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NOTAS DE ALGEBRA Y ANALISIS

14

REINHARD MENNICKEN and MANFRED MÖLLER

BOUNDARY EIGENVALUE PROBLEMS

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BOUNDARY EIGENVALUE PROBLEMS

Reinhard Mennicken and Manfred Möller

0. INTRODUCTION

This paper deals with boundary eigenvalue problems for first order systems of ordinary linear differential equations and also for n-th order ordinary linear differential equations. The boundary conditions may contain interface conditions and also an integral term. The coefficients of the differential equation as well as of the boundary conditions are allowed to be holomorphically or polynomially dependent on the eigenvalue parameter.

The aim of this paper is to establish expansion theorems for the above mentioned nonlinear eigenvalue problems. This will be achieved in two steps: first by establishing the formal expansion and secondly by proving the convergence of the formal expansion. Formal expansion is achieved by the application of a theorem of Keldyš [19], [20], cf. also Gohberg and Sigal [13]. The theorem states that the principal parts of the inverse of a holomorphic Fredholm-valued operator function having non-empty resolvent set can be expressed by root functions (eigenvectors and associated vectors) of the given operator function itself and its adjoint operator function. For this application the explicit form of the adjoint boundary eigenvalue problem is needed. The proof of the convergence of the formal expansion is based on criteria for regularity. The proofs of these criteria are obtained by making a careful analytic study of the Green's matrix or Green's function associated with the given boundary eigenvalue problem.

Section 1 contains some basic functional analytic notations, definitions, and results.

The sections 2-6 are concerned with boundary eigenvalue problems of the form

$$(0.1)\qquad \begin{cases} y'(x) - A(x,\lambda)y(x) = 0 & (x \in [a,b]), \\ \sum\limits_{j=1}^{m} W^{(j)}(\lambda)y(a_j) + \int\limits_a^b W(t,\lambda)y(t)dt = 0 & \end{cases}$$

where $a = a_1 < a_2 < \dots < a_m = b$ and the coefficients are $n\times n$-matrices which are assumed to be sufficiently smooth in x and holomorphic in λ. In section 2 the corresponding boundary eigenvalue operator functions are defined. Formal expansions are established by applying the theorem of Keldyš mentioned above and a modification of this theorem by the authors [28], cf. also Kaashoek [17]. Section 3 contains a theorem about the existence of a suitable asymptotic fundamental matrix of the differential system in (0.1) where we assume that the coefficient matrix $A(x,\lambda)$ has the following λ-asymptotic form

$$(0.2)\qquad A(x,\lambda) = \sum_{j=-1}^{k} \lambda^{-j} A_j(x) + O(\lambda^{-k-1}) \qquad (|\lambda| \to \infty).$$

The proof of this statement is similar to that of Langer [24]; cf. also Wasow [35] and Braaksma [6]. In view of the applications, we are interested in smoothness conditions with respect to x which are as weak as possible. In section 4 the Green's matrix for (0.1) is introduced and the inverse of the operator function defined by (0.1) is represented in terms of the Green's matrix. The authors would like to point out that the boundary conditions are also allowed to be inhomogeneous. In section 5 we state criteria for regularity. We consider boundary eigenvalue problems of type (0.1) whose differential equation has an asymptotic fundamental matrix of the form

$$(0.3)\qquad C(x,\lambda)\left\{ \sum_{\kappa=0}^{k} \lambda^{-\kappa} P^{[\kappa]}(x) + o(1) \right\} E(x,\lambda) \qquad (|\lambda| \to \infty)$$

where $C(x,\lambda)$ is an $n\times n$-matrix function which is a polynomial in λ and $E(x,\lambda)$ is a diagonal matrix with entries of the form $\exp\{\lambda R_i(x)\}$. We prove a geometric regularity criterion which goes back to Cole [8] for the special case that $C(x,\lambda)$ is the identity matrix. In section 6 we study boundary eigenvalue problems of the form (0.1) with

$$(0.4) \qquad A(x,\lambda) = A_0(x) + \lambda A_1(x)$$

where $A_1(x)$ is a diagonal matrix whose elements must fulfill certain additional conditions. In contrast to Cole [8] we allow $A_1(x)$ to have elements which are identical on $[a,b]$ and we also permit some of these elements to vanish identically on $[a,b]$. Thus we do not assume that $A_1(x)$ is invertible on $[a,b]$. We prove the expandability of those vector functions having certain smoothness properties and fulfilling boundary conditions which, in contrast to the original boundary conditions in (0.1), are independent of λ. Boundary conditions of this type have already been introduced by Mennicken in [26].

The sections 7-12 are concerned with boundary eigenvalue problems for n-th order differential equations. In section 7 we study eigenvalue problems of the form

$$(0.5) \qquad \begin{cases} \sum_{i=0}^{n} p_i(x,\lambda)\eta^{(i)}(x) = 0 \qquad (x \in [a,b]), \\ \sum_{j=1}^{m} W^{(j)}(\lambda) \begin{bmatrix} \eta(a_j) \\ \vdots \\ \eta^{(n-1)}(a_j) \end{bmatrix} + \int_a^b W(x,\lambda) \begin{bmatrix} \eta(x) \\ \vdots \\ \eta^{(n-1)}(x) \end{bmatrix} dx \end{cases}$$

where again $a = a_1 < a_2 < \dots < a_m = b$, the coefficients $p_i(x,\lambda)$ are sufficiently smooth in x and holomorphic in λ, and the $n\times n$-matrices $W^{(j)}(\lambda)$ and $W(x,\lambda)$ have the same properties as in (0.1). Problems of this kind were first approached by Tamarkin [34] with most valuable results. By $y(x) := (\eta(x),\dots,\eta^{(n-1)}(x))^t$ the problem (0.5) is transformed to an eigenvalue problem of type (0.1). The corresponding boundary eigenvalue operator functions are related to each other and we state relationships between the root functions of the operator functions belonging to (0.5) and its adjoint problem on the one hand and the root functions of the operator functions defined by the corresponding problem (0.1) and its adjoint problem on the other hand. From these results we deduce a theorem which states that the principal part of the inverse of the operator function defined by (0.5) is representable by root functions belonging to (0.5) and the corresponding adjoint eigenvalue problem. This adjoint eigenvalue problem is defined in the weak sense of distribution theory. Therefore we consider the special case of two-point eigenvalue problems in section 8. We state relationships between the classical adjoint eigenvalue problem and the adjoint problem defined in this weak sense. From these results we obtain a theorem which has been stated by Naimark [29] without proof and concerns the principal part of the Green's function belonging to a two-point eigenvalue problem. A complete proof of this statement seems to have been open.

The sections 9-12 are devoted to boundary eigenvalue problems of type (0.5) where the differential equation has the special form

$$\mathbb{K}\eta - \lambda\,\mathbb{H}\eta = 0 \tag{0.6}$$

with $\mathbb{K}$ and $\mathbb{H}$ being differential operators of order $n \geq 1$ and $0 \leq p \leq n-1$. Eigenvalue problems of this type for $p = 0$ and λ-independent boundary conditions have been studied by many authors, cf. e.g. Naimark [29], Orazov [30], Kostyuchenko and Shkalikov [21], Shkalikov [32], [33], Blošanskaja [5], Il'in [16]. Benedek, Güichal and Panzone [3] and Benedek and Panzone [4] treated the case of a special second order differential equation (n=2) with λ-dependent boundary conditions. The general case of (0.6) with arbitrary order p has been studied by Eberhard and Freiling [9], [10], [11], Freiling [12] and Heisecke [14], [15]. In this paper we reprove the main part of their results by making use of the generalization of Cole's regularity criterion, which we stated in section 5. By the application of this criterion we omit the laborious estimation of the Green's function. With the aid of our results from section 6 we are able to get rid of the assumption in [12], [14] and [15] requiring that the underlying boundary eigenvalue problem has to be normal, i.e. the Green's function has only simple poles. The expansion theorem in section 12 states the expandability of functions which are sufficiently smooth and fulfill certain λ-independent boundary conditions. In contrast to Eberhard, Freiling and Heisecke we do not suppose that the functions and their derivatives up to some order are zero at the boundary points a and b.

The authors finally point out that the general results concerning the root functions and the inverse of a boundary eigenvalue operator function also are applicable to more complicated eigenvalue problems, such as general differential-boundary systems as considered by Krall in a series of papers, cf. [22].

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1. PRELIMINARIES

Let E and F be Banach spaces. $L(E,F)$ denotes the Banach space of all continuous linear operators on E to F. If $S \in L(E,F)$, $N(S)$ denotes its null-space and $R(S)$ its range. $S \in L(E,F)$ is called a Fredholm operator if both its nullity $\mathrm{nul}(S) := \dim N(S)$ and its deficiency $\mathrm{def}(S) := \mathrm{codim}\, R(S)$ are finite. $\Phi(E,F)$ denotes the set of all Fredholm operators on E to F. If $S \in \Phi(E,F)$, then $\mathrm{ind}(S) := \mathrm{nul}(S) - \mathrm{def}(S)$ is called the index of S.

E' and F' denote the dual Banach spaces of E or F respectively and $S^* \in L(F',E')$ the adjoint of $S \in L(E,F)$. We set

$$(y \otimes v)(w) := \langle w,v\rangle y \qquad (w \in F)$$

for $y \in E$ and $v \in F'$. Note that $y \otimes v \in L(F,E)$.

If U is an open subset of $\mathbb{C}$, $H(U,E)$ denotes the set of all holomorphic mappings defined on U with values in the Banach space E.

Let $T \in H(U,L(E,F))$. $\rho(T) := \{\lambda \in U : T(\lambda)$ is invertible$\}$ is called the resolvent set of T, $\sigma(T) := U \setminus \rho(T)$ its spectrum and $\sigma_p(T) := \{\lambda \in U : T(\lambda)$ is not injective$\}$ its point spectrum or the set of eigenvalues of T. We set $T^{-1}(\lambda) := T(\lambda)^{-1}$ for $\lambda \in \rho(T)$ and $T^*(\lambda) := T(\lambda)^*$ for $\lambda \in U$.

In the following we assume $\mu \in U$.

(1.1) DEFINITION. $y \in H(U,E)$ is called a *root function* of T at μ if $y(\mu) \neq 0$ and $(Ty)(\mu) = 0$. $\nu(y)$ denotes the order of the zero at μ and is called the multipilicity of y.

For $i \in \mathbb{N}$ we set

$$L_i := \{y(\mu) : y \text{ is a root function of } T \text{ at } \mu \text{ with } \nu(y) \geq i\} \cup \{0\}. \tag{1.2}$$

This set L_i is a subspace of $N(T(\mu))$. From now on we assume that $\dim N(T(\mu)) = r < \infty$ and that, for some $s > 0$, $L_s \neq \{0\}$ and $L_{s+1} = \{0\}$. The last condition is fulfilled if T^{-1} has a pole of order s at μ. We define

$$m_j := \max\{i \in \mathbb{N} : \dim L_i \geq j\} \qquad (j = 1,\ldots,r) \tag{1.3}$$

and state that $m_j \geq m_{j+1}$.

(1.4) DEFINITION. A system $\{y_1,\ldots,y_r\}$ of root functions of T at μ is called a *canonical system of root functions* (CSRF) if $y_1(\mu),\ldots,y_r(\mu)$ are linearly independent (and thus form a basis of $N(T(\mu))$ and one of the equivalent conditions

i) $\nu(y_j) = \max\{\nu(y) : y$ is a root function of T at μ and $y(\mu) \notin \operatorname{span}\{y_1(\mu),\ldots,y_{j-1}(\mu)\}\}$ $(j=1,\ldots,r)$,

ii) $\nu(y_j) = m_j$ $(j=1,\ldots,r)$,

iii) $\nu(y_j) \geq m_j$ $(j=1,\ldots,r)$

is fulfilled.

A root function corresponds to a chain of an eigenvector and associated vectors. A canonical system of root functions is related to a canonical system of eigenvectors and associated vectors. For more details see [28].

2. BOUNDARY EIGENVALUE OPERATOR FUNCTIONS

Let $n \in \mathbb{N}$ and $-\infty < a < b < \infty$. We consider the Sobolev spaces

$$W^{j,p}(a,b) := \{y \in L_p(a,b) : y^{(i)} \in L_p(a,b),\ 1 \leq i \leq j\}$$

$(j \in \mathbb{N},\ 1 \leq p \leq \infty)$ where the derivative is the weak derivative in the sense of distribution theory. $W^{j,p}(a,b)$ is a Banach space with respect to the norm

$$|y|_{W^{j,p}(a,b)} := \sum_{i=0}^{j} |y^{(i)}|_{L_p(a,b)} \qquad (y \in W^{j,p}(a,b)),$$

cf. e.g. [1], Theorem 3.2. We write $H_j(a,b)$ instead of $W^{j,2}(a,b)$. $H_1(a,b)$ is continuously embedded in $C^0([a,b])$, cf. e.g. [1].

For an arbitrary set G, $M_{n,m}(G)$ denotes the set of all $n\times m$-matrices with entries in G. If $m=n$, we briefly write $M_n(G)$. Let $A \in H(\mathbb{C}, M_n(L_\infty(a,b)))$. We define

$$(2.1) \qquad T^D(\lambda)y := y' - A(\cdot,\lambda)y \qquad (y \in H_1^n(a,b),\ \lambda \in \mathbb{C})$$

where $H_1^n(a,b)$ is the n-fold product of $H_1(a,b)$. Let $m \geq 2$, $a = a_1 < a_2 < \ldots < a_m = b$, $W^{(j)} \in H(\mathbb{C}, M_n(\mathbb{C}))$ $(j=1,2,\ldots,m)$ and $W \in H(\mathbb{C}, M_n(L_1(a,b)))$. We set

$$(2.2) \qquad T^R(\lambda)y := \sum_{j=1}^{m} W^{(j)}(\lambda)y(a_j) + \int_a^b W(\xi,\lambda)y(\xi)d\xi$$

$(y \in H_1^n(a,b),\ \lambda \in \mathbb{C})$ and

$$(2.3) \qquad T(\lambda)y := (T^D(\lambda)y, T^R(\lambda)y) \qquad (y \in H_1^n(a,b),\ \lambda \in \mathbb{C}).$$

From [28] and [26] we know

(2.4) PROPOSITION. i) $T \in H(\mathbb{C}, \Phi(H_1^n(a,b), L_2^n(a,b)\times\mathbb{C}^n))$.
ii) $\operatorname{ind} T(\lambda) = 0 \quad (\lambda \in \mathbb{C})$.
iii) *If* $\rho(T) \neq \emptyset$, *then* $\sigma(T)$ *is a discrete subset of* $\mathbb{C}$, $\sigma(T) = \sigma_p(T)$, *and* T^{-1} *is a meromorphic function whose poles are the eigenvalues of* T.

The dual of $H_1^n(a,b)$ can be identified with

$$H_{-1}^n[a,b] := \{v_0 - v_1' : v_0, v_1 \in L_2^n(\mathbb{R});\ \operatorname{supp}(v_0 - v_1') \subset [a,b]\},$$

cf. [28], (4.2). The adjoint $T^* \in H(\mathbb{C}, L(L_2^n(a,b)\times\mathbb{C}^n, H_{-1}^n[a,b]))$ of T has the form

$$T^*(\lambda)(u,d) = -u' - A^t(\cdot,\lambda)u + \sum_{j=1}^{m} W^{(j)^t}(\lambda)d\delta_{a_j} + W^t(\cdot,\lambda)d$$

$(u \in L_2^n(a,b),\ d \in \mathbb{C}^n)$, where δ_{a_j} is the Dirac distribution with support at a_j and ^{t} denotes the transposition, cf. [28], section 4.

Since

(2.5) $\quad T(\lambda)y = f \qquad (f \in L_2^n(a,b) \times \mathbb{C}^n)$

is a boundary value problem, we call T the boundary eigenvalue operator function of (2.5). We say that

(2.6) $\quad T^*(\lambda)(u,d) = h \qquad (h \in H_{-1}^n[a,b])$

is the adjoint "boundary eigenvalue problem" of (2.5).

From now on we assume that $\rho(T) \neq \emptyset$. An immediate consequence of a theorem of Keldyš, cf. e.g. [27], (2.1), is the following

(2.7) THEOREM. *Let* $\mu \in \sigma(T)$ *and* $\{y_1,\dots,y_r\}$ *be a* CSRF *of* T *at* μ.

Then there are polynomials $v_j : \mathbb{C} \to L_2^n(a,b) \times \mathbb{C}^n$ *of degree* $< m_j$ *such that*

$$D := T^{-1} - \sum_{j=1}^{r} (\cdot-\mu)^{-m_j} y_j \otimes v_j$$

is holomorphic at μ. *The* v_j *are uniquely determined by the system* $\{y_1,\dots,y_r\}$. $\{v_1,\dots,v_r\}$ *is a* CSRF *of* T^* *at* μ, $\nu(v_j) = m_j$, *and the biorthogonal relationships*

(2.8) $$\frac{1}{l!}\frac{d^l}{d\lambda^l}\langle \eta_{ih}, v_j\rangle(\mu) = \delta_{ij}\,\delta_{m_i-h,l}$$

$$(1 \leq h \leq m_i;\ 0 \leq l \leq m_j-1;\ i,j = 1,\dots,r)$$

hold where $\eta_{ih} := (\cdot-\mu)^{-h} T y_i$ *and* $\langle\ ,\ \rangle$ *is the canonical bilinear form on* $L_2^n(a,b) \times \mathbb{C}^n$.

The Theorem (2.10) in [28] leads to the following interesting modification of the foregoing theorem.

(2.9) THEOREM. *Let* $k_1 \geq k_2 \geq \dots \geq k_r$ *be positive natural numbers. Assume that* $\mu \in \sigma(T)$. *Let* $y_1,\dots,y_r$ *be root functions of* T *at* μ *and* $v_1,\dots,v_r$ *be root functions of* T^* *at* μ. *Assume that* $\nu(y_j) \geq k_j$ *and* $\nu(v_j) \geq k_j$ *for* $j \in \{1,\dots,r\}$. *Set* $\eta_i := (\cdot-\mu)^{-k_i} T y_i$ *and suppose that the biorthogonal relationships*

$$\frac{1}{l!}\frac{d^l}{d\lambda^l}<\eta_i,v_j>(\mu)=\delta_{ij}\,\delta_{0l} \qquad (0\le l\le k_j-1;\ i,j=1,\dots,r)$$

are fulfilled.

Then $\{y_1,\dots,y_r\}$ *is a* CSRF *of* T *at* μ, $\{v_1,\dots,v_r\}$ *is a* CSRF *of* T^* *at* μ, $\nu(y_j)=\nu(v_j)=k_j$ $(j=1,\dots,r)$ *and*

$$D := T^{-1}-\sum_{j=1}^{r}(\cdot-\mu)^{-k_j}\,y_j\otimes v_j$$

is holomorphic at μ.

Let $Y\in H(\mathbb{C},M_n(W^{1,\infty}(a,b)))$ be a fundamental matrix of $T^D y=0$, cf. e.g. [28], section 4. We set

$$M(\lambda) := T^R(\lambda)Y(\cdot,\lambda)$$

and call it the *characteristic matrix function* of (2.5). Obviously, $M\in H(\mathbb{C},M_n(\mathbb{C}))$ and $\sigma(M)=\sigma(T)$. In [28] the authors proved the

(2.10) THEOREM. *Let* $\mu\in\sigma(M)=\sigma(T)$. *Let* $\{c_1,\dots,c_r\}$ *be a* CSRF *of* M *at* μ *and* $\{d_1,\dots,d_r\}$ *be a* CSRF *of* $M^*=M^t$ *at* μ. *Suppose that the biorthogonal relationships*

$$(2.11)\qquad \frac{1}{l!}\frac{d^l}{d\lambda^l}<(\cdot-\mu)^{-m_i}Mc_i,d_j>(\mu)=\delta_{ij}\,\delta_{0l}$$

$(0\le l\le m_j-1;\ i,j=1,\dots,r)$ *hold where* m_i *is the multiplicity of* c_i. *We define*

$$y_i(\lambda) := Y(\cdot,\lambda)c_i(\lambda) \qquad (i=1,\dots,r;\ \lambda\in\mathbb{C})$$

and

$$u_i(\lambda)(x) := Y^t(x,\lambda)^{-1}\left\{\sum_{j=1}^{m-1}Y^t(a_j,\lambda)W^{(j)^t}(\lambda)\chi_{(a_j,b)}(x)+\int_a^x Y^t(\xi,\lambda)W^t(\xi,\lambda)\,d\xi\right\}d_i(\lambda)$$

$(i=1,\dots,r;\ \lambda\in\mathbb{C})$. *We set*

$$v_i := (u_i,d_i) \qquad (i=1,\dots,r).$$

Then $\{y_1,\dots,y_r\}$ *is a* CSRF *of* T *at* μ, $\{v_1,\dots,v_r\}$

is a CSRF *of* T^* *at* μ, *the biorthogonal relationships*

$$(2.12)\qquad \frac{1}{l!}\frac{d^l}{d\lambda^l}<(\cdot-\mu)^{-m_i}Ty_i,v_j>(\mu) = \delta_{ij}\,\delta_{0l}$$

$(0 \le l \le m_j-1;\ i,j=1,\dots,r)$ *hold, and*

$$T^{-1} - \sum_{j=1}^{r} (\cdot-\mu)^{-m_j}\, y_j \otimes v_j$$

is holomorphic at μ.

Kaashoek [17] proved that T is globally equivalent to the canonical $L_2^n(a,b)$-extension of M if T corresponds to a two-point boundary value problem ($m=2$, $W=0$).

3. ASYMPTOTIC FUNDAMENTAL MATRICES FOR FIRST ORDER SYSTEMS

In this section we assume that

$$(3.1)\qquad A(\cdot,\lambda) = \sum_{j=-1}^{k} \lambda^{-j}A_{-j} + \lambda^{-k-1}A^k(\cdot,\lambda)$$

where $k \ge 0$, $A_1 \in M_n(W^{k,\infty}(a,b))$, $A_{-j} \in M_n(W^{k-j,\infty}(a,b))$ $(j=0,\dots,k)$, $A^k(\cdot,\lambda) \in M_n(L_\infty(a,b))$ is bounded in $M_n(L_\infty(a,b))$ as $\lambda \to \infty$. Assume that A_1 has the diagonal form

$$A_1 = \begin{bmatrix} A_0^1 & & & 0 \\ & A_1^1 & & \\ & & \ddots & \\ 0 & & & A_l^1 \end{bmatrix}$$

where

$$A_\nu^1 = r_\nu I_{n_\nu} \quad (\nu = 0,\dots,l), \qquad \sum_{\nu=0}^{l} n_\nu = n,$$

and for $\nu,\mu = 0,\dots,l$

$$r_\nu(x) - r_\mu(x) = |r_\nu(x) - r_\mu(x)|\, e^{i\varphi_{\nu\mu}},$$

$$|r_\nu - r_\mu|^{-1} \in L_\infty(a,b) \qquad (\nu \ne \mu)$$

with some $\varphi_{\nu\mu} \in \mathbb{R}$.

We set

$$R_\nu(x) := \int_a^x r_\nu(\xi)d\xi \qquad (\nu = 0,\ldots,l;\ x \in [a,b]),$$

$$E_\nu(x,\lambda) := \exp(\lambda R_\nu(x)) I_{n_\nu} \qquad (\nu = 0,\ldots,l),$$

$$E(x,\lambda) = \begin{bmatrix} E_0(x,\lambda) & & & \\ & E_1(x,\lambda) & & 0 \\ & 0 & \ddots & \\ & & & E_l(x,\lambda) \end{bmatrix}.$$

We shall often deal with functions having a special asymptotic behaviour when the parameter λ tends to infinity. For this we introduce some notations.

Let $i,j \in \mathbb{N}$ and let $|\ |$ be a fixed norm on $M_{i,j}(\mathbb{C})$. Let $f(\lambda) \in M_{i,j}(\mathbb{C})$ where $\lambda \in U \subset \mathbb{C}$, and let $g : U \to \overline{\mathbb{C}}$. We write

$$f(\lambda) = O(g(\lambda))$$

if there is a $C > 0$ such that $|f(\lambda)| \leq C|g(\lambda)|$ for $\lambda \in U$, and

$$f(\lambda) = o(g(\lambda))$$

if $|f(\lambda)||g(\lambda)|^{-1} \to 0$ as $\lambda \to \infty$. Let $a \in M_{i,j}(\mathbb{C})$. We write

$$f(\lambda) = [a] \qquad \text{if} \qquad f(\lambda) - a = o(1).$$

Now let $1 \leq p \leq \infty$ and $h \in M_{i,j}(L_p(a,b))$. $|h|_p$ denotes the $L_p(a,b)$-norm of the function $|h(\cdot)|$ where the norm on $M_{i,j}(\mathbb{C})$ is as above. Let $f(\cdot,\lambda) \in M_{i,j}(L_p(a,b))$ where $\lambda \in U \subset \mathbb{C}$, and let $g : U \to \overline{\mathbb{C}}$. We write

$$f(\cdot,\lambda) = \{O(g(\lambda))\}_p \quad \text{or} \quad f(\cdot,\lambda) = O(g(\lambda)) \ \text{in} \ M_{i,j}(L_p(a,b))$$

if there is a $C > 0$ such that $|f(\cdot,\lambda)|_p \leq C|g(\lambda)|$ for $\lambda \in U$, and

$$f(\cdot,\lambda) = \{o(g(\lambda))\}_p \quad \text{or} \quad f(\cdot,\lambda) = o(g(\lambda)) \ \text{in} \ M_{i,j}(L_p(a,b))$$

if $|f(\cdot,\lambda)|_p |g(\lambda)|^{-1} \to 0$ as $\lambda \to \infty$.

Now let $h \in M_{i,j}(L_p(a,b))$. We write

$$f(\cdot,\lambda) = [h]_p \quad \text{if} \quad f(\cdot,\lambda) - h = \{o(1)\}_p .$$

For the matrices A_j and the matrices $P^{[r]}$ defined below we form the block matrices

$$A_j =: \left(A_{j,\nu\mu}\right)_{\nu,\mu=0}^{l} \quad \text{and} \quad P^{[r]} =: \left(P^{[r]}_{\nu\mu}\right)_{\nu,\mu=0}^{l}$$

according to the block structure of A_1.

(3.2) THEOREM. A. *There are* $P^{[r]} \in M_n(W^{k+1-r,\infty}(a,b))$ $(0 \le r \le k)$ *such that*

$$P^{[0]}A_1 - A_1P^{[0]} = 0, \quad P^{[0]}(a) = I_n, \tag{3.3}$$

$$P^{[r]\prime} - \sum_{j=0}^{r} A_{-j}P^{[r-j]} + P^{[r+1]}A_1 - A_1P^{[r+1]} = 0 \quad (r=0,\dots,k-1), \tag{3.4}$$

$$P^{[k]\prime}_{\nu\nu} - A_{0,\nu\nu}P^{[k]}_{\nu\nu} = \sum_{\substack{q=0\\ q\neq\nu}}^{l} A_{0,\nu q}P^{[k]}_{q\nu} + \sum_{j=1}^{k}\sum_{q=0}^{l} A_{-j,\nu q}P^{[k-j]}_{q\nu} \tag{3.5}$$

$$(\nu = 0,\dots,l).$$

B. *For any* $P^{[r]} \in M_n(W^{k+1-r,\infty}(a,b))$ $(0 \le r \le k)$ *fulfilling* (3.3), (3.4), *and* (3.5) *there are* $\gamma > 0$ *and* $B_k(\cdot,\lambda) \in M_n(W^{1,\infty}(a,b))$ $(\lambda \in \mathbb{C},\ |\lambda| \ge \gamma)$ *such that*

$$B_k(\cdot,\lambda) = o(1) \quad \textit{in} \quad M_n(L_\infty(a,b)) \quad \textit{as} \quad \lambda \to \infty,$$

$$B_k(\cdot,\lambda) = O(\tau(\lambda)) \quad \textit{in} \quad M_n(L_\infty(a,b))$$

where

$$\tau(\lambda) = \begin{cases} \max\limits_{\substack{\nu,\mu=0\\ \nu\neq\mu}}^{l} \left(1+|\mathrm{Re}(\lambda e^{i\varphi_{\nu\mu}})|\right)^{-1} & \textit{if} \quad l > 0 \\ |\lambda|^{-1} & \textit{if} \quad l = 0 \end{cases} \qquad (|\lambda| \ge \gamma)$$

and

$$Y(x,\lambda) := \left(\sum_{r=0}^{k} \lambda^{-r}P^{[r]}(x) + \lambda^{-k}B_k(x,\lambda)\right)E(x,\lambda)$$

$$=: \tilde{P}_k(x,\lambda)E(x,\lambda)$$

is a fundamental matrix of the system

$$y' - A(\cdot,\lambda)y = 0$$

for all $|\lambda| \geq \gamma$. *Furthermore,* $\frac{1}{\lambda}B_k'(\cdot,\lambda) = o(1)$ *in* $M_n(L_\infty(a,b))$ *as* $\lambda \to \infty$ *and* $\frac{1}{\lambda}B_k'(\cdot,\lambda) = O(\tau(\lambda))$ *in* $M_n(L_\infty(a,b))$.

PROOF. A. (3.3), (3.4), and (3.5) are equivalent to

$$(3.6) \qquad P_{\nu\mu}^{[0]} = 0 \quad (\nu,\mu = 0,\dots,l;\ \nu \neq \mu), \quad P_{\nu\nu}^{[0]}(a) = I_{n_\nu} \quad (\nu=0,\dots,l),$$

$$(3.7) \qquad P_{\nu\nu}^{[r]\prime} - A_{0,\nu\nu}P_{\nu\nu}^{[r]} = \sum_{\substack{q=0\\ q\neq\nu}}^{l} A_{0,\nu q}P_{q\nu}^{[r]} + \sum_{j=1}^{r}\sum_{q=0}^{l} A_{-j,\nu q}P_{q\nu}^{[r-j]}$$

$$(\nu = 0,\dots,l;\ r = 0,\dots,k),$$

$$(3.8) \qquad P_{\nu\mu}^{[r+1]} = (r_\nu - r_\mu)^{-1}\left\{P_{\nu\mu}^{[r]\prime} - \sum_{j=0}^{r}\sum_{q=0}^{l} A_{-j,\nu q}P_{q\mu}^{[r-j]}\right\}$$

$$(\nu,\mu = 0,\dots,l;\ \nu \neq \mu;\ r = 0,\dots,k-1).$$

Let $P_{\nu\nu}^{[0]} \in M_n(W^{1,\infty}(a,b))$ be the fundamental matrix of $y' - A_{0,\nu\nu}y = 0$ with $P_{\nu\nu}^{[0]}(a) = I_{n_\nu}$ $(\nu = 0,\dots,l)$, see [28], (4.18). For $\nu \neq \mu$ we set $P_{\nu\mu}^{[0]} = 0$. Then (3.6) and, for $r = 0$, (3.7) are valid. From $P_{\nu\nu}^{[0]\prime} - A_{0,\nu\nu}P_{\nu\nu}^{[0]} = 0$ $(\nu = 0,\dots,l)$ we obtain $P^{[0]} \in M_n(W^{k+1,\infty}(a,b))$. The $P^{[r+1]}$ $(r = 0,\dots,k-1)$ are recursively defined by (3.8) and a solution of (3.7).

B. For $k > 0$ we define

$$P_k(x,\lambda) := \sum_{r=0}^{k} \lambda^{-r}P^{[r]}(x) \qquad (x \in (a,b),\ \lambda \in \mathbb{C} \setminus \{0\}).$$

For $k = 0$ let $\tilde{A}_0 = (\tilde{A}_{0,\nu\mu})_{\nu,\mu=0}^{l} \in M_n(L_\infty(a,b))$ such that $\tilde{A}_{0,\nu\nu} = A_{0,\nu\nu}$ $(\nu = 0,\dots,l)$ and $(r_\nu - r_\mu)^{-1}\tilde{A}_{0,\nu\mu} \in M_{n_\nu,n_\mu}(W^{1,\infty}(a,b))$ $(\nu,\mu = 0,\dots,l;\ \nu \neq \mu)$. We set

$$P_{\nu\mu}^{[1]} := (r_\mu - r_\nu)^{-1}\tilde{A}_{0,\nu\mu}P_{\mu\mu}^{[0]} \qquad (\nu,\mu = 0,\dots,l;\ \nu \neq \mu)$$

and

$$P_{\nu\nu}^{[1]} := 0 \qquad (\nu = 0,\dots,l).$$

Then $P^{[1]} \in M_n(W^{1,\infty}(a,b))$ and, by (3.3), (3.5), and $\tilde{A}_{0,\nu\nu} = A_{0,\nu\nu}$, (3.9) $\quad P^{[0]'} - \tilde{A}_0 P^{[0]} + P^{[1]} A_1 - A_1 P^{[1]} = 0$

holds. We set

$$P_0(x,\lambda) := P^{[0]}(x) + \lambda^{-1} P^{[1]}(x) \qquad (x \in [a,b],\ \lambda \in \mathbb{C} \setminus \{0\}).$$

Let $|\ |$ be a norm on $M_n(\mathbb{C})$ which makes it a Banach algebra. Then $M_n(L_\infty(a,b))$ is a Banach algebra with respect to $|\ |_\infty$. For each $\nu = 0,\dots,l$, $P^{[0]}_{\nu\nu}$ is a fundamental matrix by (3.7) and $P^{[0]}(a) = I_n$. Hence $P^{[0]}$ is invertible in $M_n(W^{k+1,\infty}(a,b)) \subset M_n(L_\infty(a,b))$. Let $K > 0$ such that $|P^{[0]}|_\infty \le \frac{K}{2}$ and $|P^{[0]^{-1}}|_\infty \le \frac{K}{2}$. In the case $k = 0$, K does not depend on $\tilde{A}_0$. Thus we can choose $\tilde{A}_0$ with the property

(3.10) $\quad |A_0 - \tilde{A}_0|_1 \le (l+1)^{-2} K^{-8}.$

Then there is a $\gamma > 0$ such that $P_k(\cdot,\lambda)$ is invertible in $M_n(L_\infty(a,b))$ and $|P_k(\cdot,\lambda)|_\infty \le K$, $|P_k(\cdot,\lambda)^{-1}|_\infty \le K$ for $|\lambda| \ge \gamma$. For $x \in [a,b]$ and $|\lambda| \ge \gamma$ we set

$$D_0(\cdot,\lambda) := \Big\{ (\tilde{A}_0 - A_0) P^{[0]} + \lambda^{-1} (P^{[1]'} - A_0 P^{[1]} - A^0(\cdot,\lambda) P_0(\cdot,\lambda)) \Big\} P_0(\cdot,\lambda)^{-1}$$

and, if $k \ge 1$,

$$D_k(\cdot,\lambda) := \Big\{ P^{[k]'} - \sum_{r=k}^{2k} \sum_{j=r-k}^{k} \lambda^{k-r} A_{-j} P^{[r-j]} - \lambda^{-1} A^k(\cdot,\lambda) P_k(\cdot,\lambda) \Big\} P_k(\cdot,\lambda)^{-1},$$

furthermore, for $k \ge 0$,

$$S_k(x,\lambda) := P_k(x,\lambda) E(x,\lambda).$$

Let $\kappa = \max\{1,k\}$. In view of $E'(\cdot,\lambda) = \lambda A_1 E(\cdot,\lambda)$, (3.3), (3.4), and (3.9) for $k = 0$, we infer, omitting the variables,

$$S_k' - AS_k = P_k' E + P_k E' - AP_k E$$

$$= \Big\{ \sum_{r=0}^{\kappa} \lambda^{-r} P^{[r]\prime} + \lambda \sum_{r=0}^{\kappa} \lambda^{-r} P^{[r]} A_1$$

$$- \Big(\sum_{j=-1}^{k} \lambda^{-j} A_{-j} + \lambda^{-k-1} A^k \Big) \sum_{r=0}^{\kappa} \lambda^{-r} P^{[r]} \Big\} E$$

$$= \Big\{ \sum_{r=0}^{\kappa} \lambda^{-r} P^{[r]\prime} + \sum_{r=0}^{\kappa-1} \lambda^{-r} P^{[r+1]} A_1 + \lambda P^{[0]} A_1$$

$$- \sum_{r=-1}^{k+\kappa} \sum_{j=\max\{-1,r-\kappa\}}^{\min\{k,r\}} \lambda^{-r} A_{-j} P^{[r-j]} - \lambda^{-k-1} A^k P_k \Big\} E$$

$$= \Big\{ \lambda \Big(P^{[0]} A_1 - A_1 P^{[0]} \Big) + \sum_{r=0}^{\kappa-1} \lambda^{-r} \Big(P^{[r]\prime} - \sum_{j=0}^{r} A_{-j} P^{[r-j]}$$

$$+ P^{[r+1]} A_1 - A_1 P^{[r+1]} \Big) + \lambda^{-\kappa} \Big(P^{[\kappa]\prime} - \sum_{j=0}^{k} A_{-j} P^{[\kappa-j]} \Big)$$

$$- \lambda^{-\kappa-1} \sum_{r=\kappa+1}^{k+\kappa} \sum_{j=r-\kappa}^{k} \lambda^{\kappa+1-r} A_{-j} P^{[r-j]} - \lambda^{-k-1} A^k P_k \Big\} E$$

$$= \lambda^{-k} D_k S_k .$$

We refer to [28], (4.4) for the rule of the differentiation of a product of elements of Sobolev spaces.

Let $I^{(\nu)}$ be the n×n-matrix whose ν-th diagonal block is I_{n_ν} and whose components are zero elsewhere. For $|\lambda| \geq \gamma$, $\nu,\mu = 0,\ldots,1$ we set $x_{\nu\mu}(\lambda) = a$ if $\operatorname{Re}(\lambda e^{i\varphi_{\nu\mu}}) \leq 0$ and $x_{\nu\mu}(\lambda) = b$ if $\operatorname{Re}(\lambda e^{i\varphi_{\nu\mu}}) > 0$ where $\varphi_{\nu\nu}$ is arbitrary, e.g. $\varphi_{\nu\nu} = 0$.

Next we prove that for sufficiently large λ there is a $C(\cdot,\lambda) \in M_n(L_\infty(a,b))$ such that

$$(3.11) \qquad C(x,\lambda) = I_n - \lambda^{-k} \sum_{\nu,\mu=0}^{1} \int_{x_{\nu\mu}(\lambda)}^{x} S_k(x,\lambda) I^{(\nu)} S_k^{-1}(t,\lambda) D_k(t,\lambda) *$$

$$* C(t,\lambda) S_k(t,\lambda) I^{(\mu)} S_k^{-1}(x,\lambda)\, dt .$$

For this we consider the continuous operator

$$T_\lambda : M_n(L_\infty(a,b)) \to M_n(L_\infty(a,b))$$

given by

$$(T_\lambda f)(x) := I_n - \lambda^{-k} \sum_{\nu,\mu=0}^{l} \int_{x_{\nu\mu}(\lambda)}^{x} S_k(x,\lambda) I^{(\nu)} S_k^{-1}(t,\lambda) *$$

$$* D_k(t,\lambda) f(t) S_k(t,\lambda) I^{(\mu)} S_k^{-1}(x,\lambda) dt$$

($f \in M_n(L_\infty(a,b))$). From

$$E(x,\lambda) I^{(\nu)} = I^{(\nu)} E(x,\lambda) = \exp(\lambda R_\nu(x)) I^{(\nu)}$$

we infer

$$E(x,\lambda) I^{(\nu)} E(t,\lambda)^{-1} = \exp(\lambda(R_\nu(x) - R_\nu(t))) I^{(\nu)} .$$

Using

$$R_\nu(x) - R_\nu(t) + R_\mu(t) - R_\mu(x) = e^{i\varphi_{\nu\mu}} \int_t^x |r_\nu(\eta) - r_\mu(\eta)| d\eta$$

we obtain

$$(T_\lambda f)(x) - (T_\lambda g)(x) = \lambda^{-k} \sum_{\nu,\mu=0}^{l} \int_{x_{\nu\mu}(\lambda)}^{x} \exp\Big\{\lambda e^{i\varphi_{\nu\mu}} *$$

$$* \int_t^x |r_\nu(\eta) - r_\mu(\eta)| d\eta\Big\} P_k(x,\lambda) I^{(\nu)} P_k^{-1}(t,\lambda) D_k(t,\lambda) *$$

$$* (g(t) - f(t)) P_k(t,\lambda) I^{(\mu)} P_k^{-1}(x,\lambda) dt$$

for $f,g \in M_n(L_\infty(a,b))$. By the choice of $x_{\nu\mu}(\lambda)$ we obtain for each t in the compact interval with the endpoints $x_{\nu\mu}(\lambda)$ and $x \in [a,b]$ the estimate

$$|\exp(\lambda e^{i\varphi_{\nu\mu}} \int_t^x |r_\nu(\eta) - r_\mu(\eta)| d\eta)|$$

$$= \exp(\mathrm{Re}(\lambda e^{i\varphi_{\nu\mu}}) \int_t^x |r_\nu(\eta) - r_\mu(\eta)| d\eta) \leq 1 .$$

We may assume $|I^{(\nu)}|_\infty \leq K$. Then

$$|T_\lambda| \leq |\lambda|^{-k} (l+1)^2 K^7 |D_k(\cdot,\lambda) P_k(\cdot,\lambda)|_1 .$$

There is an $M > 0$, such that, for sufficiently large λ,

$$|P^{[k]'} - \sum_{r=k}^{2k} \sum_{j=r-k}^{k} \lambda^{k-r} A_{-j} P^{[r-j]} - \lambda^{-1} A^k(\cdot,\lambda) \sum_{r=0}^{k} \lambda^{-r} P^{[r]}|_1$$

$$\leq M$$

if $k > 0$ and

$$|P^{[1]'} - A_0 P^{[1]} - A^0(\cdot,\lambda) P_0(\cdot,\lambda)|_1 \leq M$$

if $k = 0$. Hence

$$|T_\lambda| \leq |\lambda|^{-k} (l+1)^2 K^7 M$$

for $k > 0$ and, if $k = 0$,

$$|T_\lambda| \leq (l+1)^2 K^7 (|A_0 - \tilde{A}_0|_1 \frac{K}{2} + |\lambda|^{-1} M) \leq \frac{1}{2} + |\lambda|^{-1} (l+1)^2 K^7 M$$

according to (3.10). Thus there is a $\delta < 1$ such that $|T_\lambda| < \delta$ for $|\lambda| \geq \gamma$ if γ is sufficiently large. By Banach's fixed-point theorem, for each $|\lambda| \geq \gamma$ there is a $C(\cdot,\lambda) \in M_n(L_\infty(a,b))$ such that (3.11) is fulfilled. In addition, the a-priori estimate of Banach's fixed-point theorem yields

(3.12) $$|C(\cdot,\lambda) - I_n|_\infty \leq (1-\delta)^{-1} |T_\lambda I_n - I_n|_\infty .$$

From (3.11) we obtain $C(\cdot,\lambda) \in M_n(W^{1,\infty}(a,b))$ and

$$C'(x,\lambda) = -\lambda^{-k} D_k(x,\lambda) C(x,\lambda)$$

$$- \lambda^{-k} \sum_{\nu,\mu=0}^{l} \int_{x_{\nu\mu}(\lambda)}^{x} S_k'(x,\lambda) I^{(\nu)} S_k^{-1}(t,\lambda) D_k(t,\lambda) C(t,\lambda) S_k(t,\lambda) *$$

$$* I^{(\mu)} S_k^{-1}(x,\lambda) dt + \lambda^{-k} \sum_{\nu,\mu=0}^{l} \int_{x_{\nu\mu}(\lambda)}^{x} S_k(x,\lambda) I^{(\nu)} S_k^{-1}(t,\lambda) *$$

$$* D_k(t,\lambda) C(t,\lambda) S_k(t,\lambda) I^{(\mu)} S_k^{-1}(x,\lambda) S_k'(x,\lambda) S_k^{-1}(x,\lambda) dt$$

$$= -\lambda^{-k}D_k(x,\lambda)C(x,\lambda) + S_k'(x,\lambda)S_k^{-1}(x,\lambda)(C(x,\lambda) - I_n)$$

$$- (C(x,\lambda) - I_n)S_k'(x,\lambda)S_k^{-1}(x,\lambda)$$

$$= -\lambda^{-k}D_k(x,\lambda)C(x,\lambda) + S_k'(x,\lambda)S_k^{-1}(x,\lambda)C(x,\lambda)$$

$$- C(x,\lambda)S_k'(x,\lambda)S_k^{-1}(x,\lambda).$$

Define

(3.13) $$Y(x,\lambda) := C(x,\lambda)S_k(x,\lambda).$$

We infer, again omitting the variables,

$$Y' = C'S_k + CS_k'$$

$$= (-\lambda^{-k}D_kC + S_k'S_k^{-1}C - CS_k'S_k^{-1})S_k + CS_k'$$

$$= -\lambda^{-k}D_kCS_k + (\lambda^{-k}D_k + A)CS_k$$

$$= AY .$$

Furthermore

$$Y(x,\lambda) = C(x,\lambda)S_k(x,\lambda)$$

$$= \left(\sum_{r=0}^{k} \lambda^{-r}P^{[r]}(x) + \lambda^{-k}B_k(x,\lambda)\right)E(x,\lambda)$$

where

$$B_0(x,\lambda) = -\sum_{\nu,\mu=0}^{1} \int_{x_{\nu\mu}(\lambda)}^{x} S_0(x,\lambda)I^{(\nu)}S_0^{-1}(t,\lambda)D_0(t,\lambda)*$$

$$* C(t,\lambda)S_0(t,\lambda)I^{(\mu)}S_0^{-1}(x,\lambda)P_0(x,\lambda)dt + \lambda^{-1}P^{[1]}(x)$$

and, for $k > 0$,

$$B_k(x,\lambda) = -\sum_{\nu,\mu=0}^{1} \int_{x_{\nu\mu}(\lambda)}^{x} S_k(x,\lambda)I^{(\nu)}S_k^{-1}(t,\lambda)D_k(t,\lambda) *$$

$$* C(t,\lambda)S_k(t,\lambda)I^{(\mu)}S_k^{-1}(x,\lambda)P_k(x,\lambda)dt.$$

It remains to prove the estimates for $B_k(\cdot,\lambda)$. We have

$$B_0(x,\lambda) = (C(x,\lambda) - I_n)P_0(x,\lambda) + \lambda^{-1}P^{[1]}(x)$$

and, for $k > 0$,

$$B_k(x,\lambda) = \lambda^k (C(x,\lambda) - I_n)P_k(x,\lambda).$$

In order to prove $B_k(\cdot,\lambda) = o(1)$ as $\lambda \to \infty$ and

$$B_k(\cdot,\lambda) = O\left(\max_{\substack{\nu,\mu=0 \\ \nu\neq\mu}}^{1} (1 + |\mathrm{Re}(\lambda e^{i\varphi_{\nu\mu}})|)^{-1}\right)$$

we thus only have to prove that these estimates hold for $\lambda^k(T_\lambda I_n - I_n)$. For this let

$$Q^{[0]} := (\tilde{A}_0 - A_0)P^{[0]}$$

and, for $k > 0$,

$$Q^{[k]} := P^{[k]'} - \sum_{j=0}^{k} A_{-j}P^{[k-j]} .$$

From (3.5) and the choice of $\tilde{A}_0$ we conclude that the block diagonal of $Q^{[k]}$ is zero. Hence

$$\lambda^k(\dot{T}_\lambda I_n - I_n)(x) = - \sum_{\substack{\nu,\mu=0 \\ \nu\neq\mu}}^{1} \int_{x_{\nu\mu}(\lambda)}^{x} \exp\left\{\lambda e^{i\varphi_{\nu\mu}} * \right.$$

$$\left. * \int_t^x |r_\nu(\eta) - r_\mu(\eta)|d\eta\right\} P^{[0]}(x)I^{(\nu)}P^{[0]^{-1}}(t)Q^{[k]}(t)I^{(\mu)} *$$

$$* P^{[0]^{-1}}(x)dt + \frac{1}{\lambda}Q(x,\lambda)$$

for $|\lambda| \geq \gamma$ where $\{Q(\cdot,\lambda) : |\lambda| \geq \gamma\} \subset M_n(L_\infty(a,b))$ is bounded. Thus the required estimates on $\lambda^k(T_\lambda I_n - I_n)$ follow from

(3.14) PROPOSITION. *Let* $g \in L_p(a,b)$ $(1 \leq p \leq \infty)$ *and* $r \in L_\infty(a,b)$ *such that* $r > 0$ *and* $r^{-1} \in L_\infty(a,b)$. Set

$$R(x) := \int_a^x r(\xi)d\xi \quad \text{and} \quad F(x,\lambda) := \int_a^x \exp(\lambda(R(x) - R(t)))g(t)dt.$$

Then $F(\cdot,\lambda) = o(1)$ *in* $L_\infty(a,b)$ *as* $\mathrm{Re}\lambda \leq 0$ *and* $\lambda \to \infty$. *If* $g \in L_\infty(a,b)$, *then* $F(\cdot,\lambda) = O((1+|\mathrm{Re}\lambda|)^{-1})$ *for* $\mathrm{Re}\lambda \leq 0$ *in* $L_\infty(a,b)$.

PROOF. Let $\varepsilon > 0$ be arbitrary. There is a function $h \in W^{1,1}(a,b)$ such that

$$|h - \tfrac{g}{r}|_1 \leq \tfrac{\varepsilon}{2}|r|_\infty^{-1}.$$

Then

$$\left|\int_a^x \exp(\lambda(R(x) - R(t))(g(t) - h(t)r(t))dt\right| \leq |h - \tfrac{g}{r}|_1 |r|_\infty \leq \tfrac{\varepsilon}{2}$$

and

$$\left|\int_a^x \exp(\lambda(R(x) - R(t))h(t)r(t)dt\right| = \left|\frac{1}{\lambda}\left\{-h(x)\right.\right.$$

$$\left.\left. + \exp(\lambda(R(x) - R(a))h(a) + \int_a^x \exp(\lambda(R(x) - R(t)))h'(t)dt\right\}\right|$$

$$\leq \frac{\varepsilon}{2}$$

if λ is sufficiently large. The first assertion is proved. Furthermore,

$$|F(x,\lambda)| \leq |\tfrac{g}{r}|_\infty \int_a^x r(t)\exp(\mathrm{Re}\lambda(R(x) - R(t)))dt$$

$$= |\tfrac{g}{r}|_\infty |\mathrm{Re}\lambda|^{-1}\Big(1 - \exp(\mathrm{Re}\lambda R(x))\Big) \leq |\tfrac{g}{r}|_\infty (1 + |\mathrm{Re}\lambda|)^{-1} O(1)$$

and the second assertion is also clear.

In order to prove the estimates for $\frac{1}{\lambda}B_k'(\cdot,\lambda)$ we consider

$$B_k(\cdot,\lambda) = \lambda^k\Big(Y(\cdot,\lambda)E^{-1}(\cdot,\lambda) - \sum_{j=0}^{k} \lambda^{-j}P^{[j]}\Big).$$

With the aid of (3.3) and (3.4) we obtain

$$B_k'(\cdot,\lambda) = \lambda^k\Big(Y'(\cdot,\lambda)E^{-1}(\cdot,\lambda) + Y(\cdot,\lambda)E^{-1\,\prime}(\cdot,\lambda) - \sum_{j=0}^{k} \lambda^{-j}P^{[j]\prime}\Big)$$

$$= \lambda^k\Big(A(\cdot,\lambda)\tilde{P}_k(\cdot,\lambda) - \lambda\tilde{P}_k(\cdot,\lambda)A_1 - \sum_{j=0}^{k} \lambda^{-j}P^{[j]\prime}\Big)$$

$$= \lambda^k\Big\{\lambda\Big(A_1P^{[0]} - P^{[0]}A_1\Big) - \sum_{r=0}^{k-1}\lambda^{-r}\Big(P^{[r]'} - \sum_{j=0}^{r} A_{-j}P^{[r-j]}$$

$$+ P^{[r+1]}A_1 - A_1P^{[r+1]}\Big) - \lambda^{-k}P^{[k]'} + \sum_{r=k}^{2k}\sum_{j=r-k}^{k}\lambda^{-r}A_{-j}P^{[r-j]}$$

$$+ \lambda^{-k-1}A^k(\cdot,\lambda)\sum_{r=0}^{k}\lambda^{-r}\dot{P}^{[r]}\Big\} + A(\cdot,\lambda)B_k(\cdot,\lambda) - \lambda B_k(\cdot,\lambda)A_1$$

$$= -P^{[k]'} + \sum_{r=0}^{k}\sum_{j=r}^{k}\lambda^{-r}A_{-j}P^{[r+k-j]} + \lambda^{-1}A^k(\cdot,\lambda)\sum_{r=0}^{k}\lambda^{-r}P^{[r]}$$

$$+ A(\cdot,\lambda)B_k(\cdot,\lambda) - \lambda B_k(\cdot,\lambda)A_1.$$

Hence $\frac{1}{\lambda}B_k'(\cdot,\lambda)$ fulfills the same estimates as $B_k(\cdot,\lambda)$.

(3.15) COROLLARY. *Assume that there is a* $\kappa \in \mathbb{N}$ *such that* $A_1 \in M_n(W^{k+\kappa,\infty}(a,b))$, $A_{-j} \in M_n(W^{k+\kappa-j,\infty}(a,b))$ $(j = 0,\dots,k)$, $A^k(\cdot,\lambda) \in M_n(W^{\kappa,\infty}(a,b))$ *is bounded in* $M_n(W^{\kappa,\infty}(a,b))$ *as* $\lambda \to \infty$.

Then $P^{[r]} \in M_n(W^{k+\kappa+1-r,\infty}(a,b))$ $(r = 0,\dots,k)$, $B_k(\cdot,\lambda) \in M_n(W^{\kappa+1,\infty}(a,b))$, *and for* $p = 0,1,\dots,\kappa+1$ *we have* $\frac{1}{\lambda^p}B_k^{(p)}(\cdot,\lambda) = o(1)$. *in* $M_n(L_\infty(a,b))$ *as* $\lambda \to \infty$ *and*
$\frac{1}{\lambda^p}B_k^{(p)}(\cdot,\lambda) = O(\tau(\lambda))$ *in* $M_n(L_\infty(a,b))$.

4. THE INVERSE OF THE BOUNDARY EIGENVALUE OPERATOR FUNCTION

Let T be given by (2.3) and suppose that $\rho(T) \neq \emptyset$. For $\lambda \in \rho(T)$ and $f_1 \in L_2^n(a,b)$, $f_2 \in \mathbb{C}^n$ we define

(4.1) $\qquad R_1(\lambda)f_1 := T^{-1}(\lambda)(f_1,0)$,

(4.2) $\qquad R_2(\lambda)f_2 := T^{-1}(\lambda)(0,f_2)$.

For $f \in H_1^n(a,b)$ and $\lambda \in \rho(T)$ we have

(4.3) $\qquad f = T^{-1}(\lambda)T(\lambda)f = R_1(\lambda)T^D(\lambda)f + R_2(\lambda)T^R(\lambda)f.$

We now give a representation of $R_1(\lambda)$ and $R_2(\lambda)$. For this we set

$$F(x,\lambda) := \sum_{\substack{j=1 \\ a_j<x}}^{m} W^{(j)}(\lambda) + \int_a^x W(t,\lambda)\,dt \qquad (a \le x < b), \tag{4.4}$$

$$F(b,\lambda) := \sum_{j=1}^{m} W^{(j)}(\lambda) + \int_a^b W(t,\lambda)\,dt. \tag{4.5}$$

It follows that

$$T^R(\lambda)y = \int_a^b dF(t,\lambda)\,y(t) \qquad (y \in H_1^n(a,b)) \tag{4.6}$$

where the integral is the Riemann-Stieltjes integral. Let $Y(\cdot,\lambda)$ be a fundamental matrix of $T^D(\lambda)y = 0$. The Green's matrix of T^D is defined by

$$G(x,\xi,\lambda) := \begin{cases} \displaystyle\int_{t=a}^{\xi} Y(x,\lambda)M^{-1}(\lambda)\,d_tF(t,\lambda)\,Y(t,\lambda)\,Y^{-1}(\xi,\lambda) & (a \le \xi \le x \le b) \\ \displaystyle -\int_{t=\xi}^{b} Y(x,\lambda)M^{-1}(\lambda)\,d_tF(t,\lambda)\,Y(t,\lambda)\,Y^{-1}(\xi,\lambda) & (a \le x < \xi \le b) \end{cases} \tag{4.7}$$

where $M(\lambda) = T^R(\lambda)Y(\cdot,\lambda)$ and $\lambda \in \rho(T)$. G is of bounded variation with respect to ξ. Hence $G(x,\cdot,\lambda) \in M_n(L_\infty(a,b))$. We set

$$\hat{G}(x,\lambda) := Y(x,\lambda)M^{-1}(\lambda) \qquad (x \in [a,b],\ \lambda \in \rho(T)) \tag{4.8}$$

and state

(4.9) PROPOSITION. *For* $f_1 \in L_2^n(a,b)$, $f_2 \in \mathbb{C}^n$ *and* $\lambda \in \rho(T)$ *we have*

i) $(R_1(\lambda)f_1)(x) = \int_a^b G(x,\xi,\lambda)f_1(\xi)\,d\xi,$

ii) $(R_2(\lambda)f_2)(x) = \hat{G}(x,\lambda)f_2,$

iii) $(T^{-1}(\lambda)(f_1,f_2))(x) = \int_a^b G(x,\xi,\lambda)f_1(\xi)\,d\xi + \hat{G}(x,\lambda)f_2.$

PROOF. we only have to prove iii). From

$$(4.10)\qquad \begin{cases} \int_a^b G(x,\xi,\lambda)f_1(\xi)d\xi \\ \quad = Y(x,\lambda)\int_a^x \int_{t=a}^{\xi} M^{-1}(\lambda)d_tF(t,\lambda)Y(t,\lambda)Y^{-1}(\xi,\lambda)f_1(\xi)d\xi \\ \quad - Y(x,\lambda)\int_x^b \int_{t=\xi}^{b} M^{-1}(\lambda)d_tF(t,\lambda)Y(t,\lambda)Y^{-1}(\xi,\lambda)f_1(\xi)d\xi \end{cases}$$

we infer

$$\int_a^b G(\cdot,\xi,\lambda)f_1(\xi)d\xi \in H_1^n(a,b).$$

Adding and subtracting the term

$$Y(x,\lambda)\int_a^x \int_{t=\xi}^{b} M^{-1}(\lambda)d_tF(t,\lambda)Y(t,\lambda)Y^{-1}(\xi,\lambda)f_1(\xi)d\xi$$

in (4.10) we obtain

$$(4.11)\qquad \begin{cases} \int_a^b G(x,\xi,\lambda)f_1(\xi)d\xi = Y(x,\lambda)\int_a^x Y^{-1}(\xi,\lambda)f_1(\xi)d\xi \\ \quad - Y(x,\lambda)\int_a^b \int_{t=\xi}^{b} M^{-1}(\lambda)d_tF(t,\lambda)Y(t,\lambda)Y^{-1}(\xi,\lambda)f_1(\xi)d\xi \end{cases}$$

since

$$M(\lambda) = \int_a^b d_tF(t,\lambda)Y(t,\lambda)$$

by (4.6). (4.11) and $T^D(\lambda)Y(\cdot,\lambda) = 0$ imply that

$$(4.12)\qquad T^D(\lambda)\left(\int_a^b G(\cdot,\xi,\lambda)f_1(\xi)d\xi + \hat{G}(\cdot,\lambda)f_2\right) = f_1.$$

Again from (4.11) we deduce that

$$T^R(\lambda)\int_a^b G(\cdot,\xi,\lambda)f_1(\xi)d\xi = \int_a^b d_xF(x,\lambda)Y(x,\lambda)\int_a^x Y^{-1}(\xi,\lambda)f_1(\xi)d\xi$$
$$- \int_a^b \int_{t=\xi}^{b} d_tF(t,\lambda)Y(t,\lambda)Y^{-1}(\xi,\lambda)f_1(\xi)d\xi = 0.$$

Thus

$$T^R(\lambda)(\hat{G}(\cdot,\lambda)f_2) = f_2$$

which proves the assertion.

5. REGULAR BOUNDARY EIGENVALUE OPERATOR FUNCTIONS

In this section we consider a boundary eigenvalue operator of the form (2.1) and (2.2) where A, $W^{(j)}$ and W are polynomials with respect to λ.

We assume that $\rho(T) \neq \emptyset$. Then $\sigma(T)$ is a discrete subset of $\mathbb{C}$ and T^{-1} is meromorphic, see (2.4). We fix closed Jordan curves $\Gamma_\nu \subset \rho(T)$ $(\nu \in \mathbb{N})$ such that $0 \in \operatorname{int}\Gamma_\nu$, $\overline{\operatorname{int}\Gamma_\nu} \subset \operatorname{int}\Gamma_{\nu+1}$, $\bigcup_{\nu\in\mathbb{N}} \operatorname{int}\Gamma_\nu = \mathbb{C}$.

(5.1) DEFINITION. Let $|\ |$ be a continuous norm on $H_1^n(a,b)$ and let $p,p' \in \mathbb{Z}$.

i) T is called $|\ |$-regular of order (p,p') with respect to $f = (f_1,f_2) \in L_2^n(a,b) \times \mathbb{C}^n$ (and with respect to the curves Γ_ν) if

$$\int_{\Gamma_\nu} |\lambda^{-p-1}R_1(\lambda)f_1|\,|d\lambda| \to 0 \qquad (\nu \to \infty)$$

and

$$\int_{\Gamma_\nu} |\lambda^{-p'-1}R_2(\lambda)f_2|\,|d\lambda| \to 0 \qquad (\nu \to \infty)$$

hold.

ii) T is called $|\ |$-regular of order (p,p') (with respect to the curves Γ_ν) if T is regular of order (p,p') with respect to all $f \in L_2^n(a,b) \times \mathbb{C}^n$.

From now on in this capter we assume that there is a $C(\cdot,\lambda) \in M_n(W^{1,\infty}(a,b))$ which is a polynomial with respect to λ and an $\tilde{A}(\cdot,\lambda)$ of the form (3.1) such that

$$(5.2) \qquad C^{-1}(\cdot,\lambda)T^D(\lambda)C(\cdot,\lambda)y = y' - \tilde{A}(\cdot,\lambda)y \qquad (y \in H_1^n(a,b))$$

holds for all sufficiently large λ. We also assume that $C(\cdot,\lambda)$ is invertible for all sufficiently large λ and $C(\cdot,\lambda)^{-1} = \{O(\lambda^{\hat{q}})\}_\infty$ with some $\hat{q} \in \mathbb{Z}$. The following statement is evident.

(5.3) PROPOSITION. *Let* $\tilde{Y}(\cdot,\lambda)$ *be a fundamental matrix of* $\tilde{T}^D(\lambda)y := y' - \tilde{A}(\cdot,\lambda)y = 0$. *Then*

$$Y(\cdot,\lambda) := C(\cdot,\lambda)\tilde{Y}(\cdot,\lambda) \tag{5.4}$$

is a fundamental matrix of $T^D(\lambda)y = 0$ *for all sufficiently large* λ.

We are going to deduce sufficient conditions for regularity. Let q_0 and q_2 be the orders of the polynomials $C(\cdot,\lambda)$ and $W(\cdot,\lambda)$, respectively, and let q_1 be the maximum of the orders of the polynomials $W^{(j)}(\lambda)$ $(j = 1,\dots,m)$. In addition to the assumptions in section 3 we assume that for $\mu = 0,\dots,l$ $r_\mu(t) = |r_\mu(t)|e^{i\varphi_\mu}$ $(t \in (a,b))$ with some $\varphi_\mu \in \mathbb{R}$ where either $r_\mu = 0$ or $r_\mu^{-1} \in L_\infty(a,b)$.

Let $\tilde{Y}(\cdot,\lambda)$ be an asymptotic fundamental matrix of $\tilde{T}^D(\lambda)y = 0$ as constructed in section 3, and let $Y(\cdot,\lambda)$ be given by (5.4). First we investigate the asymptotic behaviour of the function

$$\tilde{M}_I(\lambda) := S_I(\lambda) + Q_I(\lambda) \tag{5.5}$$

where I is a subinterval of $[a,b]$ and

$$\begin{cases} S_I(\lambda) := \sum\limits_{\substack{j=1\\ a_j \in I}}^{m} W^{(j)}(\lambda)Y(a_j,\lambda), \\ Q_I(\lambda) := \int\limits_I W(t,\lambda)Y(t,\lambda)dt. \end{cases} \tag{5.6}$$

From (3.2) we conclude that

$$S_I(\lambda) = \sum_{\substack{j=1\\ a_j\in I}}^{m} W^{(j)}(\lambda)C(a_j,\lambda)\left\{\sum_{r=0}^{k} \lambda^{-r}P^{[r]}(a_j) + \lambda^{-k}B_k(a_j,\lambda)\right\} *$$

$$* E(a_j,\lambda) = \lambda^{q_1'} \sum_{\substack{j=1\\ a_j\in I}}^{m} S_j(\lambda)E(a_j,\lambda)$$

where $q_1' \le q_0 + q_1$,

$$S_j(\lambda) = \sum_{i=0}^{k_1'} \lambda^{-i} S_{j,i} + \lambda^{-k_1'} \hat{S}_j(\lambda),$$

$k_1' = k - (q_0 + q_1 - q_1') \geq 0$, $S_{j,i} \in M_n(\mathbb{C})$, $\hat{S}_j(\lambda) = o(1)$. The $S_j(\lambda)$ and the $S_{j,i}$ depend on I as q_1' depends on I. But we may choose to consider q_1' to be independent of I (e.g. $q_1' = q_0 + q_1$). The S_j are asymptotic polynomials with respect to λ^{-1} and with coefficients in $M_n(\mathbb{C})$. Here a function of the form

$$f(\lambda) = \sum_{j=0}^{s} \lambda^{-j} f_j + \lambda^{-s} o(1)$$

is called an asymptotic polynomial of order s with respect to λ^{-1}

Let $\tilde{r}_\mu(x)$ and $\exp(\lambda \tilde{R}_\mu(x))$ $(\mu = 1,\dots,n)$ be the diagonal elements of $A_1(x)$ and $E(x,\lambda)$, respectively. Note that

$$\tilde{R}_\mu(x) = e^{i\varphi_\mu} \int_a^x |\tilde{r}_\mu(t)|\,dt \qquad (\mu = 1,\dots,n;\ x \in [a,b]).$$

Set $S_j(\lambda) =: (s_{\nu\mu}^{(j)}(\lambda))_{\nu,\mu=1}^n$. Then

$$S_j(\lambda) E(a_j,\lambda) = \left(s_{\nu\mu}^{(j)}(\lambda) \exp(\lambda \tilde{R}_\mu(a_j))\right)_{\nu,\mu=1}^n.$$

Hence

$$(5.7) \qquad S_I(\lambda) = \left(\lambda^{q_1'} \sum_{\substack{j=1 \\ a_j \in I}}^{m} s_{\nu\mu}^{(j)}(\lambda) \exp(\lambda \tilde{R}_\mu(a_j))\right)_{\nu,\mu=1}^n$$

where the $s_{\nu\mu}^{(j)}$ are asymptotic polynomials of order k_1' with respect to λ^{-1}.

Next we consider

$$Q_I(\lambda) = \int_I W(t,\lambda) C(t,\lambda) \left(\sum_{r=0}^{k} \lambda^{-r} P^{[r]}(t) + \lambda^{-k} B_k(t,\lambda) \right) E(t,\lambda)\,dt$$

$$= \lambda^{q_2'} \int_I Q(t,\lambda) E(t,\lambda)\,dt$$

where $q_2' \le q_0 + q_2$, $k_2' = k - (q_0 + q_2 - q_2') \geq 0$,

$$Q(t,\lambda) = \sum_{i=0}^{k_2'} \lambda^{-i} Q_i(t) + \lambda^{-k_2'} \hat{Q}(t,\lambda),$$

$Q_i \in M_n(L_1(a,b))$, $\hat{Q}(\cdot,\lambda) = \{o(1)\}_1$. With $Q(t,\lambda) =: \left(q_{\nu\mu}(t,\lambda)\right)_{\nu,\mu=1}^n$ and $\bar{I} =: [a_I, b_I]$ we obtain

$$\int_I Q(t,\lambda)E(t,\lambda)dt = \left(\int_{a_I}^{b_I} q_{\nu\mu}(t,\lambda)\exp(\lambda\tilde{R}_\mu(t))dt\right)_{\nu,\mu=1}^n.$$

First let $\tilde{r}_\mu = 0$. Then $\tilde{R}_\mu = 0$, and

$$\int_{a_I}^{b_I} q_{\nu\mu}(t,\lambda)\exp(\lambda\tilde{R}_\mu(t))dt = \int_{a_I}^{b_I} q_{\nu\mu}(t,\lambda)dt$$

$$=: q_{\nu\mu}^{(a_I)}(\lambda) =: q_{\nu\mu}^{(a_I)}(\lambda)\exp(\lambda\tilde{R}_\mu(a_I)) + q_{\nu\mu}^{(b_I)}(\lambda)\exp(\lambda\tilde{R}_\mu(b_I))$$

where $q_{\nu\mu}^{(a_I)}$ and $q_{\nu\mu}^{(b_I)} = 0$ are asymptotic polynomials of order k_2' with respect to λ^{-1}.

Now let $\tilde{r}_\mu \neq 0$. Set $x_\mu(\lambda) := a_I$ if $\operatorname{Re}(\lambda e^{i\tilde{\varphi}_\mu}) \le 0$ and $x_\mu(\lambda) := b_I$ if $\operatorname{Re}(\lambda e^{i\tilde{\varphi}_\mu}) > 0$. For each $t \in [a_I, b_I]$ we estimate

$$|\exp(\lambda(\tilde{R}_\mu(t) - \tilde{R}_\mu(x_\mu(\lambda))))| = \exp(\operatorname{Re}(\lambda(\tilde{R}_\mu(t) - \tilde{R}_\mu(x_\mu(\lambda))))) \le 1.$$

Let $W(\cdot,\lambda) \in M_n(W^{l_0,1}(a,b))$ and $C(\cdot,\lambda) \in M_n(W^{l_0,\infty}(a,b))$ for some $l_0 \geq 0$ and let the assumptions of Corollary (3.15) be fulfilled with some $\kappa \in \mathbb{N}$. We set $l_1 := \min\{l_0, \kappa+1, k_2'\}$. By Corollary (3.15) and the definition of Q we obtain $q_{\nu\mu}(\cdot,\lambda) \in W^{l_1,1}(a,b)$. We define

$$u_{\nu\mu}^{[0]}(x,\lambda) := q_{\nu\mu}(x,\lambda),$$

$$u_{\nu\mu}^{[j+1]}(x,\lambda) := \left(\frac{u_{\nu\mu}^{[j]}}{\tilde{r}_\mu}\right)'(x,\lambda) \qquad (j = 0,\ldots,l_1 - 1).$$

With the aid of Corollary (3.15) we infer that the $u_{\nu\mu}^{[j]}$ are asymptotic polynomials of order $k_2' - j$ with respect to λ^{-1} in $M_n(L_1(a,b))$. An integration by parts yields

$$\int_{a_I}^{b_I} u_{\nu\mu}^{[j]}(t,\lambda)\exp(\lambda\tilde{R}_\mu(t))dt = \frac{1}{\lambda}\frac{u_{\nu\mu}^{[j]}(t,\lambda)}{\tilde{r}_\mu(t)}\exp(\lambda\tilde{R}_\mu(t))\Big|_{t=a_I}^{b_I}$$

$$-\frac{1}{\lambda}\int_{a_I}^{b_I} u_{\nu\mu}^{[j+1]}(x,\lambda)\exp(\lambda\tilde{R}_\mu(t))dt$$

for $j=0,\ldots,l_1-1$. Hence we obtain

$$\int_{a_I}^{b_I} q_{\nu\mu}(t,\lambda)\exp(\lambda\tilde{R}_\mu(t))dt$$

$$= \sum_{j=0}^{l_1-1} (-1)^j\lambda^{-j-1}\frac{u_{\nu\mu}^{[j]}(t,\lambda)}{\tilde{r}_\mu(t)}\exp(\lambda\tilde{R}_\mu(t))\Big|_{t=a_I}^{b_I} + (-1)^{l_1}\lambda^{-l_1} *$$

$$* \int_{a_I}^{b_I} u_{\nu\mu}^{[l_1]}(t,\lambda)\exp(\lambda(\tilde{R}_\mu(t)-\tilde{R}_\mu(x_\mu(\lambda))dt\ \exp(\lambda\tilde{R}_\mu(x_\mu(\lambda))$$

by a recursive application of the foregoing equation. Since $u_{\nu\mu}^{[l_1]}(\cdot,\lambda) = [\bar{u}_{\nu\mu}^{[l_1]}]_1$ for some $\bar{u}_{\nu\mu}^{[l_1]} \in L_1(a,b)$, we obtain by (3.14) that the integral on the right side tends to zero as $\lambda\to\infty$ uniformly for all I. Hence there is an asymptotic polynomial $\tilde{q}_{\nu\mu}$ in $L_\infty((a,b)\times(a,b))$ of order l_1 with respect to λ^{-1} such that

$$(5.8)\qquad Q_I(\lambda) = \Big(\lambda^{q_2'}\Big\{\tilde{q}_{\nu\mu}(a_I,b_I,\lambda)\exp(\lambda\tilde{R}_\mu(a_I)) + \tilde{q}_{\nu\mu}(b_I,a_I,\lambda)\exp(\lambda\tilde{R}_\mu(b_I))\Big\}\Big)_{\nu,\mu=1}^n.$$

We now consider the characteristic matrix $M(\lambda) = T^R(\lambda)Y(\cdot,\lambda)$, where $Y(\cdot,\lambda)$ is given by (5.4). From (5.7) and (5.8) we obtain that there is a $q' \le \max\{q_1',q_2'\}$ such that

$$(5.9)\qquad M(\lambda) = \lambda^{q'}\Big(\sum_{j=1}^m \tilde{a}_{\nu\mu}^{(j)}(\lambda)\exp(\lambda\tilde{R}_\mu(a_j))\Big)_{\nu,\mu=1}^n$$

where the $\tilde{a}_{\nu\mu}^{(j)}$ $(\nu,\mu=1,\ldots,n;\ j=1,\ldots,m)$ are asymptotic polynomials with respect to λ^{-1}.

In the following we give an estimate for the determinant of $M(\lambda)$. By (5.9),

$$\det M(\lambda) = \sum_{c\in E} \tilde{b}_c(\lambda)\exp(\lambda c),$$

where

(5.10) $$E = \left\{ \sum_{l=1}^{n} \tilde{R}_l(a_{j(l)}) : j(1),\dots,j(n) \in \{1,\dots,m\} \right\}$$

and

$$\tilde{b}_c(\lambda) = \lambda^{\nu_c}[b_c]$$

with $\nu_c \le nq'$. Let P be the convex hull of E. Let $K \subset [0,1]$ and set

$$E_K := \left\{ \sum_{l=1}^{n} \alpha_l \tilde{R}_l(b) : \alpha_l \in K \right\}.$$

Let P_K be the convex hull of E_K.

(5.11) PROPOSITION. $P = P_{\{0,1\}} = P_{[0,1]}$.

PROOF. Since $\tilde{R}_l(a) = 0$ and $\tilde{R}_l(a_j) \in \overline{0,\tilde{R}_l(b)}$, the inclusions $E_{\{0,1\}} \subset E \subset E_{[0,1]}$ hold. Thus we only have to prove that $P_{[0,1]} \subset P_{\{0,1\}}$. Let $\alpha_l \in [0,1]$ $(l = 1,\dots,n)$ and choose l_i $(i = 1,\dots,n)$ such that $\{l_i : i = 1,\dots,n\} = \{1,\dots,n\}$ and $\alpha l_1 \le \dots \le \alpha l_n$. Then

$$\sum_{l=1}^{n} \alpha_l \tilde{R}_l(b) = \alpha_{l_1} \sum_{j=1}^{n} \tilde{R}_j(b) + \sum_{i=2}^{n} (\alpha_{l_i} - \alpha_{l_{i-1}}) \sum_{j=i}^{n} \tilde{R}_{l_j}(b) + (1-\alpha_{l_n})\cdot 0,$$

where $0 \in E_{\{0,1\}} \subset P_{\{0,1\}}$, $0 \le \alpha_{l_i} - \alpha_{l_{i-1}} \le 1$, $0 \le 1-\alpha_{l_n} \le 1$,

$$\alpha_{l_1} + \sum_{i=2}^{n} (\alpha_{l_i} - \alpha_{l_{i-1}}) + (1-\alpha_{l_n}) = 1, \quad \sum_{j=i}^{n} \tilde{R}_{l_j}(b) \in E_{\{0,1\}} \subset P_{\{0,1\}}.$$

Hence $P_{[0,1]} \subset P_{\{0,1\}}$ since $P_{\{0,1\}}$ is convex.

The boundary of P consists of a finite number of line segments P_s $(s = 1,\dots,S)$. We have

$$P_s \cap E = \{c_s + \alpha_s t_{s,i} : i = 1,\dots,m_s\},$$

where $c_s \in P_s \cap E$ is a fixed element, $\alpha_s \in \mathbb{C} \setminus \{0\}$, $t_{s,i} \in \mathbb{R}$. We set $\nu_{s,i} := \nu_c$ for $c = c_s + \alpha_s t_{s,i}$. Let $\tilde{E}$ be the set of the vertices of P,

$$\tilde{L}_s := \{(t_{s,i}, \nu_{s,i}) : i \in \{1,\dots,m_s\}, c_s + \alpha_s t_{s,i} \in \tilde{E} \text{ or } \tilde{b}_{c_s+\alpha_s t_{s,i}} \neq 0\}.$$

Let $L_s \subset \tilde{L}_s$ be the set of those points $(t_{s,i}, \nu_{s,i}) \in \tilde{L}_s$ for which there are no two different points (t_{s,i_1}, ν_{s,i_1}), $(t_{s,i_2}, \nu_{s,i_2}) \in \tilde{L}_s$ and no $t \in (0,1)$ such that $t_{s,i} = t t_{s,i_1} + (1-t) t_{s,i_2}$ and $\nu_{s,i} \le t\nu_{s,i_1} + (1-t)\nu_{s,i_2}$. Set

$$\hat{E} := \{c_s + \alpha_s t_{s,i} : s \in \{1,\dots,S\}, (t_{s,i}, \nu_{s,i}) \in L_s\}.$$

Note that $\tilde{E} \subset \hat{E}$.

(5.12) LEMMA (Langer). *Assume that* $b_c \neq 0$ *for all* $c \in \hat{E}$ *and set* $g := \min\{\nu_c : c \in \hat{E}\}$. *Then there is an increasing sequence* $(d_j)_1^\infty$ *of positive numbers,* $d_j \to \infty$ *as* $j \to \infty$, *and a positive constant* δ *such that for all* $\lambda \in \mathbb{C}$, $|\lambda| = d_j$ $(j \in \mathbb{N})$ *there is a* $c(\lambda) \in \tilde{E}$ *such that* $\operatorname{Re}(\lambda(c - c(\lambda))) \le 0$ *for* $c \in P$ *and the estimate*

$$|\lambda^{-g} \det M(\lambda) \exp(-\lambda c(\lambda))| \ge \delta$$

holds.

This statement stems from Langer [24], p. 176.

(5.13) REMARK. If all ν_j are equal (e.g. $\nu_j = nq'$), then $\hat{E} = \tilde{E}$.

(5.14) THEOREM. *Assume that* $b_c \neq 0$ *for all* $c \in \hat{E}$. *Then there are* $p, p' \in \mathbb{Z}$ *such that* T *is* $|\;|_{L_2^n(a,b)}$*-regular of order* (p,p') *with respect to the curves given in* (5.12).

PROOF. Let $f_1 \in L_2^n(a,b)$. From (4.7) we obtain

$$\int_a^b G(x,\xi,\lambda) f_1(\xi)\,d\xi$$

$$= Y(x,\lambda) \int_a^x \int_{t=a}^{\xi} M^{-1}(\lambda)\, d_t F(t,\lambda) Y(t,\lambda) Y^{-1}(\xi,\lambda) f_1(\xi)\,d\xi$$

$$- Y(x,\lambda) \int_x^b \int_{t=\xi}^{b} M^{-1}(\lambda)\, d_t F(t,\lambda) Y(t,\lambda) Y^{-1}(\xi,\lambda) f_1(\xi)\,d\xi$$

$$= Y(x,\lambda) \int_a^x M^{-1}(\lambda) \tilde{M}_{[a,\xi)}(\lambda) Y^{-1}(\xi,\lambda) f_1(\xi)\,d\xi$$

$$- Y(x,\lambda)\int_x^b M^{-1}(\lambda)\widetilde{M}_{[\xi,b]}(\lambda)Y^{-1}(\xi,\lambda)f_1(\xi)d\xi$$

$$= Y(x,\lambda)\int_a^b M^{-1}(\lambda)M(x,\xi,\lambda)Y^{-1}(\xi,\lambda)f_1(\xi)d\xi$$

where $\widetilde{M}_I(\lambda)$ has been defined in (5.5), (5.6) and

$$M(x,\xi,\lambda) = \begin{cases} \widetilde{M}_{[a,\xi)}(\lambda) & \text{for } a \le \xi \le x \le b \\ -\widetilde{M}_{[\xi,b]}(\lambda) & \text{for } a \le x < \xi \le b \end{cases}$$

We recall some well-known results from linear algebra. Let $M = (m_1,\ldots,m_n) \in M_n(\mathbb{C})$ be invertible. Then

$$\det M \cdot M^{-1} = \left(\det M_{\nu\mu}\right)_{\nu,\mu=1}^n$$

where $M_{\nu\mu} := (m_1,\ldots,m_{\nu-1},e_\mu,m_{\nu+1},\ldots,m_n)$ and e_μ is the μ-th unit vector in $\mathbb{C}^n$, i.e. the ν-th column of M is substituted by the μ-th unit vector. Let $A = (a_1,\ldots,a_n) \in M_n(\mathbb{C})$. Then

$$(5.15) \qquad (\det M)\, M^{-1}A = \left(\det M_{\nu\mu}^A\right)_{\nu,\mu=1}^n$$

where $M_{\nu\mu}^A := \det(m_1,\ldots,m_{\nu-1},a_\mu,m_{\nu+1},\ldots,m_n)$. For the proof of (5.15) let $A =: (a_{\alpha\mu})_{\alpha,\mu=1}^n$. We obtain

$$(\det M)\, M^{-1}A = \left(\sum_{\alpha=1}^n (\det M_{\nu\alpha})a_{\alpha\mu}\right)_{\nu,\mu=1}^n$$

$$= \left(\det(m_1,\ldots,m_{\nu-1},\sum_{\alpha=1}^n a_{\alpha\mu}e_\alpha,m_{\nu+1},\ldots,m_n)\right)_{\nu,\mu=1}^n$$

$$= \left(\det(m_1,\ldots,m_{\nu-1},a_\mu,m_{\nu+1},\ldots,m_n)\right)_{\nu,\mu=1}^n.$$

From (5.7) and (5.8) we infer that the μ-th column of $M(x,\xi,\lambda)$ has the form

$$(5.16) \qquad m_\mu(x,\xi,\lambda) = \lambda^q\left(\sum_{s\in N(x,\xi)} a_\mu^{(s)}(x,\xi,\lambda)\exp(\lambda\widetilde{R}_\mu(s))\right)$$

where $q := \max\{q_1',q_2'\}$,

$$N(x,\xi) = \begin{cases} \{a_j : j=1,\dots,m;\ a_j < \xi\} \cup \{\xi\} & (a \le \xi \le x \le b) \\ \{a_j : j=1,\dots,m;\ a_j \ge \xi\} \cup \{\xi\} & (a \le x < \xi \le b) \end{cases}$$

and the $a_\mu^{(s)}$ are asymptotic polynomials in $L_\infty^n((a,b)\times(a,b))$. Let $m_i(\lambda)$ be the i-th column of $M(\lambda)$. From (5.15) we conclude that

$$E(x,\lambda)(\det M(\lambda))\, M^{-1}(\lambda) M(x,\xi,\lambda) E^{-1}(\xi,\lambda)$$

$$= \Big(\exp(\lambda(\tilde{R}_\nu(x) - \tilde{R}_\mu(\xi)))\det(m_1(\lambda),\dots,m_{\nu-1}(\lambda), m_\mu(x,\xi,\lambda),$$

$$m_{\nu+1}(\lambda),\dots,m_n(\lambda))\Big)_{\nu,\mu=1}^n =: \Big(h_{\nu\mu}(x,\xi,\lambda)\Big)_{\nu,\mu=1}^n .$$

For $\nu=\mu$ we have

$$h_{\nu\nu}(x,\xi,\lambda) = \lambda^{nq}\Big\{\sum_{\iota\in\Lambda_\nu} h_{\nu\nu}^{[\iota]}(x,\xi,\lambda)\exp(\lambda c_\iota(x,\xi))\Big\}$$

where Λ_ν is a finite set, $h_{\nu\nu}^{[\iota]}$ $(\iota\in\Lambda_\nu)$ are asymptotic polynomials in $L_\infty((a,b)\times(a,b))$ and

$$c_\iota(x,\xi) = \tilde{R}_\nu(x) - \tilde{R}_\nu(\xi) + \sum_{\substack{l=1\\ l\ne\nu}}^n \tilde{R}_l(a_{j_\iota(l)}) + \tilde{R}_\nu(s_\iota)$$

where $s_\iota \in N(x,\xi)$. Since ξ is an element of the interval with the endpoints s_ι and x, we obtain

$$0 \le |\tilde{R}_\nu(x)| - |\tilde{R}_\nu(\xi)| + |\tilde{R}_\nu(s_\iota)| \le |\tilde{R}_\nu(b)| .$$

Set $x_\nu(\lambda) := a$ if $\mathrm{Re}(\lambda e^{i\tilde{\varphi}_\nu}) \le 0$ and $x_\nu(\lambda) := b$ if $\mathrm{Re}(\lambda e^{i\tilde{\varphi}_\nu}) > 0$. Let

$$\hat{c}_\iota(\lambda) := \tilde{R}_\nu(x_\nu(\lambda)) + \sum_{\substack{l=1\\ l\ne\nu}}^n \tilde{R}_l(a_{j_\iota(l)}) .$$

From

$$\mathrm{Re}(\lambda(c_\iota(x,\xi) - \hat{c}_\iota(\lambda)))$$

$$= \mathrm{Re}(\lambda e^{i\tilde{\varphi}_\nu})\Big\{|\tilde{R}_\nu(x)| - |\tilde{R}_\nu(\xi)| + |\tilde{R}_\nu(s_\iota)| - |\tilde{R}_\nu(x_\nu(\lambda))|\Big\} \le 0$$

we have

$$h_{\nu\nu}(x,\xi,\lambda) = \lambda^{nq}\Big\{\sum_{c\in E} h_{\nu\nu}^{c}(x,\xi,\lambda)\exp(\lambda c)\Big\}$$

where E is given by (5.10) and $h^c_{\nu\nu}(\cdot,\cdot,\lambda) = \{0(1)\}_\infty$.

Now let $\nu \neq \mu$. We define

$$\hat{M}(x,\xi,\lambda) := \begin{cases} \tilde{M}_{[\xi,b]}(\lambda) & \text{for } a \leq \xi \leq x \leq b \\ -\tilde{M}_{[a,\xi)}(\lambda) & \text{for } a \leq x < \xi \leq b \end{cases}$$

and obtain

$$M(x,\xi,\lambda) + \hat{M}(x,\xi,\lambda) = \begin{cases} M(\lambda) & \text{for } a \leq \xi \leq x \leq b \\ -M(\lambda) & \text{for } a \leq x < \xi \leq b \end{cases} .$$

As for $M(x,\xi,\lambda)$ we conclude that the μ-th column of $\hat{M}(x,\xi,\lambda)$ has the form

$$\hat{m}_\mu(x,\xi,\lambda) = \lambda^q \Big(\sum_{s \in \hat{N}(x,\xi)} \hat{a}^{(s)}_\mu (x,\xi,\lambda) \exp(\lambda \tilde{R}_\mu(s)) \Big)$$

where

$$\hat{N}(x,\xi) = \begin{cases} \{a_j : j = 1,\dots,m;\ a_j \geq \xi\} \cup \{\xi\} & (a \leq \xi \leq x \leq b) \\ \{a_j : j = 1,\dots,m;\ a_j < \xi\} \cup \{\xi\} & (a \leq x < \xi \leq b) \end{cases}$$

and the $\hat{a}^{(s)}_\mu$ are asymptotic polynomials in $L^n_\infty((a,b)\times(a,b))$. Subtracting or adding the ν-th column to the μ-th column in the determinant defining $h_{\nu\mu}(x,\xi,\lambda)$ leads to

$$h_{\nu\mu}(x,\xi,\lambda) = \exp(\lambda(\tilde{R}_\nu(x) - \tilde{R}_\mu(\xi))) *$$

$$* \det(m_1(\lambda),\dots,\pm\hat{m}_\mu(x,\xi,\lambda),\dots,m_\mu(x,\xi,\lambda),\dots,m_n(\lambda))$$

$$= \lambda^{nq} \Big\{ \sum_{\iota \in \Lambda_{\nu\mu}} h^{[\iota]}_{\nu\mu}(x,\xi,\lambda) \exp(\lambda c_\iota(x,\xi)) \Big\}$$

where $\Lambda_{\nu\mu}$ is a finite set, $h^{[\iota]}_{\nu\mu}$ $(\iota \in \Lambda_{\nu\mu})$ are asymptotic polynomials in $L_\infty((a,b)\times(a,b))$ and

$$c_\iota(x,\xi) = \tilde{R}_\nu(x) - \tilde{R}_\mu(\xi) + \sum_{\substack{l=1 \\ l \neq \nu,\mu}}^{n} \tilde{R}_l(a_{j_\iota(l)}) + \tilde{R}_\mu(x_\iota) + \tilde{R}_\mu(y_\iota)$$

where $x_\iota \in \hat{N}(x,\xi)$ and $y_\iota \in N(x,\xi)$, i.e. $a \leq y_\iota \leq \xi \leq x_\iota \leq b$ or $a \leq x_\iota \leq \xi \leq y_\iota \leq b$. Let $x_\nu(\lambda)$ be defined as for $\nu = \mu$ and set

$$\hat{c}_\iota(\lambda) := \tilde{R}_\nu(x_\nu(\lambda)) + \tilde{R}_\mu(x_\mu(\lambda)) + \sum_{\substack{l=1\\ l\neq\nu,\mu}}^{n} \tilde{R}_l(a_{j_\iota}(l)).$$

From

$$\operatorname{Re}(\lambda(c_\iota(x,\xi) - \hat{c}_\iota(\lambda)) = \operatorname{Re}(\lambda e^{i\tilde{\varphi}_\nu})(|\tilde{R}_\nu(x)| - |\tilde{R}_\nu(x_\nu(\lambda))|)$$

$$+ \operatorname{Re}(\lambda e^{i\tilde{\varphi}_\mu})(|\tilde{R}_\mu(x_\iota)| - |\tilde{R}_\mu(\xi)| + |\tilde{R}_\mu(y_\iota)| - |\tilde{R}_\mu(x_\mu(\lambda))|) \leq 0$$

we have

$$h_{\nu\mu}(x,\xi,\lambda) = \lambda^{nq}\left\{\sum_{c\in E} h^c_{\nu\mu}(x,\xi,\lambda)\exp(\lambda c)\right\}$$

where $h^c_{\nu\mu}(\cdot,\cdot,\lambda) = \{0(1)\}_\infty$.

From $\tilde{Y}(\cdot,\lambda)E(\cdot,\lambda)^{-1} = \{0(1)\}_\infty$ and the assumptions concerning $C(\cdot,\lambda)$ we infer that there are $k^c_{\nu\mu}(\cdot,\lambda) = \{0(1)\}_2$ and $q_3 \in \mathbb{Z}$ such that

$$\int_a^b G(x,\xi,\lambda)f_1(\xi)d\xi$$

$$= \lambda^{q_3}(\det M(\lambda))^{-1}\left(\sum_{c\in E}(k^c_{\nu\mu}(x,\lambda)\exp(\lambda c))\right)^n_{\nu,\mu=1}.$$

Lemma (5.12) yields

$$|(\det M(\lambda))^{-1}\exp(\lambda c)| \leq \delta^{-1}|\lambda|^{-g}|\exp(\lambda(c-c(\lambda)))|$$

$$\leq \delta^{-1}|\lambda|^{-g}$$

for $|\lambda| = d_j$. This proves

$$\int_{|\lambda|=d_j} |\lambda^{-q_3+g-2}R_1(\lambda)f_1|_{L_2^n(a,b)}|d\lambda| \to 0 \quad (j\to\infty).$$

We have

$$E(x,\lambda)(\det M(\lambda))M^{-1}(\lambda) = \Big(\exp(\lambda\tilde{R}_\nu(x)) *$$

$$* \det(m_1(\lambda),\dots,m_{\nu-1}(\lambda),e_\mu,m_{\nu+1}(\lambda),\dots,m_n(\lambda))\Big)^n_{\nu,\mu=1}$$

$$=: (\hat{h}_{\nu\mu}(x,\lambda))^n_{\nu,\mu=1}.$$

We obtain

$$\hat{h}_{\nu\mu}(x,\lambda) = \lambda^{(n-1)q}\left\{\sum_{\iota\in\Gamma_{\nu\mu}} \hat{h}^{[\iota]}_{\nu\mu}(\lambda)\exp(\lambda c_\iota(x))\right\}$$

where $\Gamma_{\nu\mu}$ $(\subset E)$ is a finite set, $\hat{h}^{[\iota]}_{\nu\mu}$ are asymptotic polynomials and

$$c_\iota(x) = \tilde{R}_\nu(x) + \sum_{\substack{l=1\\ l\neq\nu}}^{n} \tilde{R}_l(a_{j_\iota(l)}).$$

We define $x_\nu(\lambda)$ as above and set

$$\hat{c}_\iota(\lambda) := \tilde{R}_\nu(x_\nu(\lambda)) + \sum_{\substack{l=1\\ l\neq\nu}}^{n} \tilde{R}_l(a_{j_\iota(l)}).$$

From

$$\mathrm{Re}(\lambda(c_\iota(x)-\hat{c}_\iota(\lambda))) = \mathrm{Re}(\lambda e^{i\tilde{\varphi}_\nu})(|\tilde{R}_\nu(x)|-|\tilde{R}_\nu(x_\nu(\lambda))|) \leq 0$$

we infer

$$\hat{h}_{\nu\mu}(x,\lambda) = \lambda^{(n-1)q}\left\{\sum_{c\in E} \hat{h}^{c}_{\nu\mu}(x,\lambda)\exp(\lambda c)\right\}$$

where $\hat{h}^{c}_{\nu\mu}(\cdot,\lambda) = \{O(1)\}_\infty$.

Let $f_2 \in \mathbb{C}^n$. From $\tilde{Y}(\cdot,\lambda)E(\cdot,\lambda)^{-1} = \{O(1)\}_\infty$ and the assumptions concerning $C(\cdot,\lambda)$ we infer that there are $\hat{k}^{c}_{\nu\mu}(\cdot,\lambda) = \{O(1)\}_\infty$ and $q_4 \in \mathbb{Z}$ such that

$$Y(x,\lambda)M^{-1}(\lambda)f_2 = \lambda^{q_4}(\det M(\lambda))^{-1}\left(\sum_{c\in E} \hat{k}^{c}_{\nu\mu}(x,\lambda)\exp(\lambda c)\right)_{\nu,\mu=1}^{n}.$$

This representation and (5.12) yield

$$\int_{|\lambda|=d_j} |\lambda^{-q_4+g-2}R_2(\lambda)f_2|_{L_2^n(a,b)}|d\lambda| \to 0 \quad (j\to\infty),$$

and the theorem is proved.

6. AN EXPANSION THEOREM

In this section we assume that $A(\cdot,\lambda) = A_0 + \lambda A_1$ where $A_0, A_1 \in M_n(W^{k,\infty}(a,b))$ for some $k \in \mathbb{N}$ and

$$A_1 = \mathrm{diag}(\tilde{r}_1,\ldots,\tilde{r}_n)$$

where $\tilde{r}_1 = \ldots = \tilde{r}_p = 0$ for some $0 \le p \le n-1$ and $\tilde{r}_{p+1}^{-1}, \ldots, \tilde{r}_n^{-1} \in W^{k,\infty}(a,b)$. We set $l := n-p$, $\Omega_l := \mathrm{diag}(\tilde{r}_{p+1}, \ldots, \tilde{r}_n)$, and write A_0 as a block matrix in $\mathbb{C}^p \times \mathbb{C}^l$

$$A_0 =: \begin{bmatrix} A_{11}^{[0]} & A_{12}^{[0]} \\ A_{21}^{[0]} & A_{22}^{[0]} \end{bmatrix} .$$

(6.1) PROPOSITION. *Let* T *be* $|\ |_{L_2^n(a,b)}$*-regular of order* $(p',p")$ *where* $p' \ge 0$ *and* $\int_{\Gamma_\nu} |\lambda|^{-2} |d\lambda| \to \infty$ $(\nu \to \infty)$.

Then T *is* $|\ |_{H_1^n(a,b)}$*-regular of order* $(p'+1, p"+1)$.

PROOF. Let $f_1 \in L_2^n(a,b)$ and $f_2 \in \mathbb{C}^n$. The assertion of the proposition immediately follows from

$$\begin{aligned} (R_1(\lambda)f_1)' &= T^D(\lambda)R_1(\lambda)f_1 + (A_0 + \lambda A_1)(R_1(\lambda)f_1) \\ &= f_1 + (A_0 + \lambda A_1)(R_1(\lambda)f_1) \end{aligned}$$

and

$$(R_2(\lambda)f_2)' = (A_0 + \lambda A_1)(R_2(\lambda)f_2).$$

(6.2) PROPOSITION. *Let* $f \in H_{k+1}^n(a,b)$, $f =: (y^{[0]}, z^{[0]}) \in H_{k+1}^p(a,b) \times H_{k+1}^l(a,b)$ *and assume that* $y^{[0]'} - A_{11}^{[0]}y^{[0]} - A_{12}^{[0]}z^{[0]} = 0$.

Then there are $f^{[j]} \in H_{k+1-j}^n(a,b)$ $(j = 1,\ldots,k+1)$ *such that*

(6.3) $$f^{[j]'} - A_0 f^{[j]} - A_1 f^{[j+1]} = 0 \quad (j = 0,\ldots,k).$$

PROOF. For $j = 1,\ldots,k$ we recursively choose a solution $y^{[j]} \in H_{k+1-j}^p(a,b)$ of the differential equation

$$y^{[j]'} - A_{11}^{[0]}y^{[j]} - A_{12}^{[0]}z^{[j]} = 0$$

and define

$$z^{[j+1]} := \Omega_l^{-1}\left\{z^{[j]'} - A_{21}^{[0]}y^{[j]} - A_{22}^{[0]}z^{[j]}\right\} \quad (j = 0,\ldots,k).$$

We set $y^{[k+1]} := 0$ and $f^{[j]} := (y^{[j]}, z^{[j]})$. The $f^{[j]}$ have the desired properties.

In addition, we assume that T^R is a polynomial, i.e.

$$T^R(\lambda) = \sum_{r=0}^{q} \lambda^r T_r^R$$

for some $q \in \mathbb{N}$.

(6.4) THEOREM. *Let* T *be* $|\ |$*-regular of order* (p',p'') *with respect to the curves* Γ_ν $(\nu \in \mathbb{N})$ *and assume that* $k \geq \max\{p',p''\}$. *Suppose that* $f \in H^n_{k+1}(a,b)$ *fulfills the assumptions of* (6.2) *and that*

$$\sum_{r=0}^{\min\{q,k-j+1\}} T_r^R f^{[j+r-1]} = 0 \qquad (j = 1,\dots,p'') \tag{6.5}$$

where the $f^{[j]} \in H^n_{k+1-j}(a,b)$ *are defined according to* (6.2). *Then*

$$f = \lim_{\nu\to\infty} \sum_{\mu\in\sigma(T)\cap \mathrm{int}\Gamma_\nu} \Big\{-(\mathrm{res}_\mu R_1)A_1 f$$

$$+ \mathrm{res}_{\lambda=\mu}\Big(R_2(\lambda) \sum_{j=0}^{q-1} \lambda^j \sum_{r=j+1}^{\min\{q,k+j+1\}} T_r^R f^{[r-j-1]}\Big)\Big\}$$

holds in $(H^n_1(a,b),|\ \ |)$.

PROOF. The relationship (4.3) and Proposition (6.2) lead to

$$R_1(\lambda)A_1 f^{[j]} = -\frac{1}{\lambda} f^{[j]} + \frac{1}{\lambda} R_1(\lambda)A_1 f^{[j+1]} + \frac{1}{\lambda} R_2(\lambda)T^R(\lambda)f^{[j]}$$

for $j = 0,\dots,k$. A recursive substitution for $j = 0,\dots,k$ yields

$$R_1(\lambda)A_1 f = -\sum_{j=0}^{k} \lambda^{-j-1} f^{[j]} + \lambda^{-k-1} R_1(\lambda)A_1 f^{[k+1]}$$

$$+ \sum_{j=0}^{k} \lambda^{-j-1} R_2(\lambda)T^R(\lambda)f^{[j]}.$$

We integrate along the curves Γ_ν and obtain

$$(6.6)\quad \begin{cases} f = -\displaystyle\sum_{\mu\in\sigma(T)\cap\mathrm{int}\Gamma_\nu} (\mathrm{res}_\mu R_1)A_1 f + \frac{1}{2\pi i}\int_{\Gamma_\nu} \sum_{j=0}^{k} \lambda^{-j-1} R_2(\lambda) * \\ \quad * T^R(\lambda) f^{[j]} d\lambda + \displaystyle\frac{1}{2\pi i}\int_{\Gamma_\nu} \lambda^{-k-1} R_1(\lambda)A_1 f^{[k+1]} d\lambda. \end{cases}$$

The calculation

$$\sum_{j=0}^{k} \lambda^{-j-1} T^{R}(\lambda) f^{[j]} = \sum_{j=0}^{k} \sum_{r=0}^{q} \lambda^{r-j-1} T_r^R f^{[j]}$$

$$= \sum_{r=0}^{q} \sum_{j-r+1=-r+1}^{k-r+1} \lambda^{r-j-1} T_r^R f^{[j]}$$

$$= \sum_{r=0}^{q} \sum_{j=-r+1}^{k-r+1} \lambda^{-j} T_r^R f^{[j+r-1]}$$

$$= \sum_{j=-q+1}^{k+1} \lambda^{-j} \sum_{r=\max\{0,-j+1\}}^{\min\{q,k-j+1\}} T_r^R f^{[j+r-1]}$$

$$= \sum_{j=-q+1}^{0} \lambda^{-j} \sum_{r=-j+1}^{\min\{q,k-j+1\}} T_r^R f^{[j+r-1]}$$

$$+ \sum_{j=1}^{k+1} \lambda^{-j} \sum_{r=0}^{\min\{q,k-j+1\}} T_r^R f^{[j+r-1]}$$

and (6.5) yield

$$f = \sum_{\mu\in\sigma(T)\cap\operatorname{int}\Gamma_\nu} \Big\{-(\operatorname{res}_\mu R_1)A_1 f + \operatorname{res}_{\lambda=\mu}\Big(R_2(\lambda)\sum_{j=0}^{q-1}\lambda^j \, *$$

$$* \sum_{r=j+1}^{\min\{q,k+j+1\}} T_r^R f^{[r-j-1]}\Big)\Big\} + \frac{1}{2\pi i}\int_{\Gamma_\nu} \sum_{j=p''+1}^{k+1} \lambda^{-j} R_2(\lambda) \, *$$

$$* \sum_{r=0}^{\min\{q,k-j+1\}} T_r^R f^{[j+r-1]} d\lambda + \frac{1}{2\pi i}\int_{\Gamma_\nu} \lambda^{-k-1} R_1(\lambda) A_1 f^{[k+1]} d\lambda .$$

Thus the regularity of T proves the theorem.

(6.7) REMARK. Let $\max\{q,q+p''\} \le k+1$. Then (6.5) reads

(6.5') $$\sum_{r=0}^{q} T_r^R f^{[j+r-1]} = 0 \qquad (j = 1,\dots,p'')$$

and

$$\cdot f = \lim_{\nu\to\infty} \sum_{\mu\in\sigma(T)\cap\operatorname{int}\Gamma_\nu} \Big\{-(\operatorname{res}_\mu R_1)A_1 f$$

$$+ \operatorname{res}_{\lambda=\mu}\Big(R_2(\lambda)\sum_{j=0}^{q-1}\lambda^j \sum_{r=j+1}^{q} T_r^R f^{[r-j-1]}\Big)\Big\}.$$

7. n-TH ORDER DIFFERENTIAL EQUATIONS AND FIRST ORDER SYSTEMS

Let $p_i \in H(\mathbb{C}, L_\infty(a,b))$ $(i = 0,\dots,n-1)$, $p_n = 1$. We define

(7.1) $$L^D(\lambda)\eta := \sum_{i=0}^{n} p_i(\cdot,\lambda)\eta^{(i)} \qquad (\eta \in H_n(a,b),\ \lambda \in \mathbb{C}).$$

Let $m \geq 2$, $a = a_1 < a_2 < \dots < a_m = b$, $W^{(j)} \in H(\mathbb{C}, M_n(\mathbb{C}))$ $(j = 1,\dots,m)$, $W \in H(\mathbb{C}, M_n(L_1(a,b)))$. We define

(7.2) $$L^R(\lambda)\eta := \sum_{j=1}^{m} W^{(j)}(\lambda)\begin{bmatrix}\eta(a_j)\\ \vdots\\ \eta^{(n-1)}(a_j)\end{bmatrix} + \int_a^b W(\xi,\lambda)\begin{bmatrix}\eta(\xi)\\ \vdots\\ \eta^{(n-1)}(\xi)\end{bmatrix}d\xi$$

$(\eta \in H_n(a,b),\ \lambda \in \mathbb{C})$. We set

(7.3) $$L(\lambda)\eta := (L^D(\lambda)\eta, L^R(\lambda)\eta) \qquad (\eta \in H_n(a,b),\ \lambda \in \mathbb{C})$$

and assert

(7.4) PROPOSITION. i) $L \in H(\mathbb{C}, \Phi(H_n(a,b), L_2(a,b)\times\mathbb{C}^n))$, ii) $\operatorname{ind} L(\lambda) = 0$ $(\lambda \in \mathbb{C})$.

The proof of (7.4) is similar to the proof of (2.4) and thus omitted.

We want to apply the results of the foregoing sections to the differential-boundary operator L. Therefore we set

(7.5) $$\hat{A}(x,\lambda) := \begin{bmatrix} 0 & 1 & & \\ 0 & \ddots & \ddots & 0 \\ & & 0 & 1 \\ -p_0(x,\lambda) & \dots & & -p_{n-1}(x,\lambda)\end{bmatrix}$$

and define

(7.6) $$\tilde{T}^D(\lambda)\tilde{y} := \tilde{y}' - \hat{A}(\cdot,\lambda)\tilde{y} \qquad (\tilde{y} \in H_1^n(a,b),\ \lambda \in \mathbb{C})$$

and

$$(7.7)\qquad \tilde{T}^R(\lambda)\tilde{y} := \sum_{j=1}^{m} W^{(j)}(\lambda)\tilde{y}(a_j) + \int_a^b W(\xi,\lambda)\tilde{y}(\xi)d\xi .$$

Proposition (2.4) holds for $\tilde{T} := (\tilde{T}^D, \tilde{T}^R)$ and the operator functions L and $\tilde{T}$ are "equivalent" in the following sense:

(7.8) THEOREM. A. *Let* $\eta \in H(\mathbb{C}, H_n(a,b))$ *be a root function of* L *of order* $\nu > 0$ *at* $\mu \in \mathbb{C}$. *Then*

$$\tilde{y} := \begin{bmatrix} \eta \\ \eta' \\ \cdot \\ \cdot \\ \cdot \\ \eta^{(n-1)} \end{bmatrix}$$

is a root function of $\tilde{T}$ *of order* ν *at* μ.

B. *Let* $\tilde{y} \in H(\mathbb{C}, H_1^n(a,b))$ *be a root function of* $\tilde{T}$ *of order* $\nu > 0$ *at* μ. *We set* $(\tilde{y}_1, \ldots, \tilde{y}_n)^t := \tilde{y}^t$ *and*

$$\eta(\cdot,\lambda) := \sum_{j=0}^{\nu-1} \frac{1}{j!}(\lambda-\mu)^j \left(\frac{d^j}{d\lambda^j}\tilde{y}_1\right)(\cdot,\mu).$$

Then $\eta \in H(\mathbb{C}, H_n(a,b))$ *is a root function of* L *of order* $\geq \nu$ *at* μ, *and* $\eta^{(i)} - \tilde{y}_{i+1} \in H(\mathbb{C}, H_1(a,b))$ $(i = 0, \ldots, n-1)$ *has a zero of order* $\geq \nu$ *at* μ.

PROOF. A. From the shape of the matrix $\hat{A}$ we infer

$$(7.9)\qquad \tilde{T}^D(\lambda)\tilde{y}(\lambda) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ L^D(\lambda)\eta(\lambda) \end{bmatrix}.$$

$\tilde{T}^R(\lambda)\tilde{y}(\lambda) = L^R(\lambda)\eta(\lambda)$ is clear from the definition of $\tilde{T}^R$ and L^R. This proves A.

B. We set

$$\eta_i(\cdot,\lambda) := \sum_{j=0}^{\nu-1} \frac{1}{j!}(\lambda-\mu)^j \left(\frac{d^j}{d\lambda^j}\tilde{y}_i\right)(\cdot,\mu) \qquad (i = 1, \ldots, n).$$

Note that $\eta = \eta_1$. The functions

$$(\widetilde{T}^D(\lambda)\widetilde{y}(\lambda))_i = \widetilde{y}_i'(\lambda) - \widetilde{y}_{i+1}(\lambda) \qquad (1 \le i \le n-1)$$

and

$$(\widetilde{T}^D(\lambda)\widetilde{y}(\lambda))_n = \widetilde{y}_n'(\lambda) + \sum_{i=0}^{n-1} p_i(\cdot,\lambda)\widetilde{y}_{i+1}(\lambda)$$

have a zero of order $\geq \nu$ at μ. Hence

$$\eta_i' = \eta_{i+1} \qquad (1 \le i \le n-1).$$

Thus we have $\eta^{(i)} = \eta_{i+1}$ $(1 \le i \le n-1)$ which proves that $\eta \in H_n(a,b)$. We conclude from the definition of η_{i+1} that $\eta^{(i)} - \widetilde{y}_{i+1} = \eta_{i+1} - \widetilde{y}_{i+1}$ $(0 \le i \le n-1)$ has a zero of order $\geq \nu$ at μ. This proves that

$$L^D(\lambda)\eta(\lambda) = \widetilde{y}_n'(\lambda) + \sum_{i=0}^{n-1} p_i(\cdot,\lambda)\widetilde{y}_{i+1}(\lambda)$$

$$+\left(\eta^{(n-1)}(\lambda) - \widetilde{y}_n(\lambda)\right)' + \sum_{i=0}^{n-1} p_i(\cdot,\lambda)\left(\eta^{(i)}(\lambda) - \widetilde{y}_{i+1}(\lambda)\right)$$

has a zero of order $\geq \nu$ at μ. In the same way we obtain that

$$L^R(\lambda)\eta(\lambda) = \widetilde{T}^R(\lambda)\widetilde{y}(\lambda) + \widetilde{T}^R(\lambda)\begin{bmatrix} \eta(\lambda)-\widetilde{y}_1(\lambda) \\ \vdots \\ \eta^{(n-1)}(\lambda)-\widetilde{y}_n(\lambda) \end{bmatrix}$$

has a zero of order $\geq \nu$ at μ. Finally $\eta(\mu) \neq 0$ because $\eta(\mu) = 0$ would imply $\widetilde{y}_i(\mu) = \eta^{(i-1)}(\mu) = 0$ $(i = 1,\ldots,n)$ which contradicts $\widetilde{y}(\mu) \neq 0$.

(7.10) COROLLARY. *Assume that in* (7.8)B $\widetilde{y}_1 \in H(\mathbb{C}, H_n(a,b))$. *Then* $\widetilde{y}_1$ *is a root function of* L *of order* $\geq \nu(\widetilde{y})$ *at* μ.

The result is obvious since $\eta - \widetilde{y}_1$ has a zero of order $\geq \nu(\widetilde{y})$ at μ.

$L^{-1}(\lambda)$ can be expressed in terms of $\widetilde{T}^{-1}(\lambda)$:

(7.11) THEOREM. *Let* $\lambda \in \rho(L)$, $f_1 \in L_2(a,b)$ *and* $f_2 \in \mathbb{C}^n$. *Then* $\lambda \in \rho(\widetilde{T})$ *and*

$$L^{-1}(\lambda)(f_1,f_2) = e_1^t\widetilde{T}^{-1}(\lambda)\begin{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_1 \end{bmatrix}, f_2\end{bmatrix}$$

where e_1 *is the first unit vector in* $\mathbb{C}^n$.

PROOF. Assume that $\lambda \notin \rho(\tilde{T})$. Then there is an eigenvector y of $\tilde{T}(\lambda)$ since, with (2.4), $\tilde{T}(\lambda)$ is a Fredholm operator with index zero. With (7.8)B, the first component y_1 of y would be an eigenvector of $L(\lambda)$. This contradicts $\lambda \in \rho(L)$. We set

$$\tilde{y} := \tilde{T}^{-1}(\lambda)\left[\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_1 \end{bmatrix}, f_2\right], \qquad \eta := \tilde{y}_1 .$$

We have to show $\eta \in H_n(a,b)$ and $L(\lambda)\eta = (f_1,f_2)$. From the definition of $\tilde{y}$ we obtain

$$\tilde{T}^D(\lambda)\tilde{y} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_1 \end{bmatrix}$$

Hence $\tilde{y}_i' = \tilde{y}_{i+1}$ $(i = 1,\dots,n-1)$, which proves $\eta \in H_n(a,b)$, and $L^D(\lambda)\eta = f_1$. The equality $L^R(\lambda)\eta = \tilde{T}^R(\lambda)\tilde{y} = f_2$ is also clear.

For $\lambda \in \rho(L)$ and $f_1 \in L_2(a,b)$, $f_2 \in \mathbb{C}^n$ we define

(7.12) $$\hat{R}_1(\lambda)f_1 := L^{-1}(\lambda)(f_1,0),$$

(7.13) $$\hat{R}_2(\lambda)f_2 := L^{-1}(\lambda)(0,f_2).$$

Let $Y(\cdot,\lambda)$ be a fundamental matrix of $\tilde{T}^D(\lambda)y = 0$ and define

(7.14) $$\hat{G}(x,\xi,\lambda) := \begin{cases} \displaystyle\int_{t=a}^{\xi} e_1^t Y(x,\lambda)M^{-1}(\lambda)d_tF(t,\lambda)Y(t,\lambda)Y^{-1}(\xi,\lambda)e_n & (a \le \xi \le x \le b) \\ \displaystyle -\int_{t=\xi}^{b} e_1^t Y(x,\lambda)M^{-1}(\lambda)d_tF(t,\lambda)Y(t,\lambda)Y^{-1}(\xi,\lambda)e_n & (a \le x < \xi \le b), \end{cases}$$

(7.15) $$\hat{G}_R(x,\lambda) := e_1^t Y(x,\lambda)M^{-1}(\lambda).$$

$\hat{G}$ is called the Green's function of L. From (4.9) and (7.11) we obtain

(7.16) COROLLARY. *For* $f_1 \in L_2(a,b)$, $f_2 \in \mathbb{C}^n$ *and*

$\lambda \in \rho(L)$ *we have*

i) $(\hat{R}_1(\lambda)f_1)(x) = \int_a^b \hat{G}(x,\xi,\lambda)f_1(\xi)d\xi,$

ii) $(\hat{R}_2(\lambda)f_2)(x) = \hat{G}_R(x,\lambda)f_2,$

iii) $(L^{-1}(\lambda)(f_1,f_2))(x) = \int_a^b \hat{G}(x,\xi,\lambda)f_1(\xi)d\xi + \hat{G}_R(x,\lambda)f_2.$

Now we shall construct a biorthogonal canonical system of root functions of L and L^*. For this, let $z_1(\cdot,\lambda),\dots,$ $z_n(\cdot,\lambda)$ be a set of fundamental solutions of $L^D(\lambda)$. This means that

(7.17) $$Y(\cdot,\lambda) := \left(z_j^{(i-1)}(\cdot,\lambda)\right)_{i,j=1}^n$$

is a fundamental matrix of $\tilde{T}^D(\lambda)$. We may suppose that $Y \in H(\mathbb{C},M_n(W^{1,\infty}(a,b)))$, cf. [28]. The matrix

$$M(\lambda) = \left(L^R(\lambda)z_j(\cdot,\lambda)\right)_{j=1}^n = \tilde{T}^R(\lambda)Y(\cdot,\lambda)$$

is called the characteristic matrix of $L(\lambda)$.

(7.18) THEOREM. *Let* $\mu \in \sigma(L)$. *Let* $\{c_1,\dots,c_r\}$ *be a* CSRF *of* M *at* μ *and* $\{d_1,\dots,d_r\}$ *be a* CSRF *of* M^* *at* μ. *Suppose that the biorthogonal relationships*

$$\frac{1}{l!}\frac{d^l}{d\lambda^l} \langle (\cdot-\mu)^{-m_i}Mc_i, d_j\rangle(\mu) = \delta_{ij}\,\delta_{0l}$$

$(0 \le l \le m_j-1;\ i,j = 1,\dots,r)$ *hold where* m_i *is the multiplicity of* c_i. *We define*

$$\eta_i(\lambda) := e_1^t Y(\cdot,\lambda)c_i(\lambda) \qquad (i = 1,\dots,r;\ \lambda \in \mathbb{C})$$

and

$$\xi_i(\lambda)(x) := e_n^t Y^t(x,\lambda)^{-1}\Bigl(\sum_{j=1}^{m-1} Y^t(a_j,\lambda)W^{(j)^t}(\lambda)\chi_{(a_j,b)}(x) + \int_a^x Y^t(\xi,\lambda)W^t(\xi,\lambda)d\xi\Bigr)d_i(\lambda).$$

We set

$$\zeta_i := (\xi_i, d_i) \qquad (i = 1,\dots,r).$$

Then $\{\eta_1,\dots,\eta_r\}$ *is a* CSRF *of* L *at* μ, $\{\zeta_1,\dots,\zeta_r\}$ *is a* CSRF *of* L* *at* μ, $\nu(\eta_j) = \nu(\zeta_j) = m_j$ $(j = 1,\dots,r)$, *the biorthogonal relationships*

$$(7.19) \qquad \frac{1}{l!}\frac{d^l}{d\lambda^l} \langle (\cdot - \mu)^{-m_i} L\eta_i, \zeta_j \rangle(\mu) = \delta_{ij}\,\delta_{0l}$$

$(0 \le l \le m_j - 1\ ;\ i,j = 1,\dots,r)$ *hold, and*

$$L^{-1} - \sum_{j=1}^{r} (\cdot - \mu)^{-m_j} \eta_j \otimes \zeta_j$$

is holomorphic at μ.

PROOF. According to Theorem (2.10) we define the root functions $\tilde{y}_i$ of $\tilde{T}$ at μ and $\tilde{v}_i = (\tilde{u}_i, d_i)$ of $\tilde{T}^*$ at μ $(i = 1,\dots,r)$. We have $\eta_i = e_1^t \tilde{y}_i$ and $\zeta_i = (e_n^t \tilde{u}_i, d_i)$ $(i = 1,\dots,r)$. With (7.17) and (7.10) we obtain that $\eta_1,\dots,\eta_r \in H(\mathbb{C}, H_n(a,b))$ are root functions of L at μ with $\nu(\eta_i) \ge \nu(\tilde{y}_i) = m_i$ $(i = 1, \dots, r)$. Let $\eta \in H_n(a,b)$ and set $y := (\eta, \eta', \dots, \eta^{(n-1)})^t$. With the aid of (7.9) we infer

$$\begin{aligned}
&\langle \eta, L^*(\lambda)\zeta_i(\lambda) \rangle_{(H_n(a,b),(H_n(a,b))')} \\
&= \langle L^D(\lambda)\eta, e_n^t \tilde{u}_i(\lambda) \rangle_{(L_2(a,b),L_2(a,b))} + \langle L^R(\lambda)\eta, d_i(\lambda) \rangle_{(\mathbb{C}^n,\mathbb{C}^n)} \\
&= \langle \tilde{T}^D(\lambda) y, \tilde{u}_i(\lambda) \rangle_{(L_2^n(a,b),L_2^n(a,b))} + \langle \tilde{T}^R(\lambda) y, d_i(\lambda) \rangle_{(\mathbb{C}^n,\mathbb{C}^n)} \\
&= \langle y, \tilde{T}^*(\lambda)\tilde{v}_i(\lambda) \rangle_{(H_1^n(a,b),(H_1^n(a,b))')} .
\end{aligned}$$

Hence

$$\langle \eta, \frac{d^l}{d\lambda^l}(L^*\zeta_i)(\mu) \rangle = 0 \qquad (l = 0,\dots,m_i - 1).$$

This proves

$$\frac{d^l}{d\lambda^l}(L^*\zeta_i)(\mu) = 0 \qquad (l = 0,\dots,m_i - 1)$$

since $\eta \in H_n(a,b)$ is arbitrary. Thus ζ_i $(i = 1,\dots,r)$ is a root function of L* of order $\ge m_i$ at μ. By (7.17) and the definition of $\tilde{y}_i$ we have $\tilde{y}_i = (\eta_i, \eta_i', \ldots, \eta_i^{(n-1)})^t$. Thus the above calculation leads to

$$\langle L(\lambda)\eta_i(\lambda), \zeta_j(\lambda)\rangle = \langle \tilde{T}(\lambda)\tilde{y}_i(\lambda), \tilde{v}_j(\lambda)\rangle \quad (i,j=1,\ldots,r).$$

This and (2.12) gives the biorthogonal relationships

$$\frac{1}{l!}\frac{d^l}{d\lambda^l}\langle(\cdot-\mu)^{-m_i}L\eta_i, \zeta_j\rangle(\mu) = \delta_{ij}\,\delta_{0l}$$

$(0 \le l \le m_j-1;\ i,j=1,\ldots,r)$. The theorem is proved on the basis of Theorem (2.10) in [28].

We conclude this section with a technical result which we will need in the next section. For a distribution $u \in \mathcal{D}'(\mathbb{R})$ let $u_r \in \mathcal{D}'(a,b)$ be its restriction to (a,b), i.e. $\langle\varphi, u_r\rangle := \langle\varphi, u\rangle$ $(\varphi \in C_0^\infty(a,b))$. Obviously $u_r' = (u')_r$.

(7.20) PROPOSITION. *Let* $k,l \ge 0$, $p_i \in W^{i+k,\infty}(a,b)$ $(0 \le i \le l)$, $p_l = 1$ *and* $\zeta \in L_2(a,b)$. *Assume that*

$$\sum_{i=0}^{l}(p_i\zeta)_r^{(i)} \in H_k(a,b).$$

Then $\zeta \in H_{k+1}(a,b)$.

PROOF. For $l=0$ nothing is to be proved. Let $l \ge 1$ and assume that the statement holds for $l-1$. Assume that $\zeta \notin H_{k+1}(a,b)$. Then there is a $0 \le j < k+1$ such that $\zeta \in H_j(a,b) \setminus H_{j+1}(a,b)$. Let $j' := \min\{j,k\}$. Then

$$\left(\sum_{i=0}^{l-1}(p_{i+1}\zeta)_r^{(i)}\right)' = \sum_{i=0}^{l}(p_i\zeta)_r^{(i)} - (p_0\zeta)_r \in H_{j'}(a,b)$$

whence

$$\sum_{i=0}^{l-1}(p_{i+1}\zeta)_r^{(i)} \in H_{j'+1}(a,b).$$

By assumption we have $\zeta \in H_{j'+1+l-1}(a,b)$. If $j'=j$, we obtain the contradiction $\zeta \in H_{j+1}(a,b)$. Hence $j'=k$ and we obtain the contradiction $\zeta \in H_{k+1}(a,b)$.

A similar proof leads to

(7.21) COROLLARY. *Let* $k,l \ge 0$, $p_i \in C^{i+k}[a,b]$ $(0 \le i \le l)$, $p_l = 1$ *and* $\zeta \in L_2(a,b)$. *Assume that*

$$\sum_{i=0}^{l} (p_i\zeta)_r^{(i)} \in C^k[a,b].$$

Then $\zeta \in C^{k+1}[a,b]$.

Note that we have $\zeta \in H_{k+1}(a,b) \subset C^0[a,b]$ for $l > 0$ by (7.20).

8. THE ADJOINT L^* AND THE CLASSICAL ADJOINT L^+

By [25], Ch. 1, Th. 2.2, there is a continuous linear map $\tau : H_n(a,b) \to H_n(\mathbb{R})$ such that $\tau f|_{(a,b)} = f$ for all $f \in H_n(a,b)$. The Sobolev space

$$H_{-n}(\mathbb{R}) := \left\{ \sum_{i=0}^{n} (-1)^i v_i^{(i)} : v_i \in L_2(\mathbb{R}) \right\}$$

is a representation of the dual space of $H_n(\mathbb{R})$ with respect to the bilinear form

$$\langle g,v\rangle_{(H_n(\mathbb{R}),H_{-n}(\mathbb{R}))} := \sum_{i=0}^{n} \langle g^{(i)},v_i\rangle_{(L_2(\mathbb{R}),L_2(\mathbb{R}))}$$

where

$$\langle f,h\rangle_{(L_2(\mathbb{R}),L_2(\mathbb{R}))} := \int_{\mathbb{R}} f(x)h(x)dx ,$$

cf. e.g. [1], Th. 3.10 and Cor. 3.19. We set

$$H_{-n}[a,b] := \{v \in H_{-n}(\mathbb{R}) : \operatorname{supp} v \subset [a,b]\}.$$

(8.1) PROPOSITION. $H_{-n}[a,b]$ *is a representation of the dual space of* $H_n(a,b)$ *with respect to the bilinear form*

$$\langle y , v\rangle_{(H_n(a,b),H_{-n}[a,b])} := \langle \tau y,v\rangle_{(H_n(\mathbb{R}),H_{-n}(\mathbb{R}))}.$$

For $n = 1$ this proposition is Proposition (4.2) in [28]. The proof of the general case is analogous and therefore omitted here.

We consider $L^D(\lambda)$ which has been defined in section 7. Let $\eta \in H_n(a,b)$ and $\zeta \in L_2(a,b)$. We obtain

$$\langle \eta,L^{D*}(\lambda)\zeta\rangle_{(H_n(a,b),H_{-n}[a,b])}$$

$$= \langle L^D(\lambda)\eta,\zeta\rangle_{(L_2(a,b),L_2(a,b))}$$

$$= \sum_{i=0}^{n} \langle p_i(\cdot,\lambda)\eta^{(i)},\zeta\rangle_{(L_2(a,b),L_2(a,b))}$$

$$= \sum_{i=0}^{n} \langle(\tau\eta)^{(i)},p_i(\cdot,\lambda)\zeta\rangle_{(L_2(\mathbb{R}),L_2(\mathbb{R}))}$$

$$= \sum_{i=0}^{n} (-1)^i\langle\tau\eta,(p_i(\cdot,\lambda)\zeta)^{(i)}\rangle_{(H_n(\mathbb{R}),H_{-n}(\mathbb{R}))}$$

$$= \sum_{i=0}^{n} (-1)^i\langle\eta,(p_i(\cdot,\lambda)\zeta)^{(i)}\rangle_{(H_n(a,b),H_{-n}[a,b])}.$$

This proves

(8.2) $$L^{D*}(\lambda)\zeta = \sum_{i=0}^{n}(-1)^i(p_i(\cdot,\lambda)\zeta)^{(i)} \qquad (\zeta\in L_2(a,b))$$

where the derivative is taken in $\mathcal{D}'(\mathbb{R})$, the space of distributions on $\mathbb{R}$. For $\eta\in H_n(a,b)$ and $d\in\mathbb{C}^n$ we infer

$$\langle\eta,L^{R*}(\lambda)d\rangle_{(H_n(a,b),H_{-n}[a,b])} = \langle L^R(\lambda)\eta,d\rangle_{(\mathbb{C}^n,\mathbb{C}^n)}$$

$$= \left\langle \sum_{j=1}^{m} W^{(j)}(\lambda)\begin{bmatrix}\eta(a_j)\\ \vdots\\ \eta^{(n-1)}(a_j)\end{bmatrix} + \int_a^b W(\xi,\lambda)\begin{bmatrix}\eta(\xi)\\ \vdots\\ \eta^{(n-1)}(\xi)\end{bmatrix}d\xi, d\right\rangle_{(\mathbb{C}^n,\mathbb{C}^n)}$$

$$= \sum_{k=0}^{n-1}\sum_{j=1}^{m}\eta^{(k)}(a_j)e_{k+1}^tW^{(j)}(\lambda)^td + \sum_{k=0}^{n-1}\int_a^b\eta^{(k)}(\xi)e_{k+1}^tW(\xi,\lambda)^td\,d\xi$$

$$= \sum_{k=0}^{n-1}\sum_{j=1}^{m}\langle\eta,(-1)^ke_{k+1}^tW^{(j)}(\lambda)^td\delta_{a_j}^{(k)}\rangle_{(H_n(a,b),H_{-n}[a,b])}$$

$$+ \sum_{k=0}^{n-1}\langle\eta,(-1)^ke_{k+1}^t(W(\cdot,\lambda)^t)^{(k)}d\rangle_{(H_n(a,b),H_{-n}[a,b])}$$

where e_i is the i-th unit vector in $\mathbb{C}^n$ and $^{(k)}$ denotes the k-th derivative in the sense of distributions. Hence, for $d\in\mathbb{C}^n$,

(8.3) $$L^{R*}(\lambda)d = \sum_{k=0}^{n-1}(-1)^ke_{k+1}^t\left(\sum_{j=1}^{m}W^{(j)}(\lambda)^td\delta_{a_j}^{(k)} + (W(\cdot,\lambda)^t)^{(k)}d\right).$$

Next we consider a two-point boundary value problem

$$(8.4)\qquad \begin{cases} L^D(\lambda) = \sum_{i=0}^{n} p_i(\cdot,\lambda)\eta^{(i)} \\ L^R(\lambda)\eta = W^a(\lambda)\begin{bmatrix}\eta(a)\\ \vdots \\ \eta^{(n-1)}(a)\end{bmatrix} + W^b(\lambda)\begin{bmatrix}\eta(b)\\ \vdots \\ \eta^{(n-1)}(b)\end{bmatrix}\end{cases}$$

$(\eta \in H_n(a,b))$ where $\operatorname{rank}(W^a(\lambda),W^b(\lambda)) = n$ for all $\lambda \in \mathbb{C}$. We intend to prove that there is an invertible matrix $Q(\lambda) \in M_{2n}(\mathbb{C})$ which depends holomorphically on λ such that

$$(8.5)\qquad Q(\lambda) = \begin{bmatrix} W^a(\lambda)^t & \tilde{A}(\lambda) \\ W^b(\lambda)^t & \tilde{B}(\lambda)\end{bmatrix}$$

with suitable $\tilde{A},\tilde{B} \in H(\mathbb{C},M_n(\mathbb{C}))$.

(8.6) LEMMA. *Let* $k > l \geq 1$, $a_j \in H(\mathbb{C},\mathbb{C}^k)$ $(j = 1,\dots,l)$ *and assume that* $a_1(\lambda),\dots,a_l(\lambda)$ *are linearly independent for all* $\lambda \in \mathbb{C}$.

Then there are $a_j \in H(\mathbb{C},\mathbb{C}^k)$ $(j = l+1,\dots,k)$ *such that* $a_1(\lambda),\dots,a_k(\lambda)$ *are linearly independent for all* $\lambda \in \mathbb{C}$.

PROOF. Set

$$A := (a_1,\dots,a_l).$$

Then $A \in H(\mathbb{C},M_{k,l}(\mathbb{C}))$. With respect to the decomposition $\mathbb{C}^k = \mathbb{C}^l \oplus \mathbb{C}^{k-l}$ we write $A^t =: (A_0,A_1)$. Since $\operatorname{rank} A(\lambda) = l$ for all $\lambda \in \mathbb{C}$, we may assume without loss of generality that $\operatorname{rank} A_0(\lambda_0) = l$ for some $\lambda_0 \in \mathbb{C}$. Thus the function $\det A_0$ is not identically zero. Let $B_0(\lambda)$ be the transpose of the matrix of the cofactors of $A_0(\lambda)$. Then B_0 is holomorphic and fulfills

$$A_0B_0 = (\det A_0)\, I_l.$$

Let $c \in \{0\} \times (\mathbb{C}^{k-l} \setminus \{0\})$ and define $z \in H(\mathbb{C},\mathbb{C}^k)$ by

$$z(\lambda) := \begin{bmatrix} B_0(\lambda) & -B_0(\lambda)A_1(\lambda) \\ 0 & (\det A_0(\lambda))I_{k-l}\end{bmatrix} c \qquad (\lambda \in \mathbb{C}).$$

Then $z \neq 0$ since the matrix

$$\begin{bmatrix} B_0(\lambda) & -B_0(\lambda)A_1(\lambda) \\ 0 & (\det A_0(\lambda))I_{k-1} \end{bmatrix}$$

is invertible iff $\det A_0(\lambda) \neq 0$. For all $\lambda \in \mathbb{C}$ we have

$$A^t(\lambda)z(\lambda) = (A_0(\lambda), A_1(\lambda)) \begin{bmatrix} B_0(\lambda) & -B_0(\lambda)A_1(\lambda) \\ 0 & (\det A_0(\lambda))I_{k-1} \end{bmatrix} c$$

$$= ((\det A_0(\lambda))I_1, 0)c = 0.$$

By Weierstrass' theorem there is a holomorphic function $\gamma : \mathbb{C} \to \mathbb{C}$ such that the set of the zeros and their multiplicities coincide for γ and z. Hence

$$a_{l+1} := \frac{z}{\gamma}$$

is a holomorphic function. From $a_{l+1}(\lambda) \neq 0$ and

$$a_{l+1}(\lambda) \in N(A^t(\lambda)) = (R(A(\lambda)))^{\perp}$$

we see that $a_1(\lambda), \ldots, a_{l+1}(\lambda)$ are linearly independent for all $\lambda \in \mathbb{C}$. The statement of the lemma follows by induction.

The idea of the above proof can be found in Wedderburn's article [36] on page 331. The authors thank H. Bart who drew our attention to Wedderburn's article. H. Bart also told us that the lemma follows from a more general result of Bart [2], Theorem 2.1 on holomorphic left inverses and a result of Saphar [31], Proposition 14 on global holomorphic bases.

From now on we additionally assume that the coefficients p_i in (8.4) belong to $H(\mathbb{C}, W^{i,\infty}(a,b))$ $(i = 0, \ldots, n-1)$. For a fixed $\lambda \in \mathbb{C}$ we define the linear operators $l(\lambda)$ and $l^+(\lambda)$ in $L_2(a,b)$ by

$$D(l(\lambda)) := H_n(a,b)$$

$$l(\lambda)\eta := \sum_{i=0}^{n} p_i(\cdot,\lambda)\eta^{(i)} \qquad (\eta \in H_n(a,b)),$$

$$D(l^+(\lambda)) := H_n(a,b),$$

$$l^{+}(\lambda)\zeta := \sum_{i=0}^{n} (-1)^{i}(p_i(\cdot,\lambda)\zeta)^{(i)} \qquad (\zeta \in H_n(a,b)).$$

Integrations by parts yield that there is a matrix $H(\lambda) \in M_{2n}(\mathbb{C})$ such that for all $\eta,\zeta \in H_n(a,b)$

(8.7) $$\hat{\eta}^{t} H(\lambda)\hat{\zeta} =$$

$$<l(\lambda)\eta,\zeta>_{(L_2(a,b),L_2(a,b))} - <\eta,l^{+}(\lambda)\zeta>_{(L_2(a,b),L_2(a,b))}$$

where

$$\hat{\eta}^{t} = (\eta(a),\ldots,\eta^{(n-1)}(a),\eta(b),\ldots,\eta^{(n-1)}(b)),$$
$$\hat{\zeta}^{t} = (\zeta(a),\ldots,\zeta^{(n-1)}(a),\zeta(b),\ldots,\zeta^{(n-1)}(b)).$$

The matrix $H(\lambda)$ is invertible (see e.g [29], p. 9 or [7], p. 288) and obviously depends holomorphically on λ. By Lemma (8.6) there is a matrix function of the form (8.5) which is invertible for all $\lambda \in \mathbb{C}$. We define

(8.8) $$\begin{bmatrix} \tilde{C}(\lambda) & \tilde{D}(\lambda) \\ \tilde{W}^{a}(\lambda) & \tilde{W}^{b}(\lambda) \end{bmatrix} := \begin{bmatrix} W^{a}(\lambda)^{t} & \tilde{A}(\lambda) \\ W^{b}(\lambda)^{t} & \tilde{B}(\lambda) \end{bmatrix}^{-1} H(\lambda) ,$$

where the matrix on the left side is divided into $n\times n$-block-matrices.

We define the operator $L_0(\lambda)$ in $L_2(a,b)$ by

(8.9) $$D(L_0(\lambda)) := \{\eta \in H_n(a,b) : (W^{a}(\lambda),W^{b}(\lambda))\hat{\eta} = 0\} \subset L_2(a,b)$$

and

(8.10) $$L_0(\lambda)\eta := l(\lambda)\eta \qquad (\eta \in D(L_0(\lambda))),$$

and, analogously, the operator $L_0^{+}(\lambda)$ in $L_2(a,b)$ by

(8.11) $$D(L_0^{+}(\lambda)) := \{\zeta \in H_n(a,b) : (\tilde{W}^{a}(\lambda),\tilde{W}^{b}(\lambda))\hat{\zeta} = 0\} \subset L_2(a,b)$$

and

(8.12) $$L_0^{+}(\lambda)\zeta := l^{+}(\lambda)\zeta \qquad (\zeta \in D(L_0^{+}(\lambda))).$$

Since $C_0^{\infty}(a,b) \subset D(L_0(\lambda))$ is dense in $L_2(a,b)$, the adjoint $L_0(\lambda)^*$ of $L_0(\lambda)$ is a well-defined operator on $D(L_0(\lambda)^*) \subset L_2(a,b)$ to $L_2(a,b)$.

(8.13) PROPOSITION. *For* $\zeta \in L_2(a,b)$ *we have* $\zeta \in D(L_0(\lambda)^*)$ *iff there is a* $d \in \mathbb{C}^n$ *such that* $L^*(\lambda)(\zeta,d) \in L_2(a,b)$.

For such ζ, d *we have* $L_0(\lambda)^*\zeta = L^*(\lambda)(\zeta,d)$.

PROOF. Let $\eta \in D(L_0(\lambda))$ and $\zeta \in L_2(a,b)$, $d \in \mathbb{C}^n$ such that $L^*(\lambda)(\zeta,d) \in L_2(a,b)$. By definition of $D(L_0(\lambda))$ we have $L^R(\lambda)\eta = 0$. Thus we obtain

$$<L_0(\lambda)\eta,\zeta>_{(L_2(a,b),L_2(a,b))} = <l(\lambda)\eta,\zeta>_{(L_2(a,b),L_2(a,b))}$$

$$= <L^D(\lambda)\eta,\zeta>_{(L_2(a,b),L_2(a,b))} + <L^R(\lambda)\eta,d>_{(\mathbb{C}^n,\mathbb{C}^n)}$$

$$= <L(\lambda)\eta,(\zeta,d)>_{(L_2(a,b)\times\mathbb{C}^n,L_2(a,b)\times\mathbb{C}^n)}$$

$$= <\eta,L^*(\lambda)(\zeta,d)>_{(H_n(a,b),H_{-n}[a,b])}$$

$$= <\eta,L^*(\lambda)(\zeta,d)>_{(L_2(a,b),L_2(a,b))}.$$

This proves $\zeta \in D(L_0(\lambda)^*)$ and $L_0(\lambda)^*\zeta = L^*(\lambda)(\zeta,d)$.

Conversely, let $\zeta \in D(L_0(\lambda)^*)$. Then there is a $\zeta^* \in L_2(a,b)$ such that

$$<L_0(\lambda)\eta,\zeta> = <\eta,\zeta^*>$$

for all $\eta \in D(L_0(\lambda))$. Hence, for all $\eta \in D(L_0(\lambda))$

$$0 = <\eta,\zeta^*>_{(L_2(a,b),L_2(a,b))} - <L^D(\lambda)\eta,\zeta>_{(L_2(a,b),L_2(a,b))}$$

$$= <\eta,\zeta^* - L^{D*}(\lambda)\zeta>_{(H_n(a,b),H_{-n}[a,b])}.$$

Since $L^R(\lambda)$ is a finite dimensional operator, the range $R(L^{R*}(\lambda))$ is finite dimensional and thus closed. This implies

$$\zeta^*-L^{D*}(\lambda)\zeta \in D(L_0(\lambda))^{\perp(H_n(a,b),H_{-n}[a,b])}$$

$$= N(L^R(\lambda))^{\perp(H_n(a,b),H_{-n}[a,b])} = R(L^{R*}(\lambda))$$

cf. [18], p. 234. Thus there is a $d \in \mathbb{C}^n$ such that $\zeta^* - L^{D*}(\lambda)\zeta = L^{R*}(\lambda)d$ which proves $\zeta^* = L^{D*}(\lambda)\zeta + L^{R*}(\lambda)d = L^*(\lambda)(\zeta,d)$.

(8.14) REMARK. The element $d \in \mathbb{C}^n$ in (8.13) is unique.

PROOF. Let $d_1 \in \mathbb{C}^n$ such that $\zeta^* = L^*(\lambda)(\zeta,d) = L^*(\lambda)(\zeta,d_1)$. Then

$$L^{R*}(\lambda)d_1 = \zeta^* - L^{D*}(\lambda)\zeta = L^{R*}(\lambda)d.$$

Hence, by (8.3),

$$0 = \sum_{k=0}^{n-1} (-1)^k e_{k+1}^t \Big(W^a(\lambda)^t (d_1 - d)\delta_a^{(k)} + W^b(\lambda)^t (d_1 - d)\delta_b^{(k)} \Big).$$

The elements $\delta_a, \delta_b, \ldots, (-1)^{n-1}\delta_a^{(n-1)}, (-1)^{n-1}\delta_b^{(n-1)} \in H_{-n}[a,b]$ are linearly independent, whence

$$\begin{bmatrix} W^a(\lambda)^t \\ W^b(\lambda)^t \end{bmatrix} (d_1 - d) = 0.$$

From $\operatorname{rank}(W^a(\lambda), W^b(\lambda)) = n$ we infer $d_1 = d$.

(8.15) THEOREM. i) $L_0^+(\lambda) = L_0^*(\lambda)$,

ii) $(L_0^+(\lambda))^* = L_0(\lambda)$.

PROOF. i) Let $\eta \in D(L_0(\lambda))$ and $\zeta \in D(L_0^+(\lambda))$. We infer

$$<L_0(\lambda)\eta, \zeta> = <l(\lambda)\eta, \zeta> = <\eta, l^+(\lambda)\zeta> + \hat{\eta}^t H(\lambda)\hat{\zeta}.$$

By the definitions (8.7), (8.8), (8.9) and (8.11) we have

$$(8.16) \qquad \hat{\eta}^t H(\lambda)\hat{\zeta} = \hat{\eta}^t \begin{bmatrix} W^a(\lambda)^t & \tilde{A}(\lambda) \\ W^b(\lambda)^t & \tilde{B}(\lambda) \end{bmatrix} \begin{bmatrix} \tilde{C}(\lambda) & \tilde{D}(\lambda) \\ \tilde{W}^a(\lambda) & \tilde{W}^b(\lambda) \end{bmatrix} \hat{\zeta} = (0,*) \begin{bmatrix} * \\ 0 \end{bmatrix} = 0.$$

Thus

$$<L_0(\lambda)\eta, \zeta> = <\eta, l^+(\lambda)\zeta>$$

which proves $\zeta \in D(L_0(\lambda)^*)$ and $L_0(\lambda)^*\zeta = l^+(\lambda)\zeta = L^+(\lambda)\zeta$.

Conversely, let $\zeta \in D(L_0(\lambda)^*)$. Then we have to prove $\zeta \in D(L_0^+(\lambda))$. By (8.3) and the special choice of the boundary conditions (8.4) we conclude that $(L^{R*}(\lambda)d)_r = 0$ for all $d \in \mathbb{C}^n$. Hence $(L^{D*}(\lambda)\zeta)_r \in L_2(a,b)$ by (8.13). From (8.2) and (7.20) we infer $\zeta \in H_n(a,b)$. For $\eta \in H_n(a,b)$ we obtain

$$(8.17) \qquad \begin{cases} <\eta, L^{D*}(\lambda)\zeta>_{(H_n(a,b), H_{-n}[a,b])} = <l(\lambda)\eta, \zeta>_{(L_2(a,b), L_2(a,b))} \\ = <\eta, l^+(\lambda)\zeta>_{(L_2(a,b), L_2(a,b))} + \hat{\eta}^t H(\lambda)\hat{\zeta}. \end{cases}$$

The definition of $\hat{\eta}$ and (8.8) give

$$(8.18) \qquad \hat{\eta}^t H(\lambda)\hat{\zeta} = <\eta, \Big(\sum_{k=0}^{n-1} (-1)^k e_{k+1}^t (W^a(\lambda)^t \delta_a^{(k)} + W^b(\lambda)^t \delta_b^{(k)}),$$

$$\sum_{k=0}^{n-1}(-1)^k e_{k+1}^t\left(\tilde{A}(\lambda)\delta_a^{(k)}+\tilde{B}(\lambda)\delta_b^{(k)}\right)\Bigg)\begin{bmatrix}\tilde{C}(\lambda) & \tilde{D}(\lambda)\\ \tilde{W}^a(\lambda) & \tilde{W}^b(\lambda)\end{bmatrix}\hat{\zeta}>.$$

Let $d \in \mathbb{C}^n$. By (8.3), (8.4), (8.17) and (8.18) we infer that

$$(8.19)\quad \begin{cases} L^*(\lambda)(\zeta,d) - l^+(\lambda)\zeta = \sum\limits_{k=0}^{n-1}(-1)^k e_{k+1}^t\, * \\ *\Big\{\Big(W^a(\lambda)^t(\tilde{C}(\lambda),\tilde{D}(\lambda))\hat{\zeta}+\tilde{A}(\lambda)(\tilde{W}^a(\lambda),\tilde{W}^b(\lambda))\hat{\zeta}+W^a(\lambda)^t d\Big)\delta_a^{(k)} \\ +\Big(W^b(\lambda)^t(\tilde{C}(\lambda),\tilde{D}(\lambda))\hat{\zeta}+\tilde{B}(\lambda)(\tilde{W}^a(\lambda),\tilde{W}^b(\lambda))\hat{\zeta}+W^b(\lambda)^t d\Big)\delta_b^{(k)}\Big\}. \end{cases}$$

Now let us choose d according to (8.13). Then $L^*(\lambda)(\zeta,d) - l^+(\lambda)\zeta \in L_2(a,b)$ and thus also the right side of equation (8.19) belongs to $L_2(a,b)$. The elements $\delta_a,\delta_b,\ldots,(-1)^{n-1}\delta_a^{(n-1)},\ldots,(-1)^{n-1}\delta_b^{(n-1)} \in H_{-n}[a,b]$ are linearly independent and

$$\operatorname{span}\{\delta_a,\delta_b,\ldots,(-1)^{n-1}\delta_a^{(n-1)},(-1)^{n-1}\delta_b^{(n-1)}\}\cap L_2(a,b)=\{0\}.$$

Therefore

$$\begin{bmatrix}W^a(\lambda)^t & \tilde{A}(\lambda)\\ W^b(\lambda)^t & \tilde{B}(\lambda)\end{bmatrix}\begin{bmatrix}\tilde{C}(\lambda) & \tilde{D}(\lambda)\\ \tilde{W}^a(\lambda) & \tilde{W}^b(\lambda)\end{bmatrix}\hat{\zeta}+\begin{bmatrix}W^a(\lambda)^t\\ W^b(\lambda)^t\end{bmatrix}d = 0.$$

From

$$(\tilde{W}^a(\lambda),\tilde{W}^b(\lambda)) = (0,I_n)\begin{bmatrix}\tilde{C}(\lambda) & \tilde{D}(\lambda)\\ \tilde{W}^a(\lambda) & \tilde{W}^b(\lambda)\end{bmatrix},$$

$$\begin{bmatrix}W^a(\lambda)^t\\ W^b(\lambda)^t\end{bmatrix} = \begin{bmatrix}W^a(\lambda)^t & \tilde{A}(\lambda)\\ W^b(\lambda)^t & \tilde{B}(\lambda)\end{bmatrix}\begin{bmatrix}I_n\\ 0\end{bmatrix}$$

we infer

$$(\tilde{W}^a(\lambda),\tilde{W}^b(\lambda))\hat{\zeta} = -(0,I_n)\begin{bmatrix}I_n\\ 0\end{bmatrix}d = 0$$

which proves $\zeta \in D(L_0^+(\lambda))$.

ii) For $\eta \in H_n(a,b)$ we have

$$l^{++}(\lambda)\eta = \sum_{i=0}^{n}(-1)^i\sum_{j=0}^{i}\binom{i}{j}(-1)^j\left(p_i^{(i-j)}(\cdot,\lambda)\eta\right)^{(j)}$$

$$= \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{i} \binom{i}{j} (-1)^j \sum_{k=0}^{j} \binom{j}{k} p_i^{(i-k)}(\cdot,\lambda)\eta^{(k)}$$

$$= \sum_{k=0}^{n} \sum_{i=k}^{n} (-1)^i \binom{i}{k} p_i^{(i-k)}(\cdot,\lambda)\eta^{(k)} \sum_{j=k}^{i} (-1)^j \frac{(i-k)!}{(i-j)!(j-k)!}$$

$$= \sum_{k=0}^{n} p_k(\cdot,\lambda)\eta^{(k)}$$

since

$$\sum_{j=k}^{i} (-1)^j \frac{(i-k)!}{(i-j)!(j-k)!} = (-1)^k \sum_{j=0}^{i-k} (-1)^j \binom{i-k}{j} = (-1)^k \delta_{ik}.$$

This proves $l^{++}(\lambda) = l(\lambda)$. From (8.7) we infer

$$\hat{\zeta}^t(-H(\lambda)^t)\hat{\eta} = \langle l^+(\lambda)\zeta,\eta\rangle - \langle \zeta, l^{++}(\lambda)\eta\rangle,$$

and (8.8) leads to

$$\begin{bmatrix} -\tilde{A}(\lambda)^t & -\tilde{B}(\lambda)^t \\ -W^a(\lambda) & -W^b(\lambda) \end{bmatrix} = \begin{bmatrix} \tilde{W}^a(\lambda)^t & \tilde{C}(\lambda)^t \\ \tilde{W}^b(\lambda)^t & \tilde{D}(\lambda)^t \end{bmatrix}^{-1} (-H(\lambda))^t.$$

Hence we have $L_0^{++}(\lambda) = L_0(\lambda)$. And finally, we apply part i) to $L_0^+(\lambda)$ and obtain $(L_0^+(\lambda))^* = L_0(\lambda)$.

(8.20) DEFINITION. Let $\eta \in H(\mathbb{C}, H_n(a,b))$ and $\mu \in \mathbb{C}$. η is called a root function of L_0 at μ iff $\eta(\mu) \neq 0$, $(l\eta)(\mu) = 0$ and $(W^a(\mu), W^b(\mu))\hat{\eta}(\mu) = 0$. The minimum of the orders of the zero of $l\eta$ and $(W^a, W^b)\hat{\eta}$ at μ is called the multiplicity of η.

From $l\eta = L^D\eta$ and $(W^a, W^b)\hat{\eta} = L^R\eta$, we obtain

(8.21) REMARK. Let $\eta \in H(\mathbb{C}, H_n(a,b))$, $\mu \in \mathbb{C}$ and $\nu \in \mathbb{N}$. Then η is a root function of L_0 of order ν at μ iff η is a root function of L of order ν at μ.

Canonical systems of root functions of L_0 are defined in the same way as for L. Hence a system of root functions is a canonical system of root functions of L_0 at μ iff it is a canonical system of root functions of L at μ.

The situation is different for $L_0^+ = L_0^*$ and L^*.

(8.22) PROPOSITION. *Let* $(\zeta, d) \in H(\mathbb{C}, L_2(a,b) \times \mathbb{C}^n)$ *be a* *root function of* L^* *of multiplicity* ν *at* μ*. We may assume that* ζ *is a polynomial of order* $\leq \nu-1$.

Then $\zeta \in H(\mathbb{C}, H_n(a,b))$, ζ *is a root function of* L_0^+ *of multiplicity* $\geq \nu$ *at* μ *and* $d + (\tilde{C}, \tilde{D})\hat{\zeta}$ *has a zero of order* $\geq \nu$ *at* μ.

PROOF. By assumption

$$\zeta(\lambda) = \sum_{i=0}^{\nu-1} (\lambda-\mu)^i \zeta_i \qquad (\lambda \in \mathbb{C})$$

where $\zeta_i \in L_2(a,b)$. First we shall show that $\zeta_i \in H_n(a,b)$. For this, define

$$d_i := \frac{1}{i!}\left(\frac{d^i}{d\lambda^i} d\right)(\mu) \qquad (i = 0,\ldots,\nu-1).$$

Since (ζ,d) is a root function of L^* of multiplicity ν at μ we have

$$(8.23) \qquad L^*(\mu)(\zeta_i, d_i) = -\sum_{j=1}^{i} \frac{1}{j!}\left(\frac{d^j}{d\lambda^j} L^*\right)(\mu)(\zeta_{i-j}, d_{i-j}) \quad (i=0,\ldots,\nu-1).$$

Especially $L^*(\mu)(\zeta_0, d_0) = 0$. By (8.13) and (8.15) we have

$$\zeta_0 \in D(L_0^+(\mu)) \subset H_n(a,b).$$

Now assume that $\zeta_0,\ldots,\zeta_{i-1} \in H_n(a,b)$ for some $i \leq \nu-1$. From (8.2), (8.3) and (8.4) it immediately follows that the restriction of the right side of (8.23) to $\mathcal{D}'(a,b)$ belongs to $L_2(a,b)$. Hence $(L^{D*}(\mu)\zeta_i)_r = (L^*(\mu)(\zeta_i, d_i))_r \in L_2(a,b)$. By (7.20) we obtain $\zeta_i \in H_n(a,b)$.

Since (ζ,d) is a root function of L^* of multiplicity ν at μ, we see that the Taylor coefficients of $L^*(\zeta,d) - 1^+\zeta$ at μ are in $L_2(a,b)$ up to the order $\nu-1$. As in the proof of (8.15) we see that

$$\begin{bmatrix} W^{a^t} & \tilde{A} \\ W^{b^t} & \tilde{B} \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \\ \tilde{W}^a & \tilde{W}^b \end{bmatrix} \hat{\zeta} + \begin{bmatrix} W^{a^t} \\ W^{b^t} \end{bmatrix} d$$

has a zero of order $\geq \nu$ at μ. From this we conclude that $1^+\zeta$ has a zero of order $\geq \nu$ at μ. Continuing as in the proof of (8.15) we see that $(\tilde{W}^a, \tilde{W}^b)\hat{\zeta}$ has a zero of order $\geq \nu$ at μ.

Finally we obtain that

$$\begin{bmatrix} W^{a^t} \\ W^{b^t} \end{bmatrix} (\tilde{C},\tilde{D})\hat{\zeta} + \begin{bmatrix} W^{a^t} \\ W^{b^t} \end{bmatrix} d$$

has a zero of order $\geq \nu$ at μ. This proves the last assertion since $(W^a, W^b)^t$ is injective.

(8.24) PROPOSITION. *Let* $\zeta \in H(\mathbb{C}, H_n(a,b))$ *be a root function of* L_0^+ *of multiplicity* ν *at* μ. *Set* $d := -(\tilde{C},\tilde{D})\hat{\zeta}$. *Then* (ζ, d) *is a root function of* L^* *of multiplicity* $\geq\nu$ *at* μ.

PROOF. By assumption, $(\tilde{W}^a, \tilde{W}^b)\hat{\zeta}$ has a zero of order $\geq \nu$ at μ. Hence this also holds for the term on the left side of (8.19) by the definition of d. Because of the equation (8.19) the assertion is clear.

(8.25) PROPOSITION. *For* $\lambda \in \mathbb{C}$, $L(\lambda)$ *is bijective iff* $L_0(\lambda)$ *is bijective, and* $L_0(\lambda)^{-1} f = L^{-1}(\lambda)(f,0)$ *for* $\lambda \in \rho(L_0)$ *and* $f \in L_2(a,b)$.

PROOF. Let $L_0(\lambda)$ be bijective. Then $L(\lambda)$ is injective by (8.21). Hence $L(\lambda)$ is bijective since $\operatorname{ind} L(\lambda) = 0$ by (7.4). Conversely, let $L(\lambda)$ be bijective. Then $L_0(\lambda)$ is injective by (8.21). For $f \in L_2(a,b)$ we have $\eta := L^{-1}(\lambda)(f,0) \in H_n(a,b)$, $l(\lambda)\eta = L^D(\lambda)\eta = f$ and $(W^a(\lambda), W^b(\lambda))\hat{\eta} = L^R(\lambda)\eta = 0$. This proves that $L_0(\lambda)$ is surjective and that $L_0(\lambda)^{-1} f = L^{-1}(\lambda)(f,0)$.

A mapping $Y = (y_{i,j})_{i,j=1}^n \in H(\mathbb{C}, M_n(L_2(a,b))$ is called a holomorphic fundamental matrix of L or l, respectively, if $y_{1,1}(\lambda), \ldots, y_{1,n}(\lambda)$ is a basis of $N(L^D(\lambda))$, and if

$$y_{i,j}(\lambda) = (y_{1,j}(\lambda))^{(i-1)} \quad (i = 2,\ldots,n;\ j = 1,\ldots,n).$$

Obviously, Y is a holomorphic fundamental matrix of L iff Y is a holomorphic fundamental matrix of $\tilde{T}^D$. We set

$$M(\lambda) := L^R(\lambda) e_1^t Y(\cdot,\lambda) = W^a(\lambda) Y(a,\lambda) + W^b(\lambda) Y(b,\lambda).$$

We have

$$\sigma(L_0) = \sigma(L) = \sigma(\tilde{T}) = \sigma(M).$$

We assume that $\rho(L_0) \neq \emptyset$. The function

$$(8.26)\qquad \hat{G}(x,\xi,\lambda) := \begin{cases} e_1^t Y(x,\lambda)M^{-1}(\lambda)W^a(\lambda)Y(a,\lambda)Y^{-1}(\xi,\lambda)e_n \\ \qquad\qquad\qquad\qquad (a \le \xi \le x \le b) \\ -e_1^t Y(x,\lambda)M^{-1}(\lambda)W^b(\lambda)Y(b,\lambda)Y^{-1}(\xi,\lambda)e_n \\ \qquad\qquad\qquad\qquad (a \le x < \xi \le b) \end{cases}$$

$(\lambda \in \rho(L_0))$ is called the Green's function of L_0.

(8.27) PROPOSITION. *Let* $\lambda \in \rho(L_0)$ *and* $f \in L_2(a,b)$. *Then*

$$(L_0(\lambda)^{-1}f)(x) = \int_a^b \hat{G}(x,\xi,\lambda)f(\xi)d\xi.$$

PROOF. This follows from (8.25) and (7.16) since the functions $\hat{G}$ defined in (8.26) and (7.14) have different values only for $\xi = a$ and $\xi = b$, i.e. on a zero set.

(8.28) THEOREM. *Let* $\mu \in \sigma(L_0)$ *and let* $\{\eta_{i,h} : 1 \le i \le r, 0 \le h \le m_i - 1\}$ *be a canonical system of eigenvectors and associated vectors of* L_0 *at* μ.

Then there is a canonical system of eigenvectors and associated vectors $\{\zeta_{i,h} : 1 \le i \le r, 0 \le h \le m_i - 1\}$ *of* L_0^+ *at* μ *such that the principal part of* $\hat{G}(x,\xi,\cdot)$ *at* μ *has the form*

$$(8.29)\qquad \sum_{i=1}^{r}\sum_{j=0}^{m_i-1}(\lambda-\mu)^{j-m_i}\sum_{k=0}^{j}\eta_{i,k}(x)\zeta_{i,j-k}(\xi).$$

If W^a *and* W^b *do not depend on* λ, *then the biorthogonal relationships*

$$(8.30)\qquad \sum_{k=0}^{m}\frac{1}{k!}\int_a^b\left(\frac{d^k}{d\lambda^k}\psi_{ih}\right)(x,\mu)\zeta_{j,m-k}(x)dx = \delta_{ij}\,\delta_{m_i-h,m}$$

$(1 \le h \le m_i;\ 0 \le m \le m_j - 1;\ i,j = 1,\dots,r)$ *hold where*

$$\psi_{ih}(\cdot,\lambda) := 1(\lambda)\sum_{m=0}^{m_i-1}(\lambda-\mu)^{m-h}\eta_{i,m}.$$

PROOF. We set

$$\eta_i(\lambda) := \sum_{h=0}^{m_i-1}(\lambda-\mu)^h\eta_{i,h}.$$

$\{\eta_1,\ldots,\eta_r\}$ is a CSRF of L at μ by (8.21). By [27],(2.1), cf. also (2.7), there are polynomials $(\zeta_i,d_i):\mathbb{C}\to L_2(a,b)\times\mathbb{C}^n$ of degree $<m_i$ such that $\{(\zeta_1,d_1),\ldots,(\zeta_r,d_r)\}$ is a CSRF of L^* at μ,

$$(8.31)\qquad L^{-1}-\sum_{i=1}^{r}(\cdot-\mu)^{-m_i}\eta_i\otimes(\zeta_i,d_i)$$

is holomorphic at μ and the biorthogonal relationships

$$(8.32)\qquad \frac{1}{m!}\frac{d^m}{d\lambda^m}\langle\tilde{\psi}_{ih},(\zeta_j,d_j)\rangle(\mu)=\delta_{ij}\,\delta_{m_i-h,m}$$

$(1\le h\le m_i;\ 0\le m\le m_j-1;\ i,j=1,\ldots,r)$ hold where $\tilde{\psi}_{ih}:=(\cdot-\mu)^{-h}L\eta_i$. By (8.22), $\zeta_1,\ldots,\zeta_r$ are root functions of L_0^+ at μ and $d_i(\mu)+(\tilde{C}(\mu),\tilde{D}(\mu))\hat{\zeta}_i(\mu)=0$, i.e. $\zeta_1(\mu),\ldots,\zeta_r(\mu)$ are linearly independent. This and (8.24) show that $\{\zeta_1,\ldots,\zeta_r\}$ is a CSRF of L_0^+ at μ. We set

$$\zeta_i(\lambda)=:\sum_{h=0}^{m_i-1}(\lambda-\mu)^h\zeta_{i,h}\qquad (i=1,\ldots,r)$$

and infer that $\{\zeta_{i,h}:1\le i\le r,\ 0\le h\le m_i-1\}$ is a canonical system of eigenvectors and associated vectors of L_0^+ at μ. By (8.25) and (8.31), the principal part of L_0^{-1} at μ is equal to the principal part of

$$\sum_{i=1}^{r}(\cdot-\mu)^{-m_i}\eta_i\otimes\zeta_i$$

at μ. (8.27) yields that the principal part of $\hat{G}(x,\xi,\cdot)$ at μ is

$$\sum_{i=1}^{r}\sum_{j=0}^{m_i-1}(\lambda-\mu)^{j-m_i}\frac{1}{j!}\frac{d^j}{d\lambda^j}(\eta_i(x,\cdot)\zeta_i(\xi,\cdot))(\mu)$$

$$=\sum_{i=1}^{r}\sum_{j=0}^{m_i-1}(\lambda-\mu)^{j-m_i}\sum_{k=0}^{j}\eta_{i,k}(x)\zeta_{i,j-k}(\xi).$$

If W^a and W^b are constant, then $L^R\eta_i$ is a polynomial of order $\le m_i-1$ and has a zero of order $\ge m_i$ at μ. Hence $L^R\eta_i=0$ for $i=1,\ldots,r$. Thus $\tilde{\psi}_{ih}=(\psi_{ih},0)$, and (8.32) leads to (8.29).

Now we additionally assume that $p_i \in H(\mathbb{C}, C^i[a,b])$. We define $\hat{L}_0 := L_0|_{C^n[a,b]}$ and $\hat{L}_0^+ := L_0^+|_{C^n[a,b]}$. A chain of an eigenvector and associated vectors of $\hat{L}_0$ at $\mu \in \mathbb{C}$ obviously is a chain of an eigenvector and associated vectors of L_0 at μ. Conversely, let $\{\eta_0, \ldots, \eta_{\nu-1}\}$ be a chain of an eigenvector and associated vectors of L_0 at μ. Then

$$L_0(\mu)\eta_i = -\sum_{j=1}^{i} \frac{1}{j!}\left(\frac{d^j}{d\lambda^j} l\right)(\mu)\eta_{i-j} \qquad (i = 0, \ldots, \nu-1).$$

By induction we obtain $\eta_i \in C^n[a,b]$ $(i = 0, \ldots, \nu-1)$ from (7.21). From (8.27) we obtain

$$(\hat{L}_0(\lambda)^{-1} f)(x) = \int_a^b \hat{G}(x,\xi,\lambda) f(\xi)\, d\xi \qquad (\lambda \in \rho(L_0),\ f \in C^0[a,b]).$$

In particular we have $\rho(\hat{L}_0) = \rho(L_0)$. (8.28) yields

(8.33) THEOREM (Naimark). *Let* $\mu \in \sigma(\hat{L}_0)$ *and let* $\{\eta_{i,k} : 1 \le i \le r,\ 0 \le h \le m_i - 1\}$ *be a canonical system of eigenvectors and associated vectors of* $\hat{L}_0$. *at* μ. *Then there is a canonical system of eigenvectors and associated vectors* $\{\zeta_{i,h} : 1 \le i \le r,\ 0 \le h \le m_i - 1\}$ *of* $\hat{L}_0^+$ *at* μ *such that the principal part of* $\hat{G}(x,\xi,\cdot)$ *at* μ *has the form*

$$\text{(8.34)} \qquad \sum_{i=1}^{r} \sum_{j=0}^{m_i - 1} (\lambda - \mu)^{j - m_i} \sum_{k=0}^{j} \eta_{i,k}(x)\zeta_{i,j-k}(\xi).$$

If W^a *and* W^b *do not depend on* λ, *then the biorthogonal relationships*

$$\text{(8.35)} \qquad \sum_{k=0}^{m} \frac{1}{k!} \int_a^b \left(\frac{d^k}{d\lambda^k}\psi_{ih}\right)(x,\mu)\zeta_{j,m-k}(x)\, dx = \delta_{ij}\,\delta_{m_i - h, m}$$

$(1 \le h \le m_i;\ 0 \le m \le m_j - 1;\ i,j = 1, \ldots, r)$ *hold where*

$$\psi_{ih}(\cdot,\lambda) := l(\lambda) \sum_{m=0}^{m_i - 1} (\lambda - \mu)^{m-h} \eta_{i,m}.$$

The foregoing theorem is stated in Naimark [29] on page 41 without proof. For a proof Naimark refers to Kamke giving the number [45a] of his bibliography. But Kamke was not concerned with theorem (8.33) either in his books or in his papers. Therefore the authors believe that number [45a] may be a misprint. It might be that Naimark wanted to refer to [34] in his bibliography, namely Keldyš' famous paper on nonlinear eigenvalue problems published in 1951. But neither this paper nor Keldyš' paper of 1971, cited here under number [47], contain such a theorem. It might also be that Naimark intended to refer to Tamarkin, cited in Naimark's book under number [110]. The papers which are cited there under b) and c) do not state the theorem (8.33) in its complete form. Since Tamarkin's doctoral thesis, cited under a) in Naimark's book was not available to the authors, it could not be determined whether the complete form of the theorem (8.33) is to be found there.

9. ASYMPTOTIC FUNDAMENTAL SYSTEMS FOR $\mathbb{K}\eta - \lambda\mathbb{H}\eta = 0$

We consider the differential operator

(9.1) $$L^D(\lambda) = \mathbb{K} - \lambda\mathbb{H}$$

where, for $\eta \in H_n(a,b)$

$$\mathbb{K}\eta = \eta^{(n)} + \sum_{i=0}^{n-1} k_i\eta^{(i)},$$

$$\mathbb{H}\eta = \eta^{(p)} + \sum_{i=0}^{p-1} h_i\eta^{(i)},$$

$k_i, h_i \in L_\infty(a,b)$ and $0 \leq p \leq n-1$. We set $l := n-p$ and

$$\omega_j := \exp\frac{2\pi(j-1)i}{l} \qquad (j = 1,\dots,l).$$

(9.2) THEOREM. *Let* $k \in \mathbb{N}$ *and suppose that* $k \geq \max\{1,p-1\}$ *if* $p > 0$. *Assume that*

α) $k_j \in L_\infty(a,b)$ $(j = 0,\dots,n-1-k)$ *and* $k_{n-1-j} \in W^{k-j,\infty}(a,b)$ $(j = 0,\dots,\min\{k-1,n-1\})$ *if* $p = 0$,

β) $h_0,\dots,h_{p-1} \in W^{k,\infty}(a,b)$, $k_0,\dots,k_{p-1} \in W^{k-1,\infty}(a,b)$ *and* $k_{n-1-j} \in W^{k-j,\infty}(a,b)$ $(j = 0,\dots,l-1)$ *if* $p > 0$.

For sufficiently large ρ *the differential equation*

$\mathbb{K}\eta - \rho^l \mathbb{H}\eta = 0$ *has a fundamental system* $\{\eta_1(\cdot,\rho),\dots,\eta_n(\cdot,\rho)\}$ *with the following properties:*

i) *There are a fundamental system* $\{\pi_1,\dots,\pi_p\} \subset W^{k+p,\infty}(a,b)$ *of* $\mathbb{H}\eta = 0$ *and functions* $\pi_{\nu r} \in W^{k+p-lr,\infty}(a,b)$ $(1 \le \nu \le p,\ 1 \le r \le [\frac{k}{l}])$ *such that*

$$(9.3)\qquad \eta_\nu^{(\mu)}(\cdot,\rho) = \pi_\nu^{(\mu)} + \sum_{r=1}^{[\frac{k}{l}]} \rho^{-lr}\pi_{\nu r}^{(\mu)} + \{o(\rho^{-k})\}_\infty$$

$$(\nu = 1,\dots,p;\ \mu = 0,\dots,p-1),$$

$$(9.4)\qquad \eta_\nu^{(\mu)}(\cdot,\rho) = \pi_\nu^{(\mu)} + \sum_{r=1}^{[\frac{k-\mu+p-1}{l}]} \rho^{-lr}\pi_{\nu r}^{(\mu)} + \{o(\rho^{-k+\mu-p+1})\}_\infty$$

$$(\nu = 1,\dots,p;\ \mu = p,\dots,n-1).$$

ii) *Set* $\tilde{k} := \min\{k, k+1-p\}$. *For* $r = 0,\dots,\tilde{k}$ *there are functions* $\varphi_r \in W^{k+1-r,\infty}(a,b)$ *such that*

α) φ_0 *is the solution of the initial value problem*

$$\eta' - \frac{1}{l}(h_{p-1} - k_{n-1})\eta = 0, \quad \eta(a) = 1,$$

β)

$$(9.5)\qquad \eta_\nu^{(\mu)}(x,\rho) = \left[\frac{d^\mu}{dx^\mu}\right]\left\{\sum_{r=0}^{\tilde{k}} (\rho\omega_{\nu-p})^{-r}\varphi_r(x)e^{\rho\omega_{\nu-p}x}\right\}$$

$$+ \{o(\rho^{-\tilde{k}+\mu})\}_\infty e^{\rho\omega_{\nu-p}x}$$

$$(\nu = p+1,\dots,n;\ \mu = 0,\dots,n-1),$$

where $\left[\frac{d^\mu}{dx^\mu}\right]$ *means that we drop those terms of the Leibniz expansion which contain a function* $\varphi_r^{(j)}$ *with* $j > \tilde{k}-r$.

PROOF. We denote the i-th unit vector in $\mathbb{C}^n$, $\mathbb{C}^p$, $\mathbb{C}^l$ by e_i, ϵ_i, ε_i. For $i \in \mathbb{Z}\setminus\{1,\dots,n\}$ or $i \in \mathbb{Z}\setminus\{1,\dots,p\}$ or $i \in \mathbb{Z}\setminus\{1,\dots,l\}$ we set $e_i := 0$, $\epsilon_i := 0$, $\varepsilon_i := 0$, respectively. As in section 7 we consider the associated first order system

$$(9.6)\qquad \tilde{T}^D(\lambda)\tilde{y} = \tilde{y}' - \hat{A}(\cdot,\lambda)\tilde{y} \qquad (\tilde{y} \in H_1^n(a,b),\ \lambda \in \mathbb{C}).$$

In this case (9.1), the coefficient matrix $\hat{A}$ has the block representation

$$\hat{A}(\cdot,\lambda) = \begin{bmatrix} J_p & \epsilon_p\varepsilon_1^t \\ \varepsilon_1 a_1^t(\lambda) & \varepsilon_1 a_2^t + J_l + \lambda\varepsilon_1\varepsilon_1^t \end{bmatrix}$$

according to the decomposition $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^l$, where

(9.7) $\quad a_1^t(\lambda) := \lambda(h_o,\dots,h_{p-1}) - (k_o,\dots,k_{p-1}) =: \lambda a_{11}^t + a_{12}^t,$

(9.8) $\quad a_2^t := -(k_p,\dots,k_{n-1}),$

$$J_r := \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \in M_r(\mathbb{C}).$$

We set

$$\varepsilon^t := \sum_{i=1}^{l} \varepsilon_i^t = (1,\dots,1) \in \mathbb{C}^l,$$

$$\Omega_l := \operatorname{diag}(\omega_1,\dots,\omega_l),$$

$$\Delta_r(\rho) := \operatorname{diag}(1,\rho,\dots,\rho^{r-1}) \in M_r(\mathbb{C}),$$

$$V := \sum_{i=1}^{l} \varepsilon_i \varepsilon^t \Omega_l^{i-1} = \begin{bmatrix} 1 & \dots & 1 \\ \omega_1 & \dots & \omega_l \\ \vdots & & \vdots \\ \omega_1^{l-1} & \dots & \omega_l^{l-1} \end{bmatrix}.$$

It is easy to check that V is invertible,

(9.9) $$V^{-1} = \frac{1}{l}\sum_{k=1}^{l} \Omega_l^{1-k} \varepsilon \varepsilon_k^t$$

and

(9.10) $$\varepsilon_i^t V = \varepsilon^t \Omega_l^{i-1} \qquad (i = 1,\dots,l).$$

We set

(9.11) $$C(\rho) := \begin{bmatrix} \rho^{-1} I_p & \rho^{-1} \epsilon_p \varepsilon^t \Omega_l^{-1} \\ 0 & \Delta_l(\rho) V \end{bmatrix} \qquad (\rho \in \mathbb{C} \setminus \{0\}).$$

For $\rho \in \mathbb{C} \setminus \{0\}$ and $y \in H_1^n(a,b)$ we obtain

(9.12) $$C^{-1}(\rho)\tilde{T}^D(\rho^l)C(\rho)y = y' - A(\cdot,\rho)y$$

with

(9.13) $$A(\cdot,\rho) = \sum_{j=-1}^{l} \rho^{-j} A_{-j}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & \Omega_l \end{bmatrix} =: \Omega,$$

$$(9.14)\qquad A_0 = \begin{bmatrix} J_p - \epsilon_p a_{11}^t & \left(\epsilon_{p-1} - (h_{p-1} - k_{n-1})\epsilon_p\right)\varepsilon^t \Omega_l^{-1} \\ \frac{1}{l}\Omega_l \varepsilon a_{11}^t & \frac{1}{l}(h_{p-1} - k_{n-1})\Omega_l \varepsilon\varepsilon^t \Omega_l^{-1} \end{bmatrix}.$$

$$(9.15)\qquad A_{-j} = \begin{bmatrix} 0 & k_{n-1-j}\epsilon_p \varepsilon^t \Omega_l^{-1-j} \\ 0 & -\frac{1}{l}k_{n-1-j}\Omega_l \varepsilon\varepsilon^t \Omega_l^{-1-j} \end{bmatrix} \qquad (j = 1,\dots,l-1),$$

$$(9.16)\qquad A_{-l} = \begin{bmatrix} -\epsilon_p a_{12}^t & k_{p-1}\epsilon_p \varepsilon^t \Omega_l^{-1} \\ \frac{1}{l}\Omega_l \varepsilon a_{12}^t & -\frac{1}{l}k_{p-1}\Omega_l \varepsilon\varepsilon^t \Omega_l^{-1} \end{bmatrix}.$$

For the proof of (9.12) we set

$$C_1(\rho) := \begin{bmatrix} \rho^{-1} I_p & 0 \\ 0 & \Delta_l(\rho)V \end{bmatrix}, \qquad C_2 := \begin{bmatrix} I_p & \epsilon_p \varepsilon^t \Omega_l^{-1} \\ 0 & I_l \end{bmatrix}$$

and obtain $C(\rho) = C_1(\rho)C_2$. A simple calculation leads to

$$C_1^{-1}(\rho)\hat{A}(\cdot,\rho^l)C_1(\rho) =$$

$$\begin{bmatrix} J_p & \rho\epsilon_p \varepsilon_1^t \Delta_l(\rho)V \\ \rho^{-1}V^{-1}\Delta_l^{-1}(\rho)\varepsilon_1 a_1^t(\rho^l) & V^{-1}\Delta_l^{-1}(\rho)(\varepsilon_1 a_2^t + J_l + \rho^l \varepsilon_1 \varepsilon_1^t)\Delta_l(\rho)V \end{bmatrix}.$$

From (9.9) and (9.10) we obtain the equations

$$\varepsilon_1^t \Delta_l(\rho)V = \varepsilon_1^t V = \varepsilon^t, \qquad V^{-1}\varepsilon_1 = \frac{1}{l}\Omega_l \varepsilon,$$

$$V^{-1}\Delta_l^{-1}(\rho)(J_l + \rho^l \varepsilon_1 \varepsilon_1^t)\Delta_l(\rho)V = \rho V^{-1}(J_l + \varepsilon_1 \varepsilon_1^t)V = \rho\Omega_l$$

which yields the representation

$$C_1^{-1}(\rho)\hat{A}(\cdot,\rho)C_1(\rho)$$

$$= \begin{bmatrix} J_p & 0 \\ \frac{1}{l}\rho^{-l}\Omega_l \varepsilon a_1^t(\rho^l) & \frac{1}{l}\rho^{1-l}\Omega_l \varepsilon a_2^t \Delta_l(\rho)V \end{bmatrix} + \rho \begin{bmatrix} 0 & \epsilon_p \varepsilon^t \\ 0 & \Omega_l \end{bmatrix}.$$

The assertion (9.12) follows from

$$\begin{bmatrix} I_p & -\epsilon_p \varepsilon^t \Omega_l^{-1} \\ O & I_l \end{bmatrix} \begin{bmatrix} O & \epsilon_p \varepsilon^t \\ O & \Omega_l \end{bmatrix} \begin{bmatrix} I_p & \epsilon_p \varepsilon^t \Omega_l^{-1} \\ O & I_l \end{bmatrix} = \begin{bmatrix} O & O \\ O & \Omega_l \end{bmatrix},$$

$$\begin{bmatrix} I_p & -\epsilon_p \varepsilon^t \Omega_l^{-1} \\ O & I_l \end{bmatrix} \begin{bmatrix} J_p & O \\ \frac{1}{l}\rho^{-l}\Omega_l \varepsilon a_1^t(\rho^l) & \frac{1}{l}\rho^{l-l}\Omega_l \varepsilon a_2^t \Delta_l(\rho) V \end{bmatrix} *$$

$$* \begin{bmatrix} I_p & \epsilon_p \varepsilon^t \Omega_l^{-1} \\ O & I_l \end{bmatrix} =$$

$$\begin{bmatrix} J_p - \rho^{-l}\epsilon_p a_1^t(\rho^l) & \epsilon_{p-1}\varepsilon^t \Omega_l^{-1} - \rho^{-l}\epsilon_p a_1^t(\rho^l)\epsilon_p \varepsilon^t \Omega_l^{-1} - \rho^{l-l}\epsilon_p a_2^t \Delta_l(\rho) V \\ \frac{1}{l}\rho^{-l}\Omega_l \varepsilon a_1^t(\rho^l) & \frac{1}{l}\rho^{-l}\Omega_l \varepsilon a_1^t(\rho^l)\epsilon_p \varepsilon^t \Omega_l^{-1} + \frac{1}{l}\rho^{l-l}\Omega_l \varepsilon a_2^t \Delta_l(\rho) V \end{bmatrix},$$

the definitions (9.7), (9.8), and the relationship (9.10).

By the assumptions α) and β) we have $A_{-j} \in M_n(W^{k-j,\infty}(a,b))$ $(j = 0,\dots,\min\{k,l\})$. According to (9.13),... ,(9.16) the coefficient matrix $A(\cdot,\rho)$ of (9.12) has the form (3.1) with respect to ρ instead of λ. From theorem (3.2) we obtain that $y' - A(\cdot,\rho)y = 0$ has a fundamental system

$$(9.17) \qquad Y(\cdot,\rho) = \left(\sum_{r=0}^{k} \rho^{-r} P^{[r]} + \{o(\rho^{-k})\}_\infty \right) E(x,\rho)$$

if ρ is sufficiently large, where $P^{[r]} \in M_n(W^{k+1-r,\infty}(a,b))$ and

$$(9.18) \qquad E(x,\rho) = \operatorname{diag}(1,\dots,1,e^{\rho\omega_1 x},\dots,e^{\rho\omega_l x}).$$

From (9.12) we infer that

$$(9.19) \qquad (\eta_{\mu-1,\nu})_{\mu,\nu=1}^{n} := \tilde{Y}(\cdot,\rho) := C(\rho)Y(\cdot,\rho) \begin{bmatrix} \rho I_p & O \\ O & \rho^p \Omega_l^p \end{bmatrix}$$

is a fundamental matrix of $\tilde{T}^D(\rho^l)\tilde{y} = 0$ if ρ is sufficiently large. We set

(9.20) $$\sum_{r=0}^{k} \rho^{-r} P^{[r]} + \{o(\rho^{-k})\}_\infty =: \begin{bmatrix} \tilde{Q}_{11}(\cdot,\rho) & \tilde{Q}_{12}(\cdot,\rho) \\ \tilde{Q}_{21}(\cdot,\rho) & \tilde{Q}_{22}(\cdot,\rho) \end{bmatrix}$$

and obtain from (9.11), (9.18) and (9.19) that

(9.21) $$\left\{ \tilde{Y}(\cdot,\rho) = \begin{bmatrix} \tilde{Q}_{11}(\cdot,\rho) + \epsilon_p \varepsilon^t \Omega_1^{-1} \tilde{Q}_{21}(\cdot,\rho) & \rho^{p-1}\tilde{Q}_{12}(\cdot,\rho)\Omega_1^p + \rho^{p-1}\epsilon_p \varepsilon^t \Omega_1^{-1} \tilde{Q}_{22}(\cdot,\rho)\Omega_1^p \\ \rho\Delta_1(\rho)V\tilde{Q}_{21}(\cdot,\rho) & \rho^p \Delta_1(\rho)V\tilde{Q}_{22}(\cdot,\rho)\Omega_1^p \end{bmatrix} E(\cdot,\rho). \right.$$

We set $\eta_\nu := \eta_{0,\nu}$. Then $\{\eta_1,\ldots,\eta_n\}$ is a fundamental system of $\mathbb{K}\eta - \rho^l \mathbb{H}\eta = 0$ and $\eta_\nu^{(\mu)} = \eta_{\mu,\nu}$ $(\nu = 1,\ldots,n;\ \mu = 0,\ldots,n-1)$. We shall show that the η_ν have the properties stated in Theorem (9.2). According to (9.20) we have

$$\tilde{Q}_{ij}(\cdot,\rho) = \sum_{r=0}^{k} \rho^{-r} Q_{ij}^{[r]} + \{o(\rho^{-k})\}_\infty \qquad (i,j = 1,2)$$

where the elements of $Q_{ij}^{[r]}$ belong to $W^{k+1-r,\infty}(a,b)$. We set $Q_{ij}^{[r]} := 0$ for $r < 0$.

We infer that, in the case we consider here, the relationships (3.3), (3.4) and (3.5) are equivalent to the following equations:

(9.22) $$Q_{11}^{[0]\prime} - (J_p - \epsilon_p a_{11}^t) Q_{11}^{[0]} = 0, \qquad Q_{11}^{[0]}(a) = I_p,$$

(9.23) $$Q_{22}^{[0]\prime} - \frac{1}{l}(h_{p-1} - k_{n-1}) Q_{22}^{[0]} = 0, \qquad Q_{22}^{[0]}(a) = I_l,$$

(9.24) $$Q_{12}^{[0]} = 0, \qquad Q_{21}^{[0]} = 0,$$

(9.25) $$\left\{ \begin{aligned} & Q_{11}^{[r]\prime} - (J_p - \epsilon_p a_{11}^t) Q_{11}^{[r]} = \left\{\epsilon_{p-1} - (h_{p-1} - k_{n-1})\epsilon_p\right\} \varepsilon^t \Omega_1^{-1} Q_{21}^{[r]} \\ & + \sum_{j=1}^{l} k_{n-1-j} \epsilon_p \varepsilon^t \Omega_1^{-1-j} Q_{21}^{[r-j]} - \epsilon_p a_{12}^t Q_{11}^{[r-1]} \qquad (r = 1,\ldots,k), \end{aligned} \right.$$

(9.26) $$\begin{cases} Q_{21}^{[r]} = \Omega_1^{-1} Q_{21}^{[r-1]'} - \frac{1}{l}\varepsilon a_{11}^t Q_{11}^{[r-1]} - \frac{1}{l}(h_{p-1}-k_{n-1})\varepsilon\varepsilon^t \Omega_1^{-1} Q_{21}^{[r-1]} \\ + \frac{1}{l}\sum_{j=1}^{l} k_{n-1-j}\varepsilon\varepsilon^t \Omega_1^{-1-j} Q_{21}^{[r-1-j]} - \frac{1}{l}\varepsilon a_{12}^t Q_{11}^{[r-l-1]} \qquad (r=1,\dots,k), \end{cases}$$

(9.27) $$\begin{cases} Q_{12}^{[r]} = -Q_{12}^{[r-1]'}\Omega_1^{-1} + (J_p - \epsilon_p a_{11}^t) Q_{12}^{[r-1]}\Omega_1^{-1} \\ + \left\{\epsilon_{p-1} - (h_{p-1}-k_{n-1})\epsilon_p\right\}\varepsilon^t \Omega_1^{-1} Q_{22}^{[r-1]}\Omega_1^{-1} \\ + \sum_{j=1}^{l} k_{n-1-j}\epsilon_p \varepsilon^t \Omega_1^{-1-j} Q_{22}^{[r-1-j]}\Omega_1^{-1} - \epsilon_p a_{12}^t Q_{12}^{[r-l-1]}\Omega_1^{-1} \\ \qquad (r = 1,\dots,k), \end{cases}$$

(9.28) $$\begin{cases} \Omega_1 Q_{22}^{[r]} - Q_{22}^{[r]}\Omega_1 = Q_{22}^{[r-1]'} - \frac{1}{l}\Omega_1 \varepsilon a_{11}^t Q_{12}^{[r-1]} \\ -\frac{1}{l}(h_{p-1}-k_{n-1})\Omega_1\varepsilon\varepsilon^t \Omega_1^{-1} Q_{22}^{[r-1]} + \frac{1}{l}\sum_{j=1}^{l} k_{n-1-j}\Omega_1\varepsilon\varepsilon^t\Omega_1^{-1-j} Q_{22}^{[r-1-j]} \\ -\frac{1}{l}\Omega_1\varepsilon a_{12}^t Q_{12}^{[r-l-1]} \qquad (r = 1,\dots,k), \end{cases}$$

(9.29) $$\begin{cases} 0 = \varepsilon_\nu^t\Big\{Q_{22}^{[k]'} - \frac{1}{l}\Omega_1\varepsilon a_{11}^t Q_{12}^{[k]} - \frac{1}{l}(h_{p-1}-k_{n-1})\Omega_1\varepsilon\varepsilon^t\Omega_1^{-1} Q_{22}^{[k]} \\ + \frac{1}{l}\sum_{j=1}^{l} k_{n-1-j}\Omega_1\varepsilon\varepsilon^t \Omega_1^{-1-j} Q_{22}^{[k-j]} - \frac{1}{l}\Omega_1\varepsilon a_{12}^t Q_{12}^{[k-l]}\Big\}\varepsilon_\nu \\ \qquad (\nu = 1,\dots,l). \end{cases}$$

The equations (9.22) and (9.24),...,(9.29) are derived from (3.3), (3.4) and (3.5) by direct computation. The relationships (3.3) imply

(9.30) $$Q_{22}^{[0]}\Omega_1 - \Omega_1 Q_{22}^{[0]} = 0, \qquad Q_{22}^{[0]}(a) = I.$$

The first of these equations implies that $Q_{22}^{[0]}$ is a diagonal matrix. The diagonal elements of $Q_{22}^{[0]}$ have to satisfy the initial value problem

$$\eta' - \frac{1}{l}(h_{p-1}-k_{n-1})\eta = 0, \quad \eta(a) = 1.$$

We conclude this fact from the equation (9.28) for $r = 1$. For this purpose we have to observe that the diagonal elements of

$\Omega_1 Q_{22}^{[1]} - Q_{22}^{[1]}\Omega_1$ are zero, that $Q_{12}^{[0]} = 0$ by (9.24) and that the diagonal elements of $\Omega_1 \varepsilon\varepsilon^t \Omega_1^{-1}$ have the value 1. Hence $Q_{22}^{[0]}$ fulfills the conditions (9.23). Conversely it follows that $Q_{22}^{[0]}$ is a diagonal matrix if it satisfies the equations (9.23) whence (9.30) holds. This proves the asserted equivalence.

From (9.20) and (9.21) we infer that there are $\pi_{\nu r\mu} \in W^{k+1-r,\infty}(a,b)$ $(r = 0,\ldots,k;\ \nu = 1,\ldots,p;\ \mu = 0,\ldots,p-1)$ such that

$$(9.31)\qquad \eta_\nu^{(\mu)}(\cdot,\rho) = \sum_{r=0}^{k} \rho^{-r} \pi_{\nu r\mu} + \{o(\rho^{-k})\}_\infty$$

$$(\nu = 1,\ldots,p;\ \mu = 0,\ldots,p-1)$$

where $(\pi_1,\ldots,\pi_p) := (\pi_{100},\ldots,\pi_{p00}) = \epsilon_1^t Q_{11}^{[0]}$ is a fundamental system of $\mathbb{H}\eta = 0$ by (9.22). From (9.22) and (9.25) we obtain

$$\epsilon_i^t Q_{11}^{[r]\prime} = \epsilon_{i+1}^t Q_{11}^{[r]} + \epsilon_{i+1}^t \epsilon_p \varepsilon^t \Omega_1^{-1} Q_{21}^{[r]}$$

for $i = 1,\ldots,p-1;\ r = 0,\ldots,k$. These equations and (9.21) show that $\pi_{\nu r\mu} = \pi_{\nu r0}^{(\mu)}$ for $\mu = 0,\ldots,p-1;\ \nu = 1,\ldots,p;\ r = 0,\ldots,k$. Since $\pi_{\nu r,p-1} \in W^{k+1-r,\infty}(a,b)$, it follows that $\pi_{\nu r0} \in W^{k+p-r,\infty}(a,b)$. From (9.20) and (9.21) we infer that there are $\pi_{\nu r\mu} \in W^{k+p-\mu-r,\infty}(a,b)$ for $\nu = 1,\ldots,p;\ \mu = p,\ldots,n-1$ and $r = p-\mu-1,\ldots,k+p-\mu-1$ such that

$$(9.32)\qquad \eta_\nu^{(\mu)}(\cdot,\rho) = \sum_{r=p-\mu-1}^{k+p-\mu-1} \rho^{-r} \pi_{\nu r\mu} + \{o(\rho^{-k-p+\mu+1})\}_\infty$$

$$(\nu = 1,\ldots,p;\ \mu = p,\ldots,n-1).$$

According to Theorem (3.2), differentiation leads to

$$(9.33)\qquad \eta_\nu^{(\mu+1)}(\cdot,\rho) = \sum_{r=p-\mu-1}^{k+p-\mu-2} \rho^{-r} \pi'_{\nu r\mu} + \{o(\rho^{-k-p+\mu+2})\}_\infty$$

$$(\nu = 1,\ldots,p;\ \mu = p,\ldots,n-1).$$

Again by Theorem (3.2) we obtain

$$(9.34)\qquad \eta_\nu^{(p)}(\cdot,\rho) = \sum_{r=0}^{k-1} \rho^{-r} \pi'_{\nu r,p-1} + \{o(\rho^{-k+1})\}_\infty$$

from (9.31) for $\mu = p-1$. From (9.32), (9.33) and (9.34) we deduce $\pi_{\nu r\mu} = 0$, if $r < 0$, and $\pi_{\nu r\mu} = \pi'_{\nu r,\mu-1}$, if $r \geq 0$, and μ runs from p to $n-1$. The last equations lead to

$$\pi_{\nu r\mu} = \pi^{(\mu-p+1)}_{\nu r,p-1} = \pi^{(\mu)}_{\nu r0}$$

for $\nu = 1,\dots,p$; $\mu = p,\dots,n-1$ and $r = 0,\dots,k+p-\mu-1$. Thus part i) of Theorem (9.2) is proved if we show that $\pi_{\nu r0} = 0$ for $r = 1,\dots,k$ if r is not a multiple of l. This is a consequence of the following proposition because

(9.35) $\quad \varepsilon^t \Omega_l^j \varepsilon = 0$ if j is not a multiple of l.

(9.36) PROPOSITION. *Let* $Q_{11}^{[r]}(a) = 0$ *for* $r = 1,\dots,k$. *We assert:*

i) $Q_{11}^{[r]} = 0$ *for* $r = 1,\dots,k$ *if* r *is not a multiple of* l.

ii) *For* $r = 1,\dots,k$ *there are* $q^{[r]} \in M_{1,p}(W^{k+1-r,\infty}(a,b))$ *such that* $Q_{21}^{[r]} = \Omega_l^{1-r}\varepsilon q^{[r]}$.

PROOF. The assertion immediately follows from (9.25) and (9.26) by induction where we make use of (9.35) and the equation $\Omega_l^{-1} = I_l$.

Now we shall prove the assertion ii) of (9.2). First let $\hat{Q}_{12}^{[r]}$ and $\hat{Q}_{22}^{[r]}$ $(r = 0,\dots,k)$ be arbitrary solutions of (9.23), (9.27), (9.28) and (9.29). We shall show that the matrix functions

(9.37) $$Q_{12}^{[r]} := \sum_{m=1}^{l} \hat{Q}_{12}^{[r]} \varepsilon_1 \varepsilon_m^t \Omega_l^{-r-1}$$

and

(9.38) $$Q_{22}^{[r]} := \sum_{m=1}^{l} (J_l^t + \varepsilon_1\varepsilon_l^t)^{m-1} \hat{Q}_{22}^{[r]} \varepsilon_1 \varepsilon_m^t \Omega_l^{-r}$$

where $r = 0,\dots,k$ also fulfill (9.23), (9.27), (9.28) and (9.29). The elements of $Q_{12}^{[r]}$ and $Q_{22}^{[r]}$ belong to $W^{k+1-r,\infty}(a,b)$ because $\hat{Q}_{12}^{[r]}$ and $\hat{Q}_{22}^{[r]}$ have this property. $\hat{Q}_{12}^{[0]} = 0$ implies $Q_{12}^{[0]} = 0$. A simple calculation shows that (9.23) holds for $Q_{22}^{[0]}$. The equality $\omega_i = \omega_{i-1}\omega_l^{-1}$ $(i = 2,\dots,l)$ leads to

$$\Omega_l^s(J_l^t + \varepsilon_1\varepsilon_l^t) = \Omega_l^s\left(\sum_{i=2}^{l} \varepsilon_i\varepsilon_{i-1}^t + \varepsilon_1\varepsilon_l^t\right) = \sum_{i=2}^{l} \omega_i^s \varepsilon_i \varepsilon_{i-1}^t + \varepsilon_1\varepsilon_l^t$$

$$= \omega_1^{-s}\left(\sum_{i=2}^{l} \omega_{i-1}^{s}\varepsilon_i\varepsilon_{i-1}^{t} + \omega_1^{s}\varepsilon_1\varepsilon_1^{t}\right) = \omega_1^{-s}(J_1^t + \varepsilon_1\varepsilon_1^t)\Omega_1^s$$

for $s \in \mathbb{Z}$ and further to

$$\Omega_1^s(J_1^t + \varepsilon_1\varepsilon_1^t)^{m-1} = \omega_m^s(J_1^t + \varepsilon_1\varepsilon_1^t)^{m-1}\Omega_1^s \qquad (s \in \mathbb{Z}; m=1,\dots,l) \tag{9.39}$$

by induction with respect to m. Since

$$\Omega_1^{-1}\varepsilon_1 = \varepsilon_1, \quad \Omega_1^{l} = I_l, \quad \varepsilon_m^t\Omega_1^{-r+j} = \omega_m^{j+1}\varepsilon_m^t\Omega_1^{-r-1},$$

$$\varepsilon^t(J_1^t + \varepsilon_1\varepsilon_1^t) = \varepsilon^t$$

and because of (9.39) we obtain

$$Q_{12}^{[r]} = \sum_{m=1}^{l}\Big\{-\hat{Q}_{12}^{[r-1]'}\Omega_1^{-1}\varepsilon_1\varepsilon_m^t\Omega_1^{-r-1} + (J_p - \epsilon_p a_{11}^t)\hat{Q}_{12}^{[r-1]}\Omega_1^{-1}\varepsilon_1\varepsilon_m^t\Omega_1^{-r-1}$$

$$+ \Big(\epsilon_{p-1} - (h_{p-1}-k_{n-1})\epsilon_p\Big)\varepsilon^t\Omega_1^{-1}\hat{Q}_{22}^{[r-1]}\Omega_1^{-1}\varepsilon_1\varepsilon_m^t\Omega_1^{-r-1}$$

$$+ \sum_{j=1}^{l} k_{n-1-j}\epsilon_p\varepsilon^t\Omega_1^{-1-j}\hat{Q}_{22}^{[r-1-j]}\Omega_1^{-1}\varepsilon_1\varepsilon_m^t\Omega_1^{-r-1}$$

$$- \epsilon_p a_{12}^t\hat{Q}_{12}^{[r-l-1]}\Omega_1^{-1}\varepsilon_1\varepsilon_m^t\Omega_1^{-r-1}\Big\}$$

$$= \sum_{m=1}^{l}\Big\{-\hat{Q}_{12}^{[r-1]'}\varepsilon_1\varepsilon_m^t\Omega_1^{-r-1} + (J_p - \epsilon_p a_{11}^t)\hat{Q}_{12}^{[r-1]}\varepsilon_1\varepsilon_m^t\Omega_1^{-r-1}$$

$$+ \Big(\epsilon_{p-1} - (h_{p-1}-k_{n-1})\epsilon_p\Big)\varepsilon^t\Omega_1^{-1}(J_1^t+\varepsilon_1\varepsilon_1^t)^{m-1}\hat{Q}_{22}^{[r-1]}\varepsilon_1\varepsilon_m^t\Omega_1^{-r}$$

$$+ \sum_{j=1}^{l} k_{n-1-j}\epsilon_p\varepsilon^t\Omega_1^{-1-j}(J_1^t + \varepsilon_1\varepsilon_1^t)^{m-1}\hat{Q}_{22}^{[r-1-j]}\varepsilon_1\varepsilon_m^t\Omega_1^{-r+j}$$

$$- \epsilon_p a_{12}^t\hat{Q}_{12}^{[r-l-1]}\varepsilon_1\varepsilon_m^t\Omega_1^{-r+l-1}\Big\}$$

for $r = 1,\dots,k$. We conclude that the matrices $Q_{12}^{[r]}$ and $Q_{22}^{[r]}$ fulfill (9.27) because these equations hold for the matrices $\hat{Q}_{12}^{[r]}$ and $\hat{Q}_{22}^{[r]}$. In a similar way we obtain

$$\Omega_1 Q_{22}^{[r]} - Q_{22}^{[r]}\Omega_1 =$$

$$\sum_{m=1}^{l}\left\{\Omega_1(J_1^t+\varepsilon_1\varepsilon_1^t)^{m-1}\hat{Q}_{22}^{[r]}\varepsilon_1\varepsilon_m^t\Omega_1^{-r}-(J_1^t+\varepsilon_1\varepsilon_1^t)^{m-1}\hat{Q}_{22}^{[r]}\varepsilon_1\varepsilon_m^t\Omega_1^{-r+1}\right\}$$

$$=\sum_{m=1}^{l}(J_1^t+\varepsilon_1\varepsilon_1^t)^{m-1}(\Omega_1\hat{Q}_{22}^{[r]}-\hat{Q}_{22}^{[r]}\Omega_1)\varepsilon_1\varepsilon_m^t\Omega_1^{-r+1}$$

$$=\sum_{m=1}^{l}\Big\{(J_1^t+\varepsilon_1\varepsilon_1^t)^{m-1}\hat{Q}_{22}^{[r-1]'}\varepsilon_1\varepsilon_m^t\Omega_1^{-r+1}$$

$$-\frac{1}{l}(J_1^t+\varepsilon_1\varepsilon_1^t)^{m-1}\Omega_1\varepsilon a_{11}^t\hat{Q}_{12}^{[r-1]}\varepsilon_1\varepsilon_m^t\Omega_1^{-r+1}$$

$$-\frac{1}{l}(h_{p-1}-k_{n-1})(J_1^t+\varepsilon_1\varepsilon_1^t)^{m-1}\Omega_1\varepsilon\varepsilon^t\Omega_1^{-1}\hat{Q}_{22}^{[r-1]}\varepsilon_1\varepsilon_m^t\Omega_1^{-r+1}$$

$$+\frac{1}{l}\sum_{j=1}^{l}k_{n-1-j}(J_1^t+\varepsilon_1\varepsilon_1^t)^{m-1}\Omega_1\varepsilon\varepsilon^t\Omega_1^{-1-j}\hat{Q}_{22}^{[r-1-j]}\varepsilon_1\varepsilon_m^t\Omega_1^{-r+1}$$

$$-\frac{1}{l}(J_1^t+\varepsilon_1\varepsilon_1^t)^{m-1}\Omega_1\varepsilon a_{12}^t\hat{Q}_{12}^{[r-1-l]}\varepsilon_1\varepsilon_m^t\Omega_1^{-r+1}\Big\}$$

$$=Q_{22}^{[r-1]'}-\frac{1}{l}\Omega_1\varepsilon a_{11}^tQ_{12}^{[r-1]}-\frac{1}{l}(h_{p-1}-k_{n-1})\Omega_1\varepsilon\varepsilon^t\Omega_1^{-1}Q_{22}^{[r-1]}$$

$$+\frac{1}{l}\sum_{j=1}^{l}k_{n-1-j}\Omega_1\varepsilon\varepsilon^t\Omega_1^{-1-j}Q_{22}^{[r-1-j]}-\frac{1}{l}\Omega_1\varepsilon a_{12}^tQ_{12}^{[r-1-l]}$$

for $r=1,\dots,k$. Finally the relationships

$$\varepsilon_\nu^t\Omega_1\varepsilon=\omega_\nu\varepsilon_1^t\varepsilon=\omega_\nu\varepsilon_1^t\Omega_1\varepsilon,\qquad \varepsilon_m^t\Omega_1^{-r}\varepsilon_\nu=\omega_\nu^{-r}\delta_{m\nu}$$

and again (9.39) yield

$$\varepsilon_\nu^t\Big\{Q_{22}^{[k]'}-\frac{1}{l}\Omega_1\varepsilon a_{11}^tQ_{12}^{[k]}-\frac{1}{l}(h_{p-1}-k_{n-1})\Omega_1\varepsilon\varepsilon^t\Omega_1^{-1}Q_{22}^{[k]}$$

$$+\frac{1}{l}\sum_{j=1}^{l}k_{n-1-j}\Omega_1\varepsilon\varepsilon^t\Omega_1^{-1-j}Q_{22}^{[k-j]}-\frac{1}{l}\Omega_1\varepsilon a_{12}^tQ_{12}^{[k-l]}\Big\}\varepsilon_\nu$$

$$=\omega_\nu^{-k}\varepsilon_1^t\Big\{\hat{Q}_{22}^{[k]'}-\frac{1}{l}\Omega_1\varepsilon a_{11}^t\hat{Q}_{12}^{[k]}-\frac{1}{l}(h_{p-1}-k_{n-1})\Omega_1\varepsilon\varepsilon^t\Omega_1^{-1}\hat{Q}_{22}^{[k]}$$

$$+\frac{1}{l}\sum_{j=1}^{l}k_{n-1-j}\Omega_1\varepsilon\varepsilon^t\Omega_1^{-1-j}\hat{Q}_{22}^{[k-j]}-\frac{1}{l}\Omega_1\varepsilon a_{12}^t\hat{Q}_{12}^{[k-l]}\Big\}\varepsilon_1=0.$$

(9.40) PROPOSITION. *Let* $p \geq 2$. *For* $r = 0,\dots,p-2$ *and* $i = 1,\dots,p-r-1$ *we have* $\epsilon_i^t Q_{12}^{[r]} = 0$. *For* $r = 1,\dots,p-1$ *we have* $\epsilon_{p-r}^t Q_{12}^{[r]} \varepsilon_1 = \varepsilon_1^t Q_{22}^{[0]} \varepsilon_1$.

PROOF. The first assertion is clear for $r = 0$ by (9.24). Assume that it holds for $0 \leq r-1 < p-2$. We have to prove the assertion for r and $i = 1,\dots,p-r-1$ which is at most $p-2$. From (9.27) and the induction hypothesis we obtain

$$\epsilon_i^t Q_{12}^{[r]} = -\epsilon_i^t Q_{12}^{[r-1]\,'} \Omega_1^{-1} + \epsilon_{i+1}^t Q_{12}^{[r-1]} \Omega_1^{-1} = 0.$$

The second assertion holds for $r = 1$ since, by (9.27),

$$\epsilon_{p-1}^t Q_{12}^{[1]} \varepsilon_1 = \varepsilon^t \Omega_1^{-1} Q_{22}^{[0]} \Omega_1^{-1} \varepsilon_1 = \varepsilon_1^t Q_{22}^{[0]} \varepsilon_1.$$

Assume that the assertion holds for $1 \leq r-1 < p-1$. Then the first assertion and (9.27) yield

$$\epsilon_{p-r}^t Q_{12}^{[r]} \varepsilon_1 = \epsilon_{p-r+1}^t Q_{12}^{[r-1]} \Omega_1^{-1} \varepsilon_1 = \varepsilon_1^t Q_{22}^{[0]} \varepsilon_1.$$

Let $p \geq 2$. From (9.40) we obtain $\epsilon_1^t Q_{12}^{[r]} = 0$ for $r = 0,\dots,p-2$. Hence, by (9.20) and (9.21), there are $\varphi_{\nu r} \in W^{k+2-p-r,\infty}(a,b)$ $(\nu = p+1,\dots,n;\ r = 0,\dots,k+1-p)$ such that, for $\nu = p+1,\dots,n$,

$$\eta_\nu(x,\rho) = \left\{ \sum_{r=0}^{k+1-p} \rho^{-r} \varphi_{\nu r}(x) + \{o(\rho^{-k-1+p})\}_\infty \right\} e^{\rho\omega_{\nu-p} x}.$$

From (9.20) and (9.21) we immediately infer that this representation also holds for $p=1$ and that, for $p=0$ and $\nu = 1,\dots,n$,

$$\eta_\nu(x,\rho) = \left\{ \sum_{r=0}^{k} \rho^{-r} \varphi_{\nu r}(x) + \{o(\rho^{-k})\}_\infty \right\} e^{\rho\omega_{\nu-p} x}$$

where $\varphi_{\nu r} \in W^{k+1-r,\infty}(a,b)$.

If $p > 0$ and $p+1 \leq \nu \leq n$, $0 \leq r \leq k+1-p$, we have

$$(9.41)\quad \left\{ \begin{aligned} \varphi_{\nu r} &= \epsilon_1^t \left\{ Q_{12}^{[r+p-1]} \Omega_1^p + \epsilon_p \varepsilon^t \Omega_1^{-1} Q_{22}^{[r+p-1]} \Omega_1^p \right\} \varepsilon_{\nu-p} \\ &= \epsilon_1^t \left\{ \hat{Q}_{12}^{[r+p-1]} + \epsilon_p \varepsilon^t \Omega_1^{-1} \hat{Q}_{22}^{[r+p-1]} \right\} \varepsilon_1 \omega_{\nu-p}^{-r} \end{aligned} \right.$$

by (9.21), (9.37), (9.38) and (9.39). If $p=0$, $1 \leq \nu \leq n$ and $0 \leq r \leq k$, we have, by (9.21), (9.10) and (9.38),

$$(9.42)\quad \varphi_{\nu r} = \varepsilon_1^t V Q_{22}^{[r]} \varepsilon_{\nu-p} = \omega_{\nu-p}^{-r} \varepsilon^t \hat{Q}_{22}^{[r]} \varepsilon_1.$$

This leads to

$$\varphi_{\nu r} = \omega_{\nu-p}^{-r}\varphi_{p+1,r}$$

for $\nu = p+1,\ldots,n$ and $r = 0,\ldots,\tilde{k}$. Hence, for $\nu = p+1,\ldots,n$,

$$(9.43) \qquad \eta_\nu(x,\rho) = \left\{\sum_{r=0}^{\tilde{k}} (\rho\omega_{\nu-p})^{-r}\varphi_r(x) + \{o(\rho^{-\tilde{k}})\}_\infty\right\} e^{\rho\omega_{\nu-p}x}$$

where $\varphi_r := \varphi_{p+1,r}$.

For $p = 0$, (9.42) yields

$$\varphi_0 = \varphi_{10} = \varepsilon_1^t Q_{22}^{[0]} \varepsilon_1 .$$

For $p = 1$, (9.41) yields

$$\varphi_0 = \varphi_{20} = \varepsilon^t \Omega_1^{-1} Q_{22}^{[0]} \Omega_1 \varepsilon_1 = \varepsilon_1^t Q_{22}^{[0]} \varepsilon_1 .$$

For $p \geq 2$, (9.41) and the second assertion of (9.40) yield

$$\varphi_0 = \varphi_{p+1,0} = \epsilon_1^t Q_{12}^{[p-1]} \Omega_1^p \varepsilon_1 = \varepsilon_1^t Q_{22}^{[0]} \varepsilon_1 .$$

From (9.23) we thus obtain $\varphi_0' - \frac{1}{l}(h_{p-1} - k_{n-1})\varphi_0 = 0$ and $\varphi_0(a) = 1$.

If $p = 0$ or $p = 1$, we have $\varphi_r = \varphi_{p+1,r} \in W^{k+1-r,\infty}(a,b)$ for $r = 0,\ldots,k$. Next we will prove that this also holds for $p \geq 2$ and $r = 0,\ldots,k+1-p$. From (9.20) and (9.21) we infer for $\nu = p+1,\ldots,n$ and $\mu = 0,\ldots,p-1$ that

$$(9.44) \qquad \eta_\nu^{(\mu)}(x,\rho) = \left\{\sum_{r=1-p}^{k+1-p} \rho^{-r}\varphi_{\nu r\mu}(x) + \{o(\rho^{-k-1+p})\}_\infty\right\} e^{\rho\omega_{\nu-p}x}$$

where

$$(9.45) \qquad \varphi_{\nu r\mu} = \omega_{\nu-p}^{p}\epsilon_{\mu+1}^{t}\left(Q_{12}^{[r+p-1]} + \epsilon_p \varepsilon^t \Omega_1^{-1} Q_{22}^{[r+p-1]}\right)\varepsilon_{\nu-p} \in W^{k+2-p-r,\infty}(a,b)$$

for $r = 1-p,\ldots,k+1-p$ and $\varphi_{\nu r0} = \varphi_{\nu r}$. For $\nu = p+1,\ldots,n$, $\mu = 1,\ldots,p-1$ and $r = 2-p,\ldots,k+1-p$ the equations (9.27) yield

$$(9.46) \qquad \begin{cases} \varphi_{\nu r,\mu-1} = \omega_{\nu-p}^{p}\epsilon_\mu^t Q_{12}^{[r+p-1]}\varepsilon_{\nu-p} = \\ \omega_\nu^{p-1}\left(-\epsilon_\mu^t Q_{12}^{[r+p-2]'} + \epsilon_{\mu+1}^t Q_{12}^{[r+p-2]} + \epsilon_\mu^t \epsilon_{p-1}\varepsilon^t \Omega_1^{-1} Q_{22}^{[r+p-2]}\right)\varepsilon_{\nu-p} \\ = \omega_{\nu-p}^{-1}(-\varphi_{\nu,r-1,\mu-1}' + \varphi_{\nu,r-1,\mu}) \ . \end{cases}$$

We will prove that (9.46) leads to

$$(9.47) \qquad \varphi_{\nu r\mu} \in W^{k+1-r-\mu,\infty}(a,b)$$

for $\nu = p+1,\ldots,n$, $\mu = 0,\ldots,p-1$ and $r = 1-p,\ldots,k+1-p$. For

$\mu = p-1$ this is true because of (9.45). And (9.45) yields $\varphi_{\nu,1-p,\mu} = 0$ for $\mu = 0,\dots,p-2$. Assume that (9.47) holds for $r-1$ where $1-p < r \leq k+1-p$. A recursive application of (9.46) yields $\varphi_{\nu r \mu} \in W^{k+1-r-\mu,\infty}(a,b)$ for $\mu = p-2, p-3,\dots,0$.

From (9.47) we obtain $\varphi_r = \varphi_{p+1,r0} \in W^{k+1-r,\infty}(a,b)$.

It remains to prove (9.5) for $\mu = 1,\dots,n-1$. The equations (9.20) and (9.21) yield

(9.48) $$\eta_\nu^{(\mu)}(x) = \left\{ \sum_{r=-\mu}^{k-\mu} \rho^{-r} \varphi_{\nu r \mu}(x) + \{o(\rho^{-k+\mu})\}_\infty \right\} e^{\rho\omega_{\nu-p} x}$$

for $\nu = p+1,\dots,n$ and $\mu = p,\dots,n-1$ with $\varphi_{\nu r\mu} \in W^{k+1-r-\mu,\infty}(a,b)$. According to (9.5) we have to prove that

(9.49) $$\eta_\nu^{(\mu)}(x,\rho) = \left\{ \sum_{r=-\mu}^{\tilde{k}-\mu} (\rho\omega_{\nu-p})^{-r} \sum_{j=0}^{r+\mu} \binom{\mu}{j} \varphi_{r+\mu-j}^{(j)} + \{o(\rho^{-\tilde{k}+\mu})\}_\infty \right\} e^{\rho\omega_{\nu-p} x}$$

for $\nu = p+1,\dots,n$ and $\mu = 0,\dots,n-1$. This representation is true for $\mu = 0$ by (9.43). Suppose that it holds for some $\mu < n-1$. Since the right side of (9.49) is equal to the right side of (9.43), (9.44) or (9.48), we infer from theorem (3.2) that differentiation is allowed and leads to

$$\begin{aligned}
\eta_\nu^{(\mu+1)}(x,\rho) &= \left\{ \sum_{r=-\mu}^{\tilde{k}-\mu-1} (\rho\omega_{\nu-p})^{-r} \sum_{j=0}^{r+\mu} \binom{\mu}{j} \varphi_{r+\mu-j}^{(j+1)}(x) \right. \\
&\qquad \left. + \{o(\rho^{-\tilde{k}+\mu+1})\}_\infty \right\} e^{\rho\omega_{\nu-p} x} \\
&\quad + \left\{ \sum_{r=-\mu}^{\tilde{k}-\mu} (\rho\omega_{\nu-p})^{-r+1} \sum_{j=0}^{r+\mu} \binom{\mu}{j} \varphi_{r+\mu-j}^{(j)}(x) \right. \\
&\qquad \left. + \{o(\rho^{-\tilde{k}+\mu+1})\}_\infty \right\} e^{\rho\omega_{\nu-p} x} \\
&= \left\{ \sum_{r=-(\mu+1)}^{\tilde{k}-(\mu+1)} (\rho\omega_{\nu-p})^{-r} \sum_{j=0}^{r+(\mu+1)} \binom{\mu+1}{j} \varphi_{r+(\mu+1)-j}^{(j)}(x) \right. \\
&\qquad \left. + \{o(\rho^{-\tilde{k}+(\mu+1)})\}_\infty \right\} e^{\rho\omega_{\nu-p} x}
\end{aligned}$$

which completes the proof of Theorem (9.2).

From (9.2) and (9.21) we infer

(9.50) COROLLARY. *Let the assumptions of* (9.2) *be fullfilled. Then there is a* $Q_2 \in M_{1,p}(W^{1,\infty}(a,b))$ *such that*

$$\tilde{Y}(\cdot,\rho) = \begin{bmatrix} [Q_{11}^{[0]}]_\infty & \varphi_o \Delta_p(\rho)[\hat{V}]_\infty \\ [Q_2]_\infty & \rho^p \varphi_o \Delta_1(\rho)[V\Omega_1^p]_\infty \end{bmatrix} E(\cdot,\rho)$$

where

$$\hat{V} = \begin{bmatrix} 1 & \dots & 1 \\ \omega_1 & \dots & \omega_l \\ \vdots & & \vdots \\ \omega_1^{p-1} & & \omega_l^{p-1} \end{bmatrix} \in M_{p,l}(\mathbb{C}).$$

Theorem (9.2) stems from Eberhard and Freiling [11] apart from the smoothness conditions on the coefficients k_i and h_i which they don't specify explicitly. In [9] they proved a weaker form of Theorem (9.2) which is more or less equal to Corollary (9.50). This weaker form was also given by Wasow [35], pp. 229-232. The foregoing proof of Theorem (9.2) is different from that of Eberhard and Freiling, but rather similar to Wasow's proof of Corollary (9.50).

Eberhard and Freiling, cf. also Naimark [29], subdivided the ρ-plane into certain sectors. We do not need this subdivision in our proof. Eberhard and Freiling state their asymptotic expansions with respect to these sectors. Consequently, their regularity criteria depend a priori on these sectors and they have to prove the independence of the regularity criteria from the sectors.

10. THE CHARACTERISTIC DETERMINANT

From now on we will be concerned with the boundary eigenvalue problem

$$(10.1)\quad \begin{cases} L^D(\lambda)\eta = \mathbb{K}\eta - \lambda\mathbb{H}\eta = 0 \\ L^R(\lambda)\eta = \sum_{j=1}^{m} W^{(j)}(\lambda)\begin{bmatrix} \eta(a_j) \\ \vdots \\ \eta^{(n-1)}(a_j) \end{bmatrix} + \int_a^b W(\xi,\lambda)\begin{bmatrix} \eta(\xi) \\ \vdots \\ \eta^{(n-1)}(\xi) \end{bmatrix} d\xi = 0 \end{cases}$$

$(\eta \in H_n(a,b))$ where

$$\mathbb{K}\eta = \eta^{(n)} + \sum_{i=0}^{n-1} k_i \eta^{(i)}, \qquad \mathbb{H}\eta = \eta^{(p)} + \sum_{i=0}^{p-1} h_i \eta^{(i)},$$

$n \in \mathbb{N} \setminus \{0\}$, $0 \le p \le n-1$, $l := n-p$, $m \ge 2$, $a = a_1 < \ldots < a_m = b$, $W^{(j)}(\lambda) \in M_n(\mathbb{C})$ $(j = 1,\ldots,m)$, $W(\cdot,\lambda) \in M_n(L_1(a,b))$. Furthermore, we require that W and the $W^{(j)}$ $(j = 1,\ldots,m)$ are polynomials with respect to λ and that the coefficients $h_0,\ldots,h_{p-1},k_0,\ldots,k_{n-1}$ fulfill the assumptions of Theorem (9.2).

Let $\hat{R}_1$ and $\hat{R}_2$ be defined as in (7.12) and (7.13). For the following definition we assume that $\rho(L) \neq \emptyset$. We fix closed Jordan curves $\Gamma_\nu \subset \rho(L)$ $(\nu \in \mathbb{N})$ such that $0 \in \operatorname{int} \Gamma_\nu$, $\overline{\operatorname{int} \Gamma_\nu} \subset \operatorname{int} \Gamma_{\nu+1}$, $\bigcup_{\nu \in \mathbb{N}} \operatorname{int} \Gamma_\nu = \mathbb{C}$.

(10.2) DEFINITION. Let $|\ |$ be a continuous norm on $H_n(a,b)$ and let $\tilde{p},\tilde{p}' \in \mathbb{Z}$.

i) L is called $|\ |$-regular of order $(\tilde{p},\tilde{p}')$ with respect to $f = (f_1,f_2) \in L_2(a,b) \times \mathbb{C}^n$ (and with respect to the curves Γ_ν) if

$$\int_{\Gamma_\nu} |\lambda^{-\tilde{p}-1}\hat{R}_1(\lambda) f_1|\,|d\lambda| \to 0 \qquad (\nu \to \infty)$$

and

$$\int_{\Gamma_\nu} |\lambda^{-\tilde{p}'-1}\hat{R}_2(\lambda) f_2|\,|d\lambda| \to 0 \qquad (\nu \to \infty)$$

hold.

ii) L is called $|\ |$-regular of order $(\tilde{p},\tilde{p}')$ (with respect to the curves Γ_ν) if L is $|\ |$-regular of order $(\tilde{p},\tilde{p}')$ with respect to all $f \in L_2(a,b) \times \mathbb{C}^n$.

For the following we substitute λ by ρ^l. In section 9 we proved that the corresponding differential system (7.6) $\tilde{T}^D(\rho^l)\tilde{y} = 0$ can be transformed to a system of type (3.1) so that the regularity Theorem (5.14) can be applied to the boundary eigenvalue problem

$$(10.3)\qquad \begin{cases} \tilde{T}^D(\rho^l)\tilde{y} = \tilde{y}' - \hat{A}(\cdot,\rho^l)\tilde{y} = 0 \\[2mm] \tilde{T}^R(\rho^l)\tilde{y} = \sum\limits_{j=1}^{m} W^{(j)}(\rho^l)\tilde{y}(a_j) + \int\limits_a^b W(\xi,\rho^l)\tilde{y}(\xi)\,d\xi = 0 \end{cases}$$

$(y \in H_1^n(a,b))$ and thus also to (10.1), cf. Theorem (7.11). For this purpose we have to determine the asymptotic behaviour of the "characteristic" determinant.

(10.4) PROPOSITION. *Let* $u_j \in \mathbb{C}^n$ $(j = 1,\ldots,p)$, $v_j^{(s)} \in \mathbb{C}^n$,

$d_j^{(s)} \in \mathbb{C}$ $(j=1,\dots,l;\ s=1,\dots,m)$. *We define* $M \in M_n(\mathbb{C})$ *by*

$$M := (u_1,\dots,u_p, \sum_{s=1}^{m} d_1^{(s)} v_1^{(s)},\dots, \sum_{s=1}^{m} d_l^{(s)} v_l^{(s)}).$$

We set

$$\Theta := \{1,\dots,m\}^l,$$
$$d^\theta := d_1^{(\theta_1)} \cdot \ldots \cdot d_l^{(\theta_l)} \qquad (\theta = (\theta_1,\dots,\theta_l) \in \Theta),$$
$$v_j^\theta := v_j^{(\theta_j)},$$
$$v^\theta := \det(u_1,\dots,u_p,v_1^\theta,\dots,v_l^\theta)$$

and assert that

$$\det M = \sum_{\theta\in\Theta} d^\theta v^\theta.$$

PROOF. The assertion is clear for $l=1$. Assume that it holds for $l-1$. Set

$$M_s := (u_1,\dots,u_p, d_1^{(s)} v_1^{(s)}, \sum_{s=1}^{m} d_2^{(s)} v_2^{(s)},\dots, \sum_{s=1}^{m} d_l^{(s)} v_l^{(s)})$$

$(s=1,\dots,m)$. By the induction hypothesis we have

$$\det M_s = \sum_{\substack{\theta\in\Theta \\ \theta_1 = s}} d^\theta v^\theta.$$

Hence

$$\det M = \sum_{s=1}^{m} \det M_s = \sum_{\theta\in\Theta} d^\theta v^\theta.$$

Now we suppose that the assumptions of Theorem (9.2) are fulfilled. Let $Y(\cdot,\rho)$ be the fundamental matrix of $\tilde{T}^D(\rho^l)\tilde{y} = 0$ with $Y(a,\rho) = I_n$. Then $Y \in H(\mathbb{C}, M_n(H_1(a,b)))$, see for example [28].

$$D(\rho) := \det(\tilde{T}^R(\rho^l) Y(\cdot,\rho))$$

is called the characteristic determinant of the boundary eigenvalue problem (10.1). Let $\tilde{Y}(\cdot,\rho)$ be the fundamental matrix established in Theorem (9.2). We set

$$\hat{Y}(\cdot,\rho) = \tilde{Y}(\cdot,\rho) E^{-1}(a,\rho).$$

Obviously

$$Y(\cdot,\rho) = \hat{Y}(\cdot,\rho)\hat{Y}^{-1}(a,\rho)$$

if ρ is sufficiently large. Hence

(10.5) $$D(\rho) = \det(\tilde{T}^R(\rho^l)\hat{Y}(\cdot,\rho))\cdot\det\hat{Y}^{-1}(a,\rho).$$

From Corollary (9.50) we obtain

$$\det\hat{Y}(a,\rho) = \rho^p\cdot\ldots\cdot\rho^{n-1}\left\{\det\begin{bmatrix} Q_{11}^{[0]}(a) & 0 \\ 0 & \varphi_o(a)V\Omega_1^p\end{bmatrix} + o(1)\right\}.$$

Hence

(10.6) $$\det\hat{Y}^{-1}(a,\rho) = \rho^{-(lp+\frac{l(l-1)}{2})}[d]$$

where $d \neq 0$.

We set

$$P(x,\rho) := \tilde{Y}(x,\rho)E^{-1}(x,\rho) = \hat{Y}(x,\rho)E^{-1}(x-a,\rho)$$

and

$$U(x,\rho) := F(x,\rho^l)P(x,\rho) =:$$

$$(\tilde{u}_1(x,\rho),\ldots,\tilde{u}_p(x,\rho),\tilde{v}_1(x,\rho),\ldots,\tilde{v}_l(x,\rho))$$

where F is defined by (4.4) and (4.5). We define

$$\tilde{u}_\nu =: (\tilde{u}_{i\nu})_{i=1}^n \quad (\nu = 1,\ldots,p), \quad \tilde{v}_\nu =: (\tilde{v}_{i\nu})_{i=1}^n \quad (\nu = 1,\ldots,l),$$

$$F(\cdot,\rho^l) =: \sum_{r=0}^q \rho^{lr}F_r \ , \ F_r =: (f_{ijr})_{i,j=1}^n \quad (r = 0,\ldots,q)$$

and obtain

$$\tilde{u}_{i\nu}(\cdot,\rho) = \rho^{lq}\sum_{j=1}^n\sum_{r=0}^q \rho^{l(r-q)} f_{ijr}[\pi_\nu^{(j-1)}]_\infty$$

and

$$\tilde{v}_{i\nu}(\cdot,\rho) = \sum_{j=1}^n \omega_\nu^{j-1}\sum_{r=0}^q \rho^{lr+j-1} f_{ijr}\pi_{j,\nu+p}(\cdot,\rho)$$

where $\pi_{j,\nu+p} = [\varphi_o]_\infty$. We set

$$u_{i\nu} := \sum_{j=1}^n \int_a^b d_x f_{ijq}(x)\pi_\nu^{(j-1)}(x) \qquad (1 \le i \le n,\ 1 \le \nu \le p),$$

$$u_\nu := (u_{i\nu})_{i=1}^n \qquad (1 \le \nu \le p)$$

$$u_{i\nu}(\rho) := \sum_{j=1}^n \omega_{\nu-p}^{j-1}\sum_{r=0}^q \rho^{lr+j-1}\int_a^b d_x f_{ijr}(x)\pi_{j,\nu+p}(x,\rho)e^{\rho\omega_{\nu-p}(x-a)}$$

$$(1 \le i \le n,\ p+1 \le \nu \le n),$$

$$u_\nu(\rho) := (u_{i\nu})_{i=1}^{n} \qquad (p+1 \le \nu \le n).$$

For $i = 1,\dots,n$ we set

(10.7) $$g_i := \max\{lr+j-1 : 1 \le j \le n,\ 0 \le r \le q,\ f_{ijr} \ne 0\} \ (\ge 0),$$

where $g_i := 0$ if $f_{ijr} = 0$ for all $1 \le j \le n$, $0 \le r \le q$. Let

$$W^{(\kappa)}(\rho^l) =: \sum_{r=0}^{q} \rho^{lr} W_r^{(\kappa)},\quad W_r^{(\kappa)} =: (w_{ijr}^{(\kappa)})_{i,j=1}^{n} \qquad (\kappa=1,\dots,m),$$

$$W(\cdot,\rho^l) =: \sum_{r=0}^{q} \rho^{lr} W_r,\quad W_r =: (w_{ijr})_{i,j=1}^{n},$$

(10.8) $$\tilde{\gamma}_i^{(\kappa)} := \sum_{lr+j-1=g_i} w_{ijr}^{(\kappa)} \qquad (i = 1,\dots,n;\ \kappa = 1,\dots,m),$$

(10.9) $$\gamma_i^{(\kappa)} := \tilde{\gamma}_i^{(\kappa)} \varphi_o(a_\kappa) \qquad (i = 1,\dots,n;\ \kappa = 1,\dots,m).$$

For $i = 1,\dots,n$ and $\nu = p+1,\dots,n$ we infer that

$$\begin{aligned} &u_{i\nu}(\rho) \\ &= \rho^{g_i}\Big\{\sum_{lr+j-1=g_i} \omega_{\nu-p}^{j-1} \sum_{\kappa=1}^{m}\Big(w_{ijr}^{(\kappa)}[\varphi_o(a_\kappa)] + o(1)\Big)e^{\rho\omega_{\nu-p}(x_\kappa-a)}\Big\} \\ &+\rho^{g_i}\Big\{\sum_{lr+j-1=g_i} \omega_{\nu-p}^{j-1}\int_a^b\Big(w_{ijr}(x)+[O(\rho^{-1})]_1\Big)[\varphi_o(x)]_\infty e^{\rho\omega_{\nu-p}(x-a)}dx\Big\} \\ &= (\rho\omega_{\nu-p})^{g_i}\Big\{\sum_{\kappa=1}^{m}[\gamma_i^{(\kappa)}]e^{\rho\omega_{\nu-p}(x_\kappa-a)} \\ &\qquad + \sum_{lr+j-1=g_i}\int_a^b [w_{ijr}(x)\varphi_o(x)]_1 e^{\rho\omega_{\nu-p}(x-a)}dx\Big\} \end{aligned}$$

where the relationship $\omega_\nu^{j-1} = \omega_\nu^{g_i-lr} = \omega_\nu^{g_i}$ has been used. This equation would not hold if we allow the boundary conditions to depend on ρ instead of ρ^l. In this case the boundary conditions in (10.1) would be l-th roots of λ (cf. Benedek and Panzone [4]). With the aid of Proposition (3.14) we conclude that

$$\int_a^b [w_{ijr}(x)\varphi_o(x)]_1 e^{\rho\omega_{\nu-p}(x-a)}dx = o(1)e^{\rho\omega_{\nu-p}(c-a)}$$

where $c=a$ if $\operatorname{Re}(\rho\omega_{\nu-p}) \le 0$ and $c=b$, if $\operatorname{Re}(\rho\omega_{\nu-p}) > 0$. Thus the above calculation leads to

$$u_{i\nu}(\rho)=(\rho\omega_{\nu-p})^{g_i} \sum_{\kappa=1}^{m} [\gamma_i^{(\kappa)}] e^{\rho\omega_{\nu-p}(a_\kappa - a)} \quad (1\le i\le n,\ p+1\le\nu\le n).$$

In a similar way we obtain

$$\sum_{j=1}^{n} \sum_{r=0}^{q} \rho^{l(r-q)} \int_a^b d_x f_{ijr}(x)\, \eta_\nu^{(j-1)}(x,\rho) = [u_{i\nu}] \quad (1\le i\le n,\ 1\le\nu\le p).$$

Thus we have

$$\det\Big(\tilde{T}^R(\rho^l)\hat{Y}(\rho)\Big) = \det\Big(\int_a^b d_x F(x,\rho^l)\hat{Y}(x,\rho)\Big)$$

$$= \rho^{lqp} \det([u_1],\ldots,[u_p],u_{p+1}(\rho),\ldots,u_n(\rho)).$$

Proposition (10.4) yields

$$\det\Big(\tilde{T}^R(\rho^l)\hat{Y}(\rho)\Big) = \sum_{\theta\in\Theta} e^{\rho c_\theta} v^\theta(\rho) \tag{10.10}$$

where, if $\theta = (\theta_1,\ldots,\theta_l) \in \Theta$,

$$c_\theta = \sum_{\nu=1}^{l} \omega_\nu (a_{\theta_\nu} - a),$$

$$v^\theta(\rho) = \rho^{lqp} \det([u_1],\ldots,[u_p],v_1^\theta(\rho),\ldots,v_l^\theta(\rho)),$$

$$v_\nu^\theta(\rho) = (v_{i\nu}^\theta(\rho))_{i=1}^{n}, \quad v_{i\nu}^\theta(\rho) = (\rho\omega_\nu)^{g_i} [\gamma_i^{(\theta_\nu)}]$$

$$(1\le i\le n,\ 1\le\nu\le l).$$

For $1\le i\le n$ and $1\le\nu\le l$ we set $\gamma_{i\nu}^\theta := \gamma_i^{(\theta_\nu)}$ and obtain

$$v^\theta(\rho)=\rho^{lqp}\det\begin{bmatrix} [u_{11}]\ldots[u_{1p}] & (\rho\omega_1)^{g_1}[\gamma_{11}^\theta]\ldots(\rho\omega_l)^{g_1}[\gamma_{1l}^\theta] \\ \vdots \qquad \vdots & \vdots \qquad\qquad \vdots \\ [u_{n1}]\ldots[u_{np}] & (\rho\omega_1)^{g_n}[\gamma_{n1}^\theta]\ldots(\rho\omega_l)^{g_n}[\gamma_{nl}^\theta] \end{bmatrix}.$$

Now let

$$\hat{g} := \max\{g_{i_1}+\ldots+g_{i_l} : i_j \in \{1,\ldots,n\},\ i_j \ne i_k\ (j\ne k)\} \tag{10.11}$$

and

$$\hat{k} := lqp + \hat{g}. \tag{10.12}$$

Let $J := \{I \subset \{1,\dots,n\} : \#I = l, \sum_{i\in I} g_i = \hat{g}\}$. For $I = \{i_1,\dots,i_l\} \in J$ we set $\tilde{I} := \{1,\dots,n\} \setminus I =: \{i_1',\dots,i_p'\}$,

$$d_I^\theta := \det\begin{bmatrix} \omega_1^{g_{i_1}}\gamma_{i_1 1}^\theta & \cdots & \omega_l^{g_{i_1}}\gamma_{i_1 l}^\theta \\ \vdots & & \vdots \\ \omega_1^{g_{i_l}}\gamma_{i_l 1}^\theta & \cdots & \omega_l^{g_{i_l}}\gamma_{i_l l}^\theta \end{bmatrix},$$

$$u_{\tilde{I}} := \det\begin{bmatrix} u_{i_1' 1} & \cdots & u_{i_1' p} \\ \vdots & & \vdots \\ u_{i_p' 1} & \cdots & u_{i_p' p} \end{bmatrix}.$$

Then the expansion of the determinant v^θ yields

(10.13) PROPOSITION. $v^\theta(\rho) = \rho^{\hat{k}}[D_\theta]$, *where*

$D_\theta = \sum_{I\in J} \sigma_I d_I^\theta u_{\tilde{I}}$ *and* $\sigma_I \in \{+1,-1\}$.

Now let us consider two special cases:

I) Without loss of generality we may assume $g_i \geq g_{i+1}$ for $1 \leq i \leq n-1$. Furthermore let $p \neq 0$ and $g_l > g_{l+1}$. Then $J = \{\{1,\dots,l\}\}$, $\hat{g} = g_1+\dots+g_l$ and

$$D_\theta = \pm\det\begin{bmatrix} \omega_1^{g_1}\gamma_{11}^\theta & \cdots & \omega_l^{g_1}\gamma_{1l}^\theta \\ \vdots & & \vdots \\ \omega_1^{g_l}\gamma_{l1}^\theta & \cdots & \omega_l^{g_l}\gamma_{ll}^\theta \end{bmatrix} \det\begin{bmatrix} u_{l+1,1} & \cdots & u_{l+1,p} \\ \vdots & & \vdots \\ u_{n,1} & \cdots & u_{n,p} \end{bmatrix}.$$

If $p = 0$, then $\hat{k} = g_1+\dots+g_n$ and

$$D_\theta = \det\begin{bmatrix} \omega_1^{g_1}\gamma_{11}^\theta & \cdots & \omega_n^{g_1}\gamma_{1n}^\theta \\ \vdots & & \vdots \\ \omega_1^{g_n}\gamma_{n1}^\theta & \cdots & \omega_n^{g_n}\gamma_{nn}^\theta \end{bmatrix}.$$

II) Separable boundary conditions

Here we suppose that we have a two-point boundary operator

$$\tilde{T}^R\tilde{y} = W^a\tilde{y}(a) + W^b\tilde{y}(b)$$

with $e_i^tW^a = 0$ or $e_i^tW^b = 0$ for $i = 1,\dots,n$. We may assume $e_i^tW^a = 0$ $(i = 1,\dots,s)$ and $e_i^tW^b = 0$ $(i = s+1,\dots,n)$ for some

$s \in \{1,\ldots,n-1\}$. Let

$$|\theta| := \#\{\nu \in \{1,\ldots,l\} : \theta_\nu = 2\},$$

$$\{i_1,\ldots,i_{|\theta|}\} := \{\nu \in \{1,\ldots,l\} : \theta_\nu = 2\},$$

and

$$\{i_{|\theta|+1},\ldots,i_l\} := \{\nu \in \{1,\ldots,l\} : \theta_\nu = 1\}.$$

Since

$$e_i^t v_\nu^\theta = 0 \quad \text{for} \quad 1 \le i \le s,\ \theta_\nu = 1 \quad \text{or} \quad s+1 \le i \le n,\ \theta_\nu = 2$$

we obtain

$$v^\theta(\rho) = \pm\rho^{lqp}\det\begin{bmatrix} [u_{11}]\ldots[u_{1p}] & (\rho\omega_{i_1})^{g_1}[\gamma_1^{(2)}]\ldots(\rho\omega_{i_{|\theta|}})^{g_1}[\gamma_1^{(2)}] & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & (\rho\omega_{i_1})^{g_s}[\gamma_s^{(2)}]\ldots(\rho\omega_{i_{|\theta|}})^{g_s}[\gamma_s^{(2)}] & 0 & \ldots & 0 \\ \vdots & 0 \qquad\qquad 0 & (\rho\omega_{i_{|\theta|+1}})^{g_{s+1}}[\gamma_{s+1}^{(1)}] & \ldots & (\rho\omega_{i_l})^{g_{s+1}}[\gamma_{s+1}^{(1)}] \\ \vdots & \vdots \qquad\qquad \vdots & \vdots & & \vdots \\ [u_{n1}]\ldots[u_{np}] & 0 \qquad\qquad 0 & (\rho\omega_{i_{|\theta|+1}})^{g_n}[\gamma_n^{(1)}] & \ldots & (\rho\omega_{i_l})^{g_n}[\gamma_n^{(1)}] \end{bmatrix}.$$

The latter matrix contains two zero block matrices of the size $(n-s)\times|\theta|$ and $s\times(l-|\theta|)$. From the definition of the determinant we thus obtain $v^\theta = 0$ if $(n-s)+|\theta| > n$ or $s+(l-|\theta|) > n$. Hence

$$v^\theta = 0 \quad \text{if} \quad |\theta| < s-p \quad \text{or} \quad |\theta| > s.$$

Now let $s-p \le |\theta| \le s$. We may assume $g_1 \ge \cdots \ge g_s$ and $g_{s+1} \ge \cdots \ge g_n$. If, in addition, $g_1 > \ldots > g_s$ and $g_{s+1} > \ldots > g_n$ then

$$v^\theta(\rho) = \pm\rho^{k^\theta}\left\{ \det\begin{bmatrix} u_\theta^1 & \tilde{d}_\theta^1 & 0 \\ u_\theta^2 & 0 & 0 \\ u_\theta^3 & 0 & \tilde{d}_\theta^2 \\ u_\theta^4 & 0 & 0 \end{bmatrix}\begin{matrix} \}\ |\theta| \\ \}\ s-|\theta| \\ \}\ l-|\theta| \\ \}\ n-s-l+|\theta| \end{matrix} + o(1) \right\}$$

$$\underbrace{}_{p} \quad \underbrace{}_{|\theta|} \quad \underbrace{}_{l-|\theta|}$$

where

$$k^{\theta} := lqp + g_1 + \dots + g_{|\theta|} + g_{s+1} + \dots + g_{s+l-|\theta|} ,$$

$$\begin{bmatrix} u_{\theta}^1 \\ u_{\theta}^2 \\ u_{\theta}^3 \\ u_{\theta}^4 \end{bmatrix} = \begin{bmatrix} u_{11} & \dots & u_{1p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ u_{n1} & \dots & u_{np} \end{bmatrix},$$

$$\tilde{d}_{\theta}^1 = \operatorname{diag}(\gamma_1^{(2)},\dots,\gamma_{|\theta|}^{(2)}) \begin{bmatrix} \omega_{i_1}^{g_1} & \dots & \omega_{i_{|\theta|}}^{g_1} \\ \vdots & & \vdots \\ \omega_{i_1}^{g_{|\theta|}} & \dots & \omega_{i_{|\theta|}}^{g_{|\theta|}} \end{bmatrix}$$

$$\tilde{d}_{\theta}^2 = \operatorname{diag}(\gamma_{s+1}^{(1)},\dots,\gamma_{s+l-|\theta|}^{(1)}) \begin{bmatrix} \omega_{i_{|\theta|+1}}^{g_{s+1}} & \dots & \omega_{i_l}^{g_{s+1}} \\ \vdots & & \vdots \\ \omega_{i_{|\theta|+1}}^{g_{s+l-|\theta|}} & \dots & \omega_{i_l}^{g_{s+l-|\theta|}} \end{bmatrix}.$$

From

$$\det \begin{bmatrix} u_{\theta}^1 & \tilde{d}_{\theta}^1 & 0 \\ u_{\theta}^2 & 0 & 0 \\ u_{\theta}^3 & 0 & \tilde{d}_{\theta}^2 \\ u_{\theta}^4 & 0 & 0 \end{bmatrix} = \pm \det \begin{bmatrix} u_{\theta}^1 & \tilde{d}_{\theta}^1 & 0 \\ u_{\theta}^2 & 0 & 0 \\ u_{\theta}^4 & 0 & 0 \\ u_{\theta}^3 & 0 & \tilde{d}_{\theta}^2 \end{bmatrix}$$

$$= \pm \det \begin{bmatrix} u_{\theta}^1 & \tilde{d}_{\theta}^1 \\ u_{\theta}^2 & 0 \\ u_{\theta}^4 & 0 \end{bmatrix} \det(\tilde{d}_{\theta}^2) = \pm \det \begin{bmatrix} u_{\theta}^2 \\ u_{\theta}^4 \end{bmatrix} \det(\tilde{d}_{\theta}^1) \det(\tilde{d}_{\theta}^2)$$

we infer

(10.14) $\quad v^{\theta}(\rho) = \rho^{k^{\theta}} [\tilde{D}_{\theta}]$

where $\tilde{D}_{\theta} = \pm d_{\theta}^1 d_{\theta}^2 u_{\theta}$, $d_{\theta}^i = \det(\tilde{d}_{\theta}^i)$ $(i = 1,2)$ and

$$u_\theta = \det \begin{bmatrix} u_{|\theta|+1,1} & \cdots & u_{|\theta|+1,p} \\ \vdots & & \vdots \\ u_{s,1} & & u_{s,p} \\ u_{s+1-|\theta|+1,1} & \cdots & u_{s+1-|\theta|+1,p} \\ \vdots & & \vdots \\ u_{n,1} & \cdots & u_{n,p} \end{bmatrix}.$$

In the case $p=0$ we have $l=n$, $|\theta|=s$ and $\tilde{D}_\theta = d_\theta^1 d_\theta^2$ without any restriction on the g_i.

For the sake of simplicity we only made use of the first terms of the asymptotic expansion of $\tilde{Y}(x,\rho)$. If we would apply the full asymptotics which we stated in Theorem (9.2), we would obtain more precise asymptotic expansions for the characteristic determinant, cf. for example Freiling [12] and Heisecke [14]. More complete asymptotic formulas for $D(\rho)$ yield more general regularity criteria than those which we will state in the following section.

11. CRITERIA FOR REGULARITY FOR BOUNDARY EIGENVALUE PROBLEMS OF EBERHARD TYPE

In this section we shall apply Theorem (5.14) to the eigenvalue problem (10.3) and, by Theorem (7.11), also to the eigenvalue problem (10.1). According to (10.5), (10.6), (10.10) and (10.13) the characteristic determinant has the asymptotic representation

(11.1) $$D(\rho) = \rho^{-(lp+\frac{l(l-1)}{2})}[d] \sum_{\theta\in\Theta} e^{\rho c_\theta} \rho^{\hat{k}}[D_\theta]$$

where

$$c_\theta = \sum_{\nu=1}^{l} \omega_\nu (a_{\theta_\nu} - a)$$

for $\theta = (\theta_1,\ldots,\theta_l) \in \Theta = \{1,\ldots,m\}^l$. According to (5.10)

$$E = \{c_\theta : \theta \in \Theta\}$$

and P is the convex hull of E. By Proposition (5.11), $P = P_{\{0,1\}}$ which is the convex hull of

$$E_{\{0,1\}} = \{ \sum_{\nu\in\phi} \omega_{\nu+1}(b-a) : \phi \subset \mathbb{Z}_l \}.$$

As in section 5 we denote the set of all vertices of P by $\tilde{E}$. In order to apply Theorem (5.12) we have to calculate $\tilde{E}$ (and $\hat{E}$). For the sake of simplicity we suppose that $b-a=1$. For $r=0,\ldots,l$ and $h \in \mathbb{Z}_l$ we set

$$\phi_r := \{0,1,\ldots,r-1\} \subset \mathbb{Z}_l,$$
$$\phi_r^h := h + \phi_r = \{h,h+1,\ldots,h+r-1\} \subset \mathbb{Z}_l.$$

For $\phi \subset \mathbb{Z}_l$ we set

$$\phi\omega := \sum_{\kappa\in\phi} \omega_{\kappa+1}.$$

(11.2) LEMMA. i) *Let* l *be even. Then*

$$\tilde{E} = \{\phi_{\frac{l}{2}}^h \omega : h \in \mathbb{Z}_l\},$$

and a point $\phi\omega$ $(\phi \subset \mathbb{Z}_l)$ *belongs to the boundary of* P *iff* $\phi = \phi_r^h$ *where* $r \in \{\frac{l}{2}-1, \frac{l}{2}, \frac{l}{2}+1\}$ *and* $h \in \mathbb{Z}_l$.

ii) *Let* l *be odd. Then*

$$\tilde{E} = \{\phi_r^h \omega : r \in \{\tfrac{l-1}{2}, \tfrac{l+1}{2}\},\ h \in \mathbb{Z}_l\},$$

and a point $\phi\omega$ $(\phi \subset \mathbb{Z}_l)$ *belongs to the boundary of* P *iff* $\phi\omega \in \tilde{E}$.

PROOF. i) For $l=2$ we have $\tilde{E} = \{-1,+1\}$, $P = [-1,+1]$ and the points

$$\{\phi_r^h\omega : r \in \{0,1,2\},\ h \in \mathbb{Z}_2\} = \{-1,0,+1\}$$

belong to the boundary of P (in $\mathbb{C}$).

In the following suppose $l \geq 4$. Set $\beta_h := \phi_{\frac{l}{2}}^h \omega$ $(h \in \mathbb{Z}_l)$.

We have

$$\beta_h = \sum_{\kappa=h}^{h+\frac{l}{2}-1} \omega_{\kappa+1} = \omega_{h+1} \sum_{\kappa=0}^{\frac{l}{2}-1} \exp\left(\frac{2\pi i\kappa}{l}\right)$$

$$= \omega_{h+1} \frac{2}{1-\exp\left(\frac{2\pi i}{l}\right)}.$$

Hence the boundary of the convex hull of $\{\phi^h_{\frac{l}{2}}\omega : h \in \mathbb{Z}_l\}$ is a regular polygon with the l sides $\overline{\beta_h, \beta_{h+1}}$ $(h \in \mathbb{Z}_l)$. For every $h \in \mathbb{Z}_l$ we have $\omega_{h+1} + \omega_{h+\frac{l}{2}+1} = 0$. Hence

$$\phi^h_{\frac{l}{2}+1}\omega = \sum_{\kappa=h}^{h+\frac{l}{2}} \omega_{\kappa+1} = \sum_{\kappa=h+1}^{h+\frac{l}{2}-1} \omega_{\kappa+1} + \omega_{h+1} + \omega_{h+\frac{l}{2}+1} = \phi^{h+1}_{\frac{l}{2}-1}\omega .$$

From

$$\phi^h_{\frac{l}{2}-1}\omega = \sum_{\kappa=h}^{h+\frac{l}{2}-2} \omega_{\kappa+1} = \frac{1}{2}\left\{ \sum_{\kappa=h-1}^{h+\frac{l}{2}-2} \omega_{\kappa+1} + \sum_{\kappa=h}^{h+\frac{l}{2}-1} \omega_{\kappa+1} - (\omega_h + \omega_{h+\frac{l}{2}}) \right\}$$

$$= \frac{1}{2}\left\{ \phi^{h-1}_{\frac{l}{2}}\omega + \phi^h_{\frac{l}{2}}\omega \right\}$$

we conclude that the point $\phi^h_{\frac{l}{2}-1}\omega = \phi^{h-1}_{\frac{l}{2}+1}\omega$ is the midpoint of the line segment $\overline{\beta_{h-1}, \beta_h}$. Note that any point $z \in \mathbb{C}$ with $|z| < |\phi_{\frac{l}{2}-1}\omega|$ lies in the interior of the regular polygon with the vertices β_h $(h \in \mathbb{Z}_l)$. Now let $\phi \in \mathbb{P}(\mathbb{Z}_l) \setminus \{\phi^h_r : r \in \{\frac{l}{2}-1, \frac{l}{2}, \frac{l}{2}+1\},\ h \in \mathbb{Z}_l\}$. The assertion i) is proved if we show that $|\phi\omega| < |\phi_{\frac{l}{2}-1}\omega|$. Since

$$\phi\omega + (\mathbb{Z}_l \setminus \phi)\omega = \mathbb{Z}_l\omega = \sum_{\kappa=0}^{l-1} \omega_{\kappa+1} = 0 ,$$

we may assume that $\#\phi \le \frac{l}{2}$. Furthermore we suppose that $\phi\omega \neq 0$. Define

$$\phi^+ := \{\kappa \in \mathbb{Z}_l : \mathrm{Re}\{(\phi\omega)^{-1}\omega_{\kappa+1}\} > 0\}.$$

There is a $h \in \mathbb{Z}_l$ such that $\phi^+ = \phi^h_{\frac{l}{2}-1}$ or $\phi^+ = \phi^h_{\frac{l}{2}}$.

α) Here we consider the case that $\mathrm{Re}\{(\phi\omega)^{-1}\omega_{\kappa+1}\} > 0$ for all $\kappa \in \phi$. We conclude that ϕ is a proper subset of ϕ^+.

α_1) Let $\phi^+ = \phi^h_{\frac{l}{2}-1}$. Then

$$1 = \sum_{\kappa\in\phi} (\phi\omega)^{-1}\omega_{\kappa+1} = \mathrm{Re}\sum_{\kappa\in\phi} (\phi\omega)^{-1}\omega_{\kappa+1}$$

$$< \mathrm{Re}\sum_{\kappa\in\phi^+} (\phi\omega)^{-1}\omega_{\kappa+1} \leq |\phi\omega|^{-1}|\phi^+\omega|$$

whence

$$|\phi\omega| < |\phi^h_{\frac{1}{2}-1}\omega| = |\phi_{\frac{1}{2}-1}\omega| .$$

α_2) Let $\phi^+ = \phi^h_{\frac{1}{2}}$ and let $h \notin \phi$ or $h+\frac{1}{2}-1 \notin \phi$. We set $\phi' := \phi^+ \setminus \{h\}$ if $h \notin \phi$ and $\phi' := \phi^+ \setminus \{h+\frac{1}{2}-1\}$ if $h+\frac{1}{2}-1 \notin \phi$. Then $\phi' = \phi^{h'}_{\frac{1}{2}-1}$ for some $h' \in \mathbb{Z}_1$ and $\phi \subsetneqq \phi^{h'}_{\frac{1}{2}-1}$. Thus according to α_1) we have

$|\phi\omega| < |\phi_{\frac{1}{2}-1}\omega|$.

α_3) Let $\phi^+ = \phi^h_{\frac{1}{2}}$ and $\{h, h+\frac{1}{2}-1\} \subset \phi$. Since $\phi \neq \phi^h_{\frac{1}{2}}$ there is a $j \in \phi^h_{\frac{1}{2}} \setminus \phi$. Let $\kappa' \in \{h, h+\frac{1}{2}-1\}$ such that

$$\mathrm{Re}\{(\phi\omega)^{-1}\omega_{\kappa'+1}\} = \min\{\mathrm{Re}\{(\phi\omega)^{-1}\omega_{h+1}\}, \mathrm{Re}\{(\phi\omega)^{-1}\omega_{h+\frac{1}{2}}\}\}$$

and set $\phi' := \phi^+ \setminus \{\kappa'\}$. Then

$$\mathrm{Re}\{(\phi\omega)^{-1}\omega_{\kappa'+1}\} < \mathrm{Re}\{(\phi\omega)^{-1}\omega_{j+1}\}.$$

Therefore

$$1 = \mathrm{Re}\sum_{\kappa\in\phi} (\phi\omega)^{-1}\omega_{\kappa+1} < \mathrm{Re}\left\{\sum_{\substack{\kappa\in\phi \\ \kappa\neq\kappa'}} (\phi\omega)^{-1}\omega_{\kappa+1} + (\phi\omega)^{-1}\omega_{j+1}\right\}$$

$$\leq \mathrm{Re}\sum_{\kappa\in\phi'} (\phi\omega)^{-1}\omega_{\kappa+1} \leq |\phi\omega|^{-1}|\phi'\omega|$$

and hence again

$$|\phi\omega| < |\phi_{\frac{1}{2}-1}\omega|$$

Thus part i) is proved if ϕ fulfills the assumptions of α).

β) Here we consider the case that $\phi \not\subset \phi^+$. We choose some $\kappa_1 \in \phi \setminus \phi^+$ and set $\phi' := \phi \setminus \{\kappa_1\}$. Since

$$1 = \sum_{\kappa\in\phi} (\phi\omega)^{-1}\omega_{\kappa+1},$$

ϕ' is nonvoid and $\operatorname{Im} \sum_{\kappa \in \phi} (\phi\omega)^{-1} \omega_{\kappa+1} = 0$. We infer that

$$1 = \left\{ \operatorname{Re} \sum_{\kappa \in \phi} (\phi\omega)^{-1} \omega_{\kappa+1} \right\}^2$$

$$< \left\{ \operatorname{Re} \sum_{\kappa \in \phi'} (\phi\omega)^{-1} \omega_{\kappa+1} \right\}^2 + \left\{ \operatorname{Im} \sum_{\kappa \in \phi'} (\phi\omega)^{-1} \omega_{\kappa+1} \right\}^2$$

$$= |\phi\omega|^{-2} \, |\phi'\omega|^2 .$$

Now we have to repeat the procedure from the beginning with ϕ' instead of ϕ. After a finite number of steps we obtain some ϕ'' with the following properties:

$$\#\phi'' < \#\phi, \quad |\phi\omega| < |\phi''\omega|$$

and either

$$\phi'' = \phi^h_{\frac{l}{2}-1} \quad \text{for some} \quad h \in \mathbb{Z}_l$$

or

$$\phi'' \in \mathbb{P}(\mathbb{Z}_l) \setminus \{\phi^h_r : r \in \{\tfrac{l}{2}-1, \tfrac{l}{2}, \tfrac{l}{2}+1\},\ h \in \mathbb{Z}_l\}$$

and fulfills the assumption in α). In both cases we infer that

$$|\phi\omega| < |\phi''\omega| \lesssim |\phi_{\frac{l}{2}-1}\omega| .$$

ii) For $l = 1$ we have $\tilde{E} = \{0,1\}$ and $P = [0,1]$. In the following we suppose that $l \geq 3$. Let $h \in \mathbb{Z}_l$. A simple calculation shows that

$$\phi^h_{\frac{l-1}{2}}\omega = \sum_{\kappa=h}^{h+\frac{l-1}{2}-1} \omega_{\kappa+1} = \omega_{h+1} \sum_{\kappa=0}^{\frac{l-1}{2}-1} \exp\left(\frac{2\pi i \kappa}{l}\right)$$

$$= \omega_{h+1} \frac{1 + \exp\left(-\frac{\pi i}{l}\right)}{1 - \exp\left(\frac{2\pi i}{l}\right)} .$$

We set

$$\gamma := \frac{1 + \exp\left(-\frac{\pi i}{l}\right)}{1 - \exp\left(\frac{2\pi i}{l}\right)}$$

and $\beta_{2h} := \omega_{h+1}\gamma = \phi^h_{\frac{l-1}{2}}\omega$. Since $\mathbb{Z}_l\omega = 0$ we infer that

$$\phi^{h}_{\frac{l+1}{2}}\omega = -(\mathbb{Z}_l \setminus \phi^{h}_{\frac{l+1}{2}})\omega = -\phi^{h+\frac{l+1}{2}}_{\frac{l-1}{2}}\omega = -\omega_{h+\frac{l+1}{2}+1}\gamma$$

$$= \exp(-\pi i)\exp\left(\frac{2\pi i(h+\frac{l+1}{2})}{l}\right)\gamma = \omega_{h+1}\gamma\exp(\tfrac{\pi i}{l}).$$

We define $\beta_{2h+1} := \omega_{h+1}\gamma\exp(\frac{\pi i}{l})$. Thus the boundary of the convex hull of $\{\phi^{h}_{\frac{l-1}{2}}\omega, \phi^{h}_{\frac{l+1}{2}}\omega : h \in \mathbb{Z}_l\}$ is a regular polygon with $2l$ vertices and the sides $\overline{\beta_h, \beta_{h+1}}$ $(h \in \mathbb{Z}_{2l})$. The inradius of this regular polygon is $|\gamma|\cos(\frac{\pi}{2l})$. Now let
$\phi \in \mathbb{P}(\mathbb{Z}_l) \setminus \{\phi^{h}_{r} : r \in \{\frac{l-1}{2}, \frac{l+1}{2}\},\ h \in \mathbb{Z}_l\}$; we have to prove that $|\phi\omega| < |\gamma|\cos(\frac{\pi}{2l})$. Since $\phi\omega = -(\mathbb{Z}_l \setminus \phi)\omega$, we may assume that $\#\phi \leq \frac{l-1}{2}$. Furthermore we suppose that $\phi\omega \neq 0$. Define

$$\phi^{+} := \{\kappa \in \mathbb{Z}_l : \mathrm{Re}\{(\phi\omega)^{-1}\omega_{\kappa+1}\} > 0\}.$$

There is a $h \in \mathbb{Z}_l$ such that $\phi^{+} = \phi^{h}_{\frac{l-1}{2}}$ or $\phi^{+} = \phi^{h}_{\frac{l+1}{2}}$.

α) As in part i) we first consider the case that $\mathrm{Re}\{(\phi\omega)^{-1}\omega_{\kappa+1}\} > 0$ for all $\kappa \in \phi$. Also here ϕ is a proper subset of ϕ^{+}.

α_1) Let $\phi^{+} \setminus \phi = \{\kappa_1\}$. Then $\phi^{+}\omega = \phi\omega(1 + (\phi\omega)^{-1}\omega_{\kappa_1+1})$ and thus

$$|\phi^{+}\omega|^2 = |\phi\omega|^2\left\{\left(\mathrm{Im}((\phi\omega)^{-1}\omega_{\kappa_1+1})\right)^2 + \left(1+\mathrm{Re}((\phi\omega)^{-1}\omega_{\kappa_1+1})\right)^2\right\}$$

$$= |\phi\omega|^2\left\{\left(\mathrm{Im}((\phi\omega)^{-1}\omega_{\kappa_1+1})\right)^2 + \left(\mathrm{Re}((\phi\omega)^{-1}\omega_{\kappa_1+1})\right)^2 + 2\,\mathrm{Re}((\phi\omega)^{-1}\omega_{\kappa_1+1}) + 1\right\}$$

$$> |\phi\omega|^2\left\{|(\phi\omega)^{-1}\omega_{\kappa_1+1}|^2 + 1\right\} = 1 + |\phi\omega|^2.$$

Since $|\phi^{+}\omega| = |\gamma|$, we infer that

$$|\phi\omega|^2 < |\gamma|^2 - 1 = |\gamma|^2(1 - \frac{1}{|\gamma|^2}).$$

It is easy to see that $|\gamma| = (2\sin(\frac{\pi}{2l}))^{-1}$. Hence

$$|\phi\omega|^2 < |\gamma|^2(1 - 4\sin^2(\tfrac{\pi}{2l})) < |\gamma|^2\cos^2(\tfrac{\pi}{2l}).$$

α_2) Let $\#(\phi^+ \smallsetminus \phi) \geq 2$ and assume there are $h \in \mathbb{Z}_l$, $r \in \{0,\dots,\frac{l+1}{2}\}$ such that $\phi = \phi_r^h$. From $\#\phi \leq \#\phi^+ - 2 \leq \frac{l+1}{2} - 2$ we infer $2r+1 \leq l-2$. We have

$$\phi\omega = \sum_{\kappa=h+1}^{h+r} \omega_\kappa = \omega_{h+1} \sum_{\kappa=0}^{r-1} \exp(\frac{2\pi i\kappa}{l}) = \omega_{h+1} \frac{1-\exp(\frac{2\pi i r}{l})}{1-\exp(\frac{2\pi i}{l})}$$

and thus

$$|\phi\omega| = \frac{\sin(\frac{\pi r}{l})}{\sin(\frac{\pi}{l})}.$$

Hence

$$|\phi\omega| = |\gamma| \frac{\sin(\frac{\pi r}{l})}{\cos(\frac{\pi}{2l})} = |\gamma| \frac{\sin(\frac{\pi}{2l}(2r+1))\cos(\frac{\pi}{2l}) - \cos(\frac{\pi}{2l}(2r+1))\sin(\frac{\pi}{2l})}{\cos(\frac{\pi}{2l})}$$

$$< |\gamma| \sin(\frac{\pi}{2l}(2r+1)) = |\gamma|\cos(\frac{\pi}{2l}(l-(2r+1)))$$

$$< |\gamma|\cos(\frac{\pi}{2l}).$$

α_3) Let $\#(\phi^+ \smallsetminus \phi) \geq 2$ and assume $\phi \notin \{\phi_r^h : r \in \{0,\dots,l\},\ h \in \mathbb{Z}_l\}$. Set

$$d(\phi) := \min\{r \in \{0,\dots,l\} : \exists h \in \mathbb{Z}_l \quad \phi \subset \phi_r^h\}.$$

Since $\phi \subset \phi^+$, $d(\phi)$ is at most $\frac{l+1}{2}$. Choose r and h such that $r = d(\phi)$ and $\phi \subset \phi_r^h$. We have $r \geq 2$. Because of the minimality of r the two different numbers h and $h+r-1$ belong to ϕ. We assert that $\phi_r^h \subset \phi^+$. Otherwise $\mathbb{Z}_l \smallsetminus \phi^+$ and $\{h,h+r-1\}$ would be disjoint subsets of ϕ_r^h and this would lead us to the contradiction

$$\frac{l+1}{2} \geq r \geq l - \frac{l+1}{2} + 2 = \frac{l+3}{2}.$$

Hence the above assertion is clear. It follows that we can choose $\kappa_1 \in \{h,h+r-1\}$ such that

$$\mathrm{Re}\{(\phi\omega)^{-1}\omega_{\kappa_1+1}\} = \min\{\mathrm{Re}\{(\phi\omega)^{-1}\omega_{\kappa+1} \quad : \kappa \in \phi_r^h\}.$$

Furthermore, choose $\kappa_2 \in \phi_r^h \smallsetminus \phi$ and set $\phi' := (\phi \smallsetminus \{\kappa_1\}) \cup \{\kappa_2\}$. Then $\#\phi' = \#\phi$, $d(\phi') < d(\phi)$ and $\phi\omega = \phi'\omega + \omega_{\kappa_1+1} - \omega_{\kappa_2+1}$, whence

$$1 = \mathrm{Re}\{(\phi\omega)^{-1}\phi'\omega\} + \mathrm{Re}\{(\phi\omega)^{-1}\omega_{\kappa_1+1}\} - \mathrm{Re}\{(\phi\omega)^{-1}\omega_{\kappa_2+1}\}$$

$$\leq \mathrm{Re}\{(\phi\omega)^{-1}\phi'\omega\} \leq |\phi\omega|^{-1}|\phi'\omega|$$

and thus $|\phi\omega| \leq |\phi'\omega|$.

β) Here we consider the case that $\phi \not\subset \phi^+$. As in the case i),β) we choose some $\kappa_1 \in \phi \setminus \phi^+$, set $\phi' := \phi \setminus \{\kappa_1\}$ and infer that $\#\phi' < \#\phi$ and $|\phi\omega| < |\phi'\omega|$.

We continue our proof of assertion ii) by treating the cases α_3) and β) together. In both cases we have $\#\phi' \leq \#\phi$, $d(\phi') \leq d(\phi)$, $\#\phi' + d(\phi') < \#\phi + d(\phi)$ and $|\phi\omega| < |\phi'\omega|$. As in the proof of i) we have to repeat the complete procedure with ϕ' instead of ϕ. After a finite number of steps, we obtain some ϕ'' which fulfills the asssumption α_1) or α_2).

$\hat{E}$ has been defined in section 5. Since in (11.1) the exponent $\hat{k}$ does not depend on θ, we have $\hat{E} = \tilde{E}$ by Remark (5.13). For $\phi_r^h \subset \mathbb{Z}_l$ we define $\hat{\theta}_r^h = (\theta_1, \ldots, \theta_l) \in \{1,m\}^l \subset \Theta$ by setting $\theta_j = m$, if $j-1 \in \phi_r^h$, and $\theta_j = 1$, if $j-1 \notin \phi_r^h$. We abbreviate $\hat{\theta}_r^o$ by $\hat{\theta}_r$.

(11.3) THEOREM. *Let the assumptions of Theorem* (9.2) *be fulfilled and suppose that*

i) $D_{\hat{\theta}_{\frac{l}{2}}} \neq 0$ *if* l *is even,*

ii) $D_{\hat{\theta}_{\frac{l-1}{2}}} \neq 0$ *and* $D_{\hat{\theta}_{\frac{l+1}{2}}} \neq 0$ *if* l *is odd.*

Then there are $\tilde{p},\tilde{p}' \in \mathbb{Z}$ *such that the operator function* L, *given by* (10.1), *is* $|\ |_{H^j(a,b)}$*-regular of order* $(\tilde{p},\tilde{p}')$ *for* $j=0,\ldots,n$.

For the proof we state the following

(11.4) PROPOSITION. *Let* $0 \leq r \leq l$ *and* $h \in \mathbb{Z}_l$. *There are* $a_{rh} \in \mathbb{C} \setminus \{0\}$ *such that* $D_{\hat{\theta}_r^h} = a_{rh} D_{\hat{\theta}_r}$.

PROOF. The assertion is trivial for $r = 0$ or $r = l$ because we have $\hat{\theta}_0^h = \{1,\ldots,1\}$ and $\hat{\theta}_l^h = \{m,\ldots,m\}$ for all $h \in \mathbb{Z}_l$. Suppose now that $0 < r < l$ and let J be defined as in section 10. By Proposition (10.13) it is sufficient to prove that $d_I^{\hat{\theta}_r^h} = a_{rh} d_I^{\hat{\theta}_r}$ for all $I \in J$ where a_{rh} does not depend on I. We have

$$d_I^{\hat{\theta}_r^h} = (-1)^{h(l-h)} \det \begin{bmatrix} \omega_{h+1}^{g_{i_1}} \gamma_{i_1}^b \cdots \omega_{h+r}^{g_{i_1}} \gamma_{i_1}^b & \omega_{h+r+1}^{g_{i_1}} \gamma_{i_1}^a \cdots \omega_{h+l}^{g_{i_1}} \gamma_{i_1}^a \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ \omega_{h+1}^{g_{i_l}} \gamma_{i_l}^b \cdots \omega_{h+r}^{g_{i_l}} \gamma_{i_l}^b & \omega_{h+r+1}^{g_{i_l}} \gamma_{i_l}^a \cdots \omega_{h+l}^{g_{i_l}} \gamma_{i_l}^a \end{bmatrix}$$

$$= (-1)^{h(l-h)} \omega_{h+1}^{g_{i_1}+\dots+g_{i_l}} \det \begin{bmatrix} \omega_1^{g_{i_1}} \gamma_{i_1}^b \cdots \omega_r^{g_{i_1}} \gamma_{i_1}^b & \omega_{r+1}^{g_{i_1}} \gamma_{i_1}^a \cdots \omega_l^{g_{i_1}} \gamma_{i_1}^a \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ \omega_1^{g_{i_l}} \gamma_{i_l}^b \cdots \omega_r^{g_{i_l}} \gamma_{i_l}^b & \omega_{r+1}^{g_{i_l}} \gamma_{i_l}^a \cdots \omega_l^{g_{i_l}} \gamma_{i_l}^a \end{bmatrix}$$

$$= a_{rh} d_I^{\hat{\theta}_r} .$$

Now we are ready for the proof of Theorem (11.3): We have already stated in section 10 that the regularity Theorem (5.14) can be applied to the operator function $\tilde{T}(\rho^l)$ defined by (10.3). $D(\rho)$ is the corresponding characteristic determinant. By Lemma (11.2) and Proposition (11.4), The assumption i) or ii) yields that $D_\theta \neq 0$ for all $c_\theta \in \hat{E} = \tilde{E}$. According to Theorem (5.14) $\tilde{T}$ is $L_2^n(a,b)$-regular. By the proof of Proposition (6.1) we obtain the $H_1^n(a,b)$-regularity of $\tilde{T}$: there are $\hat{p}, \hat{p}' \in \mathbb{Z}$ and an increasing sequence $(d_\nu)_0^\infty$ of positive numbers with $d_\nu \to \infty$ for $\nu \to \infty$ such that

$$\int_{|\rho|=d_\nu} |\rho^{-\hat{p}-1} R_1(\rho^l) f_1|_{H_1^n(a,b)} |d\rho| \to 0 \qquad (\nu \to \infty)$$

and

$$\int_{|\rho|=d_\nu} |\rho^{-\hat{p}'-1} R_2(\rho^l) f_2|_{H_1^n(a,b)} |d\rho| \to 0 \qquad (\nu \to \infty)$$

for all $f_1 \in L_2^n(a,b)$ and $f_2 \in \mathbb{C}^n$. Choose $\tilde{p}, \tilde{p}' \in \mathbb{Z}$ such that $l(\tilde{p}+1) \geq \hat{p}+1$ and $l(\tilde{p}'+1) \geq \hat{p}'+1$. The substitution $\lambda = \rho^l$ yields

$$\int_{|\lambda|=d_\nu^l} |\lambda^{-\tilde{p}-1} R_1(\lambda) f_1 e_n|_{H_1^n(a,b)} |d\lambda| \to 0 \qquad (\nu \to \infty)$$

and

$$\int_{|\lambda|=d_\nu^l} |\lambda^{-\tilde{p}'-1} R_2(\lambda) f_2|_{H_1^n(a,b)} |d\lambda| \to 0 \qquad (\nu \to \infty)$$

for all $f_1 \in L_2(a,b)$ and $f_2 \in \mathbb{C}^n$. From $\tilde{T}^D(\lambda)R_1(\lambda) = id_{L_2^n(a,b)}$ and $\tilde{T}^D(\lambda)R_2(\lambda) = 0$ we infer

$$(e_j^t R_1(\lambda) f_1 e_n)' = e_j^t \tilde{T}^D(\lambda) R_1(\lambda) f_1 e_n + e_j^t \hat{A}(\cdot,\lambda) R_1(\lambda) f_1 e_n$$
$$= e_{j+1}^t R_1(\lambda) f_1 e_n$$

and

$$(e_j^t R_2(\lambda) f_2)' = e_j^t \hat{A}(\cdot,\lambda) R_2(\lambda) f_2 = e_{j+1}^t R_2(\lambda) f_2$$

for $j = 1,\ldots,n-1$, $f_1 \in L_2(a,b)$ and $f_2 \in \mathbb{C}^n$. These relationships and (4.1),(4.2), (7.11), (7.12), (7.13) lead to

$$\int_{|\lambda|=d_\nu^1} |\lambda^{-\tilde{p}-1} (\hat{R}_1(\lambda) f_1)^{(j)}|_{H_1(a,b)} |d\lambda|$$

$$= \int_{|\lambda|=d_\nu^1} |\lambda^{-\tilde{p}-1} e_{j+1}^t R_1(\lambda) e_n f_1|_{H_1(a,b)} |d\lambda| \to 0 \qquad (\nu \to \infty)$$

for $j = 0,\ldots,n-1$ and $f_1 \in L_2(a,b)$. Analogously we obtain

$$\int_{|\lambda|=d_\nu^1} |\lambda^{-\tilde{p}'-1} (\hat{R}_2(\lambda) f_2)^{(j)}|_{H_1(a,b)} |d\lambda| \to 0 \qquad (\nu \to \infty)$$

for $j = 0,\ldots,n-1$ and $f_2 \in \mathbb{C}^n$, whereby Theorem (11.3) is proved.

For separable boundary conditions we would have $D_\theta = 0$ for nearly all indices θ if we would use the representation of the characteristic determinant considered above. Therefore we return to the special representation given in section 10. We assume as in section 10 that $g_1 > \ldots > g_s$ and $g_{s+1} > \ldots > g_n$. Then

$$(11.5) \qquad D(\rho) = \rho^{-(lp + \frac{l(l-1)}{2})} [d] \sum_{\theta \in \Theta} e^{\rho c_\theta} \rho^{k^\theta} [\tilde{D}_\theta]$$

where $k^\theta = lqp + g_1 + \ldots + g_{|\theta|} + g_{s+1} + \ldots + g_{s+l-|\theta|}$. Since k^θ depends on θ it may happen here that $\hat{E} \neq \tilde{E}$.

In order to assure regularity we must have $\tilde{D}_\theta \neq 0$ for $c_\theta \in \tilde{E}$. From the result in section 10 we infer that it is necessary for regularity that $s-p \le \frac{l}{2} \le s$, if l is even, and $s-p \le \frac{l-1}{2}$, $\frac{l+1}{2} \le s$, if l is odd. Therefore we shall require that $\frac{l}{2} \le s \le \frac{l}{2}+p$.

(11.7) THEOREM. *Let the assumptions of Theorem (9.2) be fulfilled and let* k *be given as in Theorem* (9.2). *Assume that the boundary eigenvalue problem* (10.1) *has separable boundary conditions as defined in section 10. Suppose that*

$$\frac{l}{2} \leq s \leq \frac{l}{2}+p$$

and

$$g_1 > \ldots > g_s, \quad g_{s+1} > \ldots > g_n.$$

a) *If* l *is even, we assume in addition:*

i) $u^{\wedge}_{\theta_{\frac{l}{2}}} \neq 0$,

ii) $\tilde{\gamma}^{(2)}_1 \neq 0, \ldots, \tilde{\gamma}^{(2)}_{\frac{l}{2}} \neq 0, \; \tilde{\gamma}^{(1)}_{s+1} \neq 0, \ldots, \tilde{\gamma}^{(1)}_{s+\frac{l}{2}} \neq 0$,

iii) $g_i - g_j \notin l\mathbb{Z}$ *if* $i,j \in \{1,\ldots,\frac{l}{2}\}$ *or* $i,j \in \{s+1,\ldots,s+\frac{l}{2}\}$ *and* $i \neq j$,

iv) *for* $p > 0$, *the number* $\kappa := k - \max\{l,p-1\}$ (≥ 0) *satisfies the inequality*

$$\kappa \geq \kappa' := \begin{cases} g_{\frac{l}{2}+1} - g_{s+\frac{l}{2}} & \text{if} \quad s = \frac{l}{2}+p \\ g_{s+\frac{l}{2}+1} - g_{\frac{l}{2}} & \text{if} \quad s = \frac{l}{2} \\ \max\{g_{\frac{l}{2}+1} - g_{s+\frac{l}{2}}, \; g_{s+\frac{l}{2}+1} - g_{\frac{l}{2}}\} & \text{if} \quad \frac{l}{2} < s < \frac{l}{2}+p. \end{cases}$$

b) *If* l *is odd, we suppose in addition:*

i) $u^{\wedge}_{\theta_{\frac{l-1}{2}}} \neq 0, \; u^{\wedge}_{\theta_{\frac{l+1}{2}}} \neq 0$,

ii) $\tilde{\gamma}^{(2)}_1 \neq 0, \ldots, \tilde{\gamma}^{(2)}_{\frac{l+1}{2}} \neq 0, \; \tilde{\gamma}^{(1)}_{s+1} \neq 0, \ldots, \tilde{\gamma}^{(1)}_{s+\frac{l+1}{2}} \neq 0$,

iii) $g_i - g_j \notin l\mathbb{Z}$ *if* $i,j \in \{1,\ldots,\frac{l+1}{2}\}$ *or* $i,j \in \{s+1,\ldots,s+\frac{l+1}{2}\}$ *and* $i \neq j$.

Then there are $\tilde{p},\tilde{p}' \in \mathbb{Z}$ *such that the operator function* L *is* $|\;|_{H_j(a,b)}$*-regular of order* $(\tilde{p},\tilde{p}')$ *for* $j = 0,\ldots,n$.

PROOF. First we show that $\tilde{D}_\theta \neq 0$ if $c_\theta \in \tilde{E}$. The number u_θ only depends on $|\theta|$. As in the proof of Proposition (11.4) we see that there are $a_{rh}^{(i)} \in \mathbb{C} \setminus \{0\}$ such that

$$d^i_{\hat{\theta}_r^h} = a_{rh}^{(i)} d^i_{\hat{\theta}_r} \qquad (0 \le r \le l,\ h \in \mathbb{Z}_l,\ i = 1,2).$$

Hence the formula (10.14) yields that it is sufficient to prove that $\tilde{D}_{\hat{\theta}_r} \neq 0$ for $r = \frac{l}{2}$, if l is even and for $r = \frac{l-1}{2}$ and $r = \frac{l+1}{2}$, if l is odd. These conditions are fulfilled by the assumptions on the numbers u_θ, $\tilde{\gamma}_i^{(j)}$ and g_i: $\tilde{\gamma}_i^{(j)} \neq 0$ implies $\gamma_i^{(j)} \neq 0$ because $\varphi_o(x) \neq 0$ for all $x \in [a,b]$. The determinants $d^i_{\hat{\theta}_r}$ are different from zero since, for arbitrary $\alpha_i \in \mathbb{Z}$ $(i = 1,\dots,r)$,

$$\begin{bmatrix} \omega_1^{\alpha_1} & \dots & \omega_r^{\alpha_1} \\ \vdots & & \vdots \\ \omega_1^{\alpha_r} & \dots & \omega_r^{\alpha_r} \end{bmatrix} = \begin{bmatrix} 1 & \omega_2^{\alpha_1} \dots & (\omega_2^{\alpha_1})^{r-1} \\ \vdots & \vdots & \vdots \\ 1 & \omega_2^{\alpha_r} \dots & (\omega_2^{\alpha_r})^{r-1} \end{bmatrix}$$

and thus a Vandermonde matrix. The last equation holds because $\omega_j = \omega_2^{j-1}$.

If l is odd, $\hat{E} = \tilde{E}$ by Lemma (11.2). If l is even and $p = 0$, $s = \frac{l}{2}$ and, according to section 10, v_θ vanishes if $|\theta| \neq s$. It follows that $v_\theta = 0$ if $\theta = \hat{\theta}^h_{\frac{l}{2}-1}$ or $\theta = \hat{\theta}^{h'}_{\frac{l}{2}+1}$ for some $h \in \mathbb{Z}_l$. Therefore, also in this case, $\hat{E} = \tilde{E}$.

Suppose now that l is even and $p > 0$. Let $c \in \hat{E} \setminus \tilde{E}$. Then $c = c_{\hat{\theta}^h_{\frac{l}{2}-1}} = c_{\hat{\theta}^{h-1}_{\frac{l}{2}+1}}$ with some $h \in \mathbb{Z}_l$. Set

$$k_r := lqp + g_1 + \dots + g_r + g_{s+1} + \dots + g_{s+l-r} - lp - \frac{l(l-1)}{2}$$

and let $\tilde{b}_c$ be the function as defined in section 5. If $s = \frac{l}{2}+p$, then $v_{\hat{\theta}^h_{\frac{l}{2}-1}} = 0$ and thus

$$\tilde{b}_c(\rho) = \rho^{k_{\frac{l}{2}+1}} [d\tilde{D}_{\hat{\theta}^{h-1}_{\frac{l}{2}+1}}](\rho).$$

If $s = \frac{l}{2}$, then $v_{\hat{\theta}^{h-1}_{\frac{l}{2}+1}} = 0$ and therefore

$$\tilde{b}_c(\rho) = \rho^{\frac{k_1}{2}-1}[d\tilde{D}_\theta^\wedge h_{\frac{1}{2}-1}](\ )$$

If $\frac{1}{2} < s < \frac{1}{2}+p$, then

$$\tilde{b}_c(\rho) = \rho^{\frac{k_1}{2}-1}[d\tilde{D}_\theta^\wedge h_{\frac{1}{2}-1}](\rho) + \rho^{\frac{k_1}{2}+1}[d\tilde{D}_\theta^\wedge h_{\frac{1}{2}+1}-1](\rho)$$

From Theorem (9.2) we infer that each function $[D_\theta]$ has an asymptotic expansion of the form

$$[D_\theta](\rho) = \sum_{i=0}^{\kappa} \rho^{-i}\alpha_i^\theta + o(\rho^{-\kappa}).$$

Hence

$$\tilde{b}_c(\rho) = \rho^{\frac{k_1}{2}+\kappa'}\left(\sum_{i=0}^{\kappa} \rho^{-i}\beta_i + o(\rho^{-\kappa})\right).$$

By assumption a),iv) $\kappa' \leq \kappa$. It follows that $\tilde{b}_c(\rho) = \rho^{\frac{k_1}{2}} o(1)$ if all coefficients β_i would be zero for $i = 0,\ldots,\kappa$; this would lead to the contradiction $c \notin \hat{E}$. Hence we have $\tilde{b}_c = \rho^{k'}[b_c]$ with $b_c \neq 0$.

Altogether we have shown that $b_c \neq 0$ for all $c \in \hat{E}$. Continuing as in the proof of Theorem (11.3) we obtain the regularity.

12. AN EXPANSION THEOREM FOR $\mathbb{K} - \lambda\mathbb{H}$ WITH REGULAR BOUNDARY CONDITIONS

First we consider again the differential operator (9.1).

(12.1) PROPOSITION. *Let* $\kappa \geq 0$. *Assume* $k_i \in W^{l\kappa,\infty}(a,b)$ *for* $i = 0,\ldots,n-1$ *and* $h_i \in W^{l\kappa+i,\infty}(a,b)$ *for* $i = 0,\ldots,p-1$.

Then for any $f \in H_{n+l\kappa}(a,b)$ *there are* $f^{[j]} \in H_{n+l(\kappa-j)}(a,b)$ $(j = 0,\ldots,\kappa+1)$ *such that* $f^{[0]} = f$ *and* $\mathbb{H}f^{[j]} = \mathbb{K}f^{[j-1]}$ $(j = 1,\ldots,\kappa+1)$.

PROOF. For $j = 0$ nothing has to be proved. Suppose that the assertion holds for $j = 0,\ldots,\kappa'$ with $\kappa' \leq \kappa$. Then $\mathbb{K}f^{[\kappa']} \in H_{l(\kappa-\kappa')}(a,b)$. Let $f^{[\kappa'+1]}$ be a solution of the differential equation $\mathbb{H}f^{[\kappa'+1]} = \mathbb{K}f^{[\kappa']}$. With the aid of (7.20) we conclude that $f^{[\kappa'+1]} \in H_{l(\kappa-\kappa')+p}(a,b)$. This yields the assertion for $j=\kappa'+1$ since $l(\kappa-\kappa')+p = l(\kappa-(\kappa'+1))+n$.

(12.2) THEOREM. *Suppose that the boundary eigenvalue operator*

$$L(\lambda)\eta = (L^D(\lambda)\eta, L^R(\lambda)\eta),$$

defined by (10.1), *fulfills the assumptions of Theorem* (11.3) *in the case of general multipoint integral boundary conditions or the assumptions of theorem* (11.7) *in the case of separable boundary conditions. Let* $\tilde{p},\tilde{p}'$ *be the regularity numbers according to Theorem* (11.3) *or* (11.7), *respectively. Assume in addition that, for some* $\kappa \geq \max\{\tilde{p},\tilde{p}',0\}$, $k_i \in W^{l\kappa,\infty}(a,b)$ *for* $i=0,\dots,n-1$ *and* $h_i \in W^{l\kappa+i,\infty}(a,b)$ *for* $i=0,\dots,p-1$. *Let* $q \in \mathbb{N}$ *and*

$$L^R(\lambda) = \sum_{r=0}^{q} \lambda^r L_r^R.$$

Suppose that $f \in H_{n+l\kappa}(a,b)$ *fulfills the "boundary conditions"*

$$\sum_{r=0}^{\min\{q,\kappa-j+1\}} L_r^R f^{[j+r-1]} = 0 \qquad (j=1,\dots,\tilde{p}'). \tag{12.3}$$

Then there is a sequence $(d_\nu)_1^\infty$ *of positive numbers converging to* ∞ *such that*

$$(12.4)\qquad \left\{\begin{aligned} f = \lim_{\nu\to\infty} \sum_{\mu\in\sigma(L),\,|\mu|<d_\nu} \Big\{ &-(\operatorname{res}_\mu \hat{R}_1)\mathbb{H}f \\ &+ \operatorname{res}_{\lambda=\mu}\Big(\hat{R}_2(\lambda)\sum_{j=0}^{q-1}\lambda^j \sum_{r=j+1}^{\min\{q,\kappa+j+1\}} L_r^R f^{[r-j-1]}\Big)\Big\}\end{aligned}\right.$$

holds in $H_n(a,b)$.

PROOF. Let $f^{[j]} \in H_{n+l(\kappa-j)}(a,b)$ $(j=0,\dots,\kappa+1)$ be defined according to (12.1). This definition and (7.12), (7.13) lead to

$$\hat{R}_1(\lambda)\mathbb{H}f^{[j]} = -\frac{1}{\lambda}f^{[j]} + \frac{1}{\lambda}\hat{R}_1(\lambda)\mathbb{H}f^{[j+1]} + \frac{1}{\lambda}\hat{R}_2(\lambda)L^R(\lambda)f^{[j]}$$

for $j=0,\dots,\kappa$. A recursive substitution yields

$$\hat{R}_1(\lambda)\mathbb{H}f = -\sum_{j=0}^{\kappa}\lambda^{-j-1}f^{[j]} + \lambda^{-\kappa-1}\hat{R}_1(\lambda)\mathbb{H}f^{[\kappa+1]} \tag{12.5}$$

$$+\sum_{j=0}^{\kappa} \lambda^{-j-1}\hat{R}_2(\lambda)L^R(\lambda)f^{[j]}.$$

As in section 6 we obtain

$$(12.6)\quad \begin{cases} \sum_{j=0}^{\kappa} \lambda^{-j-1}L^R(\lambda)f^{[j]} = \sum_{j=0}^{q-1} \lambda^j \sum_{r=j+1}^{\min\{q,\kappa+j+1\}} L_r^R f^{[r-j-1]} \\ \qquad + \sum_{j=\tilde{p}'+1}^{\kappa+1} \lambda^{-j} \sum_{r=0}^{\min\{q,\kappa-j+1\}} L_r^R f^{[j+r-1]}. \end{cases}$$

By assumption the boundary eigenvalue operator function L is $|\ |_{H_n(a,b)}$-regular of order $(\tilde{p},\tilde{p}')$. According to the proof of Theorems (11.3) and (11.7) this regularity holds with respect to circles $|\lambda| = d_\nu$ where $d_\nu \to \infty$ as $\nu \to \infty$. Thus Theorem (12.2) is proved if we integrate (12.5) along these circles and use (12.6) and the definition of the regularity.

By Theorem (7.18) the residues in (12.4) can be expressed in terms of canonical systems of root functions of L and the adjoint operator function L^*. We omit the details and leave the exact formulation to the reader.

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