this moment we make some comments on problem (III) borrowing results from [H], part I.

The eigenvalues are of the form

$$s_n = n + \varphi_n \quad , \quad \varphi_n = -\frac{1}{\pi}\tan^{-1}a + O(\tfrac{1}{n}) \quad , \quad n = 0,\pm 1,\pm 2,\ldots \; ,$$

and the corresponding eigenfunctions $y_n(x)$, that satisfy the boundary conditions and $y_n'(0) = 1$, verify

$$y_n(x) = \frac{\sin s_n x}{s_n} + O(\frac{1}{n^2}) \quad , \quad n = 0,\pm 1,\pm 2,\ldots .$$

For the remarks that follow, it will be enough to consider the case $q \equiv 0$. The two O's appearing above are now identically zero, and $\varphi_n \equiv \varphi$. The normalized eigenfunctions are

$$\text{(IV)}\quad V_n(x) = y_n(x)/\|y_n\|_2 = \sin (n+\varphi)x/\left(\frac{\pi}{2} - \frac{1}{4}\,\frac{\sin 2\pi(n+\varphi)}{n+\varphi}\right)^{1/2},$$
$$n = 0,\pm1,\pm2,\dots .$$

Firstly, we observe that the sistems $\{\sqrt{2/\pi}\cos nx: n = 1,2,\dots\}$, $\{\sqrt{2/\pi}\sin nx: n = 1,2,\dots\}$ are orthonormal and the second one is complete in $L^2(0,\pi)$. We call

$$1 + C_n = \frac{\|\sin nx\|_2}{\|\sin (n+\varphi)x\|_2}\quad,\qquad C = \sup_{n>0} |C_n|$$

and for d_n complex numbers , $n = 1,2,\dots,N$, $\|d\| = \left(\sum_{n=1}^{N} |d_n|^2\right)^{1/2}$.
Then, we have

$$\left\|\sum_{n=1}^{N} d_n\,(V_n-\sqrt{2/\pi}\sin nx)\right\|_2 \leqslant \sqrt{2/\pi}\left\|\sum_{n=1}^{N} d_n(1+C_n)(\sin(n+\varphi)x-\sin nx)\right\|_2 +$$

$$+\left\|\sum_{n=1}^{N} d_n\, C_n\sqrt{2/\pi}\sin nx\right\|_2 \leqslant \sqrt{2/\pi}\left\|\sum_1^N d_n(1+C_n)\sin nx\,(\cos\varphi x-1)\right\|_2 +$$

$$+\sqrt{2/\pi}\left\|\sum_1^N d_n(1+C_n)\cos nx\sin\varphi x\right\|_2 + C.\|d\| = R + S + C.\|d\| .$$

Therefore,

$$R \leqslant \|\cos\varphi x-1\|_\infty.\left\|\sum_1^N \sqrt{2/\pi}(1+C_n)d_n\sin nx\right\|_2 \leqslant (1+C)(\cosh\pi|\varphi|-1).\|d\| ,$$

$$S \leqslant \|\sin\varphi x\|_\infty.\left\|\sqrt{2/\pi}\sum_1^N d_n(1+C_n)\cos nx\right\|_2 \leqslant \sinh\pi|\varphi|.(1+C)\,\|d\| .$$

In consequence,

$$\text{(V)}\quad \left\|\sum_1^N d_n(V_n-\sqrt{2/\pi}\sin nx)\right\|_2 \leqslant [\,C+(1+C)(\sinh|\varphi\pi|+\cosh|\pi\varphi|-1)].\|d\|=$$
$$= K.\|d\| .$$

If the number a in (III) tends to zero, so do φ and C. Therefore if $|a|$ is small enough, (V) holds with $K < 1$. A theorem due to Paley and Wiener assures then that system (VI):

(VI) $\{V_n : n = 1,2,\dots\}$

is *biorthogonal and complete* (cf. [RSzN], pp.205-7).Then,we have:

There is a positive number ε such that if $|a| < \varepsilon$, system (IV) is *not biorthogonal* and any function $V_j(x)$, $j = 0,-1,-2,\dots$, has at least two expansions in $L^2(0,\pi)$ with respect to the normalized system of eigenfunctions (IV), one involving *only* positive j's.

(Observe that the argument given in [H] between formulae (24) and (25) is in contradiction with the last statement).

2. NOTATION AND BASIC RESULTS.

We assume $q(x) \in L^1(a,b)$, $-\infty < a \leq x \leq b < +\infty$, and $\sigma = s^2$, s a complex number. q(x) will be assumed complex too until further notice. By a solution of $-y'' + (q-\sigma)y = 0$ will be understood an absolutely continuous function y(x) with absolutely continuous derivative that verifies the equation almost everywhere. P(s),Q(s), $\tilde{P}(s)$ and $\tilde{Q}(s)$ will be complex *polynomials verifying:*

i) at least one of them is not a constant,

ii) g.c.d.(P,Q) = 1 , g.c.d.($\tilde{P},\tilde{Q}$) = 1.

iii) if P(Q) is identically zero then Q(P) is identically one. The same property will be shared by the pair $\tilde{P},\tilde{Q}$.

In what follows we shall study the following boundary value problem:

$$(1)\qquad \begin{cases} y'' - (q-\sigma)y = 0, \\ P(s)y(a) + Q(s)y'(a) = 0, \\ \tilde{P}(s)y(b) + \tilde{Q}(s)y'(b) = 0. \end{cases}$$

A solution $\phi_s(x)$ of the differential equation satisfies the following integral equations for $s\neq 0$:

$$
(2)\quad \begin{cases} \phi_s(x) = \phi_s(a)\cos s(x-a) + \phi_s'(a)\dfrac{\sin s(x-a)}{s} + \displaystyle\int_a^x \frac{\sin s(x-y)}{s} q(y)\phi_s(y)dy \;, \\[2ex] \phi_s(x) = \phi_s(b)\cos s(b-x) - \phi_s'(b)\dfrac{\sin s(b-x)}{s} + \displaystyle\int_x^b \frac{\sin s(y-x)}{s} q(y)\phi_s(y)dy \;, \end{cases}
$$

which for s=0 reduce to

$$
(2')\quad \begin{cases} \phi_0(x) = \phi_0(a) + \phi_0'(a)(x-a) + \displaystyle\int_a^x (x-y)q(y)\phi_0(y)dy \;, \\[2ex] \phi_0(x) = \phi_0(b) - \phi_0'(b)(b-x) + \displaystyle\int_x^b (y-x)q(y)\phi_0(y)dy \;. \end{cases}
$$

Let us call

$$
(3)\quad \begin{cases} F_1(s) = P(s) - isQ(s) \;, & F_2(s) = P(s) + isQ(s) \;, \\ F_3(s) = \tilde{P}(s) - is\tilde{Q}(s) \;, & F_4(s) = \tilde{P}(s) + is\tilde{Q}(s) \;. \end{cases}
$$

Then if $m = \deg F_1 \vee \deg F_2$, $n = \deg F_3 \vee \deg F_4$, and $p = \deg P$, $q = \deg Q$, $\tilde{p} = \deg \tilde{P}$, $\tilde{q} = \deg \tilde{Q}$, we obtain for $Q \not\equiv 0$:

$$
(4)\qquad m = p \vee (q+1) \quad , \quad n = \tilde{p} \vee (\tilde{q}+1).
$$

(If $P, Q, \tilde{P}$ and $\tilde{Q}$ are real then $\deg F_1 = \deg F_2$, $\deg F_3 = \deg F_4$).

We also have:

LEMMA 1. ii) implies g.c.d.$(F_1,F_2) = 1$ or s, g.c.d.$(F_3,F_4) = 1$ or s.

The *converse* is true in case $|P(0)|+|Q(0)| \neq 0$, $|\tilde{P}(0)|+|\tilde{Q}(0)| \neq 0$.

Next we define for $s \neq 0$:

$$
(5)\quad \begin{cases} u_s(x) = (F_1(s)\, e^{is(x-a)} - F_2(s)e^{-is(x-a)})/2is \\[1ex] \tilde{u}_s(x) = (F_3(s)\, e^{-is(b-x)} - F_4(s)e^{is(b-x)})/2is \end{cases}
$$

and for s=0:

$$(5')\quad \begin{cases} u_0(x) = -Q(0) + P(0)(x-a) = \lim\limits_{s\to 0} u_s(x) \ , \\ \tilde{u}_0(x) = -\tilde{Q}(0) - \tilde{P}(0)(b-x) = \lim\limits_{s\to 0} \tilde{u}_s(x) \ . \end{cases}$$

We shall denote with $U_s(x)$ and $\tilde{U}_s(x)$ the solutions of $y''-(q-\sigma)y=0$ defined by the following initial conditions:

$$(6)\qquad U_s(a) = -Q(s) \ , \ U_s'(a) = P(s) \ ; \ \tilde{U}_s(b) = -\tilde{Q}(s) \ , \ \tilde{U}_s'(b) = \tilde{P}(s).$$

(5) and (5') correspond to the particular case $q \equiv 0$.

We shall define *the characteristic function* of problem (1) as

$$(7)\qquad w(s) = W(U_s,\tilde{U}_s) = U_s(b)\tilde{U}_s'(b) - U_s'(b)\tilde{U}_s(b).$$

Then $U_s(x)$ will be a solution of (1) if and only if $w(s) = 0$.

The zeroes of w(s) will be called *eigenvalues* although the name "eigenfrequencies" could fit them better. By the *multiplicity* of an eigenvalue s we understand its order as a zero of w(s). An *eigenfunction* is a nontrivial solution of (1) for s an eigenvalue. Observe that $U_s(x)$, $\tilde{U}_s(x)$ are entire functions of s for fixed x and continuous functions of (s,x). The same holds for $U_s'(x)$, $\tilde{U}_s(x)$. Therefore w(s) is an entire function of s (cf. Appendix IV).

The notation that we use coincides, as far as possible, with that of [G]. In this way we avoid unnecessary repetitions of proofs that the reader could find in that work. Then, if a result is not accompanied by a proof and no reference is given it will mean that the proof is very simple or can be found in that paper. Until now we have only one deviation in notation: our σ is equal to $-\lambda$ where λ in [G] is the spectral parameter for the equation $y''-(\lambda+q(x))y = 0$.

LEMMA 2. If $s = u+iv$ then

$$|U_s(x)| = O(1)\, e^{|v|(x-a)}|s|^{m-1}; \quad |\tilde{U}_s(x)| = O(1)e^{|v|(b-x)}|s|^{n-1} \ ,$$

$$|U_s'(x)| = O(1)\, e^{|v|(x-a)}|s|^{m} \ ; \quad |\tilde{U}_s'(x)| = O(1)e^{|v|(b-x)}|s|^{n} \ ,$$

uniformly on [a,b] for $|s| \longrightarrow \infty$.

In particular they hold for u_s, u'_s, $\tilde{u}_s$ and $\tilde{u}'_s$. For a proof, check this last case and use the formulae

$$(8)\quad \begin{cases} U_s(x) = u_s(x) + \displaystyle\int_a^x \frac{\sin s(x-y)}{s}\, q(y)\, U_s(y)dy\ , & s\neq 0\ , \\ \tilde{U}_s(x) = \tilde{u}_s(x) + \displaystyle\int_x^b \frac{\sin s(y-x)}{s}\, q(y)\, \tilde{U}_s(y)dy\ , & s\neq 0\ . \end{cases}$$

We shall *denote with* $\delta(s)$ the wronskian $W(u_s,\tilde{u}_s)$. We have:

LEMMA 3. $\delta(s) = (F_1(s)F_4(s)e^{is(b-a)} - F_3(s)F_2(s)e^{-is(b-a)})/2is$ and $w(s) = \delta(s) + O(e^{|v|(b-a)}.|s|^M)$, $M = m+n-2$.

Recalling that the polynomials in the boundary conditions are of the form:

$$(9)\quad P(s) = \sum_{j=0}^{p} a_j s^j\ ,\ Q(s) = \sum_{j=0}^{q} b_j s^j\ ,\ \tilde{P}(s) = \sum_{0}^{\tilde{p}} \tilde{a}_j s^j\ ,\ \tilde{Q}(s) = \sum_{0}^{\tilde{q}} \tilde{b}_j s^j\ ,$$

we shall introduce another *hypothesis* on them:

$$\text{iv)}\quad \begin{cases} \text{degree } F_1 < m & \Rightarrow \quad \text{degree } F_3 = n \\ \text{degree } F_2 < m & \Rightarrow \quad \text{degree } F_4 = n \end{cases}$$

From this and lemma 3 it follows that $\delta(s)$ *and* $w(s)$ *are nontrivial entire functions of finite order.*

Observe that if $q(x) \equiv 0$ and $P(s) = is = \tilde{P}(s)$, $Q = 1 = \tilde{Q}$, then any complex number is an eigenvalue. iv) avoids such a situation.

By the *spectrum* of problem (1) we shall understand the set of eigenvalues where each eigenvalue is repeated as many times as its multiplicity.

THEOREM 1. The characteristic function $w(s)$ is completely determined by the spectrum and the boundary conditions.

PROOF. According to lemma 3, $w(s)$ is an entire function of order one. If $\Pi(s)$ denotes its canonical product, from Hadamard's theorem we obtain: $w(s) = e^{\alpha s+\beta}.s^k\Pi(s)$. The factor $s^k.\Pi(s)$ is determined by the spectrum. If w_1 and w_2 are the characteristic functions of problem (1) with fixed boundary conditions and for $q(x)$ equal to $q_1(x)$ and $q_2(x)$ respectively, then we have: $w_1(s)/w_2(s) = Ce^{\gamma s}$ whenever the spectra coincide.

From the hypothesis it follows that if F_1 has not degree m then deg F_2 = m and deg F_3 = n, and if deg $F_2 < m$ then deg F_1 = m, deg F_4 = n. Therefore we can assume without loss of generality that in Lemma 3: deg $F_2.F_3$ = m+n. In consequence, if s = ih, $h > 0$, we obtain: $w_j(ih) \sim F_2(ih)F_3(ih)e^{h(b-a)}/2h$, j = 1,2, and also that $w_1(ih)/w_2(ih) \sim 1$ for $h \to +\infty$. This implies that $\gamma = 0$ and C = 1, and then that $w_1(s) = w_2(s)$. Q.E.D.

3. THE EIGENVALUES FOR THE REAL CASE.

From now on we assume that $P, Q, \tilde{P}$ and $\tilde{Q}$ are real polynomials and that $q(x)$ is a real function. Now m = deg F_1 = deg F_2, n = deg F_3 = deg F_4, and iv) is always satisfied.

Besides $\overline{F_1(\bar{s})} = F_2(s)$ and $\overline{F_3(\bar{s})} = F_4(s)$ and also $\overline{\delta(\bar{s})} = \delta(s)$. Moreover, $U_s(x) = \overline{U_{\bar{s}}(x)}$ and $\tilde{U}_s(x) = \overline{\tilde{U}_{\bar{s}}(x)}$, and therefore

$$(10) \qquad w(s) = \overline{w(\bar{s})} \, .$$

Next we prove some results on the behaviour of the eigenvalues. Since $\delta(s)$ is the dominant term in the expansion of $w(s)$, we begin considering that function.

$$(11) \quad \delta(s) = \frac{e^{-is(b-a)}F_1(s)F_4(s)}{2is} . [- \frac{F_2F_3}{F_1F_4} + e^{2is(b-a)}] =$$

$$= F(s).[-e^{i\varphi} + O(\tfrac{1}{s}) + e^{2is(b-a)}]$$

where $\varphi \in [0, 2\pi)$, $F_2F_3/F_1F_4 \sim e^{i\varphi}$ for $s \to \infty$. Let us define C_N, for N a positive integer, as the boundary of the square with center at $(\varphi/2(b-a),0)$ and side equal to $\pi(2N+1)/(b-a)$.

From the behaviour of $|e^{i\varphi}-e^{2is(b-a)}|$ on $C_N^+ = C_N \cap \{\mathrm{Im}\ z > 0\}$ and the relation $\overline{\delta(s)} = \delta(\bar{s})$ we obtain the following result:

LEMMA 4. If N is great enough there exists a positive constant A such that for $s \in C_N$ it holds that

$$|\delta(s)| \geq A.e^{|v|(b-a)}.|s|^{M+1}, \quad M = m+n-2, \quad s = u+iv.$$

If $C_{\varepsilon,J}$ denotes the circle of radius $\varepsilon < \pi/(b-a)$ and center at $\rho_J \doteq \frac{\varphi}{2(b-a)} + \frac{\pi J}{b-a}$, J an integer, the following results can be proved as in [G], pp.10-12.

LEMMA 5. If $|J|$ is large enough and $s \in C_{\varepsilon,J}$ then there exists $A(\varepsilon)$ independent of s and positive such that

$$|\delta(s)| \geq A(\varepsilon).|s|^{M+1}.$$

COROLLARY. If N is large enough, w(s) and $\delta(s)$ have the same number of zeroes enclosed within C_N. Idem for $C_{\varepsilon,J}$, $|J| = N$.

LEMMA 6. There is a ρ such that for $|s| > \rho$, $\delta(s)$ has only real and simple zeroes. They are of the form $\rho_J + O(1/J)$, J integer of modulus sufficiently large.

We shall denote with S the discrete set of zeroes of w(s). If $s \in S$ then also $\bar{s} \in S$.

THEOREM 2. There is a ρ such that w(s) has only real and simple zeroes for $|s| > \rho$ of the form $\rho_J + O(1/J)$ for all integers J with $|J|$ sufficiently large. The distance between consecutive zeroes tends to $\pi/(b-a)$ for $|J| \to \infty$.

Next we define an *auxiliary function* $A(z)$. First we choose two real numbers α,β such that $D(z) = \alpha U_z(a)+\beta U'_z(a) = (\beta P-\alpha Q)(z)$ verifies $D(z) \neq 0$, $\forall\ z \in S$. Then, if $\tilde{D}(z) = \alpha\tilde{U}_z(a)+\beta\tilde{U}'_z(a)$ we call

(13) $$A(z) = \tilde{D}(z)/D(z) .$$

For each value of z, $\overline{A(\bar{z})} = A(z)$ holds.

THEOREM 3. Assume that $h_1(s)$ and $h_2(s)$ are polynomials and such that the function $f(s) = h_1(s)A(s)+h_2(s)$ is null at each eigenvalue of modulus sufficiently large. Then $h_1 \equiv 0 \equiv h_2$.

PROOF. We can assume that $\overline{h_j(\bar{s})} = h_j(s)$. (In fact, $\overline{f(\bar{s})}$, $f(s) + \overline{f(\bar{s})}$ and $i(f(s)-\overline{f(\bar{s})})$ verify the hypothesis. Then if the theorem is true for these last two functions then it also holds for $f(s)$). If for $|s| \geqslant R$, $w(s) = 0$ implies $f(s) = 0$, we define $k(s)$ as the product: $k(s) = \Pi(s-s_j)(s-\bar{s}_j)$ where the s_j's involved are the zeroes of w of modulus less than R.

The function $\Phi(s) = k(s)f(s).D(s)$ is an entire function null at at each point of S and is of the form:

(14) $\Phi(s) = H_1(s).\tilde{D}(s)+H_2(s).D(s)$, with $H_1 = kh_1$ and $H_2 = kh_2$ real polynomials.

Then $\Phi(s) = O(e^{|v|(b-a)}.|s|^{\omega})$ for certain integer ω. Since $1/w(s) = O(e^{-|v|(b-a)}.|s|^{-(M+1)})$ on C_N, we obtain $\Phi(s)/w(s) = O(|s|^{\rho})$ for $s \in C_N$ and ρ an integer. Since Φ/w is entire this estimation holds inside C_N and also in all the complex plane. Therefore, there exists a polynomial $G(s)$ such that $\Phi(s) = w(s).G(s)\ \forall\ s$. (Since $\overline{\Phi(\bar{s})} = \Phi(s)$, G is real). From the preceding relation $\Phi(s) = (-Q(s)\ \tilde{U}'_s(a) - P(s)\ \tilde{U}_s(a)).G(s)$.

So, in view of (14),

(15) $$H(s) = \tilde{U}'_s(a).\Sigma(s) + \tilde{U}_s(a).\Pi(s)$$

where H,Σ and Π are the real polynomials: $H = H_2(-\beta P + \alpha Q)$, $\Sigma = GQ + \beta H_1$, $\Pi = GP + \alpha H_1$. The function $A(s)$ verifies $\tilde{U}_s(b) =$

$= A(s)U_s(b)$, $\tilde{U}_s'(b) = A(s)U_s'(b)$, $\forall\ s \in S$, and therefore $w(s) = 0$ implies $A(s) \neq 0$. If Σ and Π are identically zero then $H_2 \equiv 0$, and also $h_2 \equiv 0$. From the hypothesis, $h_2 \equiv 0$ implies that $h_1(s) = 0$ for an infinite number of eigenvalues, and so $h_1 \equiv 0$. Assume then that at least one of the polynomials Σ,Π is not identically zero. If $\tilde{M} = \text{g.c.d.}(\Sigma,\Pi)$ we get:

$$\tilde{U}_s(a).P_1(s) + \tilde{U}_s'(a).Q_1(s) = (H/\tilde{M})(s). \tag{16}$$

Assume $V_s(x)$ is the solution of $y''-(q-\sigma)y = 0$ such that $V_s'(a) = P_1(s)$, $V_s(a) = -Q_1(s)$. Then $W(V_s,\tilde{U}_s)(a)$ is the wronskian of a problem (1) with the first boundary condition replaced by $P_1(s)y(a) + Q_1(s)y'(a) = 0$. (It could occur that $\tilde{P},\tilde{Q}$, and P_1,Q_1, are constants but then in each pair not both are simultaneously null). $W(V_s,\tilde{U}_s)(a)$ is a function with an infinite number of zeroes and because of (16), equal to $-H(s)/\tilde{M}(s)$. Then $H(s) \equiv 0$ and as before $h_2 \equiv 0$ and also $h_1 \equiv 0$. Q.E.D.

4. EIGENFUNCTIONS AND KERNELS.

For $s,t \in S$, $(s^2-t^2)\int_a^b U_s(x)U_t(x)\,dx = W(U_s,U_t)\Big|_a^b$, $\tilde{U}_s(b) = -\tilde{Q}(s) = A(s)U_s(b)$ and $U_s'(b) = \tilde{P}(s) = A(s).U_s'(b)$. The following lemma then follows:

LEMMA 7. Assume $s,t \in S$ and $s^2 \neq t^2$. If we call:

$$V(s,t) = \frac{P(s)Q(t) - P(t)Q(s)}{s-t}\ ,\quad \tilde{V}(s,t) = \frac{\tilde{P}(s)\tilde{Q}(t) - \tilde{P}(t)\tilde{Q}(s)}{s-t}$$

then we have,

$$\langle U_s,U_t\rangle = \int_a^b U_s(x)U_t(x)\,dx = (U_s,U_{\bar{t}}) = -\frac{V(s,t)}{s+t} + \frac{1}{A(s)\,A(t)}\cdot\frac{\tilde{V}(s,t)}{s+t}.$$

LEMMA 8. a) If one of the boundary conditions in (1) is ordinary then for fixed $t \in S$ the number of $s \in S$ such that $(U_s,U_t) = 0$ is finite.

b) If neither boundary condition in (1) is ordinary then for fixed $t \in S$ there is an infinite number of $s \in S$ such that $(U_s,U_t) \neq 0$.

PROOF. b) If $(U_s,U_t) = 0$ for all $s \in S$ of modulus large enough and $|s| > |t|$, then: $A(s)(s+\bar{t})(U_s,U_t) = h_2(s) + h_1(s).A(s)$, where $h_2(s) = \tilde{V}(s,\bar{t})/A(\bar{t})$, $h_1(s) = -V(s,\bar{t})$, verifies the hypothesis of theorem 3. Therefore $h_1 \equiv h_2 \equiv 0$. In consequence $\tilde{V}(s,\bar{t}) \equiv 0 \equiv$ $\equiv V(s,\bar{t})$ $\forall$ s. But this is imposible because of the hypothesis i) and ii) on the polynomials $P,Q,\tilde{P},\tilde{Q}$, (cf.[G], App.II,p.68). Q.E.D.

Next we define the kernels $g_s(x,y) = g(x,y;s)$, $G_s(x,y) = G(x,y;s)$ as follows:

(17) $$g(x,y;s) = \begin{cases} \frac{s}{\delta(s)} \tilde{u}_s(x)\ u_s(y) & , \quad y < x \leqslant b , \\ \frac{s}{\delta(s)} \tilde{u}_s(y)\ u_s(x) & , \quad a \leqslant x < y , \end{cases}$$

(18) $$G(x,y;s) = \begin{cases} \frac{s}{w(s)} \tilde{U}_s(x)\ U_s(y) & , \quad y < x \leqslant b , \\ \frac{s}{w(s)} \tilde{U}_s(y)\ U_s(x) & , \quad a \leqslant x < y . \end{cases}$$

$s^{-1}.G(x,y;s)$ and $s^{-1}.g(x,y;s)$ are the Green kernels of problem (1) for $q(x) \in L^1$ and $q(x) \equiv 0$ respectively.

THEOREM 4. Assume $\Phi \in L^1(a,b)$, $a \leqslant x \leqslant b$ and $0 < \delta < b-a$. Then

i) $$\int_{C_N} [\int_a^b G(x,y;s)\ \Phi(y)\ dy]\, ds = \int_{C_N} [\int_a^b g(x,y;s)\ \Phi(y) dy]\ ds + o(1) ,$$

ii) $$\int_{C_N} [\int_a^b g(x,y;s)\Phi(y) dy]\, ds = \int_{C_N} [\int_{(x-\delta)\vee a}^{(x+\delta)\wedge b} g(x,y;s)\Phi(y) dy]\ ds + o(1) ,$$

where the o's are uniform in $x \in [a,b]$ for $N \to \infty$.

iii) If Φ is of bounded variation on $[a,b]$ and $a < x < b$, for $N \to \infty$ we have:

$$\frac{1}{2\pi i} \int_{C_N} [\int_a^b g(x,y;s)\Phi(y)\,dy]\,ds = \frac{1}{2}(\Phi(x+0) + \Phi(x-0)) + o(1).$$

If Φ is also continuous on $[a,b]$, $o(1)$ is uniform on compact sets in (a,b).

iv) If Φ is of bounded variation on $[a,b]$ then the following relation holds uniformly on compact sets in (a,b):

$$\frac{1}{2\pi i} \int_{C_N} [\int_a^b \frac{g(x,y;s)}{s} \Phi(y)dy]\,ds = o(1).$$

PROOF. i) Can be proved in the same way as Th.1,[G],p.18, and ii) as Th.2, [G], p.19. The proof of iii) follows the same pattern as Th.3,[G], pp.20-22. The proof of iv) is an easy adaptation of the proof of iii). Q.E.D.

COROLLARY. i) holds with ds replaced by ds/s, and iv) with g replaced by G.

PROOF. This follows repeating the proof of i) Th.4, and then using iv) of the same Theorem. Q.E.D.

Next result is a complement to the preceding theorem.

THEOREM 5. If Φ is of bounded variation on $[a,b]$ and real, then

$$(19) \quad \frac{1}{2\pi i} \int_{C_N} ds \int_a^b G(b,y;s)\Phi(y)dy \xrightarrow[N\to\infty]{} L(b).\Phi(b-0) ,$$

where $L(b) = 0$ if $\tilde{Q} \equiv 0$ or $\tilde{q} < \tilde{p}-1$, $=1$ if $\tilde{q} \geqslant \tilde{p}$ and $Q \not\equiv 0$, and

$$= \frac{(\tilde{b}_{\tilde{q}})^2}{(\tilde{b}_{\tilde{q}})^2 + (\tilde{a}_{\tilde{p}})^2} \quad \text{if} \quad \tilde{q} = \tilde{p}-1 \;,\; \tilde{Q} \not\equiv 0.$$

PROOF. According to theorem 4, it suffices to prove (19) with G replaced by g. For $s \in C_N$, $v = \text{Im } s \geqslant 0$, $0 < \delta < b-a$, we have:

$$\int_{b-\delta}^{b} g(b,y;s)\, dy = \frac{-s\, \tilde{Q}(s)}{\delta(s)} \int_{b-\delta}^{b} u_s(y)\, dy =$$

$$= - \frac{i\tilde{Q}(s)}{F_3(s)} \left[\frac{F_1F_3e^{is(b-a)}(1-e^{-is\delta})+F_2F_3e^{-is(b-a)}(1-e^{is\delta})}{-2is\delta(s)} \right] = O(\frac{1}{|s|}).$$

Moreover, for fixed s, [...] is bounded and tends to zero if $\delta \to 0$. Besides,

$$(20) \quad \int_{b-\delta}^{b} g(b,y;s)\Phi(y)dy = \Phi(b-0)\int_{b-\delta}^{b} g(b,y;s)dy+O(\frac{1}{|s|}).\overset{b-0}{\underset{b-\delta}{V}} \Phi(y),$$

as it can be seen making use of the second mean value theorem, (cf. [G], pp.21). Therefore,

$$(21) \quad \frac{1}{2\pi i} \int_{C_N^+} ds \int_{b-\delta}^{b} g(b,y;s)\Phi(y)dy = \frac{\Phi(b-0)}{2\pi i} \int_{C_N^+} \frac{-i\tilde{Q}(s)}{F_3(s)} [\ldots]ds + O(1).\overset{b-0}{\underset{b-\delta}{V}} \Phi(y) .$$

To prove (19), it is sufficient to considerer (21) since

$$\frac{1}{2\pi i} \int_{C_N^-} ds \int_{b-\delta}^{b} g(x,y;s)\Phi(y)dy = \overline{(\frac{1}{2\pi i} \int_{C_N^+} ds \int_{b-\delta}^{b} g(x,y;s)\overline{\Phi(y)}dy)} .$$

The theorem will be proved if we show that the integral in the right-hand side of (21) tends to L'(b) if $N \to \infty$, where L'(b)= 0 if $Q \equiv 0$ or $\tilde{q} < \tilde{p}-1$, $L'(b) = \pi i$ if $\tilde{Q} \not\equiv 0$ and $\tilde{q} \geqslant \tilde{p}$ and $L'(b) = \pi\, b_{\tilde{q}}/(a_{\tilde{p}} - ib_{\tilde{q}})$.

In fact, if $Q \not\equiv 0$, from lemma 4 we get

$$(22) \quad \int_{C_N^+} -\frac{i\tilde{Q}}{F_3} [\ldots]ds = \int_{C_N^+} -\frac{i\tilde{Q}}{F_3}.\frac{F_2F_3e^{-is(b-a)}}{-2is\delta(s)}\, ds + \int_{C_N^+} O(1)e^{2is(b-a)}(1-e^{-is\delta})\, \frac{ds}{s} +$$

$$+ \int_{C_N^+} O(1)\, e^{is\delta} \frac{ds}{s} .$$

Let us estimate for example, the third integral in the right-hand side of (22):

$$\int_{C_N^+} O(1)\, e^{is\delta} \frac{ds}{s} = \int_{C_1^+} f_N(s)\, e^{is((2N+1)\pi/2(b-a))\delta} \frac{ds}{s} ,$$

where $\{f_N(s)\}$ is uniformly bounded on C_1^+.

From Lebesgue dominated convergence theorem it follows that last integral tends to zero for $N \to \infty$. The same thing happens to the middle term in the right-hand side of (22). But the first one is equal to:

$$-i\int_{C_N^+} \frac{\tilde{Q}}{\tilde{P}-is\tilde{Q}}\left(1+ \frac{F_1F_4e^{is(b-a)}}{-2is\delta(s)}\right)ds = L'(b)+\int_{C_1^+} O(1)e^{is((2N+1)\pi/(b-a))\delta} \frac{ds}{s} =$$

$$= L'(b) + o(1).$$

From this last result, (21) and ii) of Theorem 4, (19) follows. Q.E.D.

An application of the theorem of residues shows that:

$$(23) \quad \frac{1}{2\pi i} \int_{C_N} ds \int_a^b G(x,y;s)\Phi(y)dy = \sum_{s\in S_N} \operatorname{Res}_s \int_a^b G(x,y;s)\Phi(y)dy.$$

where the summation is over the eigenvalues in S that are inside C_N. The details of the proof of equality (23) are the same as those given in [BP], pp.172.

If s is a simple zero of w(s) we have

$$(24) \quad \operatorname{Res}_{t=s} \int_a^b G(x,y;t)\Phi(y)dy = H(s).U_s(x) \ , \ H(s) = \frac{sA(s)}{w'(s)} \int_a^b U_s(y)\Phi(y)dy.$$

Then the following result follows easily from theorem 4:

THEOREM 6. If Φ is continuous and of bounded variation on $[a,b]$, null in neighborhoods of a and b, and if all the zeroes of $w(s)$ are simple, then $\Phi(x)$ can be represented on $[a,b]$ as a uniformly convergent series of eigenfunctions of problem 1.

COROLLARY. If the eigenvalues are of multiplicity one then the set of eigenfunctions is complete in $L^2(a,b)$.

5. SPECTRA WITH EIGENVALUES OF MULTIPLICITY ONE.

From now on we shall assume that if $s \in S, s \neq 0$ *, then it is a simple zero of* $w(s)$ *and if* $0 \in S$ *then it is at most a double zero of* w. According to (24) if 0 is a simple zero of w then $H(0) = 0$ and $U_0(x)$ does not appear in the right-hand side of (23).

If 0 is an eigenvalue of multiplicity two, then

$$(25) \quad \operatorname*{Res}_{t=0} \int_a^b G(x,y;t)\Phi(y)dy = \{\frac{2A(0)}{w''(0)} \int_a^b U_0(y)\Phi(y)dy\} \cdot U_0(x) =$$

$$= H(0).U_0(x)$$

and Theorem 6 and its corollary still hold in this case.

Assume that f is a summable real function. We shall denote with b_s the s-th *Fourier product*.

$$b_s = b_s(f) = (f,V_s) = \int_a^b f(y)\ \overline{V_s(y)}dy = \overline{b_{\bar{s}}(f)}\ ,$$

where $V_s = U_s/\|U_s\|_2$, and with $b(f) = \{b_s(f): s \in S\}$ the *Fourier product vector*. We have: $b(f) = 0 \Rightarrow f=0$. N will denote the infinite matrix (N_{st}), $(s,t) \in S\times S$, defined by:

$$
(26)\quad \mathcal{N}_{st} = \begin{cases} 2A(0)\cdot\|U_0\|_2^2/w''(0) & s=\bar{t}=0 \text{ if } 0 \text{ is a double zero of } w \\ sA(s)\cdot\|U_s\|_2^2/w'(s) & s=\bar{t}\;,\quad s\neq 0, \\ 0 & \text{in any other case.} \end{cases}
$$

$\mathcal{N}$ transforms Fourier product vectors in coefficient vectors. In fact it holds that (cf.(23),(24) and (25)):

$$
(27)\quad \frac{1}{2\pi i}\int_{C_N}\left[\int_a^b G(x,y;s)f(y)dy\right]ds = \sum_{s\in S_N} (\mathcal{N}b)_s\, V_s(x), \text{ for } f \text{ real.}
$$

THEOREM 7. i) $\mathcal{N}$ is a hermitian matrix,

ii) $\|U_s\| = \sqrt{\frac{b-a}{2}}\,|c_m|\,|s|^{m-1}(1+O(1/|s|))$, where c_m is the coefficient of s^m in $F_1(s)$: $c_m = \lim_{s\to\infty}\frac{F_1(s)}{s^m}$,

iii) if $s^2 \neq t^2$ and $V_s = U_s/\|U_s\|$ then

$\langle V_s, V_t\rangle = \sum_{j=1}^{M'} \frac{J_j(s)\,K_j(t)}{s+t}$ where $J_j(s) = O(1)$, $K_j(s) = O(1)$ for $|s| \to \infty$, and where $M' = m'+n'$, m' depends only on p,q, and n' only on $\tilde{p},\tilde{q}$. If besides $q \geq p$, $\tilde{q} \geq \tilde{p}$ then $J_j(s) = O(1/|s|)$, $K_j(s) = O(1/|s|)$.

PROOF. i) is left to the reader. Let us prove ii) Assume s real $s \in S$. From formulae (5) it follows that $\|u_s\|_\infty = O(\|u_s\|_2)$, $\|u_s\|_2 = O(\|u_s\|_\infty)$ for $|s| \to \infty$. We shall write this relation briefly as

$$
(28)\qquad \|u_s\|_\infty \approx \|u_s\|_2 \;.
$$

From (8) we obtain $\|U_s\|_\infty \leq \|u_s\|_\infty + O(\frac{1}{|s|})\cdot\|U_s\|_\infty$ and therefore that $\|U_s\|_\infty = O(\|u_s\|_\infty)$. On the other hand,

$$
\|u_s\|_\infty = \|U_s - \frac{1}{s}\int_a^x \sin s(x-y)q(y)U_s(y)dy\|_\infty = O(\|U_s\|_\infty) \;,
$$

and from (28) it follows that

(29) $$\|U_s\|_\infty \approx \|u_s\|_\infty .$$

Therefore, $\|U_s\|_2 = O(\|u_s\|_2)$. But

$$\|U_s-u_s\|_2 = \|\frac{1}{s}\int_a^x \sin s(x-y)q(y)U_s(y)dy\|_2 .$$

(28),(29) and Young's convolution theorem allows us to write

$$\|U_s-u_s\|_2 = O(\frac{1}{|s|})\ \|qU_s\|_1 = O(\frac{\|U_s\|_\infty}{|s|}) = O(\frac{\|u_s\|_2}{|s|}) .$$

In consequence, $|\|u_s\|_2 - \|U_s\|_2| = O(\frac{\|u_s\|_2}{|s|})$. This implies that

(30) $$\|U_s\|_2 = \|u_s\|_2\ (1 + O(\frac{1}{|s|}))\ ,\quad \|U_s\|_2 \approx \|u_s\|_2.$$

A direct calculation shows that

(31) $$(\|u_s\|_2/|s|^{m-1})^2 = (b-a)(|F_1(s)|/|s|^m)^2/2 + O(\frac{1}{|s|}).$$

ii) follows from (30) and (31).

If $d_n = \lim_{s\to\infty} F_3(s)/s^n$, in an analogous way we obtain

(32) $$\|\tilde{U}_s\| = \sqrt{\frac{b-a}{2}}\ |d_n||s|^{n-1}.(1+O(\frac{1}{|s|})).$$

iii) From Lemma 7 we get

(33) $$\langle V_s,V_t\rangle = -\frac{V(s,t)/\|U_s\|.\|U_t\|}{s+t} + \frac{\tilde{V}(s,t)/\|U_s\|A(s).\|U_t\|A(t)}{s+t}$$

Observe that $x^iy^j - x^jy^i = (x-y)(x^{i-1}y^j+x^{i-2}y^{j+1}+\ldots+x^jy^{i-1})$ for $i > j$. This relation and ii) imply that for a certain $m' = m'(p,q)$,

(34) $$\frac{V(s,t)}{\|U_s\|.\|U_t\|} = \sum_{k=1}^{m'} J_k(s)K_k(t)\ ,\quad J_k(s) = O(1)\ ,\quad K_k(s) = O(1)$$

for $|s| \to \infty$.

Taking into account that $\|A(s)U_s\|_2 = \|\tilde{U}_s\|_2$ and using (32), it follows analogously:

$$(35)\quad \frac{\tilde{V}(s,t)}{\|U_s\|A(s)\cdot\|U_t\|A(t)} = \sum_{j=1}^{n'} \tilde{J}_j(s)\tilde{K}_j(t) \ , \ \tilde{J}_j(s) = O(1) \ , \ \tilde{K}_j(s) = O(1) \ ,$$

$$n' = n'(\tilde{p},\tilde{q}).$$

(34) and (35) imply the first part of iii). If $q \geqslant p$ then $m-1=q$. With this observation, a more careful examination of the proof given above shows that the second part of iii) also holds. Q.E.D.

DEFINITIONS. Let J_0 be a positive integer such that for each integer J, $|J| > J_0$, there exists a unique eigenvalue s_J such that $s_J = \rho_J + O(\frac{1}{J})$, $\rho_J = \frac{\pi J}{b-a} + \frac{\varphi}{2(b-a)}$, $|O(\frac{1}{J})| \leqslant \frac{\pi}{4(b-a)}$, s_J real and simple. This subset of S will be denoted by $\mathbb{P}$. Let $\mathbb{F}$ be the finite subset of S: $\mathbb{F} := S\setminus\mathbb{P}$, (cf.Th.2, §3). Let us call $\mathbb{D} = \{(s,t) \in \mathbb{P}\times\mathbb{P}: s=t \text{ or } |s+t| < \pi/4(b-a)\}$, and $\mathbb{O} := \mathbb{P}\times\mathbb{P}\setminus\mathbb{D}$.

THEOREM 8. i) The application $\iota:\ell^2(S) \longrightarrow L^2(a,b)$ defined by $\iota(c) = \sum_{s\in S} c_s V_s$ is continuous, $c = \{c_s\}$.

ii) It is also onto: given $f \in L^2$, if N is the matrix (26), then the sequence $c = \{c_s\}$, $c_s = N(b(f))_s$, belongs to ℓ^2 and

$$(36)\qquad f = \sum c_s V_s .$$

We shall call, for a given $f \in L^2$, *residual coefficients* the elements of $N(b(f))$, and *residual expansion* of f the expansion (36) with residual coefficients. We could denote them as well *Carslaw coefficients* and *Orr expansion* of f, respectively. It has been shown that a dense family of L^2=functions has a residual expansion. Part ii) in the theorem asserts that this holds for any function in $L^2(a,b)$.

PROOF OF THEOREM 8. i) Assume $c = \{c_s\} \in \ell^2$ has only a finite number of non-null elements. We have,

$$(37)\quad \| \sum_{s\in\mathbf{S}} c_s V_s \| \leqslant \| \sum_{s\in\mathbf{F}} c_s V_s \| + \| \sum_{s\in\mathbf{P}} c_s V_s \| \leqslant \|c\| . (\#\mathbf{F})^{1/2} + \| \sum_{s\in\mathbf{P}} c_s V_s \| .$$

For the last term we have:

$$\| \sum_{s\in\mathbf{P}} c_s V_s \|^2 = \sum_{s,t\in\mathbf{P}} c_s \bar{c}_t (V_s, V_t) \leqslant \sum_{\mathbf{D}} |c_s c_t| + |\sum_{\mathbf{O}} c_s \bar{c}_t (V_s, V_t)| = A + B.$$

From the definition of $\mathbb{D}$ it is clear that for each $s \in \mathbb{P}$ there are at most two values of t such that $(s,t) \in \mathbb{D}$. Therefore, using proposition 1, Appendix III, we obtain $A \leqslant 2\|c\|^2$. Because of iii) Theorem 7, B can be written as

$$B = \left| \sum_{\mathbf{O}} c_s \bar{c}_t \left(\sum_{j=1}^{M'} \frac{J_j(s) K_j(t)}{s+t} \right) \right| \leqslant \sum_{j=1}^{M'} \left| \sum_{\mathbf{O}} \frac{c_s \bar{c}_t J_j(s) K_j(t)}{s+t} \right| = \sum_{j=1}^{M'} \frac{b-a}{\pi} H_j .$$

Each H_j is bounded by a sum of the following 4 terms: $\left| \sum_{O_h} \frac{c_s \bar{c}_t J_j(s) K_j(t)}{(s+t)(b-a)/\pi} \right|$,

$h = 1,2,3,4$, and where O_h is the subset of $\mathbb{O}$ in the h^{th} quadrant. These terms are of the form

$$(38)\qquad \sum_{m,n>J_0}{}' \frac{a_m \, b_n \, J_{j,m} \, K_{j,n}}{\delta(m,n)} ,$$

where $a_m = c_{s_{\pm m}}$, $b_n = \bar{c}_{s_{\pm n}}$, $\delta(m,n) = (s_{\pm m} + s_{\pm n})(b-a)/\pi$, $J_{j,m} = J_j(s_{\pm m})$, $K_{j,n} = K_j(s_{\pm n})$. Let α and β be defined as $\alpha(m) = \frac{b-a}{\pi} \cdot s_m - \frac{\varphi}{2\pi}$, and $\beta(m) = -\frac{b-a}{\pi} \cdot s_{-m} + \frac{\varphi}{2\pi}$. Then $\delta(m,n)$ is one of the following four expressions:

$$\begin{cases} \delta(m,n) = \alpha(m)+\alpha(n)+\varphi/\pi , \ (h=1) \ ; \quad \delta(m,n) = \alpha(m)-\beta(n)+\varphi/\pi , \ (h=4) \ ; \\ \delta(m,n) = -\beta(m)+\alpha(n)+\varphi/\pi , \ (h=2) \ ; \quad \delta(m,n) = -\beta(m)-\beta(n)+\varphi/\pi , \ (h=3) \ . \end{cases}$$

In cases h=1,3, the dash indicates that $m \neq n$, and in cases h=2,4, the dash indicates the omission of the terms such that $|\delta(m,n)| \leqslant 1/4$. Returning to cases h=1,3, we observe that

$$(39)\quad \sum_{m,n>J_0}{}' \left| \frac{a_m \, b_n \, J_{j,m} \, K_{j,n}}{\delta(m,n)} \right| \leqslant \sum_{\delta(m,n)\geq 1/4} \left| \frac{a_m \, b_n \, J_{j,m} \, K_{j,n}}{\delta(m,n)} \right|$$

since in the right-hand side no term is omitted.

In fact, for example, if h=1: $|\alpha(m)+\alpha(n)+\varphi/\pi| \geqslant m+n - \frac{1}{2} - |\varphi/\pi| \geqslant$
$\geqslant 7/2 - 2 > 1/4$.

Applying ii), Theorem 1 of Appendix III for the cases h=1,3, and i) of the same theorem for the cases h=2,4, we obtain:

$$|H_j| \leqslant C \; \|a\| \; \|b\| \quad , \quad j=1,\ldots, M' ,$$

with C independent of a and b. In consequence $B \leqslant C' \; \|c\|^2$ and then

$$\text{(40)} \qquad \| \sum_{s\in\mathbb{P}} c_s \; V_s\|_2 \leqslant L.\|c\|_2 .$$

Only remains to show that $\sum_{|s|<R} c_s V_s$ converges in L^2 for $R \to \infty$.

But from (40) we get: $\| \sum_{r<|s|<R} c_s V_s \| \leqslant L(\sum_{r<|s|} |c_s|^2)^{1/2} \xrightarrow[r\to\infty]{} 0$,

whatever it be R. This concludes the proof of i).

ii) Two auxiliary results will be needed to prove the second proposition in Theorem 8. Next we state them and postpone their proofs.

LEMMA 9. If $f \in L^2(a,b)$ then $b(f) \in \ell^2(S)$ and $b: L^2 \to \ell^2$ defines a linear bounded operator.

LEMMA 10. The elements in the main diagonal of the infinite matrix N are asymptotically equal to 1/2:

$$\text{(41)} \qquad N_{ss} = \frac{1}{2} + O(\frac{1}{|s|}) \quad , \quad |s| \to \infty .$$

To prove ii) assume $f \in L^2$ and that $\{f_m\}$ is a sequence of functions for which a residual expansion exists such that $f_m \to f$ in L^2. From lemma 9, $\|b(f_m)-b(f)\|_2 \to 0$. In particular, $b(f_m)_s \to b(f)_s \;\; \forall \; s \in S$, and from lemma 10, it follows that $\|N(b(f_m))-N(b(f))\|_2 \to 0$. Therefore,

$$\|f_m - \sum_S N(b(f))_s V_s\|_2 = \|\sum N(b(f) - b(f_m))_s V_s\|_2 \leqslant$$

$\leqslant L \; \| N(b(f)) - N(b(f_m)) \|_2 \xrightarrow[m\to\infty]{} 0.$

In consequence, $f = \sum_S N(b(f))_s \, V_s$. Q.E.D.

PROOF OF LEMMA 9. From (2),(6) and Th.7, ii), we obtain, for $|s| \to \infty$: $V_s(x) = O(1) \cos s(x-a) + O(1) \sin s(x-a) + O(\frac{1}{|s|})$ and from this and Th.2,

$$(42) \quad V_s(x) = O(1) \sin \frac{\pi(x-a)J}{b-a} + O(1) \cos \frac{\pi(x-a)J}{b-a} + O(\frac{1}{J}) \; ,$$

if $s = \rho_J + O(\frac{1}{J})$. Therefore

$b_s(f) = (O(1) \; f, \sin \frac{\pi(x-a)J}{b-a}) + (O(1) \; f, \cos \frac{\pi(x-a)J}{b-a}) + O(\frac{\|f\|}{J})$

and in consequence, $\|b(f)\| \leqslant C.\|f\|$ with C independent of $f \in L^2$,

Q.E.D.

PROOF OF LEMMA 10. Let be $s \in S$. From (6),(7) and the relation $\widetilde{U}_s(b) = A(s)U_s(b)$, etc., we get:

$$(43) \quad \frac{dw(s)}{ds} \frac{1}{A(s)} = \frac{dU_s(b)}{ds} \cdot U'_s(b) - \frac{dU'_s(b)}{ds} \cdot U_s(b) + \frac{U_s(b)}{A(s)} \frac{d\widetilde{P}(s)}{ds} + \frac{U'_s(b)}{A(s)} \frac{d\widetilde{Q}(s)}{ds} \; .$$

But from Lemma 2 and the fact $A^{-1}(s) = U_s(b)/-\widetilde{Q}(s) = U'_s(b)/\widetilde{P}(s) = O(|s|^{m-n})$, we obtain:

$$(44) \quad \frac{U'_s(b)}{A(s)} \frac{d\widetilde{Q}(s)}{ds} = O(|s|^{2(n-1)}) \; , \quad \frac{U_s(b)}{A(s)} \frac{d\widetilde{P}(s)}{ds} = O(|s|^{2(m-1)}) \; .$$

Therefore, if $s \in S$,

$$(45) \quad \frac{dw}{ds} \frac{1}{A(s)} = \{ \frac{dU_s(b)}{ds} U'_s(b) - \frac{dU'_s(b)}{ds} U_s(b) \} + O(|s|^{2(m-1)}) \; .$$

On the other hand we have from (8) and Lemma 2 that

$$\frac{dU_s(x)}{ds} = O(|s|^{m-1}) + \frac{1}{s}\cdot\int_a^x \sin s(x-y)\, \frac{dU_s(y)}{ds}\, q(y)dy,\ s \text{ real},\ |s| \to \infty$$

and therefore for s real, $|s| \to \infty$,

(46) $$\frac{dU_s(x)}{ds} = O(|s|^{m-1}),\ \text{uniformly on } a \leqslant x \leqslant b\ .$$

Analogously, $U'_s(x) = u'_s(x) + \int_a^x \cos s(x-y)q(y)U_s(y)dy$, and

$$\frac{dU'_s(x)}{ds} = O(|s|^m) + \int_a^x \cos s(x-y)q(y)\, \frac{dU_s(y)}{ds}\, dy\ -$$

$$- \int_a^x \sin s(x-y)q(y)(x-y)\, U_s(y)dy\ ,$$

which implies, for s real, $|s| \to \infty$, that

(47) $$\frac{dU'_s(x)}{ds} = O(|s|^m)\ ,\ \text{uniformly on } a \leqslant x \leqslant b.$$

Then observe that if $\Delta(x) = U_s(x) - u_s(x)$ we have:

(48) $$\begin{cases} \Delta = O(|s|^{m-2})\ , & \frac{d\Delta}{ds} = O(|s|^{m-2})\ , \\ \Delta' = O(|s|^{m-1})\ , & \frac{d\Delta'}{ds} = O(|s|^{m-1})\ , \end{cases}$$

for s real, $|s| \to \infty$, and uniformly on $[a,b]$.
Returning to the brackets in (45) and using formulae (48) we get

(49) $$\{\ldots\} = [\frac{du_s(b)}{ds}\, u'_s(b) - u_s(b)\, \frac{du'_s(b)}{ds}] + O(|s|^{2(m-1)})\ .$$

But the expression inside the square brackets in (49) is equal to (cf.(5) and Th.7):

(50) $$[\ldots] = F_1(s)F_2(s)(b-a)/s + O(|s|^{2(m-1)}) =$$

$$= |c_m|^2(b-a)s^{2m-1}(1 + O(1/|s|)).$$

$$(51)\qquad \frac{1}{A(s)}\,\frac{dw(s)}{ds} = |c_m|^2(b-a)\; s^{2m-1} + O(|s|^{2(m-1)}).$$

Then, from (26),(50) and (ii) Th.7, it follows that for $s \in S$, $|s| \longrightarrow \infty$, $N_{ss} = \|U_s\|_2^2 . s/(\frac{dw}{ds}\,\frac{1}{A(s)}) = \frac{1}{2} + O(\frac{1}{|s|})$, Q.E.D.

6. STRUCTURE OF THE SET OF EIGENFUNCTIONS. GRAM'S MATRIX AND DEGREES OF FREEDOM.

We recall that in this section the hypotheses stated in section 5 are assumed. Next we start with two propositions already proved in what precedes.

DEFINITION. $\mathcal{S}$ will denote the set $S\backslash\{0\}$ if 0 is a simple zero of w. $\mathcal{S} = S$ in any other case.

According to what we said at the beginning of section 5 about the appearance or not of U_0 in the residual expansions, $\mathcal{S}$ resembles more an ordinary spectrum than S does. Besides, with this distinction we avoid some small but fastidious difficulties.

(P_1) There is a denumerable set $\mathcal{S} = \{s_i\}$ - the set of eigenvalues - such that $s_i^2 = \lambda_i$, $s_i \in \mathcal{S} \Rightarrow \bar{s}_i \in \mathcal{S}$, and except for a finite subset: $s_i \in \mathcal{S} \Rightarrow s_i = \bar{s}_i$.

(P_2) For each $s \in \mathcal{S}$ there is a function U_s - the eigenfunction - such that $U_s = \bar{U}_{\bar{s}}$, and if $s,t \in \mathcal{S}$, $s^2 \neq t^2$, then

$$(52)\quad (s+t)\langle V_s,V_t\rangle = (s+t)\int_a^b V_s(x)V_t(x)dx = \sum_{i,j=0}^{\gamma^*-1} \mathbb{C}_{ij}\; k_i(s)\; k_j(t),$$

where $V_s = U_s/\|U_s\|_2$, $\mathbb{C}_{ij} = \mathbb{C}_{ji}$ real, and

$$(53)\qquad k_j(s) = \begin{cases} s^j/\|U_s\| , & 0 \leqslant j < p\vee q \\ s^{j-(p\vee q)}/A(s)\|U_s\| , & p\vee q \leqslant j < \gamma^* = p\vee q+\tilde{p}\vee\tilde{q}. \end{cases}$$

The matrix $\mathbb{C}$ is obtained as follows. From section 4 (lemma 7) we have:

$$(54)\qquad \begin{cases} V(s,t) := (P(s)Q(t)-P(t)Q(s))/(s-t) = \sum_{i,j=0}^{p\vee q-1} c_{ij} s^i t^j, \text{ in case } p\vee q \geqslant 1 \\ \tilde{V}(s,t) := (\tilde{P}(s)\tilde{Q}(t)-\tilde{P}(t)\tilde{Q}(s))/(s-t) = \sum_{i,j=0}^{\tilde{p}\vee\tilde{q}-1} \tilde{c}_{ij} s^i t^j, \text{ in case } \tilde{p}\vee\tilde{q} \geqslant 1. \end{cases}$$

Let $C = (c_{ij})$, $\tilde{C} = (\tilde{c}_{ij})$ whenever they are defined.

$$(55)\qquad \mathbb{C} = \begin{bmatrix} -C & 0 \\ 0 & \tilde{C} \end{bmatrix}$$

In case C or $\tilde{C}$ is not defined, we omit it in (55). Because of the general hypotheses stated in section 2, we have: γ^* = order of $\mathbb{C} \geqslant 1$ and $\det \mathbb{C} \neq 0$, (cf.[G], App.II, or [BP] or [Z]).

The Gramian - or Gram's matrix - is defined as the infinite matrix $\mathbb{A} = (A_{ts})$, $t,s \in \mathcal{S}$, $A_{ts} = (V_s,V_t)$. Then, for $s^2 \neq \bar{t}^2$,

$$(56)\qquad A_{ts}(s+\bar{t}) = \langle V_s,V_{\bar{t}}\rangle\ (s+\bar{t}) = \sum_{i,j=0}^{\gamma^*-1} \mathbb{C}_{ij}\ k_i(s)\ k_j(\bar{t}) =$$

$$= k'(s).\mathbb{C}.k(\bar{t}) = k'(\bar{t}).\mathbb{C}.k(s) = (s+\bar{t})\ A_{\bar{s}\bar{t}} ,$$

where $k(s)$ is the column vector $[k_0(s),k_1(s),\ldots,k_{\gamma^*-1}(s)]$ and $k'(s)$ represents its transpose.

In general, $\{k_i(s)\} \in \ell^\infty(\mathcal{S})\ \ \forall\, i$. This follows from theorem 7 and the fact, already used in the proof of Lemma 10, that

$$(57)\qquad 1/A(s) = O(|s|^{m-n}) ,\quad A(s) \neq 0 \quad \text{for} \quad s \in S.$$

k is a $\gamma^*\times\infty$ - matrix with *linearly independent rows.*

In fact, this follows from the definition of $k_j(s)$ and Th. 3. Now we define for $0 \leqslant i,j \leqslant \gamma^*-1$, *the dyad* Y_{ij} which is an $\infty\times\infty$-matrix such that

$$(58) \qquad (Y_{ji})_{ts} = Y_{ji}(t,s) = \frac{k_j(\bar{t})\ k_i(s)}{\bar{t}+s} \quad \text{if} \quad \bar{t}^2 \neq s^2 ,$$
$$= \quad 0 \quad \text{if} \quad \bar{t}^2 = s^2 .$$

Then

$$(59) \qquad L := \sum_{i,j=0}^{\gamma^*-1} \mathbb{C}_{ji}\ Y_{ji} ,$$

$$(60) \qquad D := \mathbb{A}-L\ ,\ D_{ts} \begin{cases} = A_{ts} & \text{if} \quad \bar{t} = \pm s , \\ = 0 & \text{if} \quad t \neq \pm s . \end{cases}$$

We know that there is a number r such that if $|s| > r$ then $s \in \mathcal{S}$ implies $s=\bar{s}$. Therefore, D is a matrix with non-null entries only in the diagonals $\{\pm t = s\}$ and in the finite submatrix defined by the indexes $|s| \vee |t| \leqslant r$. We shall say in this situation that D is a *quasi-diagonal* matrix. Of course

$$(61) \qquad |D_{ts}| \leqslant |A_{ts}| = |(V_s,V_t)| \leqslant 1.$$

THEOREM 9. $\mathbb{A}$: $\ell^2(\mathcal{S}) \longrightarrow \ell^2(\mathcal{S})$ and defines a linear bounded application.

PROOF. If b and c are elements of ℓ^2 such that $b_s = c_s = 0$ for $|s| > N$, then from i) Th.8 we get

$$(62) \qquad \|\sum c_s V_s\|^2 = \sum c_s (V_s,V_t)\bar{c}_t \leqslant M\ \|c\|^2 .$$

Since $\sum c_s(V_s,V_t)\bar{c}_t = c'\mathbb{A}c$ is a hermitian quadratic form we obtain from (62)

$$|b'\mathbb{A}c| \leqslant M\ \|b\|.\|c\|.$$

Therefore

(63) $$\|\mathbb{A}c\| \leqslant M \|c\|.$$

A density argument shows now that (63) holds for arbitrary $c \in \ell^2$. Q.E.D.

For each $c \in \ell^2$ there is an $f \in L^2$: $f = \sum c_s V_s$ $(L^2(a,b))$. The Fourier products of f define an element (in ℓ^2, cf.Th.9) $b = b(f)$ such that $\mathbb{A}.c = b(f)$. We denote with B the range of $\mathbb{A}$:

$$R(\mathbb{A}) = \mathbb{A}(\ell^2) = B = \{b \in \ell^2 : b = b(f) \text{ for some } f \in L^2\}.$$

Since $\{V_s\}$ is complete in L^2, $b(f) = 0$ *is equivalent* to $f=0$.

THEOREM 10. i) There exists a quasi-diagonal matrix N that defines an operator N on $B \subset \ell^2$ such that

(64) $$\mathbb{A}.N = I \quad , \quad \text{on} \quad B.$$

ii) If Γ denotes the range of N: $N(B) = \Gamma \subset \ell^2$, then $\Gamma = \overline{\Gamma}$. Besides $B = \overline{B} \neq \ell^2$.

PROOF. i) N coincides with the residual matrix (N_{st}) defined in section 5, (cf.Th.8, ii) for $(s,t) \in \mathcal{S}\times\mathcal{S}$.

ii) Assume there is an f such that for a fixed t, $|t| > r$: $b_s(f) = \delta_{st}$. Then $f = \sum N(b(f))_s V_s$. But then we have $f = n_{tt}V_t$ and therefore $b(f) = n_{tt}A_{st} = \delta_{st}$. Since $n_{tt} \neq 0$, this contradicts Lemma 8. That is: $\delta_{st} \notin B$ and $B \neq \ell^2$.

Let us prove now that B is closed. *Denote also with N the extension of N from* B *to* ℓ^2, which is possible since $n_{tt} \sim 1/2$.

Assume $b_j \to \beta$ in ℓ^2. Then $Nb_j \to N\beta$. But if $f_j = \sum (Nb_j)_s V_s$, because of theorem 8, $\{f_j\}$ is a Cauchy sequence in L^2. Let f be the function defined by : $f = \sum (N\beta)_s V_s$. Then we have:

(65) $$\begin{cases} \|f-f_j\|_2 \leqslant k\,\|Nb_j - N\beta\|_2 \to 0 \quad \text{for } j \to \infty , \\ b_s(f_j) \to b_s(f) \quad \text{in} \quad \ell^2. \end{cases}$$

Therefore, $b_s(f) = \beta_s$, and $\beta \in B$.

Assume $\gamma_j = Nb(f_j)$ tends to $\gamma \in \ell^2$. Then $\{\gamma_j\}$ is a Cauchy sequence and so is $\{f_j\}$ (Th.8). Therefore, $f_j \to f \in L^2$, and $b(f_j) \to b(f)$ in ℓ^2 (cf.Ths.8 and 9). In consequence $Nb(f_j) \to Nb(f) = \gamma$. That is, Γ is closed. Q.E.D.

COROLLARY. N defines a one-to-one continuous application from $B \subset \ell^2$ onto $\Gamma \subset \ell^2$ and $\mathbb{A}$ restricted to Γ defines the left inverse of N restricted to B. Besides, ℓ^2 is the direct sum of Γ and G, the null space of $\mathbb{A}$:

(66) $$G = \{g: \mathbb{A}.g = 0\} \quad , \quad G \dot{+} \Gamma = \ell^2(\mathcal{S}).$$

PROOF. For the last part of the corollary observe that $G \cap \Gamma = \{0\}$, and that (64) can be written as $\mathbb{A}N\mathbb{A} = \mathbb{A}$ on ℓ^2. Then for $c \in \ell^2$, $c - N\mathbb{A}c = g \in G$, i.e., $c = g + N(\mathbb{A}c) \in G+\Gamma$. Therefore, ℓ^2 is the direct sum (not necessarily orthogonal) of G and Γ.

Q.E.D.

The space G is characterized as follows:

$$g \in G \Leftrightarrow g \in \ell^2(\mathcal{S}) \text{ and } \sum_{s\in\mathcal{S}} g_s V_s = 0 \text{ in } L^2(a,b).$$

In other words, $\sum g_s V_s$ is a null series. We shall prove that $\dim G = \infty$.

THEOREM 11. N defines a biyective continuous application from ℓ^2 onto ℓ^2 and $\operatorname{def} B = \operatorname{def} \Gamma > 0$.

PROOF. Assume that N^r is the centered square submatrix of N with entries N_{st} verifying $|s| \vee |t| \leqslant r$. If N_{st} is not an entry of N^r then $N_{st} = 0$ for $s \neq t$, $N_{st} \neq 0$ for $s=t$ (cf.(26)).

Therefore, if $N.c = 0$ then $c_s = 0$ for $|s| > r$. On the other hand, $\det N^r \neq 0$, and then $N^r.c^r = 0 \Rightarrow c_s = 0$ for $|s| \leqslant r$, i.e., $c=0$. This means that N defines an inyective application from ℓ^2 into ℓ^2. Since $N = N^*$, it is also onto.

Suppose $m \perp \Gamma$, $m \neq 0$. Then, $0 = (m, \mathcal{N}b) = (\mathcal{N}^* m, b)\ \forall\ b \in B$, and since $\mathcal{N}$ is one-to-one in $\ell^2 \ominus \Gamma$ we obtain: $\mathcal{N}(\ell^2 \ominus \Gamma) = \ell^2 \ominus B$. $B \neq \ell^2$ implies def B = def $\Gamma > 0$. Q.E.D.

Theorem 11 means, in particular, that there exist always *null series*, i.e., series whose Fourier coefficients form a non-null ℓ^2-sequence and that converge to 0 in L^2. By the *degrees of freedom* g of system $\{V_s : s \in \mathcal{S}\}$ we shall understand the dimension of the subspace G. It is a measure of the number of expansions an L^2-function can have. In our situation $g = \infty$, (cf. [Z]).

THEOREM 12. $g = \dim G = \text{def } B = \text{def } \Gamma = \infty$.

PROOF. Because of Theorem 11 and the Corollary to Theorem 10 it only remains to prove that $g = \infty$. We shall consider three cases:

i) $0 \notin S$; ii) $0 \in S$ but $0 \notin \mathcal{S}$, i.e. 0 is a simple zero of w; iii) $0 \in \mathcal{S}$, that is, 0 is a double zero of w.

i) From formula (18) we have

$$\frac{G(x,y;s)}{s} = \begin{cases} \tilde{U}_s(x) U_s(y)/w(s) & \text{for} \quad y < x \leqslant b, \\ \tilde{U}_s(y) U_s(x)/w(s) & \text{for} \quad a \leqslant x < y, \end{cases} \tag{67}$$

and from the Corollary to Theorem 4 we obtain:

$$\frac{1}{2\pi i} \int_{C_N} ds \int_a^b \frac{G(x,y;s)}{s} \Phi(y)\, dy = o(1) \tag{68}$$

uniformly on compact sets in (a,b) if Φ is of bounded variation.

Then: $\operatorname*{Res}_{s=0} \int_a^a \frac{G(x,y;s)}{s} \Phi(y)\, dy = 0$, and as in (27) it follows that

$$\sum_{s \in \mathcal{S}} \{\mathcal{N}(b(\Phi)).s^{-1}\}\, V_s = 0 \tag{69}$$

uniformly on compact sets. N is a *linear, continuous and inyective application* from ℓ^2 into ℓ^2 (cf. proof of Theorem 11), and b is linear, inyective and continuous from L^2 into ℓ^2 (cf. Lemma 9). Then, $\frac{N\circ b}{s} : L^2 \longrightarrow \ell^2$ is also linear, inyective and continuous. Because of i) Theorem 8, (69) converges in L^2. Therefore $\frac{N\circ b}{s}$ defines a one-to-one application from the family of functions of bounded variation on $[a,b]$ into G. This implies dim G = ∞.

ii) In this case the proof of i) can be repeated but taking into account that

$$(70) \qquad \operatorname*{Res}_{s=0} \int_a^b \frac{G(x,y;s)}{s} \Phi(y)dy = \frac{A(0)}{w'(0)} (\Phi,U_0)\ U_0(x).$$

Because of the density in L^2 of the set of functions of bounded variation, given a positive integer n, we can find n functions $\Phi_1,\ldots,\Phi_n$, linearly independent, such that $(\Phi_j,U_0) = 0$ for $j = 1,\ldots,n$. Then, instead of (69) we can write

$$(71) \qquad 0.U_0(x) + \sum_{s\in S} N(b(\Phi_j))_s . s^{-1} . V_s = 0$$

Since $N(b(\Phi_j))$, $j = 1,\ldots,n$, is a linearly independent set of elements in $\ell^2(S)$, (71) implies that dim G = ∞.

iii) In this case, instead of (71) we obtain (cf.(23),Appendix IV):

$$(72) \quad U_1(x,0)\langle \Phi_j,c_{-2}U_0\rangle + U_0(x)\langle \Phi_j,c_{-1}U_0+c_{-2}U_1\rangle + \sum_{s\in S}\{Nb(\Phi_j)_s/s\}.V_s = 0 ,$$

where $c_{-2} \neq 0$ and U_1 is an associated function for s=0. If we take $\{\Phi_j : j = 1,\ldots,n\}$ linearly independent such that $\langle \Phi_j,U_0\rangle = 0$, the corresponding series (72) are null series (i.e. have coefficient vectors belonging to G). Thus again dim G = ∞. Q.E.D.

Given $\gamma \in \Gamma$ *we shall denote with* $\gamma^\perp$ the element in the manifold γ+G that is nearest to G. $\gamma^\perp$ is uniquely determined and verifies

(73) $$(\gamma+G) \cap G^{\perp} = \{\gamma^{\perp}\} .$$

In fact, if ξ and η belong to the set on the left-hand side then: $\eta-\xi \in G \cap G^{\perp} = \{0\}$.

PROPOSITION 1. a) There is a one-to-one correspondence between Γ and $G^{\perp}$ defined by $\Theta: \gamma \to \gamma^{\perp}$,

b) $\forall\ f \in L^2(a,b)$: $\|b(f)\|_2 \approx \|f\|_2 \approx \|\gamma\|_2 \approx \|\gamma^{\perp}\|_2$, for $\gamma = Nb(f)$.

PROOF. a) If $c \in G^{\perp}$ then $c = \gamma+g$ (cf.(66)), and because of (73) we see that Θ is onto (cf.Fig.1).

b) $\|b(f)\|_2 \approx \|\gamma\|_2$, $\gamma = Nb(f)$, follows from the Corollary to Theorem 10. Besides we have: $\|\gamma\| \geqslant \|\gamma^{\perp}\|$ and also $M.\|\gamma\| \geqslant \|f\|$ with M independent of f (cf.ii)Th.8). Using Banach inverse theorem we finally obtain: $\|\gamma^{\perp}\| \approx \|\gamma\| \approx \|f\|$. Q.E.D.

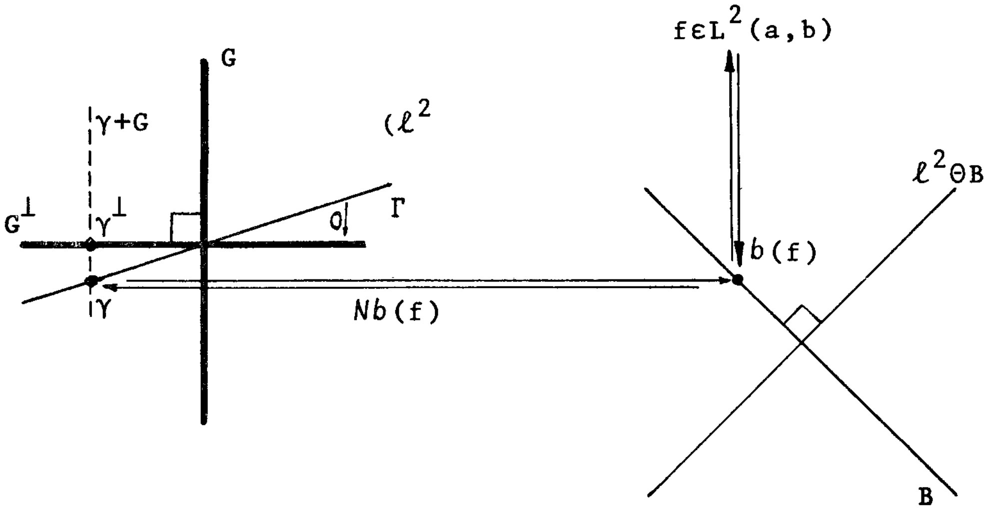

Fig.1

7. FINAL REMARKS.

We complement now sections 5 and 6 with some remarks.

a) If s and -s belong to S then it may happen that for a constant α ($\neq 0$): $U_s = \alpha U_{-s}$. The expansion of an $f \in L^2$ in eigenfunctions admits, in this case, a superfluous ambiguity.

Let us define R as the subset of S such that

$R = \{-s \in S : 0 \neq s \in S, \arg s \in [0,\pi) \text{ and } U_s = \alpha U_{-s} \text{ for a certain constant } \alpha\}$.

We shall denote with S^v the subset of S such that

$$S^v = S \setminus R.$$

PROPOSITION 1. $\{U_s : s \in S^v\}$ is linearly independent.

PROOF. Let us assume that the elements of S^v are ordered. If the proposition is false there is one U_{s_k} that is a linear combination of U_{s_j}'s with $j < k$. We can suppose k is the smallest number with such a property. Applying the operator $q(x) - \frac{d^2}{dx^2}$ to

$$U_{s_k} = \sum_{j<k} \alpha_j U_{s_j} \quad \text{we obtain} \quad s_k^2 U_{s_k} = \sum_{j<k} \alpha_j s_j^2 U_{s_j}.$$

Then, $\sum_{j<k} \alpha_j (s_j^2 - s_k^2) U_{s_j} = 0$. In consequence, $\alpha_j (s_j^2 - s_k^2) = 0$ $\forall j$, and then $\alpha_j = 0$ for any j such that $s_j^2 \neq s_k^2$. For $s_j^2 = s_k^2$, we have $s_j = -s_k$, and $\alpha_j \neq 0$ implies $U_{s_k} = \alpha_j U_{-s_k}$.
But this is impossible because of the definition of S^v. Then $\alpha_j = 0$ $\forall j$, a contradiction. Q.E.D.

If $P, Q, \tilde{P}$ and $\tilde{Q}$ are even polynomials then S contains with s also -s, and $U_s = U_{-s}$ also holds, as can be seen in [G], p.12. In that paper the set of eigenvalues is, by definition, the subset S' of S such that $\arg s \in [0,\pi)$, that is $S' = S^v$.

Now we add a new condition on the polynomials $P, Q, \tilde{P}$ and $\tilde{Q}$ to exclude the case studied by Güichal in [G], (i)-iii) § 2 are assumed).

Auxiliary hypothesis. v) $P(s).Q(-s)$ or $\tilde{P}(s).\tilde{Q}(-s)$ is not even.

PROPOSITION 2. v) holds if and only if $P, Q, \tilde{P}$ and $\tilde{Q}$ are not simultaneously even.

PROOF. It is enough to prove the following equivalence:

$$P(s).Q(-s) \text{ not even} \Leftrightarrow P \text{ or } Q \text{ not even.} \tag{74}$$

$\Rightarrow$: it is obvious. $\Leftarrow$: Let us assume, without loss of generality, that $P(s)$ is not even, and that $P(s)Q(-s)$ is an even polynomial. Then, there is a root of P, s_0, such that $-s_0$ is not a root of P. Therefore, $-s_0$ is a root of $Q(-s)$, that is: $Q(s_0) = 0$. But this is impossible, because of ii) section 2. Q.E.D.

PROPOSITION 3. If the polynomials $P, Q, \tilde{P}$ and $\tilde{Q}$ satisfy i), ii), iii) and v) then $R = \{-s \in S: 0 \neq s \in S, \arg s \in [0,\pi), U_s = \alpha U_{-s}$ for a certain constant $\alpha\}$ is a finite set.

PROOF. $U_s = \alpha U_{-s}$ implies $P(s) = \alpha P(-s)$, $Q(s) = \alpha Q(-s)$ as well as the same relations for $\tilde{P}$ and $\tilde{Q}$. Therefore,

$$P(s)Q(-s)-P(-s)Q(s) = 0 \quad , \quad \tilde{P}(s)\tilde{Q}(-s)-\tilde{P}(-s)\tilde{Q}(s) = 0. \tag{75}$$

v) means that the equalities in (75) are not both identities. From this, proposition 3 follows. Q.E.D.

When all the polynomials in the boundary conditions are even, the set $\{U_s: s \in R\}$ of "superfluous" eigenfunctions is infinite. So, it should not be a surprise to find in section 6 that $\dim G = \infty$.

Whether the boundary polynomials are even or not, we can define an application X from $\ell^2(S)$ onto $\ell^2(S^v)$:

$$X: c \longrightarrow \tilde{c}, \quad \tilde{c}_s = \begin{cases} c_s + c_{-s} & \text{if } -s \in R \text{ and } V_s = V_{-s}, \\ c_s - c_{-s} & \text{if } -s \in R \text{ and } V_s = -V_{-s}, \\ c_s & \text{if } -s \notin R. \end{cases}$$

Then, $\sum_{s \in S} c_s V_s = \sum_{s \in S^v} \tilde{c}_s V_s$ and, for $g \in \ell^2(S)$:

$$g \in G \equiv \sum_{s \in S} g_s V_s = 0 \equiv \sum_{s \in S^v} \tilde{g}_s V_s = 0.$$

In consequence, $X^{-1}(0) = \{g \in G: g_s = \pm g_{-s} \text{ if } -s \in R,\ g_s = 0 \text{ if } -s \notin R\}$.
Let $\tilde{G}$ be the set

$$\tilde{G} = \{\tilde{g} \in \ell^2(S^v): \sum_{s \in S^v} \tilde{g}_s V_s = 0\}.$$

If the boundary polynomials are all even, it is proved in [G], ch.V, that $\dim \tilde{G} < \infty$. If v) holds, observing that

α) $\dim X^{-1}(0) = \# R < \infty$

β) $\dim G = \dim \tilde{G} + \dim X^{-1}(0)$,

we obtain: $\dim \tilde{G} = \infty$. Therefore, *assuming* i) — v), *there is no advantage in using* S^v *instead of* S, *or* $\mathcal{S}$, except for the fact that is the content of proposition 1.

b) We saw that if $P, Q, \tilde{P}$ and $\tilde{Q}$ are not simultaneously even, then, except for a finite number of $s \in S$, it is not true that $U_s = \alpha U_{-s}$.

Does this imply that from some moment on, s and -s are not simultaneously in S? Not at all. If in (1), $q \equiv 0$, a=0, b=1, then

$$(76) \qquad \dot{w}(s) = (P\tilde{Q} - \tilde{P}Q) \cos s + (P\tilde{P} + s^2 Q\tilde{Q}) \frac{\sin s}{s}.$$

Therefore, if the polynomials inside the parentheses are even, w is a function of s^2. And this holds, for example, for $P \equiv \tilde{P} \equiv 1$, $Q \equiv \tilde{Q}$ an *odd* polynomial, i.e., in this case S = -S. However, it holds the following proposition.

PROPOSITION 4. Assume $p = q+1$, $\tilde{p} \neq \tilde{q}+1$, and $Q \not\equiv 0$ (or $\tilde{p} = \tilde{q}+1$, $p \neq q+1$, $\tilde{Q} \not\equiv 0$). Then

(77) $$\#\{s \in S:\ s \text{ and } -s \in S\} \text{ is finite.}$$

PROOF. Because of theorem 2 and (11) we know that (77) surely holds if $\varphi \neq \dot{\pi}$. We recall that

$$\frac{P(s)+isQ(s)}{P(s)-isQ(s)} \cdot \frac{\tilde{P}(s)-is\tilde{Q}(s)}{\tilde{P}(s)+is\tilde{Q}(s)} \xrightarrow[|s|\to\infty]{} e^{i\varphi}.$$

The second factor tends to +1 or to -1, but the first one tends to $(a_p+ib_q)/(a_p-ib_q) = e^{i\psi}$, $\psi \neq \dot{\pi}$. Q.E.D.

c) One question that seems to play a role in this theory is the following one. Does $\{c_s\}$ belong to $\ell^2(S)$ if $\sum c_s V_s$ converges in $L^2(a,b)$? In the other words, is it true or not that ·

(78) $$\sum_{\substack{N'>|s|>N \\ s\in S}} c_s V_s \xrightarrow[N',N\to\infty]{} 0 \Rightarrow \sum_{\substack{N'>|s|>N \\ s\in S}} (c_s)^2 \xrightarrow[N',N\to\infty]{} 0$$

We show that (78) does not hold in general. Precisely, we give an example satisfying the hypothesis of proposition 4 for which exists $\{c_s\}$ such that $\sum |c_s|^2 = \infty$ and $\sum_{|s|\leq N} c_s V_s$ converges in L^2.

We recall that, in case the boundary polynomials are even, (78) holds if "$s \in S$" is replaced by "$s \in S^v$", (cf.[G], p.33).

PROPOSITION 5. Assume $\varepsilon > 0$ and a real, $\neq 0$. If ε is sufficiently small, $|a| < \varepsilon$ and $\{V_n: n=0, \pm 1, \pm 2, \ldots\}$ is the system of normalized eigenfunctions defined in the introduction as system (IV), then the series

(79) $$\ldots - \frac{1}{\sqrt{h}} V_{-N_h} - \ldots - \frac{1}{\sqrt{2}} V_{-N_2} - \frac{1}{\sqrt{1}} V_{-N_1} + c_1 V_1 + c_2 V_2 + \ldots$$

converges in $L^2(0,\pi)$ if the sequence N_h, $1 = N_1 < N_2 < N_3 < \dots$ increases fast enough and the c_s' are adequate coefficients.

PROOF. Let K be the constant (less than one) appearing in formula (V) of the Introduction and let $\{g_n\}$ be the biorthogonal sequence associated to system (VI). We know that any function $f \in L^2(0,\pi)$ satisfies the following relations (cf.[RSzN],p.206):

$$(80) \qquad (1+K)^{-1}\|f\|_2 \leq \left(\sum_{n=1}^{\infty} |(f,g_n)|^2\right)^{1/2} \leq (1-K)^{-1}\|f\|_2 .$$

$$(81) \qquad f = \sum_{n=1}^{\infty} (f,g_n)V_n \quad \text{in} \quad L^2(0,\pi).$$

Let us denote with a_h^j the following scalar products

$$(82) \qquad a_h^j = (h^{-\frac{1}{2}} V_{-N_h}, g_j) \quad , \quad j=1,2,\dots, \; h=1,2,\dots .$$

The sequence N_h together with an auxiliary sequence M_h are defined by the following induction process:

$$M_1 = 0 \quad , \quad N_1 = 1 \ ,$$

Given N_h let M_{h+1} be such that

i) $\quad M_{h+1} > N_h$,

ii) $\quad \left\| \sum_{j=1}^{M_{h+1}} a_h^j V_j - \frac{1}{\sqrt{h}} V_{-N_h} \right\|_2 \leq \frac{1}{h^2}$.

Then, choose N_{h+1} so great that

j) $\quad N_{h+1} > M_{h+1}$,

jj) $\quad |a_{h+1}^j| \leq 2^{-j}/(h+1)^2$, for $1 \leq j \leq M_{h+1}$.

This is possible since

$$(V_{-N},g_j) = \sqrt{\frac{2}{\pi}}\,(1+C_N).\left[\int_0^{\pi} \sin\varphi x \; g_j(x).\cos Nx dx - \int_0^{\pi} \cos\varphi x \; g_j(x).\sin Nx \, dx\right]$$

tends to 0 if j is fixed and $N \to \infty$.

So we have defined $M_1 < N_1 < M_2 < N_2 < \dots$. From ii) and jj), it follows:

$$(83)\quad \Big\| \sum_{j=M_h+1}^{M_{h+1}} a_h^j V_j - \frac{1}{\sqrt{h}} V_{-N_h} \Big\|_2 \leqslant \frac{1}{h^2} + \sum_{j=1}^{M_h} \frac{2^{-j}}{h^2} \leqslant \frac{2}{h^2} .$$

From (80) and (82) we get

$$(84)\qquad \Big(\sum_{j=1}^{\infty} |a_h^j|^2 \Big)^{1/2} \leqslant \frac{1}{(1-K)\sqrt{h}} , \quad h=1,2,\dots .$$

Now , we define

$$(85)\qquad c_j = a_h^j , \quad M_h < j \leqslant M_{h+1} .$$

Let us prove that

$$(86)\qquad E_{AB} = - \sum_{A<N_h\leq B} \frac{V_{-N_h}}{\sqrt{h}} + \sum_{A<j\leq B} c_j V_j$$

tends to zero in L^2 for $A,B \to \infty$.

Assume first $A = M_h$, $B = M_{k+1}$. From (83) and (86) we obtain

$$(87)\quad \|E_{AB}\| \leqslant \sum_{r=h}^{k} \Big\| \frac{1}{\sqrt{r}} V_{-N_r} - \sum_{j=M_r+1}^{M_{r+1}} c_j V_j \Big\|_2 \leqslant \sum_{r=h}^{k} \frac{2}{r^2} \xrightarrow[h,k\to\infty]{} 0.$$

If $M_h \leqslant A \leqslant B \leqslant M_{h+1}$, using (80) and (84) we have:

$$(88)\quad \|E_{AB}\| \leqslant \frac{1}{\sqrt{h}} + \Big\| \sum_{j=A+1}^{B} c_j V_j \Big\|_2 \leqslant \frac{1}{\sqrt{h}} + (1+K)\Big(\sum_{A+1}^{B} |c_j|^2 \Big)^{1/2} \leqslant$$

$$\leqslant \Big(1 + \frac{1+K}{1-K}\Big) \frac{1}{\sqrt{h}} .$$

From (88) and (87), it follows that $\|E_{AB}\| \to 0$ for $A,B \to \infty$.

Q.E.D.

APPENDIX I.

ON THE ZEROES OF THE EIGENFUNCTIONS OF A BOUNDARY VALUE PROBLEM WITH BOUNDARY CONDITIONS INVOLVING LINEARLY THE SPECTRAL PARAMETER.

1. INTRODUCTION. We shall consider the second order differential equation

(q) $$y'' + (\lambda - q)y = 0 \qquad a \leqslant x \leqslant b$$

with the boundary conditions

[δ] $$y(a).(\delta_1+\lambda\delta_1') + y'(a).(\delta_2+\lambda\delta_2') = 0 \ , \ \Delta = \delta_1'.\delta_2-\delta_1.\delta_2' > 0 \ ,$$

[β] $$-y(b)(\beta_1+\lambda\beta_1') + y'(b)(\beta_2+\lambda\beta_2') = 0 \ , \ \rho = \beta_1'.\beta_2-\beta_1.\beta_2' > 0.$$

When $\delta_1' = \delta_2' = 0$, $\delta_1 = \cos\alpha$, $\delta_2 = \sin\alpha$, $0 \leqslant \alpha < \pi$, we shall write (α) instead of [δ] and if $\delta_i = \beta_i$, $\delta_i' = \beta_i'$, i=1,2, we shall write $[\tilde{\beta}]$ instead of [δ]. ($[\tilde{\beta}]$ is the boundary condition symmetric to [β]). $\phi(x,\lambda) = \phi_\lambda(x)$ and $\chi(x,\lambda) = \chi_\lambda(x)$ will denote the solutions of (q) defined by the following initial conditions:

(1) $$\begin{pmatrix}\phi_\lambda\\ \phi_\lambda'\end{pmatrix}(a) = \begin{pmatrix}\delta_2+\lambda\delta_2'\\ -\delta_1-\lambda\delta_1'\end{pmatrix} \quad ; \quad \begin{pmatrix}\chi_\lambda\\ \chi_\lambda'\end{pmatrix}(b) = \begin{pmatrix}\beta_2+\lambda\beta_2'\\ \beta_1+\lambda\beta_1'\end{pmatrix}$$

We shall assume $q \in L^1(a,b)$, real, and call a solution of (q) any absolutely continuous function with absolutely continuous derivative that satisfies the equation almost everywhere.

2. THE ZEROES OF $\phi_\lambda(x)$. First we state two fundamental results due to Sturm (for a proof see [T], pp.107-110).

THEOREM 1. Assume $g < h$ a.e., $g,h \in L^1(a,b)$. If u is a nontrivial solution of (g) and v is a solution of (h) then strictly between two consecutive zeroes of u there is a zero of v.

THEOREM 2. If besides $u(a) = v(a) = \sin\alpha$, $u'(a) = v'(a) = -\cos\alpha$, and u has m zeroes in $(a,b]$ then v has at least m zeroes there, and the i-th zero of v is less than the i-th zero of u.

Next theorem 3 is an adaptation to our situation of theorem 2.

THEOREM 3. Assume ϕ_λ and ϕ_μ defined as above, $\lambda > \mu$, and ϕ_μ has m zeroes on $(a,b]$. Then for λ sufficiently near to μ, ϕ_λ has at least m zeroes on $(a,b]$ and the i-th zero of ϕ_λ is less than the i-th zero of ϕ_μ.

PROOF. In view of Theorem 1 it is enough to see that if x_1 is the first zero of ϕ_μ on $(a,b]$ then $\phi_\lambda(x)$ has a zero in (a,x_1). Now

$$(\lambda-\mu)\int_a^{x_1}\phi_\lambda(x)\phi_\mu(x)dx = \phi_\lambda(x_1)\phi_\mu'(x_1)-W(\phi_\lambda,\phi_\mu)(a) = \phi_\lambda(x_1)\phi_\mu'(x_1)+(\lambda-\mu)(\delta_1\delta_2'-\delta_1'\delta_2),$$

and therefore

$$\phi_\lambda(x_1)\ \phi_\mu'(x_1)/(\lambda-\mu) = \Delta + \int_a^{x_1}\phi_\lambda(x)\phi_\mu(x)dx. \tag{2}$$

If $\phi_\mu(a) \neq 0$ and λ is sufficiently near to μ we have sg $\phi_\mu(a) =$ = sg $\phi_\lambda(a)$. In case $\phi_\lambda(x)$ has no zero in (a,x_1), the right hand side of (2) is positive. So $0 <$ sg $\phi_\lambda(x_1)\cdot\phi_\mu'(x_1)$ implies sg $\phi_\mu(a)\cdot\phi_\mu'(x_1) > 0$, a contradiction.

If $\Phi_\mu(a) = 0$, it follows from Theorem 1 that $\phi_{\mu+\varepsilon}$ has also a zero in (a,x_1). Q.E.D.

It can be proved by the implicit function theorem:

THEOREM 4. Assume $\phi(x_0,\lambda_0) = 0$. Then, in a certain neighborhood of (x_0,λ_0), the zeroes of $\phi(x,\lambda)$ form a continuous function of λ, $x = x(\lambda)$, such that $x(\lambda_0) = x_0$.

In other words: *the zeroes are continuous functions of the parameter*. A consequence of this, Theorem 3 and the continuity of ϕ is the following:

PROPOSITION. Any zero of $\phi(x,\lambda)$ moves continuously to the left for λ increasing (i.e. $x(\lambda)$ is a decreasing function of λ) and any new zero of $\phi(x,\lambda)$ "enters through b", (cf. fig.2).

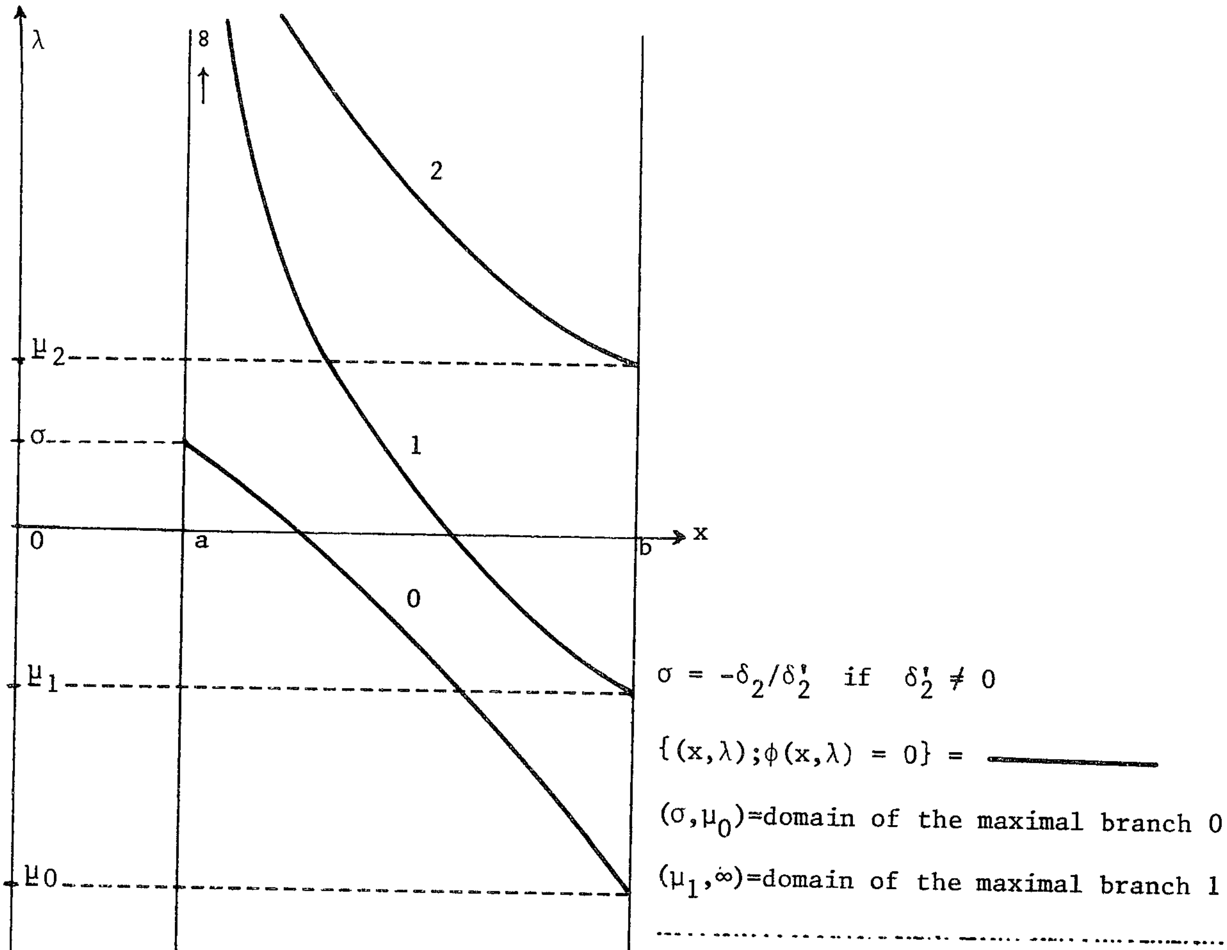

fig.2

PROOF. Suppose $\lambda > \mu$, $x_0(\mu)$ the first zero of $\phi(x,\mu)$ in (a,b). Then if λ is near enough to μ, $x_0(\lambda) < x_0(\mu)$. In fact if $\phi_\mu(a) \neq 0$, this is clear. In case $\phi_\mu(a) = 0$, we have $\mu = \sigma = -\frac{\delta_2}{\delta_2'}$ and $\phi_\mu'(a) = \frac{\Delta}{\delta_2'}$. So, sgn $\phi_\mu'(a)$ = sgn δ_2' and $\phi_\lambda(a) = (\lambda-\mu)\delta_2'$ verifies sgn $\phi_\lambda(a)$ = sgn $\phi_\mu'(a)$ = sgn $\phi_\lambda'(a)$ for λ sufficiently near to μ.

Therefore $\phi_\lambda(x)$ *does not vanish* in $(a,a+\varepsilon)$. Then Th.3 implies that $x_0(\lambda) < x_0(\mu)$; that is, $x_0(\lambda)$ is strictly decreasing. Now the first part of the proposition follows from Th.3 and the continuity of $\phi(x,\lambda)$ as a function of x and λ. For the second part, observe that if a maximal continuous branch $x = x(\lambda)$ of $\phi(x,\lambda)=0$ has domain (λ_1,λ_2), then necessarily $\lim_{\lambda\to\lambda_1} x(\lambda) = b$ and $\lim_{\lambda\to\lambda_2} x(\lambda) = a$. Q.E.D.

Making use of the estimations of $U_s(x)$ stated in section 2 we obtain:

$$(3) \quad \phi_\lambda(x) = (\delta_2+\delta_2'\lambda)\ \mathrm{ch}\ is(x-a)-(\delta_1+\lambda\delta_1')\ \frac{\mathrm{sh}\ is(x-a)}{is} + O(e^{|t|(x-a)}|s|^v)$$

where $s^2 = \lambda$, $s = \sigma+it$, $v=1$ if $\delta_2' \neq 0$, $v=0$ if $\delta_2' = 0$.

When $[\delta]$ is replaced by (α), the asymptotic formulae to use are:

$$(3') \quad \begin{cases} \phi_\lambda(x) = \cos s(x-a).\sin \alpha + O(e^{|t|(x-a)}/|s|) & \text{if } \alpha\neq 0, \\ \phi_\lambda(x) = -\dfrac{\sin s(x-a)}{s} + O(e^{|t|(x-a)}/|s|^2) & \text{if } \alpha=0. \end{cases}$$

For the derivatives we have

$$(4) \quad \phi_\lambda'(x) = is(\delta_2+\lambda\delta_2')\ \mathrm{sh}\ is(x-a) - (\delta_1+\lambda\delta_1')\ \mathrm{ch}\ is(x-a)+O(e^{|t|(x-a)}|s|^{v+1})$$

$$(4') \quad \begin{cases} \phi_\lambda'(x) = -s.\sin s(x-a).\sin \alpha + O(e^{|t|(x-a)}), & \alpha \neq 0, \\ \phi_\lambda'(x) = -\cos s(x-a) + O(e^{|t|(x-a)}/|s|), & \alpha = 0. \end{cases}$$

3. THE EIGENVALUES. From (3) it follows that for s real, $s \to +\infty$, the number of zeroes of $\phi_\lambda(x)$ in (a,b) increases without bound and for $s = it$, t positive sufficiently great, $\phi_\lambda(x)$ has no zero in the same interval. This proves that there exists a sequence

(5) $M = \{\mu_i;\ i = 0,1,2,\dots,\ \mu_i < \mu_{i+1}\}$ such that $\phi_{\mu_j}(b) = 0$ and

$\phi_\lambda(b) \neq 0$ if λ real, $\lambda \notin M$, and $\mu_{j+1}-\mu_j \sim (\frac{\pi}{b-a})^2.2j$. We call $\mu_{-1} = -\infty$.

LEMMA. $\phi'_\lambda(b)/\phi_\lambda(b)$ decreases strictly from $+\infty$ to $-\infty$ when λ increases from μ_k to μ_{k+1}, $k = -1,0,1,\ldots$.

PROOF. Assume $\mu_k < \lambda < \mu < \mu_{k+1}$ and suppose that there is a zero of $\phi_\lambda(x)$ in $[a,b]$. Let x_k be the greatest such zero. Then $x_k < b$ and $\phi_\mu(x) \neq 0$ for $x \in [x_k,b]$. From

$$(6) \qquad \frac{d}{dx}[\phi_\lambda^2(\phi'_\lambda/\phi_\lambda-\phi'_\mu/\phi_\mu)](x) = (\mu-\lambda)\phi_\lambda^2+(\phi'_\lambda-\phi_\lambda\phi'_\mu/\phi_\mu)^2(x) > 0$$

we obtain:

$$(7) \qquad \phi_\lambda^2(b)\{\phi'_\lambda/\phi_\lambda-\phi'_\mu/\phi_\mu\}(b) > \phi_\lambda^2(x_k).\{\phi'_\lambda/\phi_\lambda-\phi'_\mu/\phi_\mu\}(x_k) = 0.$$

This proves the monotonicity. Let us show that it also holds in case $\phi_\lambda(x) \neq 0$ in $[a,b]$. Then, either $\lambda < \mu_0$ or $\sigma < \lambda < \mu_1$, where $\sigma = -\delta_2/\delta'_2$ is the only zero of $\phi_\lambda(a)$, (cf.fig.2). Taking in the previous argument $x_k = a$ we obtain from (6)

$$(8) \quad \phi_\lambda^2(b)\{\phi'_\lambda/\phi_\lambda-\phi'_\mu/\phi_\mu\}(b) > \phi_\lambda^2(a)\{\ldots\}(a) = \Delta(\mu-\lambda)\,\frac{\delta_2+\lambda\delta'_2}{\delta_2+\mu\delta'_2}$$

Observing that $\phi_\lambda(a) = \delta_2+\lambda\delta'_2$ changes sign at σ and that $\sigma \notin [\lambda,\mu]$, in either case the right hand side of (8) is positive. From formulae (3) and (3'), (4) and (4'), it follows that $|(\phi'_\lambda/\phi_\lambda)(b)| \to \infty$ for $\lambda \to \mu_{-1} = -\infty$, and obviously $|(\phi'_\lambda/\phi_\lambda)(b)| \to \infty$ for $\lambda \to \mu_k$, $k \geq 0$. Q.E.D.

Now, λ is an eigenvalue if and only if

$$(9) \qquad (\lambda\beta'_1+\beta_1)\phi_\lambda(b) - (\lambda\beta'_2+\beta_2)\phi'_\lambda(b) = 0 ,$$

which for $\lambda \neq \mu_i$ reduces to

$$\frac{\phi'_\lambda(b)}{\phi_\lambda(b)} = r(\lambda) \ , \quad \text{with} \quad r(\lambda) = \frac{\lambda\beta'_1+\beta_1}{\lambda\beta'_2+\beta_2} \ .$$

When $\beta'_2 \neq 0$, $r(\lambda)$ increases strictly in $(-\infty,-\beta_2/\beta'_2)$ and $(-\beta_2/\beta'_2,+\infty)$. Then, in each interval (μ_{k-1},μ_k) there is exactly one eigenvalue with exception of the interval $(\mu_{j-1},\mu_j) \ni -\beta_2/\beta'_2$, where there are two. This exception does not appear when $-\beta_2/\beta'_2 = \mu_j$ for some j. In this case, in view of (9), μ_j is an eigenvalue.

4. MULTIPLICITY OF THE EIGENVALUES. Let us see that all the eigenvalues of the boundary problem (q),[δ],[β] are *real* and *simple*. If we call as in [F]: $R_b(u) = \beta_1 u(b)-\beta_2 u'(b)$, $R'_b(u) = \beta'_1 u(b)-\beta'_2 u'(b)$, and define:

$$R_a(u) = \delta_1 u(a) + \delta_2 u'(a) \quad , \quad R'_a(u) = \delta'_1 u(a) + \delta'_2 u'(a) \ ,$$

we have:

$$W_a(F_1,G_1) = -\frac{1}{\Delta}\,(R_a(F_1)R'_a(G_1) - R_a(G_1)R'_a(F_1)).$$

Let H be the Hilbert space $C \oplus L^2(a,b) \oplus C$ with elements $F = F_0 \oplus F_1 \oplus F_2$ and scalar product

$$(10) \qquad (F,G) = \frac{F_0\overline{G}_0}{\Delta} + \int_a^b F_1\overline{G}_1\,dx + \frac{F_2\overline{G}_2}{\rho} \ .$$

The operator A defined by

$$(11) \qquad A(F) = \begin{pmatrix} -R_a(F_1) \\ -F''_1(x)+q(x)F_1(x) \\ -R_b(F_1) \end{pmatrix} = \begin{pmatrix} -R_a(F_1) \\ \tau F_1 \\ -R_b(F_1) \end{pmatrix}$$

is *densely defined* in H if its domain is defined by

$\{F \in H\colon F_1,F'_1$ are absolutely continuous, $\tau F_1 \in L^2(a,b)$, $F_0 = R'_a(F_1), F_2 = R'_b(F_1)\}$.

Besides, it is a *symmetric operator*. The eigenvalues of the boundary problem are the zeroes of the entire function $\omega(\lambda) = \lambda R_b'(\phi_\lambda)+R_b(\phi_\lambda)$. These are the eigenvalues of A: $A\Phi = \lambda\Phi$ iff Φ_1 *satisfies* (q),[δ],[β]. Therefore they are *real*. As in [F], (3.11), we obtain:

$$(12)\qquad (\lambda_n-\lambda)\int_a^b \phi_{\lambda n}\phi_\lambda dx = -\frac{1}{k_n}(\omega(\lambda)+(\lambda_n-\lambda)R_b'(\phi_\lambda)) - W_a(\phi_{\lambda n},\phi_\lambda) =$$

$$= -\frac{1}{k_n}(\omega(\lambda)+(\lambda_n-\lambda)R_b'(\phi_\lambda)) + \frac{R_a(\phi_{\lambda n})R_a'(\phi_\lambda)-R_a(\phi_\lambda)R_a'(\phi_{\lambda n})}{\Delta},$$

where $k_n = \chi_{\lambda n}(x)/\phi_{\lambda n}(x)$ is a nonnull constant. Then

$$\int_a^b \phi_{\lambda n}\phi_\lambda\, dx = \frac{\omega(\lambda)}{k_n(\lambda-\lambda_n)} - \frac{R_b'(\phi_\lambda)}{k_n} - \frac{R_a'(\phi_{\lambda n})R_a'(\phi_\lambda)}{\Delta}.$$

It follows from this for $\lambda \longrightarrow \lambda_n$ that:

$$(13)\qquad \frac{\omega'(\lambda_n)}{k_n} = \int_a^b \phi_{\lambda n}^2(x)dx + \frac{R_b'(\phi_{\lambda n})}{k_n} + \frac{(R_a'(\phi_{\lambda n}))^2}{\Delta}.$$

Since $R_b'(\phi_{\lambda_n}) = \rho/k_n$, we get then $\omega'(\lambda_n)/k_n > 0$. Then, the zeroes are simple as in the boundary problem (q),(α),[β]. Collecting results of the preceding section and this one, we have:

THEOREM 5. The eigenvalues of problems (q),(α),[β] and (q),[δ], [β], are real, simple and bounded below. Besides,

i) if $-\beta_2/\beta_2' = \mu_p \in M$ then there is exactly one eigenvalue on each interval (μ_j,μ_{j+1}), $j = -1,0,1,\dots$. μ_p is the only eigenvalue in M.

ii) if $-\beta_2/\beta_2' \in (\mu_{p-1},\mu_p)$, $\beta_2' \neq 0$, there is exactly one eigenvalue on each interval (μ_j,μ_{j+1}) with the exception of (μ_{p-1},μ_p) where two eigenvalues occur. No μ_j is an eigenvalue.

iii) if $\beta_2' = 0$, there is exactly one eigenvalue on each interval (μ_j,μ_{j+1}). No μ_j is an eigenvalue.

5. ZEROES OF THE EIGENFUNCTIONS. CASE I: (q),(α),[β]. If $\beta_2' = 0$ the eigenfunction ψ_j has j *zeroes* on (a,b), j = 0,1,2,..., as it is easy to verify. Assume then $\beta_2' \neq 0$. It follows from the results of §1 and Th.5 that if an eigenfunction ψ has its corresponding eigenvalue on $(\mu_{j-1},\mu_j]$ then ψ has j *zeroes* on (a,b). It follows too that *there are two eigenfunctions with the same number* p *of zeroes on* (a,b).

CASE II: (q),[δ],[β], $-\delta_2/\delta_2' = -\beta_2/\beta_2' = \sigma$. This case occurs for example when $[\delta] = [\tilde{\beta}]$. Let us assume $\mu_{p-1} < \sigma < \mu_p$. If $\lambda \in (\mu_{p-2},\mu_{p-1}]$ then $\phi_\lambda(x)$ has p-1 zeroes on (a,b) and has p zeroes in the same interval if $\mu_{p-1} < \lambda < \sigma$. However when λ reaches σ, the first zero of $\phi_\lambda(x)$ reaches x=a and therefore for $\sigma \leqslant \lambda \leqslant \mu_p$, $\phi_\lambda(x)$ has p-1 zeroes on (a,b). On $(\mu_p < \lambda \leqslant \mu_{p+1}]$, $\phi_\lambda(x)$ has again p zeroes on (a,b). In consequence, *the eigenfunctions* $\psi_j(x)$ *have* 0,1,...,p-2,p-1,p,p-1,p,p+1,p+2,... *zeroes on* (a,b) *for* j = 0,1,2,... *respectively*.

Assume now $\sigma = \mu_p$. Then, when λ reaches σ the first zero of $\phi_\lambda(x)$ reaches x=a and a new zero appears at x=b. Therefore, $\phi_{\mu_p}(x)$ has p-1 zeroes on (a,b). Again $\phi_\lambda(x)$ has p zeroes on this interval if $\mu_p < \lambda \leqslant \mu_{p+1}$. So the same situation as above is present now (but where the interval $[\sigma,\mu_p]$ is reduced to the degenerate interval $[\mu_p]$): *two eigenfunctions have* p *zeroes and two* p-1 *zeroes on* (a,b) *and appear alternately and consecutively as the eigenvalue increases*.

Assume now $\beta_2' = 0$, i.e., $\sigma = \infty$. The behaviour of the eigenfunctions is the same as in case I under the the same hypothesis: ψ_j *has* j *zeroes on* (a,b), j = 0,1,2,... .

CASE III: (q),[δ],[β], $\sigma_b = -\beta_2/\beta_2' \neq -\delta_2/\delta_2' = \sigma$. All the tools to discuss this case are given. We shall only make an observation. In case II the eigenfunctions have alternately an even and an odd number of zeroes on (a,b) as $\lambda_n \to +\infty$. This is not necessarily true in case III. In fact, assume that $\sigma_b \in (\mu_{p-1},\mu_p)$ and $\sigma \in (\mu_r,\mu_{r+1})$, $\mu_p \leqslant \mu_r$. Then, on the first interval there are two eigenvalues whose eigenfunctions have p zeroes on (a,b). On the other hand, $\phi_\lambda(x)$ has r+1 zeroes if $\mu_r < \lambda < \sigma$ and r zeroes if $\sigma \leqslant \lambda < \mu_{r+1}$. Besides, if $\mu_{r+1} < \lambda < \mu_{r+2}$, $\phi_\lambda(x)$ has r+1 zeroes on (a,b). The μ_j's and σ depend only on the initial condition [δ]. Therefore, without changing [δ] it is possible to change [β] as to have an eigenvalue in (μ_r,σ) or in $[\sigma,\mu_{r+1})$. In consequence we have two consecutive eigenfunctions with r+1 zeroes on (a,b), or respectively, with r and r+1 zeroes. Assume now $\mu_p = \mu_r$. Then, in the first situation *three consecutive eigenfunctions occur with* p *zeroes on* (a,b) and in the second one *two consecutive eigenfunctions with* p *zeroes and the next two with* p+1 *zeroes*. In any case, *the evenness does not occur alternately*.

APPENDIX II.

BOUNDARY VALUES FOR A DIFFERENTIAL OPERATOR.

Let T be the symmetric operator: $T = \frac{d^2}{dx^2} + q(x)$ with domain

$\mathcal{D}(T) = \{\varphi: \varphi,\varphi'$ absolutely continuous, $\varphi'' \in L^2$, $\varphi(0)=\varphi(1)=\varphi'(0)=\varphi'(1) = 0\}$.

We assume that $q(x)$ is real, $x \in [0,1]$, and, for example, that $q \in L^{\infty}(0,1)$. Then its adjoint operator T^* has the same formal expression as T but with domain:

$\mathcal{D}(T^*) = \{\varphi: \varphi,\varphi'$ absolutely continuous, $\varphi'' \in L^2\}$, (cf.[N]).

$\mathcal{D}(T^*)$ becomes a Hilbert space H if the following scalar product is introduced: $\langle x,y \rangle = (x,y) + (T^*x,T^*y)$.

A linear functional $B \in H^*$ null on $\mathcal{D}(T) \subset H$ is called a *boundary value for* T, ([DS]). To characterize them it is convenient to have an alternative representation for H.

First observe that if $\{\varphi_j\} \subset H$ and $\varphi_j \longrightarrow 0$ then $\int_0^1 |\varphi_j|^2 dt \to 0$, $\int_0^1 |\varphi_j''+q(x)\varphi_j|^2 dt \to 0$. Therefore, $\int_0^1 |\varphi_j''|^2 dt \to 0$. From this it follows that: $\varphi_j'(x) \xrightarrow{\cdot} 0$, $\varphi_j(x) \xrightarrow{\cdot} 0$. Consider now the Hilbert space: $\mathcal{H} = L^2 \oplus R \oplus R$. The linear application $J: H \longrightarrow \mathcal{H}$,

$$J(\varphi) = (\varphi''+q\varphi, \varphi(0), \varphi'(0))$$

is then continuous. But given $f \in L^2$

$$(1) \qquad \varphi(x) = \int_0^1 G(x,y)f(y)dy + ay_1(x) + by_2(x)$$

(G is the Green kernel and $y_1(x)$, $y_2(x)$ are certain solutions of the homogeneous equation $Ty = 0$) is the solution of the non-homogeneous problem: $T\varphi = f$, $\varphi(0) = a$, $\varphi'(0) = b$. Therefore, J is onto and one-to-one. In consequence, it defines an isomorfism between H and $\mathcal{H}$. We have also proved that an expression like:

(2) $B(\varphi) = u\varphi(0) + v\varphi(1) + w\varphi'(0) + z\varphi'(1)$, u,v,w,z constants,

defines a continuous linear functional on H, null on $\mathcal{D}(T)$. Conversely, all boundary values are of this form. In fact, a linear continuous functional B on $\mathcal{H}$ can be written as

$$B(\varphi) = \int_0^1 (\varphi''+q\varphi)g\,dx + c_1\varphi(0) + c_2\varphi'(0) \quad , \quad g \in L^2,$$

and therefore, a boundary value B is of this form and such that

(3) $$B(\varphi) = \int_0^1 \chi(y)g(y)dy = 0 \quad , \ \chi = \varphi''+q\varphi \quad , \quad \text{for} \quad \varphi \in \mathcal{D}(T).$$

But χ belongs to the range of T, $\mathcal{R}(T)$, which can be characterized as the (closed) subspace given by

(4) $$\mathcal{R} = \{\chi \in L^2 : \int_0^1 G(1,t)\chi(t)dt = 0 = \int_0^1 \frac{\partial G}{\partial x}(1,t)\chi(t)dt\} .$$

In fact, this follows from (1) observing that $\int_0^1 G(x,t)\chi(t)dt$ is null at x=0 together with its derivative.

We conclude from (3) that $g \in L^2 \ominus \mathcal{R}$. Then, from (4) that

$$g = \alpha . G(1,t) + \beta . \frac{\partial G}{\partial x}(1,t) \quad , \quad \alpha,\beta \text{ constants.}$$

In consequence, for $\varphi \in H$

$$B(\varphi) = \int_0^1 (\varphi''+q\varphi)(\alpha G(1,t)+ \beta \frac{\partial G}{\partial x}(1,t))dt + c_1\varphi(0) + c_2\varphi'(0) ,$$

and using (1):

$$B(\varphi) = \alpha(\varphi(1)-\varphi(0)y_1(1)-\varphi'(0)y_2(1)) + \beta(\varphi'(1)-\varphi(0)y_1'(1)-\varphi'(0)y_2'(1))+$$
$$+ c_1\varphi(0)+c_2\varphi'(0) = u\varphi(0) + v\varphi(1) + w\varphi'(0) + z\varphi'(1).$$

We have proved then the following proposition: (2) *characterizes any boundary value for* T . ([DS], XIII, 2.27).

APPENDIX III.

ON HILBERT'S FORMS.

Let α and β be complex-valued functions defined on the positive integers Z^+ such that

(1) $\alpha(n) = n + O(1/n)$, $\beta(n) = n + O(1/n)$.

Let a and b be sequences in $\ell^2(Z^+)$, and J and K sequences in $\ell^\infty(Z^+)$.

THEOREM 1. Given $\varepsilon > 0$, there exists a constant $C = C(\alpha,\beta,\psi,J,K,\varepsilon)$ independent of N, a and b, such that

i) $$\left|\sum_{n=1}^{N} \sum_{m=1}^{N}{}' \frac{a_n J(n).b_m K(m)}{\alpha(n)-\beta(m)+\psi}\right| \leqslant C.\|a\|_2 \|b\|_2 .$$

where the dash indicates the omission of the terms in which $|\alpha(n)-\beta(m)+\psi| \leqslant \varepsilon$.

ii) $$\sum_{n=1}^{N} \sum_{m=1}^{N}{}' \left|\frac{a_n J(n).b_m K(m)}{\alpha(n)+\beta(m)+\psi}\right| \leqslant C. \|a\|_2 \|b\|_2$$

where the dash indicates the omission of the terms in which $|\alpha(n)+\beta(m)+\psi| \leqslant \varepsilon$.

In both cases, ψ is a real constant.

PROOF. We can assume a and b real, and $J \equiv K \equiv 1$.

Firstly we prove a simple auxiliary proposition:

PROPOSITION 1. Assume $K_{n,m} \geqslant 0$ and such that $\sum_n K_{nm} \leqslant M \geqslant \sum_m K_{nm}$ for a certain positive constant M. Then,

$$\left|\sum_{n=1}^{N} \sum_{m=1}^{N} K_{nm}\, a_n\, b_m\right| \leqslant M.\|a\|_2 \|b\|_2.$$

In fact, $|\sum_n K_{nm} a_n| \leqslant M^{1/2}(\sum_n K_{nm} a_n^2)^{1/2}$, and therefore

$$|\sum_m(\sum_n K_{nm} a_n)b_m| \leqslant \sum_m |b_m| M^{1/2}(\sum_n K_{nm} a_n^2)^{1/2} \leqslant M^{1/2}\|b\|.(\sum_{mn}\sum K_{nm} a_n^2)^{1/2} \leqslant$$

$$\leqslant M^{1/2} \|b\|.M^{1/2} \|a\|.$$

i) Since $\{(n,m):|\alpha(n)-\beta(m)+\psi| \leqslant \varepsilon\} \subset \{(n,m):|n-m| \leqslant H\}$ for a certain positive integer $H = H(\varepsilon)$, we have:

$$(2) \qquad \left|\sum_{n,m}{}' \frac{a_n b_m}{\alpha(n)-\beta(m)+\psi}\right| \leqslant \left|\sum_{|n-m|>H} \frac{a_n b_m}{\alpha(n)-\beta(m)+\psi}\right| + \sum_{|n-m|\leq H} \varepsilon^{-1}|a_n b_m|.$$

The last term is bounded by $(2H+1)\varepsilon^{-1}.\|a\|.\|b\|$. On the other hand, if $|n-m| > H$

$$(3) \qquad \frac{1}{\alpha(n)-\beta(m)+\psi} - \frac{1}{n-m} = \frac{O(\frac{1}{m}) + O(\frac{1}{n}) - \psi}{(\alpha(n)-\beta(m)+\psi)(n-m)} .$$

Define K_{mn} as equal to the modulus of (3) for $|n-m| > H$ and equal to zero elsewhere. If is not difficult to see that K_{mn} satisfies the hypothesis of proposition 1.

In consequence, to prove that

$$(4) \qquad \left|\sum_{|n-m|>H} \frac{a_n b_m}{\alpha(n)-\beta(m)+\psi}\right| \leqslant C'.\|a\| \|b\|$$

it suffices to prove that

$$(5) \qquad \left|\sum_{|n-m|>H} \frac{a_n b_m}{n-m}\right| \leqslant C''.\|a\| \|b\|.$$

But

$$(6) \qquad \left|\sum_{|n-m|>H} \frac{a_n \cdot b_m}{n-m}\right| \leqslant \left|\sum_{n\neq m} \frac{a_n b_m}{n-m}\right| + \sum_{|p-m|\leq H} |a_n b_m| .$$

The last term is bounded by $(2H+1)\|a\|\|b\|$.Since a well-known result for Hilbert's forms (cf.[HLP],p.212 or 235) assures that

(7) $$\left|\sum_{n\neq m} \frac{a_n b_m}{n-m}\right| \leqslant \pi . \|a\| \ \|b\| ,$$

i) follows.

ii) can be proved repeating step by step the proof of i) and using instead of (7) the following result due to Hilbert:

(8) $$\sum_{n,m=1}^{N} \left|\frac{a_n b_m}{n+m}\right| \leqslant \pi . \|a\| . \|b\| .$$ Q.E.D.

APPENDIX IV.

ASSOCIATED FUNCTIONS.

In this appendix we want to exhibit the general form of the residual expansion of a function φ. From (23) we know that it is the sum of the following terms

$$(1)\quad \operatorname{res}_s \int_a^b G(x,y;t)\varphi(y)dy = \int_a^b \varphi(y)(\operatorname{res}_s G(x,y;t))dy, \quad s \in S.$$

For a simple zero of w, (1) is equal to (24). We already mentioned that formula (1) can be proved as in [BP],p.172.

To calculate $\operatorname{res}_s G(x,y;t)$ in general we need the following.

LEMMA 1. Assume $q \in L^1(a,b)$. $y''-(q-s^2)y = 0$, $y(a) = f(s)$, $y'(a) = g(s)$, where f and g are entire functions of s, has a unique solution y(x,s) which is an entire function of s for each $x \in [a,b]$, and continuous in (x,s). Moreover, for k = 1,2,...

$$(2)\qquad \frac{\partial^k}{\partial s^k}\frac{\partial}{\partial x} y(x,s) = \frac{\partial}{\partial x}\frac{\partial^k}{\partial s^k} y(x,s)$$

$$(3)\qquad \frac{\partial^k}{\partial s^k}\frac{\partial^2}{\partial x^2} y(x,s) = \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial x}\frac{\partial^k}{\partial s^k} y(x,s)\Big).$$

The first equality holds everywhere and the second one in $\mathcal{L}\times C$ where $\mathcal{L}$ is a measurable set of [a,b] of measure b-a.

PROOF. a) y(x,s) is a solution of the problem iff

$$(4)\quad y(x,s) = [f(s)+g(s)(x-a)] + \int_a^x (x-t)(q(t)-s^2)y(t,s)\,dt .$$

Let $y_0(x,s) = f(s)+g(s)(x-a)$ and $V(x,s) = \int_a^x (x-t)|q(t)-s^2|dt$.

Then

(5) $$\begin{cases} V(x,s) \geqslant 0 \\ V'(x,s) = \int_a^x |q(t)-s^2|\,dt \geqslant 0 \\ V''(x,s) = |q(x)-s^2| \geqslant 0\ . \end{cases}$$

Let us define $y_k(x,s)$ for $k \geqslant 1$:

(6) $$y_k(x,s) = y_0(x,s) + \int_a^x (x-t)(q(t)-s^2)y_{k-1}(t,s)\,dt.$$

Then $|y_1(x,s) - y_0(x,s)| \leqslant V(x,s)\ \|y_0(.,s)\|_\infty$.

Let us prove that

(7) $$|y_k(x,s) - y_{k-1}(x,s)| \leqslant \frac{V^k(x,s)}{k!}\ \|y_0(.,s)\|_\infty.$$

Assuming it for k=h, we obtain

(8) $$|y_{h+1}(x,s) - y_h(x,s)| \leqslant \int_a^x (x-t)V''(t,s).\frac{V^h(t,s)}{h!}\ \|y_0(.,s)\|_\infty dt =$$

$$= \|y_0(.,s)\|_\infty \int_a^x (x-t)\{\frac{d^2}{dt^2}\ (\frac{V^{h+1}}{(h+1)!}) - \frac{V^{h-1}V'^2}{(h-1)!}\}dt \leqslant \text{in view of (5)} \leqslant$$

$$\leqslant \|y_0(.,s)\|_\infty \int_a^x (x-t)\ \frac{d^2}{dt^2}\ (\frac{V^{h+1}}{(h+1)!})dt = \|y_0(.,s)\|_\infty\ \frac{V^{h+1}(x,s)}{(h+1)!}\ .$$

This proves (7).

Since $V(x,s)$ and $y_0(x.s)$ are uniformly bounded in $G = G_R =$ $= [a,b] \times \{|s| < R\}$, R arbitrary,

$$y_N(x,s) = y_0(x,s) + \sum_{k=1}^{N} (y_k(x,s) - y_{k-1}(x,s))$$

converges uniformly in G to a limit $y(x,s)$, *solution of* (4). The uniform convergence assures the continuity of $y(x,s)$ in G and its analiticity. The uniqueness also follows, even for $q \in L^1$, from the fact that

(9) $$u'' - (q(t)-s^2)u = 0\ \ ,\ u(a) = u'(a) = 0 \quad \Rightarrow \quad u \equiv 0.$$

(This can be seen as follows: from

$$u(x,s) = \int_a^x (x-t)(q(t)-s^2)u(t,s)dt ,$$

it follows by induction, like (7), for every k,

$$|u(x,s)| \leqslant \frac{V^k(x,s)}{k!} \|u(.,s)\|_\infty .$$

Letting $k \longrightarrow \infty$, we obtain $u(x,s) \equiv 0$).

To prove (2) let $D^j = \frac{\partial^j}{\partial s^j}$. Then, from Cauchy's formula it follows that $D^j y(x,s)$ is continuous in G and in view of (4) equal to:

$$D^j y(x.s) = f^{(j)}(s) + (x-a)g^{(j)}(s) + \int_a^x (x-t)(q(t)-s^2)D^j y(t,s)dt +$$

$$+ \int_a^x (x-t)[-2sjD^{j-1}y(t,s) - j(j-1)D^{j-2}y(t,s)]\ dt .$$

Thus, it is absolutely continuous in $a \leqslant x \leqslant b$, and, $\forall$ j,

$$(10) \quad \frac{d}{dx}(D^j y(x,s)) = g^{(j)}(s) + \int_a^x (q(t)-s^2)D^j y(t,s)\ dt -$$

$$- \int_a^x [2sjD^{j-1}y + j(j-1)D^{j-2}y]\ dt.$$

In particular,

$$\frac{d}{dx} y(x,s) = g(s) + \int_a^x (q(t)-s^2)y(t,s)\ dt$$

is an entire function of s, continuous in G and

$$D^j(y'(x,s)) = g^{(j)}(s) + \int_a^x D^j[(q(t)-s^2)y(t,s)]\ dt .$$

Thus $D^j y'(x,s) = (D^j y(x,s))'$ which is just (2).

From (10) it is clear that $(D^j y(x,s))'$ is absolutely continuous for $a \leqslant x \leqslant b$ and fixed s, and also that

(11) $(D^j y)''(x,s) = D^j[(q(x)-s^2)y(x,s)]$ a.e.

More precisely (11) holds for any $x \in \mathcal{L} = \{x;\ \frac{1}{h}\int_0^h |q(x+t)-q(x)|dt \xrightarrow[h\to 0]{} 0\}$(*).

$\mathcal{L}$ is the set of Lebesgue points of q(x) which has measure b-a. Thus (11) holds in $\mathcal{L} \times C$.

But the right hand side of (11) is just $D^j(y''(x,s))$ for $x \in \mathcal{L}$.

Therefore $D^j y''(x,s) = (D^j y)''(x,s)$ $\forall$ s , $\forall$ $x \in \mathcal{L}$. Q.E.D.

In our situation $y(x,s) = U_s(x)$. We recall that for $s \in S, \tilde{U}_s(x) = A(s)U_s(x)$ and that A(t) is holomorphic in a neighborhood of s. Moreover. it can be seen-as (8),pp.25-27 of [G]-that *if* s *is a zero of order* k *of* w(t), *then* $\tilde{U}_t(x)-A(t)U_t(x)$ *has a zero of order* k *at* t=s. Therefore, in (1) we can make use of the equality:

(12) $$\operatorname*{res}_{t=s} G(x,y;t) = \operatorname*{res}_{t=s} \frac{t\ A(t)}{w(t)} \cdot U_t(x)U_t(y).$$

If we put: $M(t) = \frac{t.A(t)}{w(t)}$, then, following the steps of the proof given in [BP], pp.174-175 we obtain next theorem 1. For the sake of the reader we shall reproduce the proof.

Let us *define* the functions $\mathcal{U}_j(x,s)$, j = 0,1,2,... , by

(13) $$\mathcal{U}_j(x,s) = \frac{1}{j!}\ \frac{\partial^j}{\partial s^j}\ y(x,s)\ .$$

Therefore, if $s \in S$,

(14) $$y(x,t) = \mathcal{U}_0(x,s) + (t-s)\mathcal{U}_1(x,s) + \dots .$$

If M has a pole of order r at s,

(15) $$(t-s)^r M(t) = c_{-r} + c_{-r+1}(t-s) + \dots + c_{-1}(t-s)^{r-1} + \dots .$$

(*) If p(x) is continuous and $q(x) \in L^1$ then any Lebesgue point of q(x) is a Lebesgue point of (p.q)(x).

In consequence

$$(16)\quad (t-s)^r M(t).y(x,t)y(z,t) = \ldots+(t-s)^{r-1}\sum_{j=1}^{r} c_{-j}[\sum_{k=0}^{j-1} u_k(x,s)u_{j-1-k}(z,s)]+\ldots .$$

But

$$(17)\qquad (t^2-s^2)y(x,t) = -y''(x,t) + (q(x)-s^2)y(x,t) \qquad \text{a.e..}$$

Then, from (2),(3),(13) and (17) we obtain ($u_{-1}:=0$):

$$\sum_{k=0}^{\infty}[-\frac{d^2}{dx^2}u_k(x,s)+(q(x)-s^2)u_k(x,s)](t-s)^k = \sum_{k=0}^{\infty}u_k(t-s)^{k+1}.(t+s) =$$

$= \sum_{h=1}^{\infty}(u_{h-1}.2s+u_{h-2})(t-s)^h$, since $t+s = 2s+(t-s)$. This implies that the functions u_k satisfy the following equations for x in a set of measure b-a and $s \in S$:

$$(18)\qquad \begin{cases} u_0'' + (s^2-q)\ u_0 = 0 \\ u_1'' + (s^2-q)\ u_1 = -u_0.2s \\ u_2'' + (s^2-q)\ u_2 = -u_1.2s-u_0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_n'' + (s^2-q)\ u_n = -u_{n-1}.2s-u_{n-2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

If $s\neq 0$, for each n, the set $u_0,\ldots,u_n$ is *linearly independent* on $[0,1]$. In fact, $u_0(x,s) = y(x,s) \not\equiv 0$. Assume k is the first integer such that

$$u_k = \sum_{m=0}^{k-1}\alpha_m u_m .$$

Applying the operator $d^2/dx^2 - (q-s^2)$ to this identity, it follows from (18) that

$$(19)\quad -(u_{k-1}.2s+u_{k-2}) = \text{linear combination of } u_j\text{'s with } j \leqslant k-2.$$

This is a contradiction. If s=0, $u_0,\ldots,u_n$ *may be linearly dependent*. (In fact, because of (18), $u_1''-qu_1 = 0$, and from (13) we have: $u_1(0) = -Q'(0)$, $u_1'(0) = P'(0)$. If we had, for example,

$Q'(0) = 0 = P'(0)$, we would have $u_1 \equiv 0$). However, $u_0, u_2, \dots, u_{2n}$ are still *linearly independent*.

If r is the order of the pole s of M(t), $\{u_0(x,s), \dots, u_{r-1}(x,s)\}$ is called *the principal chain of functions associated to* y(x,s). From (1), (16) and (12) it follows next theorem.

THEOREM 1. If $s \in S$ and $\varphi \in L^1$ the following relations hold :

$$\text{(20)} \qquad \operatorname*{res}_{t=s} G(x,y;t) = \sum_{j=1}^{r} c_{-j}\Big[\sum_{k=0}^{j-1} u_k(x,s)\, u_{j-1-k}(y,s)\Big],$$

$$\text{(21)} \qquad \operatorname*{res}_{t=s} \int_a^b G(x,y;t)\varphi(y)\,dy = \sum_{k=0}^{r-1} u_k(x,s)\, \gamma_{k,r}(\varphi) ,$$

where

$$\text{(22)} \qquad \gamma_{k,r}(\varphi) = \int_a^b \varphi(y) \sum_{j=k+1}^{r} c_{-j} u_{j-1-k}(y,s)\, dy , \quad (\text{cf.}(15)).$$

Observe that r is the order of the pole of M at s. So, if s=0, it is equal to the order of the zero of w minus one.

Assume that in (12) instead of G(x,y;t) we had G(x,y;t)/t.

Then, the only change in the previous developements is the definition of M(t) (which now is $\frac{A(t)}{w(t)}$) and accordingly the c_j's are not the same.

Now, M(t) has a pole of order r at *every* zero of order r of w, and therefore if s=0 is a double zero of w, we have:

$$\text{(23)} \qquad \operatorname*{res}_{t=0} \int_a^b (G/t)\varphi\, dy = \gamma_{0,2}(\varphi).U_0(x) + \gamma_{1,2}.u_1(x,0) ,$$

$$\text{(24)} \qquad \begin{cases} \gamma_{0,2} = \int_a^b \varphi(x)[c_{-1}U_0(x) + c_{-2}u_1(x,0)]\, dx \\ \gamma_{1,2} = \int_a^b \varphi(x)[c_{-2}U_0(x)]\, dx , \quad c_{-2} \neq 0. \end{cases}$$

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NOTATIONS AND DEFINITIONS