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PRÓLOGO E ÍNDICE

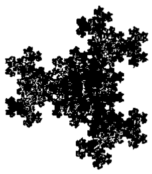
Esta Notas contienen dos trabajos de los autores y un apéndice con una conferencia (Conferencia "Alberto González Domínguez", XLIX Reunión Anual de la Unión Matemática Argentina , 24/09/1999, La Plata, Argentina) impartida por el primer autor. Esta última suministra un contexto general en el que se insertan los dos artículos originales.

R.P.

KEY WORDS: self-similar set, fractile, number system, positional representation, tessellation.

The boundary of the Eisenstein set is a log3/log2-set and a Hölder continuous curve

A. Benedek and R. Panzone



ABSTRACT. It is shown that the boundary K of the (normalized) Eisenstein set F has a bounded positive measure in its Hausdorff dimension. K is the union of six congruent copies of a self-similar Hölder continuous curve V whose convex hull is a hexagon. The set K has a box dimension that coincides with its Hausdorff and packing dimensions and is equal to $s=\log 3/\log 2 \cong 1.585$. It holds that $0.0013<H^s(V)<0.422<|V|^s \cong 0.797$.

0. INTRODUCTION. Let $b\in\mathbf{C}$, $|b|>1$, $D=\{0,d_1,d_2,\dots,d_k\}\subset\mathbf{C}$. α is said *representable in base b with ciphers* D if there exists $\{a_j \in D: j=M,M-1,\dots\}$ such that $\alpha=\sum_{-\infty}^{M} a_j b^j$. We write $\alpha=a_M\dots a_0.a_{-1}a_{-2}\dots=(e.f)_b$ and call (e) the integral part of α and (f) the fractional part of α. G denotes the set of all representable numbers. F is the set of *fractional numbers*, i.e., those numbers in G with a representation such that (e)=0. The set W of *integers* of the system is the subfamily of G with a representation such that (f)=0. A number r will be called a *rational* of the number system (b,D) if it has a finite positional representation, that is, $a_j=0$ for $j<J(r)$. U will denote the set of rationals of the system. The number system that we shall consider has base -2 and set of ciphers $D\not\subset\mathbf{R}$, $D:=\{0,1,w,w^2\}$, where $w=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $w^2=\overline{w}$. $D\setminus\{0\}$ is a multiplicative group such that $1+w+w^2=0$. E denotes the *Eisenstein's point-lattice*: $E\equiv[1,w]:=\{m.1+n.w: m,n\in Z\}$. Let

$$\sigma:=D\cup(-D)=\{0,\pm1,\pm w,\pm\overline{w}\}.$$

$$S:=D-D=\{0,\pm1,\pm w,\pm\overline{w},\pm(1-w),\pm(1-\overline{w}),\pm(w-\overline{w})\},$$

$$S':=S\setminus\sigma=\{\pm(1-w),\pm(1-\overline{w}),\pm(w-\overline{w})\}.$$

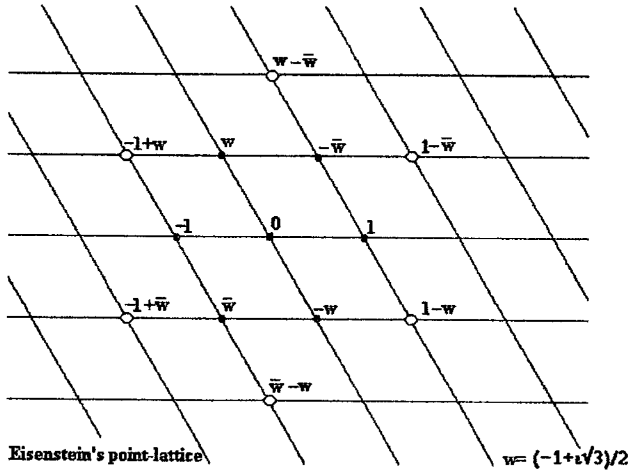
S and σ are subsets of the set E of Eisenstein "integers". The numbers in $\sigma\setminus\{0\}$ have modulus equal to 1 and those in S' have modulus equal to $\sqrt{3}$. Besides, $\alpha\in S\Rightarrow|\alpha|\le\sqrt{3}$, $|\mathrm{Re}\,\alpha|\le 3/2$, $|\mathrm{Im}\,\alpha|\le\sqrt{3}$. x used as a cipher will represent the number $\overline{w}$. m(A) will denote the plane Lebesgue measure of $A\subset\mathbf{C}$ and B(z;r) the open ball of center z and radius r.

DEFINITION 1. $F_g:=g+F$ where $g\in E$. For $j\in D=\{0,1,w,x\}$ let us define

$$\Phi_j(z)=\frac{z}{b}+\frac{j}{b}=-\frac{z+j}{2}. \bullet \tag{1}$$

$F_0 \equiv F$ is the fractional set of the number system and $F = \bigcup_{i \in D} \Phi_i(F)$. Thus, the 4-reptile F is the invariant set of the family $\{\Phi_j\}$. We shall call it the *Eisenstein set*. The preceding definition can be extended in the following way

(2) $$F_{a_M \ldots a_0 . a_{-1} a_{-2} \ldots a_{-n}} := \{z; z = a_M \ldots a_0 . a_{-1} \ldots a_{-n} \ldots\}.$$



BASIC THEOREM. i) W=E and G=**C**.

ii) Any rational point has a unique representation in $(-2, \{0, 1, w, x\})$. There are points with two and three representations but there is no point with four positional representations.

iii) If $z \in \mathbf{C}$ and $|z| \leq 1/8$ then $z \in$ F. Moreover, cl(int F)=F and int(F) is the union of an infinite denumerable family of open components.

iv) F is a connected compact set with $m(F_0) = \sqrt{3}/2$; $F \subset B(0, \sqrt{7/9})$.

v) If $g \neq 0$ then (int F)$\cap F_g = \varnothing$; the family $\{\operatorname{int} F_g : g \in E\}$ is pairwise disjoint.

vi) F is a self-similar set that satisfies the open set condition.

vii) $K = \partial F$ is not a Jordan curve and $K \subset \mathrm{cl} \bigcup \{\operatorname{int}(F_d) : 0 \neq d \in S\}$.

viii) The complement of F, F^, has infinitely many open components.

ix) The family $\{F_g : g \in E\}$ defines a *tessellation* in the sense that not only $\mathbf{R}^2 = \bigcup \{F_g : g \in E\}$ but also that any two different sets of the family have an intersection of plane Lebesgue measure zero.

x) Let H be the tessellation of the plane given by regular hexagons of apotheme 1/2 centered at the points of the Eisenstein's point-lattice. Let Ω_0 be the central hexagon of $\mathsf{H}_0 \equiv \mathsf{H}$ and $\Omega_{n+1} := \bigcup_{j \in D} \Phi_j(\Omega_n)$. Ω_n is a connected set, union of 4^n hexagons in $\mathsf{H}_n \equiv \mathsf{H}/2^n$. The sequence $\{\Omega_n\}$ converges to F in the metric of Hausdorff. There is a tessellation with central tile Ω_n, $\Delta_n := \{\Omega_n + g : g \in E\}$ that uniformly approximates the tiling $\{F_g : g \in E\}$ in the Hausdorff metric for $n \to \infty$. Each tile in Δ_n is in contact with exactly six congruent tiles (cf. Fig. 1). •

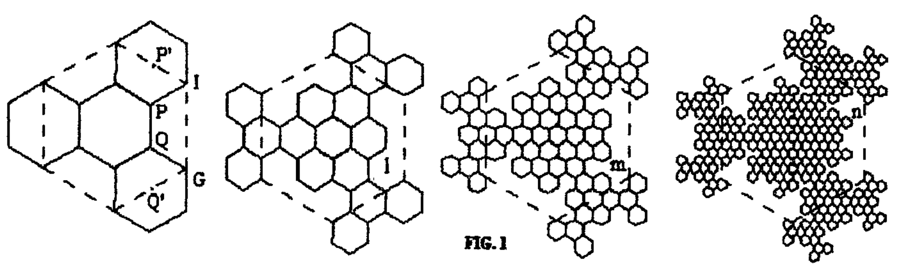


FIG. 1

Most of these results are proved in [B] and in [BP] the remaining ones.

1. THE SELF-SIMILAR SET V. We wish to prove that $\dim_H K = \log 3/\log 2$, (cf. [E], p. 166). The hexagon X of center 0 and apotheme 1/2 (dotted lines in Fig. 1) has side of length $1/\sqrt{3}$. Let us *define*:

$$I:=1/2+i/(2\sqrt{3}) \qquad G:=1/2-i/(2\sqrt{3}) \qquad P:=I/2 \qquad Q:=G/2 \qquad B:=1/3$$

$$Q' = Q - i/(2\sqrt{3}) = -w/2 \qquad P' = P + i/(2\sqrt{3}) = -x/2$$

and *the similarities*

(3) $$\Theta_0(z) = Q'z + P' \qquad \Theta_1(z) = (-z+1)/2 \qquad \Theta_2(z) = P'z + Q'.$$

We have $\Theta_2(z) = \overline{\Theta_0(\bar{z})}$. Since b=–2, we obtain

$$\Theta_0(z) = b^{-1}(wz + x) \qquad \Theta_1(z) = b^{-1}(z-1) \qquad \Theta_2(z) = b^{-1}(xz + w)$$

$$\Theta_0(z) = w\Phi_w(z) \qquad \Theta_1(z) = \Phi_1(z) + 1 \qquad \Theta_2(z) = x\Phi_x(z).$$

We *define the set* V *as the invariant set (attractor) of the family of similarities* L:= $\{\Theta_0, \Theta_1, \Theta_2\}$ (see Fig. 2). The *similarity dimension* of V is then equal to σ=log3/log2.

AUXILIARY THEOREM 1. Let M be the invariant set of an iterated function system, each function a similarity. If $d = \dim_H M$ then $\overline{\dim}_B M = d$ and $H^d(M) < \infty$, (cf. [F]). •

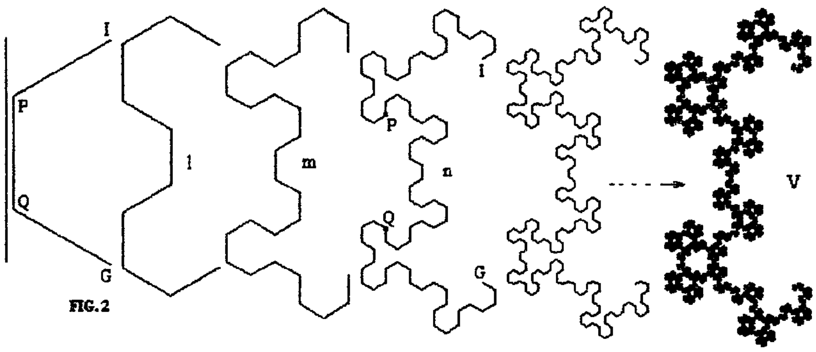


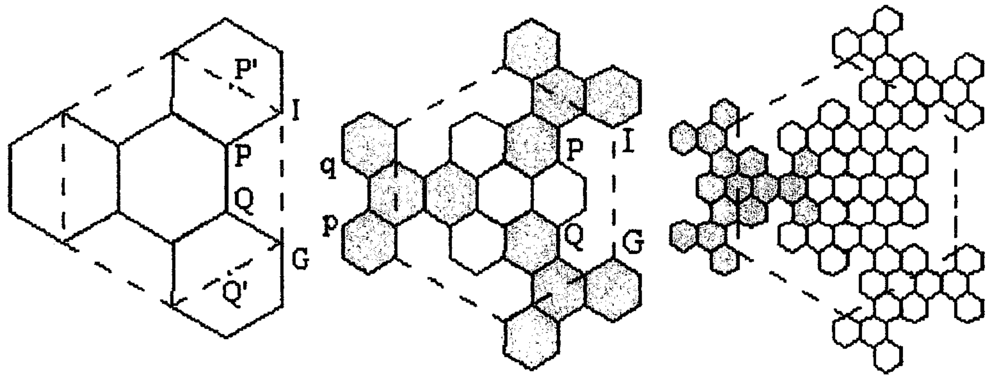
FIG.2

LEMMA 1. $\sigma \geq s = \dim_H V = \dim_B V = \overline{\dim}_B V$, $H^s(V) < \infty$. •

PROOF. It follows from the auxiliary theorem, QED.

REMARK 1. As a matter of fact σ = s because, as we shall prove, V is an s-set.

2. CHARACTERIZATIONS OF K=∂F. The hexagon X is step 0. The diagram shows



steps 1, 2 and 3 of the construction that starts with the set X and converges to the set F, the attractor of the family of similarities $\Phi_j(z)=b^{-1}(z+j)$, $j\in\{0,1,w,x\}$. Φ_1 maps I on p and G on q, so, Θ_1 maps the arc A_n between the points I and G of the boundary of Ω_n onto the piece of the boundary of Ω_{n+1} between Q and P. Φ_x inserts between P and I an arc congruent to the preceding one adequately rotated; likewise Φ_w but between Q and G.
This action of the function Φ_h, h=1,w,x, on the hexagons of the family H_n coincides, from the point of view of the boundaries, with the action of the function Θ_k, k=1,2,0, respectively. In consequence, the boundary of Ω_n is formed by six congruent poligonals arcs with extremes on the vertices of X. The arc from I to G in the boundary negatively oriented is the same obtained with the functions of L starting from the segment IG. It is shown in Fig. 2 as an approximant of the limit V. We have then that $\partial\Omega_n$ is a Jordan curve and

(4) $$\partial\Omega_n=\bigcup_{c\in D\setminus\{0\}}cA_n\cup\bigcup_{c\in D\setminus\{0\}}c(A_n-1).$$

We know from the proof of Theorem IV 2 [B] that $\Omega_n\to F$ in the Hausdorff metric and that $\{g+\Omega_n;g\in E\}$ is a tesellation of $\mathbf{R}^2$. We have also the following general result:

LEMMA 2. Let F and $\Omega\subset\mathbf{R}^2$ be compact sets and E a discret set of points such that $\{F+g;g\in E\}$ and $\{\Omega+g;g\in E\}$ are coverings of the plane. If both families consist of sets with disjoint interiors then $d(\partial F,\partial\Omega)\le d(F,\Omega)$.
PROOF. Let $\varepsilon:=d(F,\Omega)$. Because of the symmetry of the hypothesis it will suffice to prove that for $x\in\partial F$ there is a point $\tilde{x}\in\partial\Omega$ such that $|x-\tilde{x}|\le\varepsilon$. Let us assume that $x\in\partial F$, $g\in E$ and $x\in F\cap F+g$. From the definition of ε we know that there exist points $y\in\Omega$, $z\in\Omega+g$ verifying $|x-y|\le\varepsilon$, $|x-z|\le\varepsilon$. But then there is a point $\tilde{x}\in\partial\Omega$ on the segment [y,z] such that $|x-\tilde{x}|\le\varepsilon$, QED.

COROLLARY. $\partial\Omega_n$ converges to K=∂F in the Hausdorff metric. •

LEMMA 3. K= $\bigcup_{c\in D\setminus\{0\}}cV\cup\bigcup_{c\in D\setminus\{0\}}c(V-1)$ is a connected set. •
PROOF. It follows from (4) and the Corollary, QED.

DEFINITION 2. $V_\gamma:=F\cap F_\gamma$. •

LEMMA 4. $K=\bigcup V_\gamma$, $\gamma\in\{\pm1,\pm w,\pm\bar{w}\}$ and σ≥s=$\dim_H K=\dim_B K=\overline{\dim}_B K$, $H^s(K)<\infty$ •
PROOF. F is in contact with exactly twelve sets F_g distinct of F itself. But if g∈S′, $F\cap F_g$ reduces to a point, (cf. [BP]). Because of Lemma 3, K is a connected set, therefore, these six points $\{F\cap F_g:g\in S'\}$ are not isolated but already included in $\bigcup V_\gamma$,

$\gamma\in\{\pm1,\pm w,\pm\overline{w}\}$=S\(S'∪{0}). The second part follows from Lemmas 1 and 3, the finite stability of the dimensions involved and the inequalities they satisfy, QED.

3. ON THE BOX DIMENSION. Let S_r be a net of cubes in $\mathbf{R}^d$ with sides of length r parallel to the axes, $S_r=\{q+\tau:\tau\in rZ^d\}$, q a cube of side of length r. The upper box dimension of the bounded set E⊂$\mathbf{R}^d$ is defined by $\overline{\dim}_B(E):=\overline{lim}\,\dfrac{\log N(r,E)}{\log 1/r}$ for $r\to 0$ where N(r,E)=#{cubes ∈ S_r that intersect E}. Several other families can play the role of the S_r's. We wish to use families of sets T of positive measure for which there exists a constant K verifying $|T|^d/m(T)$<K, $|T|$=diam(T). Because of the isodiametric inequality a lower bound always exists for these quotients:

$$|T|^d/m(T)\geq|T|^d/m(B(0,|T|/2)=1/m(B(0,1/2))>1.$$

LEMMA 5. Suppose that E is a bounded set, E⊂$\bigcup\{T+\gamma:\gamma\in\Lambda\}$, B_1=B(0;1) and a) and b) hold:

a) q∈S_r, m(T)=m(q), $(|T|+|q|)^d/m(T)\leq K<\infty$,

b) E⊂$\bigcup\{T+\gamma:\gamma\in\Lambda\}$ a minimal covering such that $m((T+\gamma)\cap(T+\gamma'))=0$ if $\gamma\neq\gamma'$.

Let M(r,E):=#Λ. Then,

$$\mathrm{M(r,E)}\leq \mathrm{Km}(B_1).\mathrm{N(r,E)}\ ,\ \mathrm{N(r,E)}\leq \mathrm{Km}(B_1).\mathrm{M(r,E)}.\ \bullet$$

PROOF. Let E⊂$\bigcup\ \{q+\tau:(q+\tau)\cap E\neq\varnothing\}$ and for $\gamma\in\Lambda$, $\Xi=\Xi(\gamma)$ the family of τ's for which $(T+\gamma)\cap(q+\tau)\cap E\neq\varnothing$.

If y=#Ξ then $ym(q)=ym(T)\leq m(B(0,|T|+|q|)=(|T|+|q|)^d m(B_1)$.Thus, $y\leq Km(B_1)$ and

$$\mathrm{N(r,E)}\leq(\sup y)\mathrm{M(r,E)}\leq \mathrm{Km}(B_1)\mathrm{M(r,E)}$$

and the Lemma follows, QED.

LEMMA 6. Let k be a positive integer and $r_j\downarrow 0$ a decreasing sequence such that $kr_{j+1}\geq r_j$. Then, the following expressions have the same limits for $j\to\infty$ and $r\to 0$, respectively,

$$\text{(5)}\qquad \frac{\log N(r_j,E)}{\log 1/r_j}\ ,\ \frac{\log N(r,E)}{\log 1/r}.\ \bullet$$

PROOF. A cube q∈S_{r_j} intersects at most $(k+1)^d$ cubes of $S_{r_{j+1}}$. Therefore,

$$N(r_{j+1},E)\leq(k+1)^d N(r_j,E).$$

We used only that $kr_{j+1}\geq r_j$. However, if $r<r_j$ it holds that

$$N(r_j,E)\leq 2^d N(r,E).$$

Then, for $r_{j+1}<r<r_j\leq kr_{j+1}$, we obtain,

$$\text{(6)}\qquad N(r_{j+1},E)\leq(k+1)^d N(r_j,E)\leq(2(k+1))^d N(r,E)\leq 4^d(k+1)^d N(r_{j+1},E).$$

Besides, $\log 1/r_{j+1} \geq \log 1/r \geq \log 1/r_j \geq \log 1/r_{j+1} - \log k$. Thus, from some j on,

$$(7) \qquad \frac{\log N(r_{j+1},E)}{\log 1/r_{j+1}} \leq \frac{\log N(r,E)+c}{\log 1/r} \leq \frac{\log N(r_{j+1},E)+\widetilde{c}}{\log 1/r_{j+1} - \log k}.$$

From (7) we obtain, for example, $\overline{lim}_{j\to\infty} \frac{\log N(r_j,E)}{\log 1/r_j} = \overline{lim}_{r\to 0} \frac{\log N(r,E)}{\log 1/r}$, QED.

LEMMA 7. Let $r_j \downarrow 0$ with $kr_{j+1} \geq r_j$, $q \in S_{r_j}$, k a positive integer. For each j let T=T(j) be such that m(T)=m(q), $(|T|+|q|)^d / m(T) \leq K < \infty$ with K independent of j. Assume that $\{T+\gamma : \gamma \in \Lambda\}$ is a minimal covering of E verifying $\gamma \neq \gamma' \Rightarrow m((T+\gamma)\cap(T+\gamma')) = 0$. If M($r_j$,E)=#$\Lambda$ then

$$(8) \qquad \overline{\dim}_B(E) = \overline{lim}_{j\to\infty} \frac{\log M(r_j,E)}{\log 1/r_j}, \quad \underline{\dim}_B(E) = \underline{lim}_{j\to\infty} \frac{\log M(r_j,E)}{\log 1/r_j}. \bullet$$

PROOF. Because of Lemma 5, $\overline{lim} \frac{\log M(r_j,E)}{\log 1/r_j} = \overline{lim} \frac{\log N(r_j,E)}{\log 1/r_j}$. It follows from Lemma 6 that both are equal to $\overline{lim}_{r\to 0} \frac{\log N(r,E)}{\log 1/r}$, QED.

4. EVALUATION OF THE BOX DIMENSION. Next we give a method for estimating the box dimension of certain sets. To illustrate it we restrict ourselves to plane sets. This method is a slight variant of the one presented in [K], pgs. 11-12. Earlier W. J. Gilbert in his paper [G] determined the Hausdorff dimension of fractal sets that arise in the study of complex numerical systems.

H) Let the number b, $|b| > 1$, be the base of a numerical system with D=$\{0, a_1, \ldots\} \subset$ C as its set of ciphers. We *assume* that there exists a point lattice L=[1,g]:= $\{m+ng : m,n \in Z\}$ such that bL $\cup$ D $\subset$ L and D is a *complete set of residues modulo b*, (each point y of L can be written in a unique way as y=bx+c, x$\in$L, c$\in$D).

We also *assume* that the family { F_t :t$\in$L}, where F={ $z : z = 0.c_1c_2\ldots ; c_i \in D$ } and F_t:=t+F, is a tesselation of the plane. •

NB. H) implies that the set of integers of the system W=$\{(c_m \ldots c_0)_b : m \geq 0, c_j \in D\}$ $\subset$ L have a unique positional representation as integers of {b,D}.

DEFINITION 3. S°:=$\{\gamma \in L : \gamma \neq 0, F \cap F_\gamma \neq \varnothing\}$; if $\gamma \in$ S° then $V_\gamma := F \cap F_\gamma$. •

DEFINITION 4. G(S°) will denote the graph with set of nodes S° and arrows $\gamma \xrightarrow{\binom{\varepsilon}{\widetilde{\varepsilon}}} b\gamma + \widetilde{\varepsilon} - \varepsilon$ whenever $b\gamma + \widetilde{\varepsilon} - \varepsilon \in S°$, $\varepsilon, \widetilde{\varepsilon} \in D$. •

Observe that we know, from the definition of S°, that given $\gamma \in$S° there exists a point z such that $z = 0.\varepsilon... = \gamma + 0.\tilde{\varepsilon}... \in V_\gamma$. For this point, $(\gamma b + \tilde{\varepsilon} - \varepsilon) + 0.\tilde{\varepsilon}_2... = 0.\varepsilon_2...$. Since D is a complete set of residues modulo b, $\gamma b + \tilde{\varepsilon} - \varepsilon \neq 0$. Then $\gamma b + \tilde{\varepsilon} - \varepsilon \in S°$. That is, from each $\gamma \in S°$ starts (at least) an arrow which is the beginning of an infinite string in G(S°) that defines a point z=$0.\varepsilon\varepsilon_2... \in V_\gamma$.

DEFINITION 5. $F_{\sigma+0.abc...d} := \{z : z = \sigma + 0.abc...dc_1c_2...; c_j \in D\}, \sigma \in$ L, $a,b,...,d \in$ D. •

DEFINITION 6. M(n,γ):=$\#\left\{(\varepsilon_1,...,\varepsilon_n;\tilde{\varepsilon}_1,...,\tilde{\varepsilon}_n) : \exists\varepsilon_{n+1},...;\exists\tilde{\varepsilon}_{n+1},...: \sum_1^\infty \varepsilon_j b^{-j} = \gamma + \sum_1^\infty \tilde{\varepsilon}_j b^{-j}\right\}$
= number of paths in G(S°) of length n starting from $\gamma \in$ S°;
m(n,γ):=$\#\{(\varepsilon_1,...,\varepsilon_n) : \exists\varepsilon_{n+1},\varepsilon_{n+2},...;\exists\tilde{\varepsilon}_1,\tilde{\varepsilon}_2,...: 0.\varepsilon_1...\varepsilon_n... = \gamma + 0.\tilde{\varepsilon}_1...\}$. •

The family $\{ F_{t+0.a_1...a_n} : t \in L, a_i \in D \}$ is a tesselation of the plane like $\{ F_t$:$t\in$L$\}$ but with smaller tiles in a ratio $|b|^{-n}$. m(n,γ) *is the number of such tiles contained in F and in contact with* V_γ. They form a covering of V_γ by essentially disjoint sets of diameter $|F||b|^{-n}$ and measure $m(F)|b|^{-2n}$. Obviously, m(n,γ)≤M(n,γ).
On the other hand, M(n,γ) counts the number of pairs of tiles in $\{ F_{t+0.a_1...a_n} : t \in L, a_i \in D \}$ with non void intersection, one of the tiles in F and the other one in F_γ. *Calling* s=#S°, from the uniqueness properties mentioned above we obtain for fixed $\varepsilon_1,...,\varepsilon_n$ that the number of possible sets $\tilde{\varepsilon}_1,...,\tilde{\varepsilon}_n$ is not greater than s. In fact, $\{\gamma b^n + (\tilde{\varepsilon}_1,...,\tilde{\varepsilon}_n)_b - (\varepsilon_1,...,\varepsilon_n)_b\} + 0.\tilde{\varepsilon}_{n+1}... = 0.\varepsilon_{n+1}...$. If τ is the number inside the brackets then $\tau \in$ S° and $(\tilde{\varepsilon}_1,...,\tilde{\varepsilon}_n)_b = \tau - \gamma b^n + (\varepsilon_1,...,\varepsilon_n)_b \in L$. In consequence, the $\tilde{\varepsilon}_j$'s are uniquely determined by τ. Thus

(9) $$m(n,\gamma) \leq M(n,\gamma) \leq s.m(n,\gamma).$$

σ_j will *denote* the number of arrows in G(S°) starting from γ_j. We already know that σ_j>0. p_{jk} will *denote* the number of 1-paths from γ_j to γ_k. It holds then, for each γ_j, that $\sigma_j = \sum_k p_{jk} > 0$. We also have,

(10) $$M(n+1,\gamma_j) = \sum_{k=1}^{s} p_{jk} M(n,\gamma_k).$$

If $Y^{(1)} := \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_s \end{pmatrix}, Y^{(n)} = \begin{pmatrix} M(n,\gamma_1) \\ \vdots \\ M(n,\gamma_s) \end{pmatrix}, P = \begin{pmatrix} p_{11} & \cdots & p_{1s} \\ & \cdots & \\ p_{s1} & \cdots & p_{ss} \end{pmatrix}$ then $Y^{(n+1)} = PY^{(n)} = P^n Y^{(1)}$.

P is a *nonnegative* matrix that verifies $PY^{(1)} \geq tY^{(1)}$>0 for a certain t>0. Therefore, its spectral radius λ is a *positive eigenvalue* and there exists a *non null* s-dimensional vector v≥ 0 such that $Pv = \lambda v$. There also exists $\mu > 0$ such that $Y^{(1)} \geq \mu v$. In consequence, $\mu\lambda^n v = P^n \mu v \leq P^n Y^{(1)} = Y^{(n+1)}$. Then, if γ is such that v_γ (the γ th element of v) is positive then $M(n+1,\gamma) \geq \mu\lambda^n v_\gamma$>0. Therefore, for an adequate constant B, we have, $n\log\lambda + B \leq \log M(n+1,\gamma)$ and $\frac{n\log\lambda}{\log|b^n|} + o(1) \leq \frac{\log M(n,\gamma)}{\log|b^n|}$. By (9), $\frac{\log M(n,\gamma)}{\log|b^n|}$ and $\frac{\log m(n,\gamma)}{\log(1/|b|^{-n})}$ *have the same limits.*

We call $r_j = \sqrt{m(F)}|b|^{-j}$ and K'$=\frac{|F_{0.\varepsilon_1...\varepsilon_j}|^2}{m(F_{0.\varepsilon_1...\varepsilon_j})} = \frac{|F|^2}{m(F)}$. We apply Lemma 7 with K=2K' plus a constant. Since $m(j,\gamma) = M(r_j, V_\gamma)$, it follows that

(11) $$\frac{\log\lambda}{\log|b|} \leq \underline{lim}\frac{\log M(j,\gamma)}{\log 1/r_j} = \underline{lim}\frac{\log m(j,\gamma)}{\log 1/r_j} = \underline{lim}\frac{\log M(r_j,V_\gamma)}{|\log r_j|}.$$

But the last limit is equal to $\underline{\dim}_B V_\gamma$. Then we have the following result,

LEMMA 8. Assume H). Let λ be the spectral radius of P and let γ be such that v_γ>0, where v_γ is the γ^{th} element of v, the eigenvector corresponding to λ. Then, for $V_\gamma = F \cap F_\gamma$ it holds that,

(12) $$\log\lambda / \log|b| \leq \underline{\dim}_B V_\gamma . \bullet$$

THEOREM 1. Assume the hypothesis H). Let λ=spectral radius of P and assume that v_γ>0. Then, the box dimension of V_γ exists and

(13) $$\log\lambda / \log|b| = \dim_B V_\gamma . \bullet$$

PROOF. $\|Y^{(n)}\|_\infty \leq \|P^{n-1}\| \|Y^{(1)}\|_\infty$ implies that $M(n,\gamma) \leq C\|P^{n-1}\|$ and therefore $\frac{\log M(n+1,\gamma)}{n} \leq \log(\|P^n\|^{1/n}) + o(1)$. In consequence, $\overline{lim}\frac{\log M(n,\gamma)}{\log|b^n|} \leq \frac{\log\lambda}{\log|b|}$. From this we obtain, $\overline{\dim}_B V_\gamma = \overline{lim}\frac{\log m(n,\gamma)}{\log|b^n|} = \overline{lim}\frac{\log M(n,\gamma)}{\log|b^n|} \leq \frac{\log\lambda}{\log|b|} \leq \underline{\dim}_B V_\gamma$, QED.

REMARK 2. The hypothesis H) implies that b is a *quadratic* integer if it is not real, (cf. [BPP])). But if b is a real number then it is a *rational* integer. In fact, let $b \in \mathbf{R}$. Then, if 0$\neq d \in$D, $b^j d \in$ L for every $j\geq 0$. Therefore $\forall j$ $b^j \in L_1 := (L/d) \cap \mathbf{R}$. Since L_1 is a real point-lattice $L_1 = \{k/q : k \in Z\}$ for a certain positive integer q. Thus, $b^j = (k/q)^j = k(j)/q$. This implies that b is a rational integer.

5. THE MATRICES P AND Q. One can find $\lambda = \rho(P)$=the spectral radius of P, from a nonnegative matrix Q whose order is the half of the order of P and verifies $\rho(P) = \rho(Q)$. Observe that S° and the graph G(S°) are *symmetric*. In fact, given $\gamma \in$ S°, the equality $0.\varepsilon... = \gamma + 0.\tilde{\varepsilon}...$ implies $-\gamma + 0.\varepsilon... = 0.\tilde{\varepsilon}...$, so $-\gamma \in$ S°. Besides $\gamma \xrightarrow{\binom{\varepsilon}{\tilde{\varepsilon}}} \gamma' = \gamma b + \tilde{\varepsilon} - \varepsilon$ and $-\gamma \xrightarrow{\binom{\tilde{\varepsilon}}{\varepsilon}} -\gamma' = b(-\gamma) + \varepsilon - \tilde{\varepsilon}$. Therefore,

(14) $$S^\circ = -S^\circ \ , \ M(n,\gamma) = M(n,-\gamma).$$

Let S" be a subset of S° such that it does not contain opposite elements and S=S"∪(–S"). We get from (10) and (14) and for $\gamma \in$ S", that

(15) $$M(n+1,\gamma) = \sum_{\delta \in S^\circ} P_{\gamma\delta}\, M(n,\delta) = \sum_{\delta \in S''} (P_{\gamma\delta} + P_{\gamma(-\delta)})\, M(n,\delta) = \sum_{\delta \in S''} Q_{\gamma\delta}\, M(n,\delta).$$

This defines the matrix Q=[$Q_{\gamma\delta}$], $\gamma, \delta \in$ S". Using the set of indices S"∪(–S") we see that $P = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ and using only the indices in S" we have Q=A+B. Then,

$$\det(P - \lambda I) = \det\begin{pmatrix} A+B-\lambda I & B \\ A+B-\lambda I & A-\lambda I \end{pmatrix} = c(\lambda)\det(A+B-\lambda I) = c(\lambda)\det(Q - \lambda I).$$

It follows from this that the spectrum of Q is contained in the spectrum of P. For certain nonnegative vectors v,w with v+w≠0, it holds that $\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \lambda\begin{pmatrix} v \\ w \end{pmatrix}$. Thus,

$$(A+B)(v+w) = \lambda(v+w).$$

Therefore, λ is in the spectrum of Q and the two matrices have the same spectral radius.

THEOREM 2.[1] $\log\lambda / \log|b| = \dim_B V_\gamma$ for each $V_\gamma = F \cap F_\gamma$, $\gamma \in$ S°, such that it is positive the element v_γ of the eigenvector v corresponding to the eigenvalue $\lambda = \rho(P)$ of the nonnegative matrix $P = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$. λ is equal to the spectral radius of the nonnegative matrix Q=A+B. •

6. AN APPLICATION. In this paragraph we apply the method described in §4-5, specially Theorem 2, to the particular case of the boundary of the Eisenstein set. For the nomenclature used see § 1-2.

[1] In relation with Theorem 2 the reader may consult [D] and [V] and the references mentioned there.

THEOREM 3. $\sigma = \log 3/\log 2 = \dim_B V_1$. •

PROOF. We have to calculate the spectral radius of the corresponding matrix Q. P in the present situation is the following 12x12 matrix (see note following Th. 4):

$$
\begin{array}{cccccccccccc|c}
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & x \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & w \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1-w \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1-x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & w-x \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & -x \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -w \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w-1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x-1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x-w \\
\hline
1 & x & w & 1-w & 1-x & w-x & -1 & -x & -w & w-1 & x-1 & x-w &
\end{array}
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

The matrix Q associated is seen at the left. Then, $\det(Q - xI) = (1-x)^3 x^2 (3-x)$ and $\lambda = \rho(Q) = 3$. The corresponding (unique) eigenvector v, except for a constant factor, is $v^T = [1\,1\,1\,0\,0\,0]$. Thus, according to Theorem 3, $\sigma = \log 3/\log 2$ is equal to the box dimensions of $V_1, V_x, V_w, V_{-1}, V_{-x}, V_{-w}$, QED. The following final result follows from Theorem 3 and Lemma 4,

THEOREM 4. i) s = log3/log2 = $\dim_H K = \dim_B K$, $H^s(K) < \infty$,
ii) σ, the similarity dimension of V, is equal to s = $\dim_H V = \dim_B V$ and $H^s(V) < \infty$. •

NB. We already know that in our case S°=S\{0}. The existence of arrows between the elements in S° as shown in matrix P is given in [BP] and in the next paper in this volume. The graph G(S°) is in our case the digraph VΓ of these papers. Let us see why there is at most one arrow between nodes.

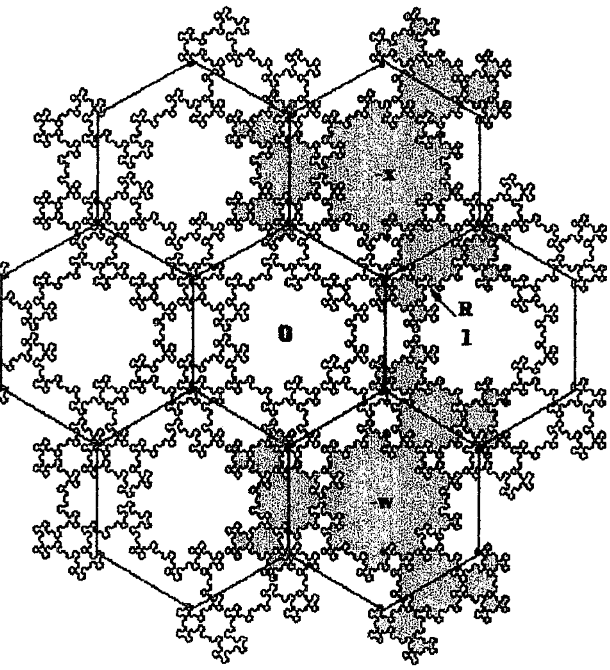
Assume that $\gamma \in S°$, $\gamma \xrightarrow{\binom{\varepsilon}{\tilde{\varepsilon}}} b\gamma + \tilde{\varepsilon} - \varepsilon$, $h = b\gamma + \tilde{\varepsilon} - \varepsilon \in S°$, $\varepsilon, \tilde{\varepsilon} \in D$ and $\gamma \xrightarrow{\binom{\eta}{\tilde{\eta}}} b\gamma + \tilde{\eta} - \eta$, $k = b\gamma + \tilde{\eta} - \eta \in S°$, $\eta, \tilde{\eta} \in D$. If $\tilde{\varepsilon} - \varepsilon = 0$ then $h = b\gamma = -2\gamma \notin S°$, a contradiction. Then, $h=k$ implies that $\tilde{\varepsilon} - \varepsilon = \tilde{\eta} - \eta \neq 0$. It is easy to verify that the numbers in S° = (D–D)/{0} can be written in a unique way as a *difference* of two numbers in D. Therefore, $\varepsilon = \eta, \tilde{\varepsilon} = \tilde{\eta}$ and there is only one arrow from γ to *h*.

7. THE SET $V_1 = F_0 \cap F_1$. It is an interesting fact that V is different of V_1.

LEMMA 9 . $V \subset V_1$, $V \neq V_1$.•

PROOF. We have L={ $\Theta_0(z) = b^{-1}(wz + x), \Theta_1(z) = b^{-1}(z-1), \Theta_2(z) = b^{-1}(xz + w)$ }. If $z\in V_1$ then $z = (0.c_1c_2...)_b = (1.d_1d_2...)_b$. Therefore, $\Theta_0(z) = 0.x(wc_1)(wc_2)... = 1.1(wd_1)(wd_2)...$, $\Theta_1(z) = 0.0d_1d_2... = 1.1c_1c_2...$, $\Theta_2(z) = 0.w(xc_1)... = 1.1(xd_1)...$. That is, $\bigcup_{i=0}^{2} \Theta_i(V_1) \subset V_1$.
Since V is the attractor of L, $V \subset V_1$. Observe that $V \subset \{z{:}z{=}(1.1...)_b{=}(0.\alpha...)_b, \alpha \neq 1\}$. In consequence, since no number has four representations (cf. [B]), the point R= $\frac{1}{2} - \frac{x}{3} = (0.x\overline{1w})_b = (1.0\overline{w1})_b = (xx.w\overline{x})_b \in V_1$ but is not in V, (cf. Fig. 3 and Fig. 7), QED.

Fig. 3

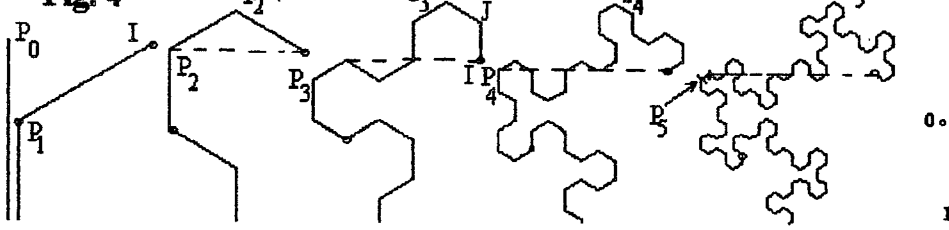


In view of Lemma 9 we note that to prove that the main dimensions of $K = \partial F$ are equal (to $s \leq \sigma$) we have used the fact that it is the union of six copies of the self-similar set V. But to prove that $\dim_B K = \sigma$ we used that K equals the union of six copies of $V_1 = F_0 \cap F_1$.

REMARK 3. One can prove that the set
$Y{:=}\{z{:}z{=}(1.1...)_b{=}(0.\alpha...)_b, \alpha \neq 1,$ *z has exactly two positional representations*$\}$
is an invariant set of the family of similarities $L{:=}\{\Theta_0, \Theta_1, \Theta_2\}$, i.e. $\bigcup \Theta_i(Y) = Y$. Therefore, $\bigcup \Theta_i(\overline{Y}) = \overline{Y}$ and $V = \overline{Y}$ follows.

8. THE CONVEX HULL OF V. Let us call $a = |I - G| = 1/\sqrt{3}$, (see Fig. 1). Then, Re(P_j)

Fig. 4



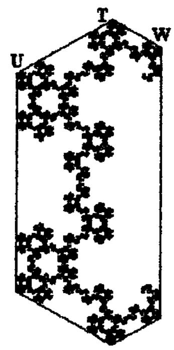


Fig. 5

(cf. Fig. 1) converges to $\frac{1}{2} - \frac{a.\sin\pi/3}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + ...\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.
On the other hand, Im(P_j)=$1/2\sqrt{3}$ if j is even or Im(P_j)=$1/2\sqrt{3} - (a/2)/2^j$ if j is odd.

Therefore P_j converges to $U=(1/6,1/(2\sqrt{3}))\in V$. Since the approximating poligonal V_j converges to V in the Hausdorff metric it follows that V has no point with real part less than 1/6 or with real part equal to 1/6 but outside the segment $U\bar{U}$. So, U and its conjugate are extreme points of the convex hull N=co(V). Since $|I-J|=|I-G|/8$, between I and J there is a copy of –V, one eighth of its size; thus, instead of the point U we find at the right side the point W∈V, $W=I+\frac{1}{8}\left(\frac{1}{3},\frac{1}{\sqrt{3}}\right)$=(13/24,5/(8$\sqrt{3}$)). Then, V is at the left of the line x=13/24 that supports N at the segment $W\bar{W}$. The poligonal V_j has as convex hull a hexagon N_j with upper vertex T_j that joins two segments with slopes $7\pi/6$ and $5\pi/6$. From the results stated above and the fact that V_j converges to V, we conclude that N_j converges to N. This is a hexagon and has $U,\bar{U},W,\bar{W}$ as vertices. Then, $T_j \to T$ where $T=(5/12,\sqrt{3}/4)=\Theta_0(U)$ and

$$\text{(16)} \qquad N=\text{co}(V)=\text{co}(U,T,W,\overline{U,T,W}).$$

In consequence, the diameter of V is equal to $|V|=|T-\bar{T}|=\sqrt{3}/2\cong 0.866$. But s=log3/log2 $\cong 1.58496$. Therefore, $H^s(V)\le|V|^s\cong 0.796$.

PROPOSITION 1. i) $\Theta_i(N)\subset N$,

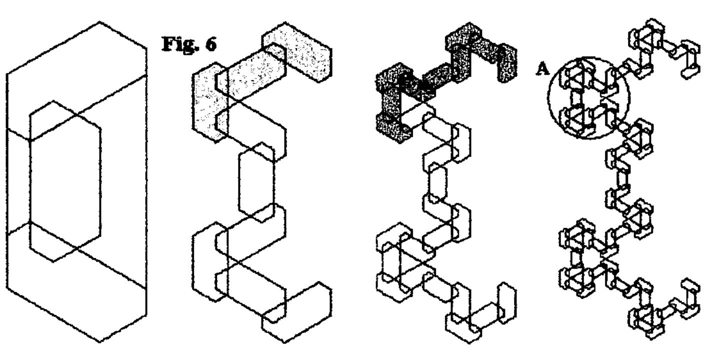
ii) $V=\bigcap_n(\bigcup\{N_{(i)}:(i)=(i_1,...,i_n)\in\{0,1,2\}^n\})$. Here, $N_{(i)}:=\Theta_{i_1}\circ\Theta_{i_2}\circ...\Theta_{i_n}(N)=\Theta_{(i)}(N)$,

iii) V is also a self-similar set in the following strict sense

$$\text{(17)} \qquad i\ne j\Rightarrow H^s(\Theta_i(V)\cap\Theta_j(V))=0. \bullet$$

PROOF. Observe that the intersection is on a decreasing family. We leave the proof of i) to the reader, (cf. Fig. 6). ii) follows immediately from i). iii) is an easy consequence of the finiteness of the measure of V and the equality of the Hausdorff and similarity dimensions, QED.

9. AN UPPER BOUND FOR THE HAUSDORFF MEASURE OF V. The circle A in Fig. 6 has a diameter equal to $\sqrt{3}/8$ and contains 17 pieces congruent to co(V)r^4, r=1/2. If $f=17r^{4s}$ then $|A|^s/f\cong$0.42148. We get $H^s(V)$<0.422 since, due to Theorem 1,§2,[Z], we have $H^s(V)\le|A|^s/f$. In fact, to prove that theorem i) and ii) of Proposition 1 are sufficient. Note



that the upper bound is almost the half of $|V|^s$. In relation with this kind of estimations and for autorships and improvements see [Ma], [P] and the references mentioned in [Z].

10. V IS AN s-SET. The following result is due to McLaughlin, ([M]). H^t represents the t-dimensional spherical Hausdorff measure.

AUXILIARY THEOREM 2. Let S be a non void compact set of Hausdorff dimension t. Suppose there exist positive constants C and r_0 such that for any open ball of radius $r < r_0$ there is an application $\varphi: S \cap B \to S$ such that for any x,y $\in S \cap B$,

(18) $$C/r \le d(\varphi(x),\varphi(y))/d(x,y).$$

Then, $(C/2)^t \le \mathsf{H}^t(S)$. •

DEFINITION 7. We say that the compact set E, the attractor set of the family of similarities $\{\varphi_1,...,\varphi_m\}$ of ratios $\{r_1,...,r_m\}$, has *property A* if there exists $\Delta > 0$ such that for any x$\in$E and any ball B=B(x;r), r < Δ, there exist y$\in$E and a similarity Y with contraction ratio one, such that:

a) Y(B(y;r) $\cap$ E) = B(x;r) $\cap$ E , b) $\exists j \in \{1,...,m\}$ such that B(y;r) $\cap$ E $\subset \varphi_j$(E). •

AUXILIARY THEOREM 3. Assume that E has property *A*. Then,

i) E satisfies (18) with C=inf$\{r_j\}.\Delta/2$,

ii) If t=$\dim_H$(E) then H^t(E) > 0. •

PROOF. i) is proved in [Z], §6. i) and Auxiliary Theorem 2 imply ii), QED.

THEOREM 5. i) V has property *A* and is an s-set, s=(log3/log2),

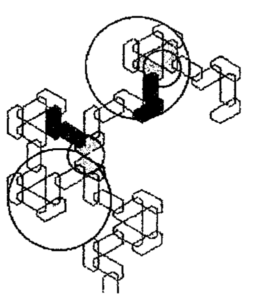
ii) K is a (log3/log2)-set,

iii) $\Delta^s \le H^s(V)$ where Δ is the parameter associated to property *A*. •

PROOF. To prove that V has property *A* for a certain Δ observe that we have to use a similarity Y different of the identity map only around the points $P_1,\overline{P_1}$, (see Fig. 4). But $E \cap B(P_1;\varepsilon))$ is congruent to $E \cap B(P_2;\varepsilon))$ for ε small enough and the assertion follows. Then i) is a consequence of the Auxiliary Theorem 3 and Theorem 4.

ii) follows from i) and Lemmas 3 and 4.

A general result proved in [P] for self-similar s-sets that satisfy (17) and have property *A* implies iii), QED.



REMARK 4. Our set *V* is a self-similar s-set with property *Z*: *V* has, by definition, *property Z* if there exists a multiindex $(t)=(t_{i_1},...,t_{i_p})$ such that for *N*=co(*V*), $N_{(t)} \subset \operatorname{int} N$ (see Figs. 5 and 6). Or equivalently (cf. [P]), *V* has *property Z* if *V* is not contained in the boundary of co(*V*). At this point we only know that $\Delta^s \le H^s$(V). But since V is a self-similar set with properties *Z* and *A* with a Δ approximately equal to 1.56/100 and all the ratios of similarity of the functions in the IFS equal to r=r_1=1/2, we can

apply a limit theorem proved in [P] to get a better *lower bound for* $H^s(V)$ *than* $\Delta^s \cong 1.37/10^3$. That theorem says that for $q \uparrow \infty$, $1/B_q \uparrow H^s(V)$, where B_q is a function defined *via* families of sets $N_{(j)}$ with $|j|$=lentgh of j = q, (cf. [Z], Th.8 for a description without proofs). To improve the lower bound shown it is *necessary* to use q>3 since $1/B_3 = \Delta^s$.

11. PARAMETRIZATION OF THE INVARIANT SET V. Let us *define*

(19) $$\theta_0(z) = \Theta_0(z) \qquad \theta_1(z) = \overline{\Theta_1(z)} \qquad \theta_2(z) = \Theta_2(z).$$

It is easy to see that the IFS formed with the similarities (19) has V as its attractor.

THEOREM 6. There exists a continuous function $f:[0,1] \to V$ which is onto and verifies, for $t, s \in [0,1]$, $|f(t) - f(s)| \le M|t-s|^\nu$, where $\nu := \frac{\log 2}{\log 3} = \frac{1}{s}$. •

PROOF. Let $t = (0.a_1a_2...)_3 \in [0,1]$, $a_i = 0,1,2$, and define $f(t) := \lim_{n\to\infty} \theta_{a_1} \circ ... \circ \theta_{a_n}(0)$.

a) *f is well defined*: Let $t = (0.a_1a_2...)_3 = \tilde{t} = (0.\tilde{a}_1\tilde{a}_2...)_3$. Then there exists a $k \in \mathrm{N}$ such that $a_j = \tilde{a}_j$ for $j < k$, $a_k = \tilde{a}_k + 1$, $a_j = 0, \tilde{a}_j = 2$ for $j > k$. Therefore,

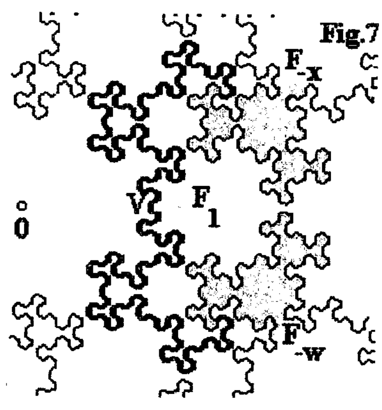
$$f(t) = \theta_{a_1} \circ ... \circ \theta_{a_k}(I) \qquad f(\tilde{t}) = \theta_{\tilde{a}_1} \circ ... \circ \theta_{\tilde{a}_k}(G)$$

where I and G, defined in paragraph 1, are the fixed points of $\Theta_0 = \theta_0$ and $\Theta_2 = \theta_2$ respectively. Since $\theta_0(G) = \theta_1(I)$=P (see Fig. 2) and $\theta_1(G) = \theta_2(I)$=Q=the conjugate of P, it follows that $\theta_{a_k}(I) = \theta_{\tilde{a}_k}(G)$, so $f(t) = f(\tilde{t})$ and *f* is well defined.

b) *f verifies the Hölder condition*: Let $t_1 = (0.a_1a_2...)_3$ and $t_2 = (0.b_1b_2...)_3$. If t_1 and t_2 belong to the same triadic interval of length 3^{-k} then they have representations such that $a_j = b_j$ for $j \le k$. Therefore $|f(t_1) - f(t_2)| \le M.2^{-k}$. Let *k* be such that

(20) $$3^{-k-1} < |t_1 - t_2| \le 3^{-k}$$

then there is a $t_0 \in [0,1]$ such that for i=1,2, t_i and t_0 belong to the same triadic interval of length 3^{-k}. So, $|f(t_i) - f(t_0)| \le M.2^{-k} = 2M.3^{-\nu(k+1)} < 2M|t_1 - t_2|^\nu$. In the last inequality we have used (20). The Hölder continuity follows, QED.



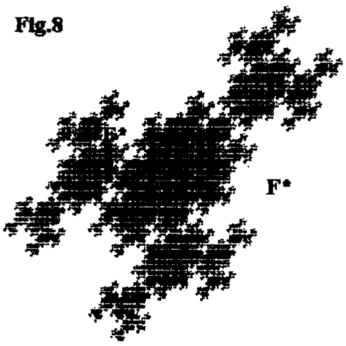
FINAL REMARK. If we conside the affine transformation $\Phi : x + iy \to \xi + i\eta$ given by

(21) $$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

then

(22) $$\Phi(0) = 0, \quad \Phi(1) = 1, \quad \Phi(i) = w, \quad \Phi(-1-i) = x.$$

Fig.8



F*

Therefore, Φ maps the gaussian integers onto the Eisenstein's point lattice and Φ^{-1} maps the base b=–2 and the ciphers $D = \{0,1,w,x\}$ into b=–2 and $D^* = \{0,1,i,-1-i\}$. The fractional set F of the system $\{-2, D\}$ is the invariant set of $f_c(z) = K \circ T_c(z)$, $c \in D$ where $T_c(z) = z + c$ and K is the contraction of ratio $-1/2$ while the fractional set F^* of the system $\{-2, D^*\}$ is the invariant set of $f_c{}^*(z) = K \circ T_c{}^*(z)$, $c \in D$ where $T_c{}^*(z) = z + c^* = z + \Phi^{-1}(c)$. It is easy to check that $f_c{}^*(\Phi^{-1}z) = \Phi^{-1}(f_c(z))$. Then $F^* = \Phi^{-1}(F)$, (cfr. Fig.8).

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NUMBERS WITH THREE REPRESENTATIONS IN THE NUMBER SYSTEM (-2,{0,1,exp2πi/3,exp4πi/3})

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ABSTRACT. In the number system (-2,D) with base b=–2 and family of ciphers $D=\{0,1,w,\overline{w}\}$ where $w=e^{2\pi i/3}$, $\overline{w}=w^2$, every complex number z is representable: $z=(a_N...a_0.a_{-1}a_{-2}...)_{-2}$, i.e., $z=\sum_{-\infty}^{N} a_j b^j$. In this paper we study the behaviour of the ciphers in the positional representations of complex numbers that are not uniquely representable in the system.

I. INTRODUCTION. Let $b\in\mathbf{C}$, $|b|>1$, $D=\{0,d_1,d_2,...,d_k\}\subset\mathbf{C}$. α is said *representable* in base b with ciphers D if there exists $\{a_j\in D: j=M,M-1,...\}$ such that $\alpha=\sum_{-\infty}^{M} a_j b^j$. We write $\alpha=a_M...a_0.a_{-1}a_{-2}...=(e.f)_b$ and call (e) the integral part of α and (f) the fractional part of α. G denotes the set of all representable numbers. F is the set of *fractional numbers,* i.e., those numbers in G with a representation such that $(e)=0$. The set W of *integers* of the system is the subfamily of G with a representation such that $(f)=0$. We study the representation of numbers in the number system with base –2 and set of ciphers $D\not\subset\mathbf{R}$, $D:=\{0,1,w,w^2\}$ where $w=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Note that x used as a cipher will represent the number $w^2=\overline{w}$. •

DEFINITION 1. $\sigma:=D\cup(-D)=\{0,\pm1,\pm w,\pm\overline{w}\}$,
E denotes the Eisenstein's point-lattice: E ≡ [1,w] := ={m.1+n.w: m, n ∈ Z},
$S:=D-D==\{0,\pm1,\pm w,\pm\overline{w},\pm(1-w),\pm(1-\overline{w}),\pm(w-\overline{w})\}$,
$S':=S\backslash\sigma=\{\pm(1-w),\pm(1-\overline{w}),\pm(w-\overline{w})\}$.•

Then, S and σ are subsets of the set E of Eisenstein "integers". The numbers in $\sigma\setminus\{0\}$ have modulus equal to 1 and those in S' have modulus equal to $\sqrt{3}$. Besides,

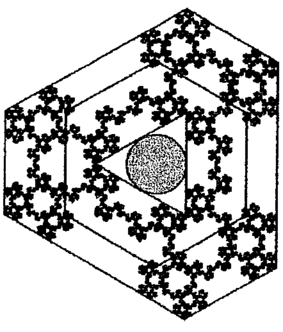
(1) $\alpha\in S\Rightarrow |\alpha|\le\sqrt{3}$, $|\mathrm{Re}\,\alpha|\le 3/2$, $|\mathrm{Im}\,\alpha|\le\sqrt{3}$ $\qquad$ $\alpha\in S\backslash\{0\}\Rightarrow 1\le|\alpha|\le\sqrt{3}$.

DEFINITION 2. $F_g:=g+F$ where $g\in E$.•

Thus, $F_0\equiv F$, the fractional set of the number system $(-2,\{0,1,w,x\})$. We shall call it the *Eisenstein set*. The definition 2 can be extended in the following form:

$$F_{a_M\ldots a_0.a_{-1}a_{-2}\ldots a_{-n}} := \{x; x = a_M\ldots a_0.a_{-1}\ldots a_{-n}\ldots\}.$$

DEFINITION 3. For $j \in D = \{0,1,w,x\}$, $\Phi_j(z) = \frac{z}{b} + \frac{j}{b} = -\frac{z+j}{2}$.•



THEOREM 1. i) W=E and m+nw has a unique representation in $(-2,\{0,1,w,x\})$.

ii) The family $\{F_g : g \in E\}$ defines a *tessellation* in the sense that $\mathbf{R}^2 = \bigcup \{F_g : g \in E\}$ and any two different sets of the family have an intersection of plane Lebesgue measure zero.

iii) $F = \bigcup_{i \in D} \Phi_i(F)$, the 4-rep tile F is the invariant set of the family $\{\Phi_j\}$. F is a compact connected set such that B(0;1/6) $\subset$ F $\subset$ B(0;1).

iv) $F_g \cap F_{g+h} \neq \emptyset \Leftrightarrow h \in$ S. •

II. STATES and TYPES. [1] Since G=C, any $\eta \in \partial F$ has at least two representations. To make clear the relations among the different representations of a given complex number let $z = (0.a_1a_2\ldots)_b \in F$ and e $\in$ W\{0} be such that $e.b_1b_2\ldots = 0.a_1a_2\ldots$. Then,

$$e = \sum_1^\infty (a_i - b_i)b^{-i} = \sum_1^\infty (-1)^i \frac{a_i - b_i}{2^i}. \tag{1}$$

Therefore, $|e| \le \sqrt{3}$ and the bound is reached, for example when $a_i - b_i = (-1)^i(w - \overline{w})$. Besides $|\mathrm{Re}\, e| \le 3/2$ and the bound is reached for $a_i - b_i = (-1)^i(1-w)$. If $|e| = 1$, (1) has several solutions. For example, $1.1w^2\overline{10} = 0.ww\overline{01}$ and $1.\overline{10} = 0.\overline{01}$ are two solutions for e=1. However, if $|e| = \sqrt{3}$ then $e \in$S' and (a) and (b) are determined: $e = b_1 - a_1$, $a_i = b_{i+1}$, $b_i = a_{i+1}$. We have then

THEOREM 1. i) The numbers in S\{0} can be written in a unique way as a difference of two ciphers.

ii) Let be $z = e.(b) = e.b_{-1}b_{-2}\ldots$, $e \in W$ and $z = 0.(a) = 0.a_{-1}a_{-2}\ldots$. Then $e \in$S. If $|e| = \sqrt{3}$ then $e \in$S', (a) and (b) are uniquely determined and $b_{-1} - a_{-1} = e$, $a_i = b_{i-1}$, $b_i = a_{i-1}$.

iii) $F \cap F_e \neq \emptyset \Rightarrow e \in S$ and $e \in S' \Rightarrow (e + F) \cap F$ contains only one point.•

The **state k of the p-expansion** of z, $z = \sum_{-\infty}^{L} p_j b^j$, is the number p(k) in W defined by

[1] The results in section II are an essential part of a paper by the authors: "Complex numbers with two positional representations ... ", Anales 5° Congreso Monteiro (1999)41-47. This part is included here for the sake of completeness. There it is also proved that there is no number with four representations.

p(k):=$\left(\sum_{k}^{L} p_j b^j\right) b^{-k}$. p(k) will also be called the **kth state of the p-representation** $p_L \cdots p_0 . p_{-1} p_{-2} \cdots$. If z has also a q-expansion z=$\sum_{-\infty}^{L} q_j b^j$ then by Theorem 1, ii), p(k)–q(k) *belongs to* S since $(p(k)-q(k)).p_{k-1} \ldots = 0.q_{k-1} \cdots$.

LEMMA 1. z= $p_L \ldots p_0 . p_{-1} p_{-2} \cdots$ and $\zeta = q_L \ldots q_0 . q_{-1} q_{-2} \cdots$ are equal if and only if $\forall k{:}p(k)-q(k) \in S$.•

PROOF. The if part follows from $\left|b^{-k}(z-\varsigma)\right| \le |p(k)-q(k)| + \sqrt{3} < 4$ letting $k \to -\infty$, QED.

We have p(k–1) = p(k)b + p_{k-1} and a similar expression for the q-expansion. Thus,

(2) $\quad (p(k)-q(k))b + (p_{k-1} - q_{k-1}) = p(k-1) - q(k-1)$.

Since b=–2, this formula can be written as

(2) $\quad p_{k-1} - q_{k-1} = (p(k-1) - q(k-1)) + 2(p(k) - q(k))$.

By the **state k of the p, q-representations** of z we mean the pair of states (p(k),q(k)) and will also refer to it as the **kth state** (p(k),q(k)). Most of the times it is not necessary to consider the kth state (p(k),q(k)) but only the difference $\Delta = p(k) - q(k)$. We call this number in S the **type** of the kth state (p(k),q(k)). That is,

DEFINITION 1. Given a number z with two positional representations p,q, we say that the kth state (p(k),q(k)) is of type Δ, <Δ>, if $\Delta = p(k) - q(k)$.•

The formula (2) gives the transition from the type Δ of the state k to the type Δ_1 of the state (*k*–1) in terms of the ciphers p_{k-1}, q_{k-1}. We shall represent it graphically as

(3) $$<\Delta> \xrightarrow[\binom{a}{c}]{} <\Delta_1>$$

where Δ_1 =*p*(*k*–1)–*q*(*k*–1) and *a*=p_{k-1}, *c*=q_{k-1}. Thus (3) stands for

(3) $\quad 2\Delta + \Delta_1 = a - c$.

According to Def. 1 I if the type Δ is not zero then neither the type Δ_1 nor *a*–*c* can be zero. So, *a*–*c* $\in$S\{0} in this case. Since any number in S\{0} can be uniquely written as a difference of two numbers in D (Th.1 i)), it follows that

(4) $\quad \Delta \ne 0 \Rightarrow a \ne c$, *a* and *c* uniquely determined.

We shall construct a digraph with nodes the types <Δ>, $\Delta \in$S, and arrows given by (3). To this end we examine the possible ciphers *a* and *c* that can occur in (3) in order that $\Delta_1 \in$S assuming that $\Delta \in$S.

THEOREM 2. i) If *a*,*c*$\in$D and Δ = *a*–*c*≠0 then $<a-c> \xrightarrow[\binom{a}{c}]{} <c-a>$.

ii) If $\Delta \in S'$ then $\Delta_1 = -\Delta = c - a$.

iii) If $\Delta = \pm 1$ then Graph 1 and Graph -1 show all the possibilities for $\Delta_1 \in S$.

iv) $<0> \xrightarrow[\binom{a}{c}]{} <a-c>$ for any a, c belonging to D.

v) the state <0> can only be reached from <0>.•

PROOF. The proofs of all the statements follow from (3). For example, $\Delta_1 = 0$ and $\Delta \neq 0$ implies $|a-c| \geq 2 >$ the modulus of any number in S. This contradiction proves v). If $\Delta = 1$ then $\Delta_1 = a-c-2$. So, $\Delta_1 \in S$ only if $\mathrm{Re}(a-c) \geq \frac{1}{2}$. This occurs in five cases, yielding the five arrows in Graph 1. We leave the details to the reader but observe that one must use repeatedly *the cyclotomic equation* $1+w+x=0$, QED.

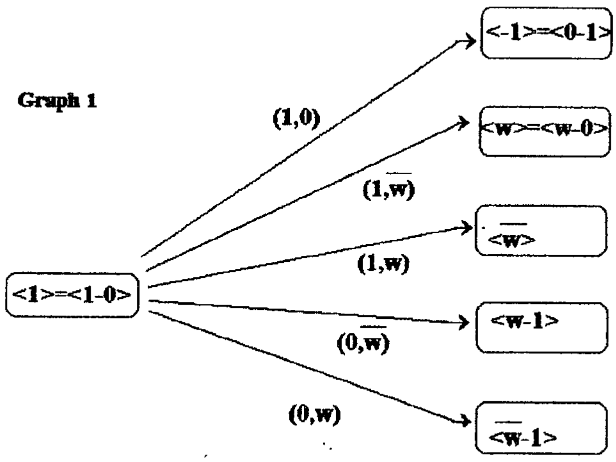
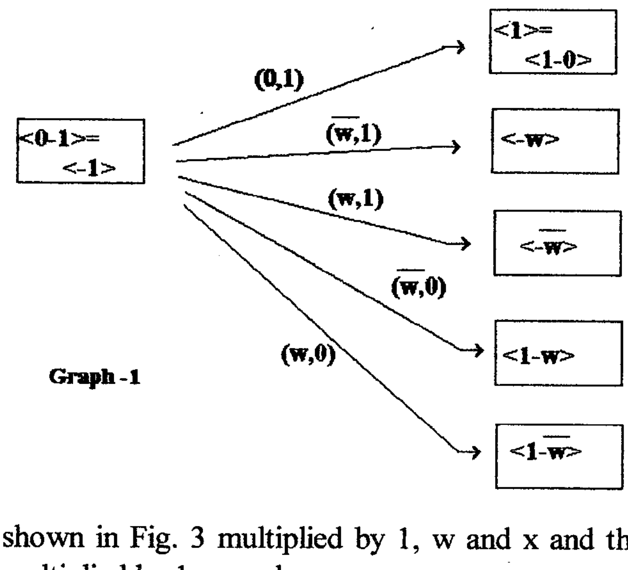


Fig. 1. We write (a,c) instead of $\binom{a}{c}$.

The types reached from $<\pm d>$, $d \in D \setminus \{0\}$, can be obtained from Graph 1 and Graph -1. In fact since $D\setminus\{0\}$ is a *multiplicative group*, $<\pm 1> \xrightarrow[(a,c)]{} <s>$ and $<\pm d> \xrightarrow[(da,dc)]{} <ds>$ are equivalent for $d \in D\setminus\{0\}$. Hence multiplying Graphs 1 and -1 in Fig. 1 by w and x we obtain the Graphs $w, -w, x$ and $-x$ with the arrows starting at $w, -w, x$ and $-x$, respectively.



DEFINITION 2. We call Γ the digraph with nodes the types and arrows from $<\Delta>$ to $<\Delta_1>$, as in (3), if $2\Delta + \Delta_1 = a-c$ with $a, c \in D$. •

The arrows of Γ starting at <0> are shown in Fig.2 except for a loop at <0>, (cf. Th. 2 v)).

THEOREM 3. The digraph Γ is obtained superposing the digraph with the arrows starting at <0> shown in Fig. 2, a loop at <0>, the three digraphs obtained from the one shown in Fig. 3 multiplied by 1, w and x and the three digraphs obtained from Fig. 3' multiplied by 1, w and x.•

For any type $<\chi>$ in Γ there is a point with a pair of representations of type $<\chi>$ (see Fig. 6). Th. 1 ii) explains the oscillation in the semicircular arcs in Figs. 3 and 4.

The edges that have arrows only in one direction are represented with dotted lines (see Figs. 3 and 3'). In Fig. 4 we show in full lines the edges of the graph Γ which have arrows in both directions. In Fig. 5 we have added, as truncated dotted lines, the remaining edges that start at $\sigma \setminus \{0\}$.

Therefore one obtains the **complete graph** Γ by superposing the **graph** $V\Gamma$ of Fig. 5 with the graph in Fig. 2 and a loop at <0>. Suppose z has **two** different positional representations. Then for some fixed k, the kth type is a node $<\Delta>$ of Γ different from <0>. The successive (k–1), (k–2), ...–th types of the representations are obtained following an infinite string starting at $<\Delta>$ in the digraph $V\Gamma$. The ciphers (p_{k-1}, q_{k-1}), (p_{k-2}, q_{k-2}), ..., are completely determined by the arrows, (cfr. (2) or (3)).

Combining this with Lemma 1 the next result is obtained.

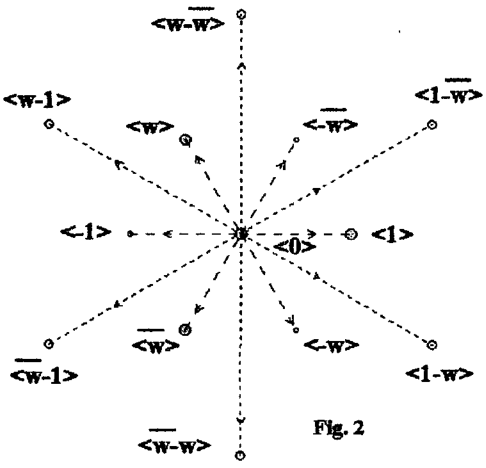


Fig. 2

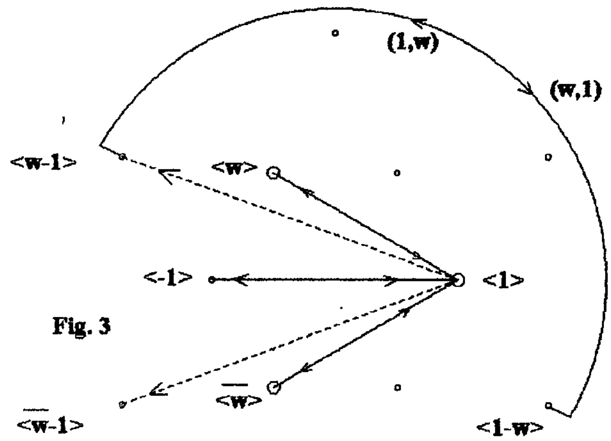
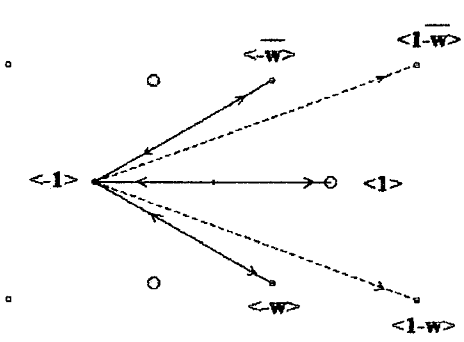


Fig. 3

THEOREM 4. The types of the successive states of a number with two positional representations p,q, follow, from some moment on, an infinite string in the graph $V\Gamma$. Conversely, given $k \in Z$ and a node $<\Delta> \neq <0>$ in the digraph $V\Gamma$ then two positional representations p,q of a number $z \in \mathbf{C}$ are obtained following an infinite string starting at $<\Delta>$, $\Delta = p(k) - q(k)$. z is defined by any of these representations.•

Fig. 3'



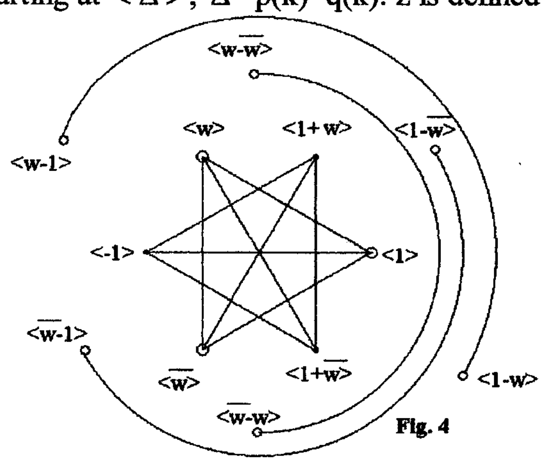
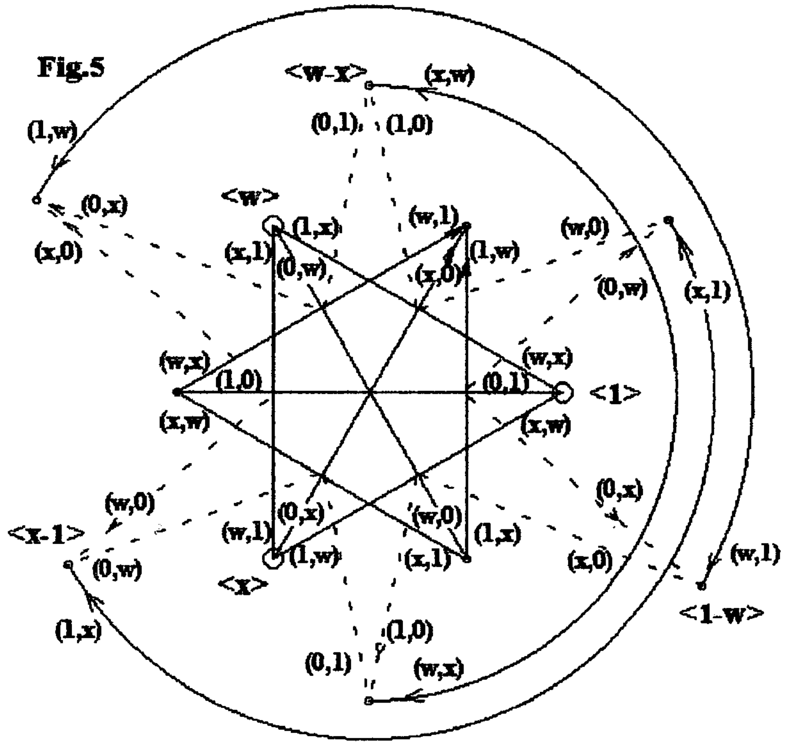


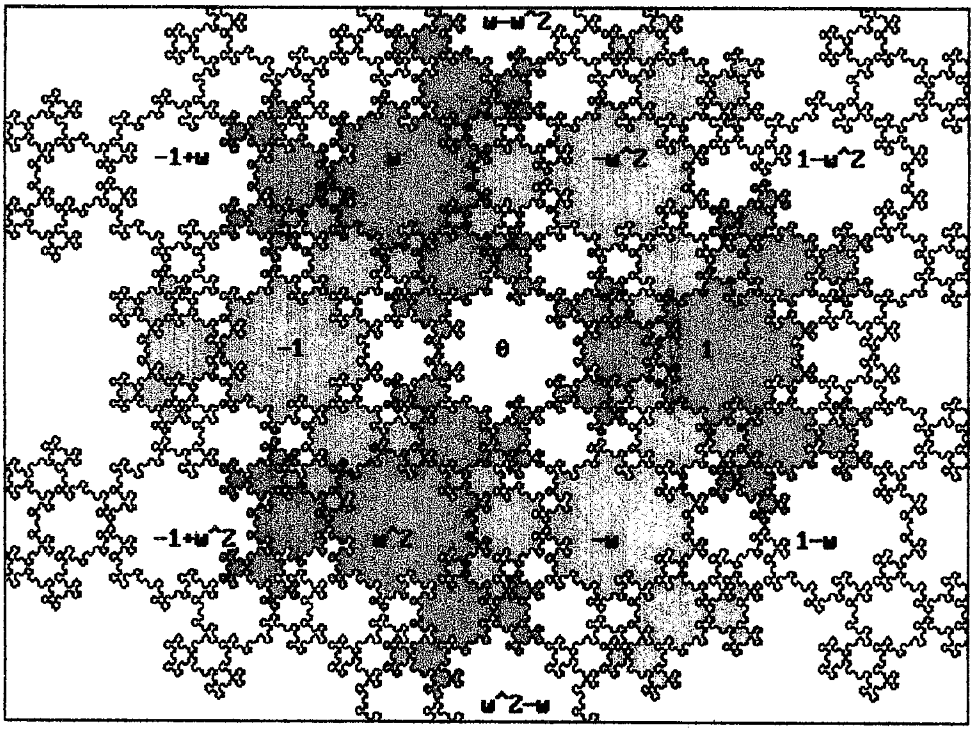
Fig. 4

We leave to the reader the remaining details of the proofs of Theorems 3 and 4. Next Fig. 5 shows the ciphers beside the arrows that permit the passage from one state to the next one.

Digraph V Γ

Fig.5



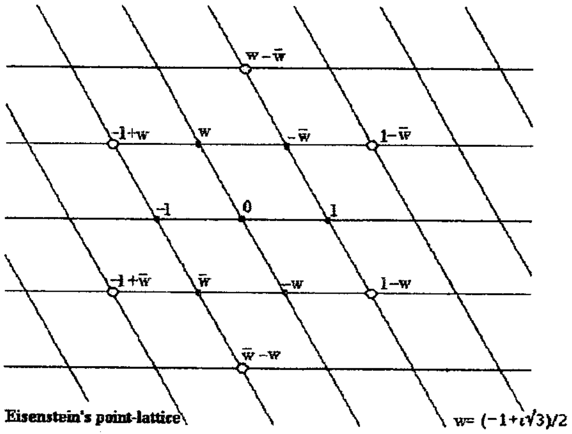


Family of tiles F_g, g in S **Fig. 6**

III. MULTIREPRESENTABLE NUMBERS. Let us assume that the complex number z has three different representations p, q, r:

(1) $\qquad (P)_b.p_{-1}p_{-2}\dots \qquad (Q)_b.q_{-1}q_{-2}\dots \qquad (R)_b.r_{-1}r_{-2}\dots$.

After multiplying by a power h of the base b the numbers P, Q, R are pairwise different (cf. II Th. 2). If this holds for h=m then it also holds for h>m. If we add $-P$ to the representations we could also assume that in (1), P=0. Then, because of I Th. 1, $Q,R\in S/\{0\}$, $Q\neq R$. We arrive in this way to a normalized situation. The possible values of Q with respect to P



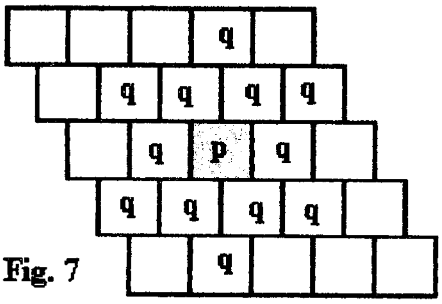


Fig. 7

are shown schematically in Fig. 7. Since these are also the values that can be taken by R with the restriction given by $Q\neq R$, $|Q-R|\leq\sqrt{3}$, we obtain the following possible configurations for the triad {P,Q,R}:

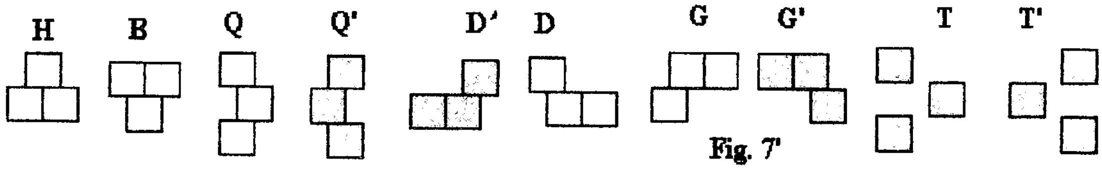
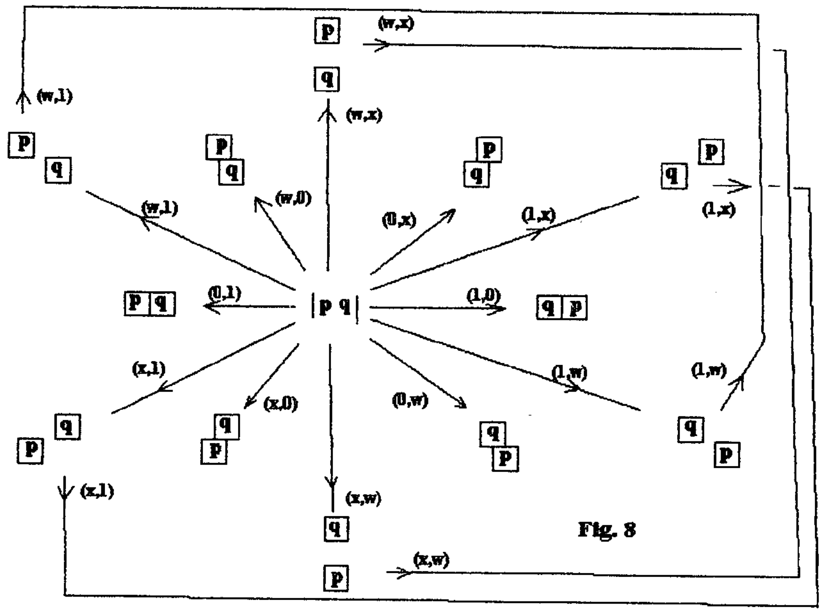


Fig. 7'



PROPOSITION 1. The configurations T, T', Q', D' and G' do not occur. •

Next we reproduce the content of previous graphs to keep them at hand in the proof of this proposition. Figure 8 is a substitute for Figure 2 with the addition of the semi-circular edges of Fig. 4. Figure 9 shows the dotted lines and the semicircular edges of graph $V\Gamma$ that start or end in S'. In Fig. 10 we see the central part of the graph $V\Gamma$.

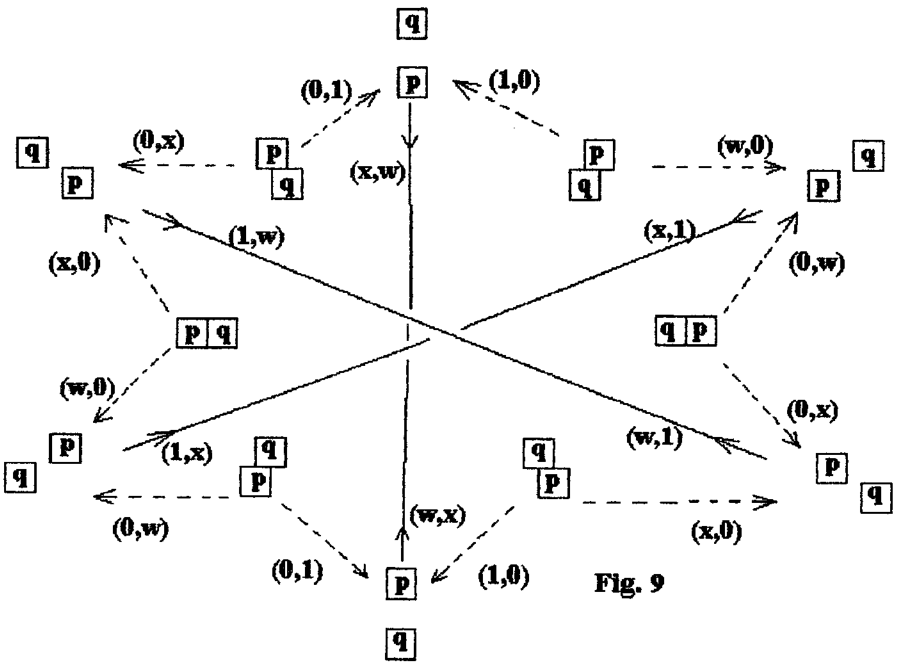


Fig. 9

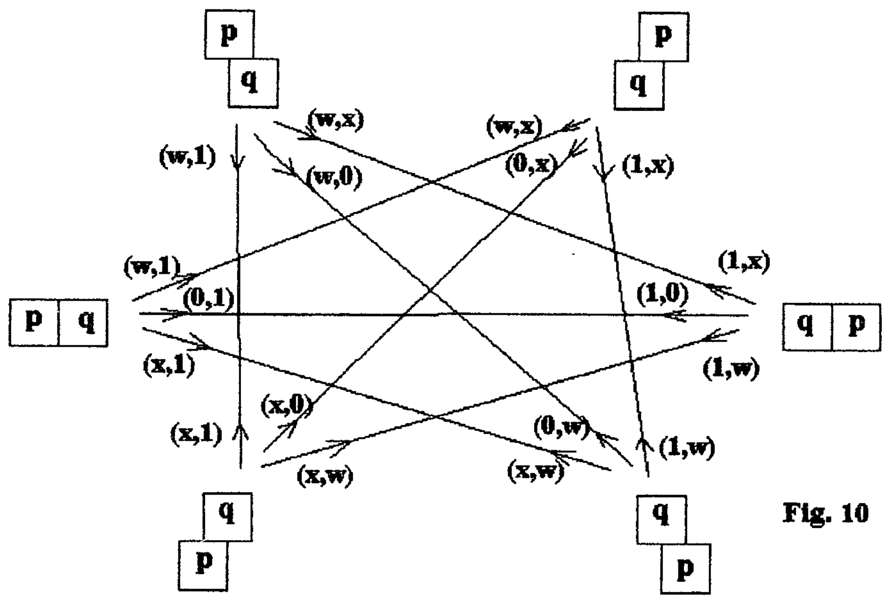


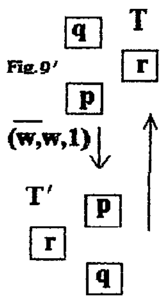
Fig. 10

In order to prove the Proposition 1 we extend some notions. By the **state k of the p,q,r-representations of z** we shall understand the triad (p(k),q(k),r(k)). (Here we assume that

P, Q and R in (1) are pairwise different and written with the same number L of ciphers. This could require to add zeroes at the beginning of one or two representations.)
Associated to the state k there is a **type** given by (Δ_1, Δ_2) to which the state belongs:

$$\Delta_1 = \Delta_{pq} = p(k) - q(k) \qquad \Delta_2 = \Delta_{qr} = q(k) - r(k).$$

So, $\Delta_3 = \Delta_{rp} = r(k) - p(k)$ is determined by the type. For i=1,2,3, Δ_i belongs to S\{0}.
This type is represented by one of the ten configurations shown schematically in Fig.7': H, B, Q, Q', D, D', G, G', T, T', the blanks filled adequately with p, q and r. The transition from state (p(k),q(k),r(k)) to (p(k−1),q(k−1),r(k−1)) is symbolized with an arrow bearing the ciphers $(p_{k-1}, q_{k-1}, r_{k-1})$.



Making use of the graph in Fig. 9 we see that from T we can go to T'. We obtain the ciphers in Fig. 9' from the full edges of that graph. This also shows that T' is the only configuration reachable from T in one step. It is seen, in the same way, that from T' one could only reach T. However, it follows easily that there are no ciphers that permit this passage from the configuration T' to the configuration T. In fact, p and q oblige to use the set of ciphers $(w, x, .)$, the pair q,r, the ciphers $(., 1, w)$ and p,r the set $(1, ., x)$ which is impossible. Then, T and T'are configurations that do not correspond to a type.

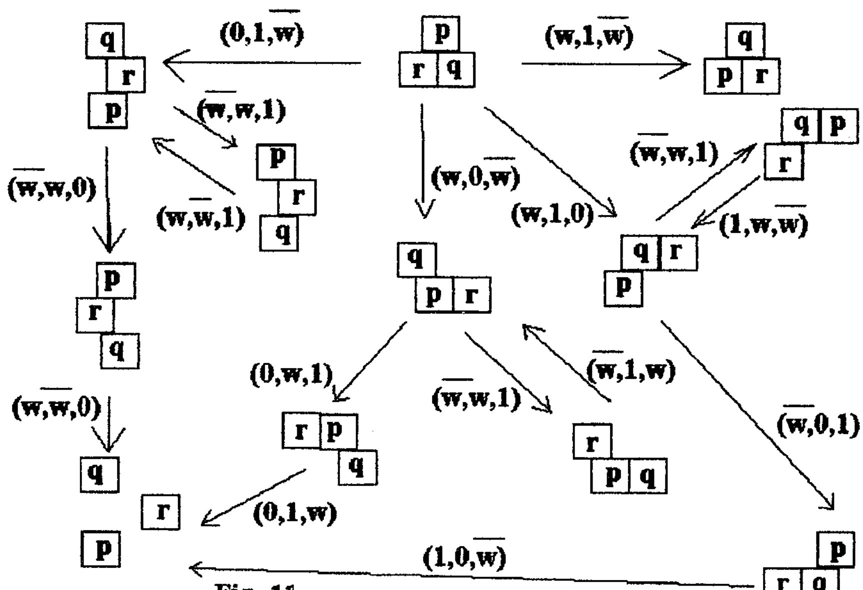


Fig. 11

The graphs in Figs. 11 and 11' are obtained using the diagrams in Figs. 9 and 10 starting in a configuration H or in a configuration B, (recall that $x = \overline{w} = w^2$). They show that from Q', G' and D' only the configuration T is reachable. Therefore Q'. D' and G' are simply configurations and not types. Therefore, the Proposition 1 follows, QED.

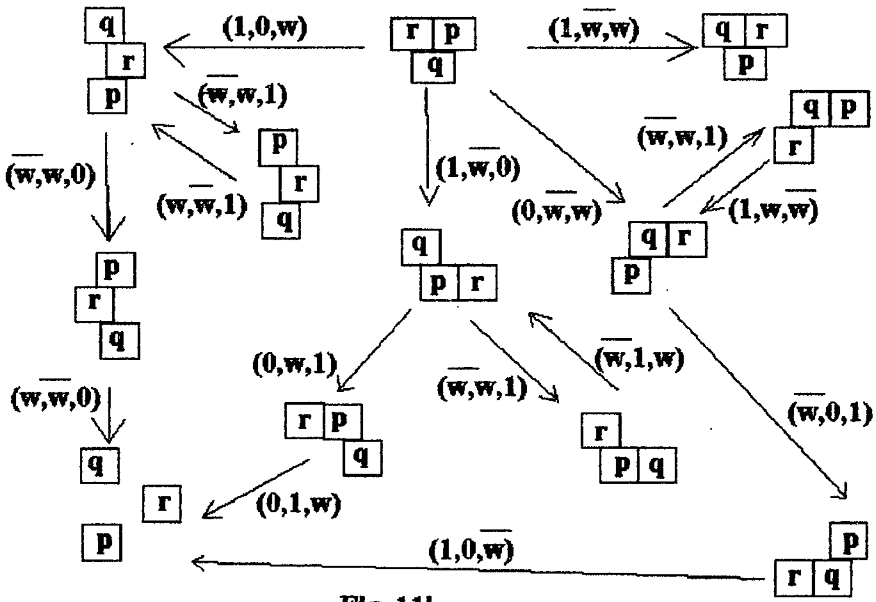
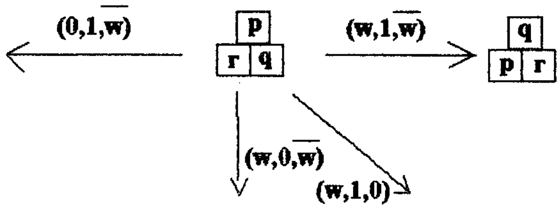
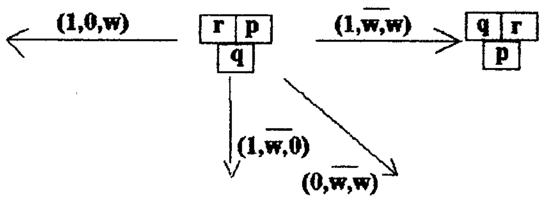


Fig. 11'

REMARK. Observe that the diagram in Fig. 11' could be obtained from that in Fig. 11 replacing

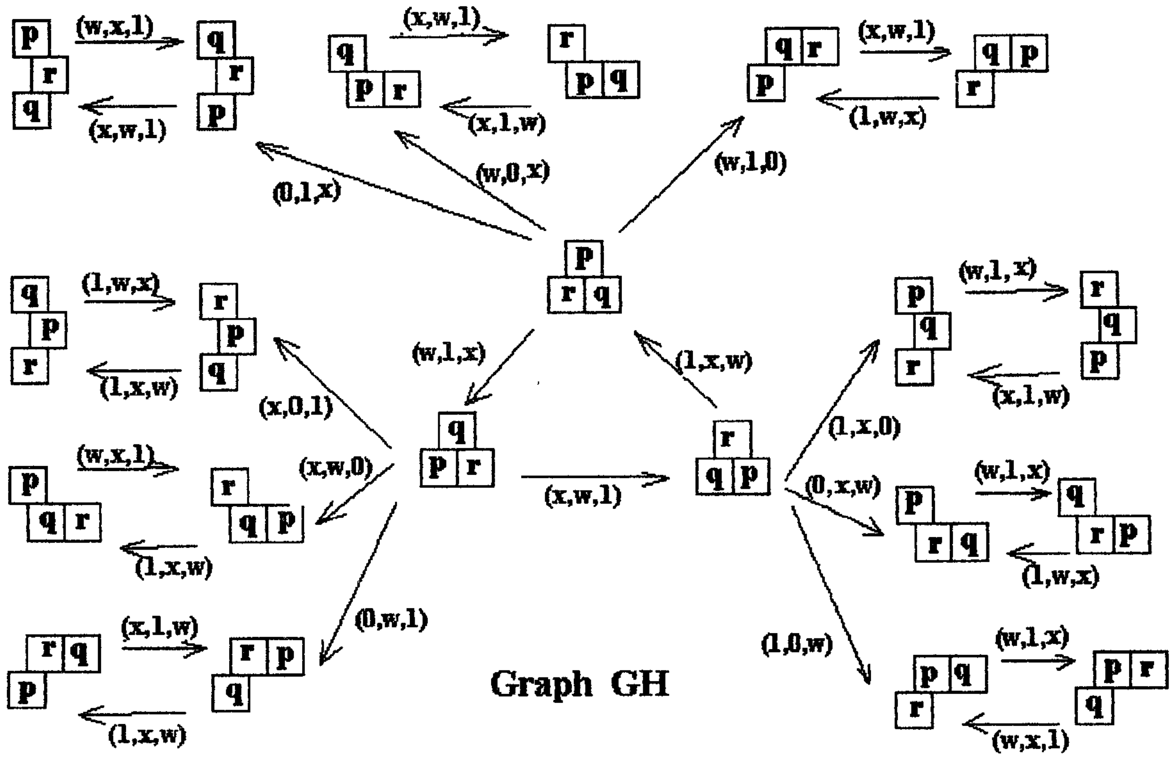


by

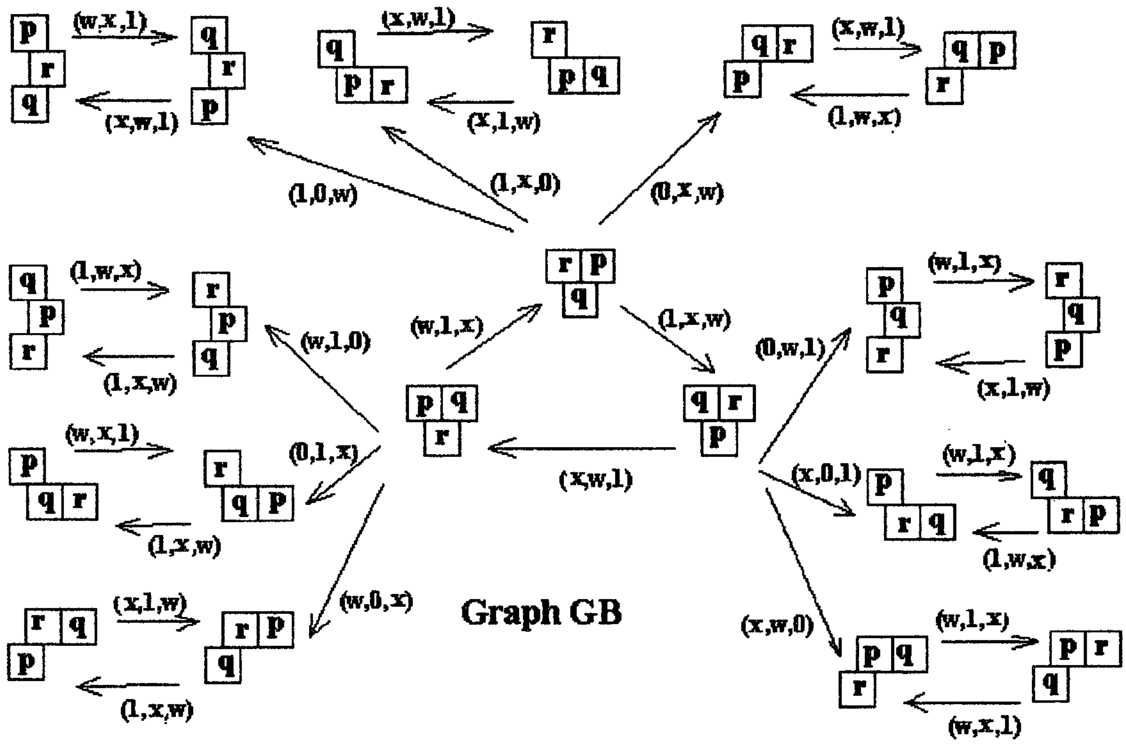


It can be shown that those configurations that were not excluded by Proposition 1 actually appear in numbers with three representations. If we do not take into account the relative position of p we have,

THEOREM 5. i) There are five and only five possible types of the representations of a number z with three different positional representations. They are: H, B, Q, D and G.
ii) From type H it is possible to go to the types H, Q, D and G and only to them.
iii) From type B it is possible to go to the types B, Q, D and G and only to them. •



Graph GH



Graph GB

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IV. THE FUNDAMENTAL GRAPHS. The diagrams of Figs. 11 and 11' lead to the *complete graphs* GH and GB. Several regularities of the behaviour of the ciphers may be discovered looking at them. For example,

THEOREM 6. i) The representations of a number with three positional representations in the number system $(-2,D)$ are finally periodic with a period of length one, two or three.
ii) The types Q, D and G are *stable* in the sense that once in one of them the type does not change.
iii) A number may be finally periodic of type H or B but these types are not stable.
iv) 0 is not a cipher in a period in the representations of a number with three positional representations. •

Observe that the complete graphs GB and GH are obtained one from the other interchanging their "kernels".
For references and bibliography the reader may consult the preceding paper of this volume.

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SOBRE LA REPRESENTACIÓN POSICIONAL DE NÚMEROS

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La Aritmética elemental contiene muchos pequeños resultados que iluminados desde una óptica de grafos y algoritmos resultan muy entretenidos y sencillos. Un ejemplo lo constituyen los sistemas numéricos generalizados.

Un sistema numérico tiene dos ingredientes. Una base b y un conjunto finito $A=\{0=a_0,a_1,...,a_{t-1}\}$ de cifras, todos ellos números complejos pertenecientes a algún anillo numérico contenido en **C** como por ejemplo **Z** o **Z**(δ), con δ un entero algebraico. Supondremos siempre que la base verifica la condición $|b|>1$.

Llamaremos *enteros del sistema* $\{b,A\}$ a los elementos del conjunto W:= $\left\{w=\sum_{j=0}^{N} c_j b^j; c_j \in A\right\}$ incluído en el anillo al cual pertenecen las cifras y la base.

Usaremos para un entero la notación $w=(c_N c_{N-1}...c_0)_b$ o bien, si la base se sobreentiende, $w=c_N...c_0$. Además sea $H:=\left\{x=\sum_{j=1}^{\infty} c_{-j} b^{-j}; c_i \in A\right\}$. H es el conjunto de los números *fraccionarios del sistema* $\{b,A\}$. Usaremos la notación $x=(0.c_{-1}c_{-2}...)_b$ o bien, si la base se sobreentiende, $x=0.c_{-1}c_{-2}...$. Finalmente llamamos *números representables en el sistema* $\{b,A\}$ a los elementos del conjunto $G:=\left\{x=\sum_{-\infty}^{N} c_j b^j; c_j \in A\right\}$.

Ejemplificaremos primero con números enteros. Sea $b\in Z$, $t:=|b|>1$ y $A\subset Z$ un sistema completo de restos módulo b que contiene al cero, esto es,

$$Z=\bigcup\{a_j+bZ; a_j \in A\} \text{ con } (a_j+bZ)\cap(a_i+bZ)=\varnothing \text{ si } i\neq j.$$

Entonces $W\subset Z$ y es natural preguntarse cuando vale la igualdad W=Z. Supongamos que nos dan un $z\in Z$ y queremos hallar su desarrollo en el sistema $\{b,A\}$. Entonces su término c_0 debe ser $c_0=a$ donde $a\in A$ es tal que $z\equiv a$(mód b). Luego, en primer lugar, los elementos de W tienen representación única. Esto permite construir un grafo dirigido por la siguiente función $J:Z\to Z$, $J(z):=(z-a)/b$, que se puede indicar $z\xrightarrow{a}\frac{z-a}{b}$. O sea, dándole el "color" a a la flecha si a es el resto de z (mód b) que está en A.

Si $z\in W$ entonces $J^h(z)=\sum_{j=h}^{N} c_j b^{j-h}$ y el camino que sale de z tiene sucesivamente los colores $c_0,c_1,...c_N,0,0,...$. O lo que es lo mismo, $J^{N+1}(z)=0$ a partir de un número natural N. Recíprocamente, si $J^{N+1}(z)=0$ para un número natural N entonces $z\in W$.

Sea $K=max\{|a|; a\in A\}$. Es inmediato que

$$\text{si } |z|>\frac{K}{|b|-1}=:L \text{ entonces } |J(z)|\leq\frac{|z|+K}{|b|}<|z|.$$

Además, si $|z| \le L$, entonces $|J(z)| \le L$. Como en todo intervalo hay sólo un número finito de enteros, resulta que el camino que sale de un entero z tiene que terminar en un ciclo, formado por enteros de módulo no mayor que L.

Definición. Denotamos con P al conjunto de enteros periódicos. Esto es

$$P = \left\{ n \in Z : \exists h \in \mathrm{N} \quad con \quad J^h(n) = n \right\}.$$

Proposición 1 (Kátai, [K]). Sea $b \in Z$, $|b|>1$ y $A \subset Z$ un sistema completo de restos módulo b. Son equivalentes las afirmaciones siguientes:

i) Z=W

ii) $P=\{0\}$.

Ejemplos: 1) $\{2,\{0,1\}\}$ $\quad P=\{0,-1\}$

2) $\{-2,\{0,1\}\}$ $\quad P=\{0\}$

3) $\{3,\{0,2,7\}\}$ $\quad P=\{0,-1\}$

4) $\{N,\{-h,-h+1,\ldots,\ -h+N-1\}\}$, N un número entero mayor que 1. Si $\max\{|h|,|N-h-1|\}<N-1$ entonces $P=\{0\}$. (En este caso $L<1$ por lo que no hay periódicos $\neq 0$.)

Son casos particulares de 4)

4') $\{3,\{-1,0,1\}\}$, $\{4,\{-1,0,1,2\}$.

No podemos escribir los enteros negativos en base 4 con las cifras $\{0,1,2,3\}$, pero basta cambiar la cifra 3 por la cifra -1 para que todo entero sea representable.

Proposición 2, ([K]). $b \in$Z, $A \subset$Z y $H := \left\{ x = \sum_{j=1}^{\infty} c_{-j} b^{-j} ; c_i \in A \right\}$. Entonces,

i) H es un conjunto compacto

ii) todo $y \in \mathbf{R}$ se puede representar en la forma $y=n+x$ con $n \in$Z, $x \in$H.

O sea siempre que $b \in Z$, $|b|>1$ y $A \subset Z$ es un sistema completo de restos módulo b, las trasladadas en enteros del conjunto H cubren la recta: $\mathbf{R} = \bigcup_{n \in Z} n + H$.

Esto puede llamar la atención cuando uno ve casos en los que el conjunto H tiene muchas lagunas. El diagrama precedente representa al conjunto H correspondiente al sistema $\{3,\{-1,0,7\}\}$. Si bien este conjunto es de segunda categoría de Baire, como todos los H que entran en la proposición 2, los intervalos abiertos que contiene tienen longitud muy pequeña y no son fáciles de detectar a simple vista, p. ej. $\left(-\frac{1}{54},\frac{1}{54}\right) \subset H$.

También uno puede preguntarse si son "muchos" los números que están en más de un conjunto $n+H$ Respecto a esta pregunta vale la siguiente proposición, donde m(Δ) representa la medida de Lebesgue del conjunto Δ y B:=$\{ c = c_1 - c_2 ; c_j \in A \} = A - A$.

Proposición 3. (Indlekofer, Kátai, Rácskó, [IKR]) Sea $b \in Z$, $|b|>1$, $A \subset Z$ un sistema completo de restos módulo b. Entonces, m($n_1 + H \cap n_2 + H$)=0 para todo par de números diferentes $n_1, n_2 \in$Z si y sólo si
(*) los enteros del sistema numérico $\{b$, B$\}$ coinciden con **Z**.
Obsérvese que B=A–A y por lo tanto contiene en particular a $A \cup -A$. Entonces es inmediato que la condición (*) se cumple en el caso $\{3,\{-1,0,7\}\}$ del gráfico pues en ese caso B$\supset \{0,1,-1\}$ y con estas cifras en base 3 se escribe todo entero, (Ej. 4'). O sea que las trasladadas en números enteros del conjunto H de la figura precedente no sólo cubren la recta sino que la cubren económicamente: sólo se superponen en conjuntos de medida cero.
(En general la condición (*) es poco restrictiva ya que el conjunto B puede contener $|b|$ restos (módulo b) para un entero z que no es múltiplo de b: si uno construye un grafo con vértices enteros correspondiente al sistema $\{b$, B$\}$ de manera que de cada entero z no divisible por b salgan flechas hacia $(z-c)/b$ donde c recorre los restos mencionados, hay más posibilidades (que con $\{b$, A$\}$) que de z salga un camino que termine en 0, (véase la Prop. 1).)
Analizando el caso b=3 y A= $\{0, a_1, a_2\}$ con a_i=i (módulo 3), i=1, 2 , se observa que

(**) $$\text{m.c.d.}(a_1, a_2)=1$$

es una condición necesaria para que se cumpla (*). En efecto, si m.c.d.(a_1, a_2)=d>1 entonces d divide a todo elemento de B y por lo tanto todos los enteros del sistema $\{3$,B$\}$ son multiplos de d. Luego no vale (*). Se conjetura que (**) también es suficiente y esto fue demostrado para los casos en que $a_2 - a_1 \in \{1,4,7,10\}$ y también verificado para $1 \le a_i \le 990$, i=1,2, sin otra limitación salvo por $a_i \equiv i$ (mód 3).

La intersección de H+n **con** H. Hemos dicho que $H+n \cap H+k$ tiene medida 0 para los sistemas numéricos que cumplen (*) de la proposición 3. Sin embargo esta intersección suele tener a menudo dimensión de Hausdorff positiva.
Sea S:=$\{n \in Z \setminus 0: H \cap H+n \neq \emptyset\}$ y para $\gamma \in$S, sea $B_\gamma := H+\gamma \cap H$. $z \in B_\gamma$ si y sólo si

(1) $$z=\gamma + \sum_{j=1}^{\infty} \varepsilon_j b^{-j} = \sum_{j=1}^{\infty} \tilde{\varepsilon}_j b^{-j} \text{ con } \varepsilon_j, \tilde{\varepsilon}_j \in A.$$

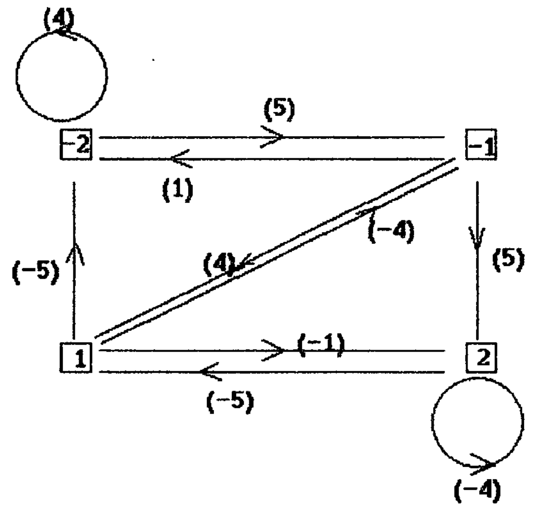
La igualdad anterior implica por un lado que $|\gamma| \le 2L = 2K/(|b|-1)$ y por otro lado que

(2) $$b\gamma + (\varepsilon_1 - \tilde{\varepsilon}_1) \in \text{S}.$$

Consideremos el grafo cuyos vértices forman S_1:={n∈Z: |n|≤2L} tal que de cada vértice $\eta \in S_1$ salgan todas las posibles flechas con "colores" $\delta \in$B y que van de η a $b\eta+\delta$, si este entero resultare ser un vértice de S_1. Si un vértice está en S entonces es claro por (1) y (2) que de él sale por lo menos una flecha que llega a un vértice de S (quizás él mismo) y por lo tanto sale de él un camino infinito que pasa sólo por vértices pertenecientes a S. Recíprocamente, si en este grafo de un vértice sale un camino infinito entonces ese vértice está en S y también lo están todos los vértices que aparecen sobre este camino infinito.

Borrando de S_1 los vértices de los cuales no sale ninguna flecha queda un conjunto S_2, borrando de S_2 los vértices de los que no salen flechas que terminan en S_2 queda un conjunto S_3, etc. Como en este proceso $S_j \supset S$ para todo j, luego de un número finito de pasos se tendrá S_j=S y se habrá encontrado S. G(S) será el grafo de vértices S y flechas desde $\gamma \in$S a $b\gamma + (\varepsilon_1 - \tilde{\varepsilon}_1) \in$S etiquetadas con $\delta = \varepsilon_1 - \tilde{\varepsilon}_1$ (habrá tantas con la misma etiqueta como representaciones que dan lugar a ese δ pero con ε_1 distintos).

Sea b=3, A={−1,0,4}. L=2, B={−5,−4,−1,0,1,4,5}. Con el proceso anterior se encuentra que S={±1,±2}. Esto es que la "baldosa" H toca, siempre en el borde, a sus trasladadas en ±1, ±2. El grafo G(S) tiene el siguiente aspecto en este caso. Veamos como se puede determinar la dimensión del contorno de H. Nos limitaremos a la dimensión box de ese conjunto (que no necesariamente coincide con la de Hausdorff aunque vale que $\dim_B \geq \dim_H$).



Una 2n-upla $(\varepsilon_1,\ldots,\varepsilon_n,\tilde{\varepsilon}_1,\ldots,\tilde{\varepsilon}_n) \subset A^{2n}$ se dirá *continuable con respecto de* B_γ si existen cifras $\varepsilon_{n+1},\varepsilon_{n+2},\ldots,\ \tilde{\varepsilon}_{n+1},\tilde{\varepsilon}_{n+2},\ldots$ tales que $z := \sum_{j=1}^{\infty} \varepsilon_j b^{-j} = \gamma + \sum_{j=1}^{\infty} \tilde{\varepsilon}_j b^{-j} \in B_\gamma$.

Sea M(n,γ) el número de 2n-uplas continuables con respecto de B_γ. Se puede ver que la dimensión box de B_γ existe y es

$$\dim_B(B_\gamma) = \lim_{n\to\infty} \frac{\log(M(n,\gamma))}{n\log|b|}.$$

Puede estimarse M(n,γ) usando el grafo G(S) pues M(n,γ) es el número de caminos de longitud n que salen de γ. Por ejemplo, en el caso anterior b=3, A={−1,0,4}, el grafo nos muestra que vale la recurrencia (por la simetría del grafo vale siempre M(n,γ)=M(n,−γ)):

M(n,1)=M(n−1,−2)+M(n−1,−1)+M(n−1,2) =M(n−1,1)+2.M(n−1,2),

M(n,2)=M(n−1,2)+M(n−1,1).

Más aún, si llamamos $y^{(n)} := \begin{vmatrix} M(n,1) \\ M(n,2) \end{vmatrix}$ y $P := \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$ entonces el sistema anterior se escribe como $y^{(n)} = Py^{(n-1)} = P^{n-1}y^{(1)}$; por inspección del grafo se ve que $y^{(1)} = \begin{vmatrix} 3 \\ 2 \end{vmatrix}$.

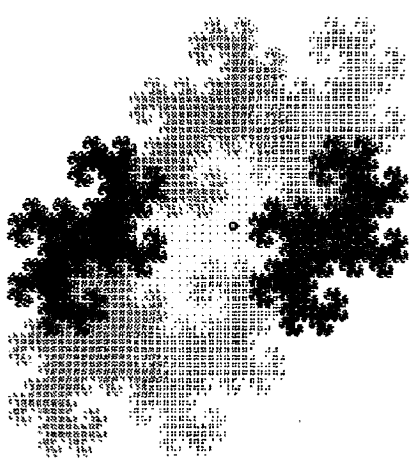
A esta ecuación de recurrencia se le aplica la teoría de Perron de matrices positivas: si ξ es el autovalor de mayor módulo de la matriz P, él y el autovector correspondiente son positivos, obteniéndose que $M(n,\gamma) \approx \xi^n$. Luego, como en este caso $\xi = 1+\sqrt{2}$, tenemos

$$\dim_B(B_\gamma) = \frac{\log\xi}{\log 3} = \frac{\log(\sqrt{2}+1)}{\log 3} \cong 0.8023.$$

Sistemas numéricos complejos en anillos de enteros algebraicos.
Ejemplificaremos ahora con sistemas numéricos en anillos de enteros algebraicos. Sea Θ un entero algebraico solución de la ecuación $f(z) = z^n + p_1 z^{n-1} + ... + p_n = 0$ donde $f(z)$ es un polinomio mónico, irreducible, de coeficientes enteros. Sea $\mathbf{Z}(\Theta)$ el anillo generado por $1, \Theta$, es decir, $\mathbf{Z}(\Theta)=\{u_0 + u_1\Theta + ... + u_{n-1}\Theta^{n-1}; u_j \in Z\}$. Sea $b\in\mathbf{Z}(\Theta)$, $|b|>1$. El conjunto $I_b := \{b\sigma; \sigma \in Z(\Theta)\}$ es un ideal en $\mathbf{Z}(\Theta)$. (Es claro que si $b=\Theta$, $u_0 + u_1\Theta + ... + u_{n-1}\Theta^{n-1} \in I_b$ si y sólo si u_0 es un múltiplo de p_n). Sea $t=\#\{clases\ laterales\ de\ Z(\Theta)/I_b\}$, (vale $t<\infty$). Eligiendo, para $j=0,1,...,t-1$, $a_j \in \mathbf{Z}(\Theta)$ de la j-ésima clase de restos en el cociente $\mathbf{Z}(\Theta)/I_b$ de manera que $a_0 \in I_b$, definimos el sistema completo de restos $A=\{a_0,...,a_{t-1}\}$. Con éste y precisando $a_0=0$, formamos el sistema numérico $\{b, A\}$ con cifras A. Los enteros de este sistema pertenecen a $\mathbf{Z}(\Theta)$ y nos podemos preguntar si todo elemento de $\mathbf{Z}(\Theta)$ se representa como un entero del sistema. Esto se formula ahora así: dado $\alpha=\sum_{j=0}^{n-1} u_j\Theta^j \in\mathbf{Z}(\Theta)$ arbitrario ¿existen elementos $c_j \in A$ tales que $\alpha = \sum_{j=0}^{N} c_j b^j$?

En analogía a lo hecho antes en **Z** se ve que de existir esos c_j ellos quedan determinados unívocamente: c_0 debe ser la cifra que está en la misma clase lateral que α, etc. Definimos la función J en $\mathbf{Z}(\Theta)$ por $J(\alpha):=(\alpha-c)/b$ donde $c\in A$ es el elemento de A en la misma clase que α. Si queremos repetir la proposición 1 tropezamos con la dificultad que ahora el grafo de la función J puede contener caminos infinitos que no terminan en ciclos pues en $\mathbf{Z}(\Theta)$ puede haber infinitos puntos en una esfera (aun en el caso en que b y sus raíces conjugadas tengan módulo mayor que 1). Un ejemplo de ello es $b=3-\sqrt{2} \in\mathbf{Z}(\Theta)$, $\Theta=\sqrt{2}$. b satisface la ecuación minimal $x^2 - 6x + 7 = 0$. $A:=\{0,1,2,...,6\}$ es un sistema completo de restos. En este caso $\mathbf{Z}(\Theta)=\{m+n\sqrt{2}\}$ es un conjunto denso en R.
Si β es no real, habrá un número finito de puntos de $\mathbf{Z}(\beta)$ en toda esfera si, por ejemplo, $\mathbf{Z}(\beta)\subset$ látice del plano complejo $\Lambda=\{mu+nv: m,n\in Z\}$, u y v linealmente independientes sobre **R**. Vale la siguiente
Proposición 4. Sea $\beta \in$C, no real y tal que para todo $j\geq 0$, β^j está en el látice Λ. Entonces β es un entero cuadrático, (cf. [BPP]).
Veamos algunos casos de esta situación.
Los enteros de Gauss. Sea $b=A+Bi\in\mathbf{Z}(i):=\{m+ni;\ m,n\in Z\}$, $|b|>1$. La ecuación minimal que satisface b es $x^2 - 2Ax + A^2 + B^2 = 0$ y un sistema completo de restos módulo b es $A_c := \{0,1,...,A^2 + B^2 - 1\}$ (el subíndice "c" indica "canónico"). $\mathbf{Z}(i)$ es un látice por lo que en cada bola sólo hay un número finito de elementos y en consecuencia la proposición 1 se extiende a este caso. Como $\mathrm{Im}(b^j)$ es un múltiplo de B para todo j, resulta que el conjunto de los enteros del sistema $\{b, A_c\}$ no coincide con $\mathbf{Z}(i)$ si $|B|>1$. Vale el siguiente teorema.

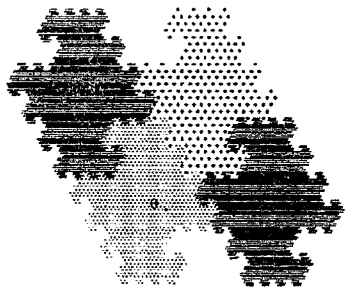
Teorema 1 (Kátai y Szabó, [KSz]). Los enteros del sistema $\{b, A_c\}$ coinciden con **Z**(i) si y sólo si Im(b)=±1, Re(b)<0.
Dado que los enteros del sistema $\{b, A_c\}$ coinciden con **Z**(i) sigue que todo número complejo es representable, por lo tanto, **C**=$\{z = c_N ...c_0.c_{-1}c_{-2}...: c_j \in A_c\} = \bigcup_{w \in W} H + w$, (véase el teorema 3). Esto dice que las trasladadas en enteros del sistema del conjunto H pavimentan el plano complejo. Entonces m(H)>0. Un argumento de empaque muestra también que m($H + w_1 \cap H + w_2$)=0 para $w_1 \neq w_2$. En el caso más simple el teorema 1 dice que con la base b=–1+i y cifras $A_c = \{0,1\}$, todo entero de Gauss es representable.



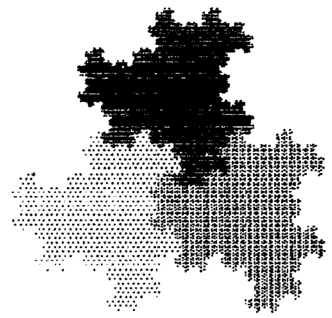
La figura muestra parte del embaldosado $\bigcup_{w \in W} H + w$ en el sistema $\{-1+i, \{0,1\}\}$. Es fácil ver que en los casos que b=2,1+i,1–i, no existen cifras A adecuadas para las cuales los enteros de $\{b,A\}$ coinciden con **Z**(i). En efecto, en estos casos $b-1$ tiene módulo 1; luego, si $c \in A \subset$**Z**(i) es una cifra no nula entonces $\gamma = c.\overline{(1-b)}$ verifica $\gamma = c + b\gamma$, de donde sigue que $J(\gamma) = \gamma$, o sea que γ es un elemento periódico no nulo que no puede representarse como entero en W.
Si **Z**(i)∍ b≠2,1+i,1–i y $|b|$>1 entonces siempre existe un sistema de cifras $A_b \subset$**Z**(i) para el cual los enteros del sistema $\{b, A_b\}$ coinciden con Z(i), (Steidl, [S]). Esto pasa, por ejemplo, para b=A+i con el sistema de cifras simétrico

$$A_s = \left\{ \left[\frac{-A^2+2}{2} \right], ..., \left[\frac{A^2+2}{2} \right] \right\} \quad \text{si } |A|>3.$$

Los enteros de Eisenstein. Consideremos el anillo **Z**(w)=$\{m + nw; m,n \in Z\}$ donde $w = \exp\left(\frac{2\pi}{3} i\right)$ es raíz cúbica de la unidad. w verifica la ecuación cuadrática $w^2 + w + 1$=0. También **Z**(w) es un látice y la proposición 1 se extiende a este caso.

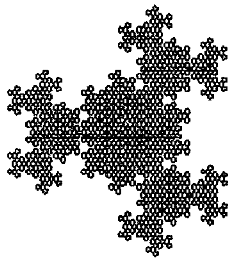


Luego, por ejemplo, las cifras A_d={0,1,w,1+w} junto con la base b =2w^2 forman un sistema numérico donde los enteros del sistema coinciden con ***Z***(w). Las baldosas H,H+1,H+1+w,H+w, son exhibidas en la figura adyacente. En este caso 0 es un punto interior de H, (cf. prop. 5). En la figura abajo a la derecha ilustramos los conjuntos H,H+1,H+1+w para b=2+w, y cifras $\tilde{A} = \{0,1,1+w\}$. En el segundo caso 0 no es un punto interior de H y el conjunto de enteros de ese sistema no coincide con **Z**(w). En efecto, $1-b$ es una unidad del anillo y se repite un argumento ya usado para probar que $\gamma = 1.(\overline{1-b}) = w^2$ es un elemento periódico y por lo tanto no representable. En ambos casos **C**= $\bigcup_{l\in Z(w)}(H+l)$ pues vale el



Teorema 2. Sea $b\in$**Z**(Θ), Θ un entero cuadrático y A un sistema completo de restos que contiene al 0. Entonces **C**= $\bigcup_{l\in Z(\Theta)}(H+l)$, (cf. prop. 2 y T. 2, [K])

Otro ejemplo de sistema numérico donde los enteros del sistema coinciden con $Z(w)$ es el de base -2 y cifras $A=\{0,1,w,-1-w\}$. En este caso también C= $\bigcup_{l\in Z(w)}(H+l)$ y 0 es un punto interior de H. El conjunto H se ve abajo a la izquierda.



Propiedades geométricas del conjunto H. Con cualquier elección de base y cifras el conjunto H es un conjunto compacto y autosemejante. En efecto, de toda sucesión de números de H se puede elegir una subsucesión que va "estabilizando las cifras" pues el conjunto de ellas es finito. Si la sucesión original converge, el límite será el elemento de H dado por las cifras estabilizadas.

La autosemejanza sigue del hecho que $H=\bigcup_{i=1}^{k} f_i(H)$ donde $A=\{a_1,\ldots,a_k\}$ y $f_i(z)=(a_i+z)/b$. Para cada i, $H_{.a_i} := f_i(H)=\{x=0.a_ic_2c_3\ldots; c_j\in A\}$ es un conjunto semejante a H resultante de una contracción de módulo $1/|b|$. Reunimos en la siguiente proposición algunos resultados generales válidos para H.

Proposición 5. Si m(H) es la medida de Lebesgue de H entonces

i) m(H)$\leq |b|^{-2} k$ m(H)

ii) Si m(H)>0 entonces $1\leq |b|^{-2} k$

iii) Si m(H)>0 entonces $|b|^2=k$ si y sólo si m($f_i(H)\cap f_j(H)$)=0 para $i\neq j$.

iv) Si 0 es un punto interior de H entonces **C**= $\bigcup_{w\in W}(H+w)$. La recíproca también es cierta si sólo hay un número finito de enteros en cada esfera.

Si recordamos que **C**= $\bigcup_{w\in W}(H+w)$ implica que m(H)>0, sigue que (por *ii*)) aquello sólo puede ocurrir si $|b^2|\leq k$. Además, por *iii*), ese cubrimiento será económico sólo si $|b^2|=k$.

Comentarios finales. 1) En primer lugar demostraremos el

Teorema 2. Sea $b\in\mathbf{Z}(\Theta)$, Θ un entero cuadrático no real y A un sistema completo de restos que contiene al 0. Entonces $\mathbf{C}=\bigcup_{l\in Z(\Theta)}(H+l)$.

Demostración. Observemos que vale, con J, K y L como antes, que si $z\in Z(\Theta)$ entonces $|J(z)|\leq\frac{|z|}{|b|}+\frac{K}{|b|}$, $|J^h(z)|\leq\frac{|z|}{|b|^h}+\frac{K}{|b|}+...+\frac{K}{|b|^h}<\frac{|z|}{|b^h|}+L$. Luego, si $|z|\leq|b|^h L$, resulta $|J^h(z)|<2L$. Sea $m:=\#\{\alpha\in Z(\Theta):|\alpha|<2L\}$. $m<\infty$ pues $\mathbf{Z}(\Theta)$ es un látice. En consecuencia, $J^{h+m}(z)\in P$. Esto dice que todo $z\in Z(\Theta)$ se escribe en base b como $z=(pc_N c_{N-1}...c_0)_b$ donde p es periódico (eventualmente 0) y $c_j\in A$. Entonces $z.b^{-M}\in\bigcup_{l\in Z(\Theta)}(H+l)$ para todo $M\leq N$, (aunque $p\neq 0$ no es representable). Como N es arbitrariamente grande, resulta que $\{z.b^{-M};z\in\mathbf{Z}(\Theta),M\in\mathbf{N}\}\subset\bigcup_{l\in Z(\Theta)}(H+l)$. Pero $\mathbf{C}=\mathrm{cl}\{z.b^{-M};z\in\mathbf{Z}(\Theta),M\in\mathbf{N}\}\subset\mathrm{cl}(\bigcup_{l\in Z(\Theta)}(H+l))=\bigcup_{l\in Z(\Theta)}(H+l)$. cl($\Delta$) indica la clausura del conjunto Δ y la última igualdad se debe a que H es compacto y $\mathbf{Z}(\Theta)$ intersecta a toda esfera en un conjunto finito de puntos, QED.

2) Esta demostración sirve para probar la proposición 2 cambiando $\mathbf{Z}(\Theta)$ por $\mathbf{Z}$ y $\mathbf{C}$ por $\mathbf{R}$.

3) La proposición 4 admite el siguiente complemento.

Proposición 6. Sea Θ un entero algebraico no real de grado $d>2$.

a) Existen en $Z(\Theta)$ elementos tan próximos entre sí como se desee.

b) Si $\mathbf{C}=\bigcup_{l\in Z(\Theta)}(H+l)$ entonces para algún par de elementos de $Z(\Theta)$, $w_1\neq w_2$, $H+w_1\cap H+w_2$ tiene medida positiva.

Demostración. a) El conjunto $B=\{c_0+c_1\Theta+...+c_{d-1}\Theta^{d-1}:|c_j|<k\}\subset\mathbf{Z}(\Theta)$ está contenido en una esfera de radio Ck. Supongamos que dos puntos arbitrarios de $\mathbf{Z}(\Theta)$ estén a distancia $>2\varepsilon>0$. B contiene $(2k-1)^d$ puntos distintos y los círculos con centro en ellos y radio ε son disjuntos. Además, ellos están en el círculo con centro 0 y radio $Ck+\varepsilon$. Luego $\varepsilon^2(2k-1)^d\leq(Ck+\varepsilon)^2$. Esto lleva a un absurdo para $k\to\infty$.

b) Por hipótesis $\mathbf{C}=\bigcup_{l\in Z(\Theta)}(H+l)$ por lo que los conjuntos $H+w$ deben contener una esfera de radio $c>0$ (T. de Baire). En consecuencia, si el par de elementos w_1, w_2 dista menos que c, $m(H+w_1\cap H+w_2)>0$, QED.

4) Para recuperar teoremas de teselado que valen para el caso $d=2$ parece más indicado asignar a $z=c_0+c_1\Theta+...+c_{d-1}\Theta^{d-1}\in\mathbf{Z}(\Theta)$ el vector $\vec{z}:=[c_0c_1...c_{d-1}]^t\in Z^d$. En esta correspondencia $\mathbf{Z}(\Theta)\to Z^d$, Z^d hereda las operaciones de $\mathbf{Z}(\Theta)$ y se convierte en un anillo. Por ejemplo, la suma será la operación común de suma entre vectores y la

multiplicación por el vector $\vec{\Theta}=[0,1,0,\ldots,0]'$, correspondiente a Θ, será multiplicar por

la matriz M= $\begin{vmatrix} 0 & 0 & . & . & . & 0 & -p_0 \\ 1 & 0 & . & . & . & 0 & -p_1 \\ . & . & . & . & . & . & . \\ & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1 & -p_{d-1} \end{vmatrix}$; es decir, $\vec{\Theta}\circ\vec{z}:=\overrightarrow{\Theta.z}=M\vec{z}$.

Los números $p_j \in Z$ provienen de la ecuación minimal $g(\Theta)=\Theta^d+p_{d-1}\Theta^{d-1}+\ldots+p_0=0$ que define a Θ. (M se denomina la matriz acompañante del polinomio $g(x)$ y su polinomio característico es $\pm g(x)$). Multiplicar por Θ^j corresponderá a multiplicar por la matriz M^j y multiplicar a z por un elemento $b\in Z(\Theta)$, $b=b_0+b_1\Theta+\ldots+b_{d-1}\Theta^{d-1}$, corresponderá a la multiplicación por la matriz $\widetilde{M}:=\sum_{j=0}^{d-1} b_j M^j$: $\overrightarrow{b.z}=\vec{b}\circ\vec{z}=\widetilde{M}\vec{z}$.

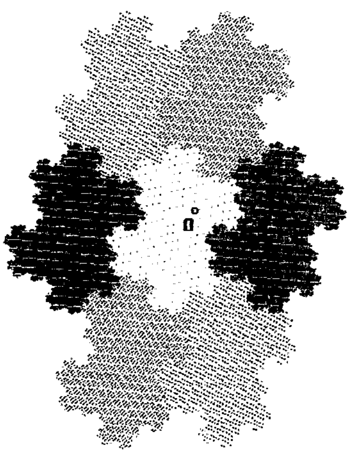
Sea $A=\{0,a_1,\ldots,a_{t-1}\}$ un sistema completo de restos módulo $b\in Z(\Theta)$. Entonces $\widetilde{A}:=\{\vec{0},\vec{a}_1,\ldots,\vec{a}_{t-1}\}$ es un sistema completo de restos de $Z^d/\widetilde{M}Z^d$ (y recíprocamente). Se definen como enteros del sistema numérico $\{\vec{b},\widetilde{A}\}=\{\widetilde{M},\widetilde{A}\}$ a los elementos del conjunto $\widetilde{W}:=\left\{\vec{z}=\sum_{j=0}^{N}\widetilde{M}^j\vec{c}_j:\vec{c}_j\in\widetilde{A}\right\}$, los fraccionarios del sistema serán los elementos del conjunto $\widetilde{H}:=\left\{\vec{z}=\sum_{-\infty}^{-1}\widetilde{M}^j\vec{c}_j:\vec{c}_j\in\widetilde{A}\right\}$ y los elementos del conjunto $\widetilde{G}:=\left\{\vec{z}=\sum_{-\infty}^{N}\widetilde{M}^j\vec{c}_j:\vec{c}_j\in\widetilde{A}\right\}$ serán los representables del sistema. Para la definición de $\widetilde{H}$ y $\widetilde{G}$ es necesario que las series involucradas converjan. Ese será el caso si todos los autovalores de $\widetilde{M}$ tienen módulo mayor que 1, o sea, si el radio espectral de $\widetilde{M}^{-1}$ es menor que 1.

Teorema 3 ([B]) Sea $\widetilde{M}$ una matriz $d\times d$ de elementos enteros, radio espectral de $\widetilde{M}^{-1}<1$ y $t:=|\det\widetilde{M}|$. Sea $\widetilde{A}:=\{\vec{0},\vec{a}_1,\ldots,\vec{a}_{t-1}\}\subset Z^d$ un sistema completo de restos módulo $\widetilde{M}$. Entonces, en el sistema $\{\widetilde{M},\widetilde{A}\}$ vale $\mathbf{R}^d=\bigcup_{\vec{z}\in Z^d}(\vec{z}+\widetilde{H})$.

5) W. J. Gilbert también ha estudiado sistemas como en el punto anterior en el caso particular en que b es un entero algebraico Θ y las cifras son $A_c=\{0,1,\ldots,t-1\}$ donde t es la norma de Θ:=|producto de todas las raíces del polinomio minimal|$=|p_0|=|\det M|$. En [G] demuestra entre otros interesantes resultados el siguiente (cf. también [KaK], [KaKo]),

Teorema 4. a) Si Θ es un entero cuadrático de polinomio minimal $\Theta^2+p_1\Theta+p_0$ entonces los enteros del sistema $\{\Theta,A\}$ coinciden con **Z**(Θ) si y sólo si $p_0\geq 2$ y $-1\leq p_1\leq p_0$.

b) En cada cuerpo cuadrático **F** puede hallarse un entero algebraico Θ tal que los enteros del sistema $\{\Theta,A\}$ coinciden con **Z**(Θ) y son precisamente los enteros algebraicos del cuerpo **F**.



6) En la figura a la izquierda mostramos el embaldosado por el conjunto fraccionario H correspondiente a la base $b=\dfrac{-1+i\sqrt{7}}{2}$ y cifras $A=\{0,1\}$ con $\Theta=b$, W=**Z**(Θ). H es una réplica del llamado "dragón domado".

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