ISSN 0078 2009

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BEHAVIOUR OF THE EIGENVALUES OF CLASSICAL BOUNDARY PROBLEMS IN THE PLANE

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REMARKS ON A THEOREM OF Å. PLEIJEL AND RELATED TOPICS, I,

BEHAVIOUR OF THE EIGENVALUES OF CLASSICAL BOUNDARY PROBLEMS IN THE PLANE.

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In memoriam Alberto R. Galmarino

RESUMEN. El problema clásico de Dirichlet en una región de Jordan plana D de contorno J admite una sucesión de autovalores y autofunciones, $w_n \in C^2(D) \cap C(\overline{D}),\ w_n = 0\ en\ J,\ -\Delta w_n = \lambda_n w_n\ en\ D,$ $(n \geq 1)$, tales que si J es C^2 vale $\sum \frac{1}{\lambda_n^z} = \frac{áreaD}{4\pi}\frac{1}{z-1} - \frac{longJ}{8\pi}\frac{1}{z-1/2} + g(z)$, $g(z)$ holomorfa en el semiplano derecho. Pleijel demuestra en particular esta fórmula pero para curvas C^∞. Sea $N(\lambda) = \#\{\lambda_j : \lambda_j \leq \lambda\}$. La fórmula asintótica de H. Weyl, $\delta(\lambda) \equiv N(\lambda) - \frac{áreaD}{4\pi}\lambda = o(\lambda)$, puede mejorarse si J es bastante regular de manera que $\delta(\lambda) = B\lambda^{1/2} + o(\lambda^{1/2})$, como conjeturó el mismo Weyl. Berry sostuvo que si el contorno no es regular esa expresión es todavía válida si se cambia 1/2 por otro exponente relacionado con la dimensión de Hausdorff de J. Pero esto no es cierto en general. Lapidus insiste en que lo que importa no es esa dimensión sino la dimensión de Minkowski del contorno que coincide con la anterior en muchos casos. En este trabajo presentamos también una ligera discusión sobre este tema.

Palabras clave: distribución de autovalores, serie de Dirichlet espectral, conjeturas de Weyl, Berry y Lapidus.

ABSTRACT. We consider Dirichlet's problem in the classical sense for the Laplacian in a plane Jordan region D with boundary J. If $\{\lambda_j\}$ is the set of eigenvalues of that problem, the counting function $N(\lambda) = \#\{\lambda_j : \lambda_j \leq \lambda\}$ satisfies H. Weyl asymptotic formula: $\delta(\lambda) := N(\lambda) - \frac{areaD}{4\pi}\lambda = o(\lambda)$.

Related to the monotonous function $N(\lambda)$ is the spectral Dirichlet series:

$P(z)=\sum \lambda_n^{-z}=\int_0^\infty x^{-z}dN(x), \operatorname{Re} z>1$. Weyl's conjecture states that if J is sufficiently regular then one should have $\delta(\lambda)=B\sqrt{\lambda}+o(\sqrt{\lambda})$. This has been proved, for example, for regions with C^∞-boundary. The behaviour of $P(z)$ follows from that of $N(\lambda)$ but results about $P(z)$ can be obtained without *a priori* knowledge of the counting function. A theorem of Å. Pleijel deals with this type of results. He proves that if D has a C^∞-boundary the following formula holds: $\sum \frac{1}{\lambda_n^z}=\frac{areaD}{4\pi}\frac{1}{z-1}-\frac{lengthJ}{8\pi}\frac{1}{z-1/2}+g(z)$, $g(z)$ holomorphic at least in $\operatorname{Re} z>0$. In fact, $g(z)$ is meromorphic in a wider region and has simple poles at some negative half integers. In this paper we discuss some points related to the Weyl (§1) and Berry conjectures (§4) and collect some results on eigenvalues, eigenfunctions and Green's kernel that hold for plane membranes (§2). We show that the preceding formula holds for a Jordan region with a C^2-boundary (§3). We present a simplified proof for this case. We show that if $N(\lambda)$ admits a second term in its asymptotic formula then it is determined by the second term in Pleijel's generalization. In §4 we also introduce the concept of ε-semiregular region. §5 and §6, are partly of expository nature. There we prove that the variational eigenvalues and eigenfunctions and the classical ones coincide for general Jordan regions proving then that in our framework we can use the results obtained by the powerful variational method.
Key words: distribution of eigenvalues, spectral Dirichlet series, Weyl, Berry and Lapidus conjectures.

1. INTRODUCTION. Our objective is the study of the behaviour of the eigenvalues of the classical Dirichlet problem in Jordan regions. In this paper *region* will always mean a bounded open connected plane set. A *Jordan region D* is a simply connected region whose boundary $J=\partial D$ is a Jordan curve. One problem is then to determine the asymptotic behaviour of the *counting function* $N(\lambda)=\#\{\lambda_j:\lambda_j\le\lambda\}$=the number of eigenvalues of Dirichlet problem not greater than λ counted according to their multiplicities. For a Jordan region D the following important result due to H. Weyl holds: if $|D|$=area of D then $\frac{4\pi}{|D|}N(\lambda)\sim\lambda$, $0<\lambda\to\infty$, i.e., $\lambda_n\sim\frac{4\pi}{|D|}n$, $n\to\infty$. Weyl's *conjecture* asserts that the second term in $N(\lambda)=\lambda|D|/4\pi+o(\lambda)$ can be determined and behaves as $\sqrt{\lambda}$ if the boundary of the region is sufficiently regular. In fact, Ivrii proved that if J is very smooth then $N(\lambda)=\frac{|D|}{4\pi}\lambda-\frac{\langle J\rangle}{4\pi}\sqrt{\lambda}+o(\sqrt{\lambda})$ where $\langle J\rangle$=length of J. Moreover, Kuznetsov proved that if the eigenvalues can be determined by the method of separation of variables then the same formula holds. For a region with a wild boundary the problem could be restated as to find an estimation of the *discrepancy* $\delta(\lambda):=N(\lambda)-\frac{|D|}{4\pi}\lambda$. Sometimes the behaviour of some function associated to $N(\lambda)$ gives information about the behaviour at infinity of the counting function. For example, the Laplace transform $Z(t)=\int_0^\infty e^{-\lambda t}dN(\lambda)=\sum_{n=1}^\infty e^{-\lambda_n t}$, also called the *partition function*, or the *spectral Dirichlet series* $\sum\lambda_n^{-s}$ defined by the function $P(s):=\int_{1+}^\infty x^{-s}dN(x), \mathrm{Re}\, s>1$. In fact, it can be shown using abelian and tauberian theorems that $N(\lambda)\sim A\lambda,\ \lambda\to\infty$ is equivalent to $Z(t)=\sum_{n=1}^\infty e^{-\lambda_n t}\sim A/t,\ 0<t\to 0+$. Here we must have $A=|D|/4\pi$. On the other hand, if $N(\lambda)=A\lambda+B\sqrt{\lambda}+o(\sqrt{\lambda}),\ \lambda\to\infty$ and $Z(t)=a/t+b/\sqrt{t}+o(t^{-1/2})$, $t\downarrow 0$ hold then $a=A,\ b=B\sqrt{\pi}/2$. We prove this proposition in the APPENDIX (§ 9).

For the spectral Dirichlet series we have the following result

THEOREM 1. *If* $N(\lambda)=A\lambda+B\sqrt{\lambda}+o(\sqrt{\lambda})$ *and* $P(s)=\frac{a'}{s-1}+\frac{b'}{s-1/2}+c'(s),\ s>1$, *with* $c'(s)$ *holomorphically extendable to* $s\geq 1/2$ *then* $a'=A,\ b'=B/2$.

PROOF. Let $F(x)=(B+o(1))\sqrt{x}=(B+f(x))\sqrt{x}$. Then, $P(s)=\frac{A}{s-1}+\lim_{K\to\infty}\int_1^K x^{-s}dF(x)$.

The limit is equal to $constant+s\int_1^\infty F(x)x^{-s-1}dx=cnt.+Bs/(s-1/2)+s\int_1^\infty f(x)x^{-s-1/2}dx=$

$B/2(s-1/2)+cnt.+s(\int_1^M+\int_M^\infty)f(x)x^{-s-1/2}dx=B/2(s-1/2)+p(s)+q(s)$ where $p(s)$ is holomorphically extendable to $\mathrm{Re}\, s>0$ and $q(s)=s\int_M^\infty f(x)x^{-s-1/2}dx$. Let $|f(x)|\leq\varepsilon$ for

$x \geq M > 1$. Then, $\left| s \int_M^\infty \cdots \right| \leq \frac{\varepsilon s}{|s-1/2|}$ if $s > 1/2$. For $s > 1$ we have

$0 = \frac{A-a'}{s-1} + \frac{B/2-b'}{s-1/2} + p(s) + q(s) - c'(s)$. It follows immediately that $A = a'$. Therefore,

$0 = \frac{B/2-b'}{s-1/2} + p(s) + q(s) - c'(s)$. For $s > 1/2$ we have

$b' - B/2 = (p(s) + q(s) - c'(s))(s - 1/2)$. That is, for $s \downarrow 1/2$, $|b' - B/2| \leq \varepsilon / 2$ holds and this inequality is valid for any $\varepsilon > 0$. In consequence, $B/2 = b'$, QED.

2. GREEN'S KERNEL. In this section we suppose that *D* is a Jordan region with boundary *J* and consider its classical Dirichlet problem. That is, we understand by *classical* eigenfunctions (eigenvalues) functions (numbers) w_n (λ_n) such that $w_n \in C^2(D) \cap C(\overline{D})$, $w_n = 0$ on ∂D, $-\Delta w_n = \lambda_n w_n$ on *D*, $(n \geq 1)$. In the following theorems, except for the continuity of the function defining the boundary, no regularity hypothesis on *J* is needed (Observe that there are Jordan regions with fractal boundary or with boundary of positive area.) Assume that $\lambda = -\chi^2$, $\chi > 0$. A radial function $u(p) = \phi(|p|)$ is a solution of $\Delta u + \lambda u = 0$ if and only if $\phi(\rho)$ is a the solution of $\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d\phi}{d\rho}\right) - \chi^2 \phi = 0$, $\rho > 0$. Let $r = \chi \rho > 0$ and $K(r) := \phi(\rho)$. Then the differential equation may be written as $\frac{1}{r}\frac{d}{dr}\left(r\frac{dK}{dr}\right) - K = K'' + \frac{1}{r}K' - K = 0$. *Kelvin's function of order* 0: $K_0(r) := \int_1^\infty \frac{e^{-rt}}{\sqrt{t^2-1}} dt$, $r > 0$, is a solution of this equation. In fact,

$K_0''(r) - K_0(r) = \int_1^\infty \sqrt{t^2-1}\, e^{-rt} dt = \int_1^\infty t\,(t^2-1)^{-1/2} e^{-rt}\, dt / r = -K_0'(r)/r$. Let us record some properties of Kelvin's function. $K_0(r) = \int_1^\infty (e^{-rt}/t) dt + \int_1^\infty (e^{-rt}/(t^3 + t^2\sqrt{t^2-1} - t)) dt$. Let $X(t,r)$ denote the last integrand. We get $K_0(r) = \int_r^\infty (e^{-t}/t) dt + \int_1^\infty X(t,r) dt =$

L $\int_r^1 (e^{-t}/t)dt + cte. + \int_1^\infty X dt = -\log r + cte. + \int_r^1 (e^{-t}-1)t^{-1}dt + \int_1^\infty X dt = -\log r + P(r)$, where $P(r) \in C^1([0,\infty))$. For $r \to 0$, $K_0(r) = -\log r + P(0) + O(r)$. If $r > \varepsilon_0 > 0$ then $0 < K_0(r) = O(e^{-r/2})$. A similar bound is valid for $|K_0'(r)|$ but $0 > K_0'(r) \geq -Me^{-r/2}$, M a positive constant. We also have $rK_0'(r) = O(1)$ for $r \to 0$ and $\int_0^\infty K_0(r)dr = \frac{\pi}{2}$.

Let $\Delta u - \chi^2 u = 0$, $-\lambda = \chi^2 > 0$. *Define* Green's kernel $G(p,q;\lambda)$ as:

$$
(1) \begin{cases} G(p,q;\lambda) := \frac{1}{2\pi} K_0(\chi|p-q|) - H(p,q;\lambda), & p,q \in D \\ (\Delta_q + \lambda) H(p,q;\lambda) = 0, \quad H(p,.;\lambda) \in C(\overline{D}) \cap C^2(D) \\ H(p,q;\lambda) = \frac{1}{2\pi} K_0(\chi|p-q|) \quad \textit{if } q \in \partial D, & p \in D \end{cases}
$$

THEOREM 2. *In a Jordan region D, $G(p,q;\lambda)$ satisfies the following properties.*

i) If $u \in C^2(D) \cap C(\overline{D})$, $\phi \in L^\infty(D)$, $(\Delta+\lambda)u = \phi$ on D, $u = 0$ on J, then for any $p \in D$, $u(p) = -\int_D G(p,q;\lambda)\phi(q)\,dq$.

ii) For any $p,q \in D$, $p \neq q$, $G(p,q;\lambda) = G(q,p;\lambda)$.

iii) Let $\phi \in L^\infty(D)$, $p \in \overline{D}$, $u(p) := \int_D G(p,q;\lambda)\phi(q)\,dq$. Then, $u \in C(\overline{D})$, $u = 0$ en ∂D and $\frac{\partial u}{\partial p_i} \in C(D)$, i=1,2, $\frac{\partial u}{\partial p_i} = \int_D \frac{\partial G}{\partial p_i}(p,q;\lambda)\phi(q)dq$.

iv) Given $\phi \in C^1(D) \cap L^\infty(D)$, let $u(p) = -\int_D G(p,q;\lambda)\phi(q)\,dq$. Then, $u \in C^2(D) \cap C(\overline{D})$, $(\Delta+\lambda)u = \phi$ on D, $u = 0$ on ∂D.

v) The following propositions are equivalent:

a) $\phi \in L^2(D)$, $\phi(p) = \mu \int_D G(p,q;\lambda)\phi(q)\,dq$,

b) $\phi \in C^2(D) \cap C(\overline{D})$, $-(\Delta+\lambda)\phi = \mu\phi$ en $D, \phi = 0$ en ∂D.

vi) If l_p denotes the distance between p and ∂D then for any $q \in \overline{D}$: $0 \leq H(p,q;\lambda) \leq \frac{K_0(\chi l_p)}{2\pi}$, $0 \leq G(p,q;\lambda) \leq \frac{K_0(\chi|p-q|)}{2\pi}$.

vii) Let $M = diam\ D$. *Then,* $\int_D G^2(p,q;\lambda)\,dq \le \frac{1}{(2\pi)^2}\int_{\{|q|\le M\}} K_0^2(\chi|q|)\,dq =: C^2(\lambda) < \infty$.

viii) $\gamma^2(\lambda) = \iint_{D\times\bar{D}} G^2(p,q;\lambda)\,dp\,dq \le C^2(\lambda).|D| < \infty$.

(As a matter of fact $H(p,q;\lambda) \in C(D\times\bar{D})$. Its derivatives admit, on compact sets, bounds with an exponential decay with respect to l_p.)

Define for $\chi = 0$, $G(p,q) := G(p,q;0)$, as:

$$(2)\quad \begin{cases} G(p,q;0)) = \dfrac{1}{2\pi}\log\dfrac{M}{|p-q|} - H(p,q), & p,q \in D, \quad M = diam\ D, \\ \Delta_q(H(p,q)) = 0 \quad en\ D, & H(p,.) \in C(\bar{D}), \\ H(p,q) = \dfrac{1}{2\pi}\log\dfrac{M}{|p-q|} & si \quad q \in \partial D, \quad p \in D. \end{cases}$$

If M denotes the diameter of D then $0 \le G(p,q) \le (2\pi)^{-1}\log(M/|p-q|)$, $(p,q) \in D\times\bar{D}$.

THEOREM 3. *Let* $\chi \ge 0$, $\lambda = -\chi^2$. *The Green's kernel* $G(p,q;\lambda)$ *defined by (1) or (2) whether* $\chi > 0$ *or* $\chi = 0$ *verifies the properties i)-viii). The functions* $E^\chi(x) := -\dfrac{K_0(\chi|x|)}{2\pi}$ *for* $\chi > 0$ *and* $E^0(x) := -\dfrac{1}{2\pi}\log\dfrac{1}{|x|}$, *defined on* $R^2\setminus\{0\}$, *satisfy, in the sense of distributions, the equation* $(\Delta - \chi^2)E^\chi = \delta$. *Besides, all the solutions of* $(\Delta + \mu)u = 0$, $\mu \in C$ *are analytic on* D.

The proposition *v)* in Theorem 1 shows that any eigenfunction ϕ of $-\Delta$ is an eigenfunction of an integral operator with a positive symmetric kernel $G(p,q) \in L^2(D\times D)$, that is, of a selfadjoint completely continuous operator $\boldsymbol{G}$ of Hilbert-Schmidt type: $\mu\,\boldsymbol{G}\phi = \phi$. Moreover,

THEOREM 4. *The eigenvalues of the classical problem* $-\Delta u = \lambda u$ *on D,* $u = 0$ *on* ∂D, *are positive of finite multiplicity and, when repeated according to their multiplicities, can be ordered in a non decreasing way:* $0 < \lambda_1 \le \lambda_2 \le \ldots$, $\lambda_n \to \infty$. *The corresponding eigenfunctions:* $\phi_n(x)$ $(-\Delta\phi_n = \lambda_n\phi_n$, $\phi_n = 0$ *on J), may be chosen real, orthogonal and* *normalized:* $\|\phi_n\|_2 = (\int_D \phi_n{}^2(p)\,dp)^{1/2} = 1$. *They belong to* $C(\overline{D}) \cap C^\infty(D)$, *and form a complete orthonormal system. Moreover, they satisfy the following equation:* $\sum_{n=1}^{\infty} \frac{\phi_n^2(p)}{\lambda_n^2} = \int_D G^2(p,q)\,dq \le C^2 < \infty$. *If* $u = \mathbf{G}f$, $f \in L^2(D)$, *then the expansion of u in eigenfunctions converges absolute and uniformly on D. This is the case, in particular, for a function u in* $C^2(D) \cap C(\overline{D})$, *null on the boundary, such that* $\Delta u \in L^\infty(D)$. *All the content of this theorem holds for the operator* $-\Delta + \chi^2$ *instead of* $-\Delta$.

The functions $\phi_n(x)$, n = 1, 2, ..., are also eigenfunctions of the χ-harmonic operator $-\Delta + \chi^2$ since $\lambda_n > 0$ and $-(\Delta + \lambda)\phi_n = (\lambda_n - \lambda)\phi_n$, $\lambda = -\chi^2$. Therefore, the spectrum of the χ-harmonic operator, $\sigma_{-\Delta+\chi^2}$, is equal to $\sigma_{-\Delta} + \chi^2$. If $p \in D$, we have $\phi_n(p)/(\lambda_n - \lambda) = \int_D G(p,q;\lambda)\,\phi_n(q)\,dq$. If $\{c_n\}$ are the Fourier coefficients of $G(p,.;\lambda)$ with respect to the complete orthonormal system $\{\phi_n\}$ then, for $\chi^2 \ge 0$, $c_n(G(p,.;\lambda)) = \phi_n(p)/(\lambda_n - \lambda)$. In consequence, from *vii)* it follows that for any $p \in D$, $\sum_{n=1}^{\infty} \frac{\phi_n^2(p)}{(\lambda_n - \lambda)^2} \le C^2(\lambda) < \infty$.

THEOREM 5. *Let* $\{\phi_n\}$ *be as above. For any* $p \in D$ *it holds that* $\sum_{\lambda_n < \lambda} (\phi_n(p))^2 / \lambda \to 1/4\pi$. *Therefore,* $\lambda_n \sim 4\pi\, n / |D|$.

In fact, $N(\lambda)/\lambda \sim |D|/4\pi$ is a consequence of the limit. Then, for $p \in D$, $n \to \infty$, $(\sum_{j=1}^{n} \phi_j^2(p))/n \to 1/|D|$.

THEOREM 6. *i) For* $p \in D$, *the following series on all the family of eigenfunctions converges on* $q \in \overline{D}$ *absolutely and uniformly to the continuous function in q,* $F(p,q;\lambda)$:

$$G(p,q;\lambda) - G(p,q) =: F(p,q;\lambda) = \lambda \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda_n - \lambda)}.$$

ii) If $F(p,\lambda):=F(p,p;\lambda)$ *then* $F(p,\lambda)=\lambda\sum_{n=1}^{\infty}\frac{\phi_n^2(p)}{\lambda_n(\lambda_n-\lambda)}=\lim_{q\to p}(G(p,q;\lambda)-G(p,q))$.

$F(p,\lambda)$ *is continuous on* $D\times(C\setminus\sigma_\Delta)$ *and is a meromorphic function of* λ *for each p.*

3. I. THE SPECTRAL DIRICHLET SERIES. In this section D is a C^2-Jordan region. In particular, J is a Jordan curve *defined* by means of functions $y_1(s), y_2(s)\in C^2$ with a tangent versor at each point. The precise definition is in II of this paragraph.

We focus on the functions $F(p;\lambda)=\lambda\sum_1^{\infty}\varphi_n^2(p)/\lambda_n(\lambda_n-\lambda)$ and $\int_D F(p;\lambda)dp$ $=\lambda\sum_1^{\infty}1/\lambda_n(\lambda_n-\lambda)$, $\lambda=-\chi^2$, $\chi>0$. By Th. 6, §2, we have $F(p,q;\lambda):=$ $G(p,q;\lambda)-G(p,q)=K_0(\chi|p-q|)/2\pi-H(p,q;\lambda)-(1/2\pi)\log(M/|p-q|)+H(p,q)$.

Using the fact that $K_0(\chi|p-q|)=\log\frac{M}{\chi|p-q|}+P(\chi|p-q|)-\log M$, we get $F(p,q;\lambda)=$ $-(\log\chi)/2\pi-H(p,q;\lambda)+H(p,q)+P(\chi|p-q|)/2\pi-(\log M)/2\pi$. Hence, for $q\to p$,

(1) $F(p;\lambda):=\lim_{q\to p\in D}F(p,q;\lambda)=$

$=-(\log\chi)/2\pi-(\log M)/2\pi+H(p,p)-H(p,p;\lambda)+P(0)/2\pi$

where

(1') $H(b,q;\lambda)=K_0(\chi|b-q|)/2\pi>0$, $q\in D$, $b\in J$.

Since $K_0(r)$ is a decreasing function $\max_{b\in J}K_0(\chi|b-q|)=K_0(\chi\, dist(q,J))$. Thus, by (1') and the maximum principle for $\chi-$harmonic functions, if $p\in\overline{D}$ and $q\in D$ then

(2) $0<H(p,q;-\chi^2)\leq K_0(\chi\, dist(q,J))/2\pi$.

To estimate $H(p,p;\lambda)$ for $\chi\to\infty$ observe that if $dist(q,J)>\delta>0$ then (cf. §2),

(3) $H(p,q;-\chi^2)\leq C(\delta)e^{-\delta\chi/2}$,

It will suffice then to estimate $H(p,p;\lambda)$ on $D\cap(\partial D)_\delta$. Our objetive is to prove the following formula (6) for Jordan regions C^2 and for this it is essential to prove next formula (4). From (6) we may conclude that the set of the eigenvalues determines the area of the region and its perimeter. *Define*, $I = \int_D H(x,x;-\chi^2)\,dx$.

From (1) and what we said above we deduce that for $w=\chi^2$ and C a constant it holds that $\int_D F(x,-w)\,dx = -\frac{|D|}{4\pi}\log w + C - I = -w\sum_1^\infty \frac{1}{\lambda_n(\lambda_n+w)}$. We shall prove in 3.II that,

(4) $I = \frac{\langle J\rangle}{8\chi} + O\left(\frac{\log\chi}{\chi^2}\right)$, $\chi\to\infty$.

Then, assuming (4),

(5) $h(w) := -w\sum_1^\infty \frac{1}{\lambda_n(\lambda_n+w)} = -\frac{|D|}{4\pi}\log w + C - \frac{\langle J\rangle}{8}\frac{1}{w^{1/2}} + O(1)\frac{\log w}{w}$.

We shall obtain next formula after integrating this equality along a curve γ of the plane w,

(6) $\sum \frac{1}{\lambda_n^z} = \int \frac{dN(\lambda)}{\lambda^z} = \frac{|D|}{4\pi}\frac{1}{z-1} - \frac{\langle J\rangle}{8\pi}\frac{1}{z-1/2} + g(z)$, $g(z)$ holomorphic on $\mathrm{Re}\, z > 0$.

Or, what is the same (cf. [Pl]),

(6') $\int_{1+}^\infty \frac{dN(\lambda)}{\lambda^z} = \frac{|D|}{4\pi}\frac{1}{z-1} - \frac{\langle J\rangle}{8\pi}\frac{1}{z-1/2} + G(z)$, $G(z)$ holomorphic on $\mathrm{Re}\, z > 0$.

γ is the contour shown in fig. 2 with r such that $0 < r < \inf(\lambda_1, 1)$. Let Γ be the region at the left of γ. Let us prove (6). The function $k(w) := w\sum_1^\infty \frac{1}{\lambda_n(\lambda_n - w)}$ is meromorphic on the plane w with simple poles at the points λ_n. For $a = -|D|/4\pi$, $b = -\langle J\rangle/8$ and C a constant it verifies on $t > 0$,

(7) $k(-t) = h(t) = a\log t + bt^{-1/2} + q(t) + C$,

where $q(t)$ is a continuous function on $t\in(0,\infty)$ such that $q(t) = O(t^{-1}\log t)$ whenever $t\to\infty$, (cf. (5)). We have on $s > 1$,

(8) $\frac{1}{2\pi i}\int_\gamma \frac{k(w)}{w^s}dw = \frac{1}{2\pi i}\int_\gamma \sum_1^\infty \frac{w^{1-s}}{\lambda_n(\lambda_n - w)}dw = \frac{-1}{2\pi i}\sum_1^\infty \int_\gamma \frac{w^{1-s}}{\lambda_n(w-\lambda_n)}dw = -\sum_1^\infty \frac{1}{\lambda_n^s}$.

In fact, let $t=|w|$. Because of Weyl's theorem we have $\sum\int\left|\frac{t^{1-s}}{\lambda_n(t+\lambda_n)}\right|dt<\infty$. This allows to prove the second equality. For the last equality in (8) observe that the region Γ is the limit, as $R\to\infty$, of the region contained between the circumferences k and K of radii r and R except for the points on the negative real axis. The integrals over K tend to zero when $R\to\infty$ since $s>1$. Thus, the last equality in (8) follows from Cauchy's theorem of residues.

Let $k'=\gamma\cap\{z:|z|<1\}$ and $l=\gamma\setminus\{z:|z|<1\}=l_1+l_2$ (see fig. 2) with l_1 from $-\infty$ to -1 where $\arg w=\pi$ and with l_2 from -1 to $-\infty$ where $\arg w=-\pi$. The function $f_0(s):=\frac{1}{2\pi i}\int_{k'}\frac{k(w)}{w^s}dw$ has an entire extension. Then,

(9) $$-\sum_1^\infty\frac{1}{\lambda_n^s}=\frac{1}{2\pi i}\int_l\frac{k(w)}{w^s}dw+f_0(s).$$

(10) $$\frac{1}{2\pi i}\int_l\frac{k(w)}{w^s}dw=\frac{1}{2\pi i}\left[-\int_{-l_1}\frac{k(w)}{w^s}dw+\int_{l_2}\frac{k(w)}{w^s}dw\right]=\frac{1}{2\pi i}\int_1^\infty k(-t)t^{-s}(e^{-i\pi s}-e^{i\pi s})dt$$

$$=\frac{-\operatorname{sen}\pi s}{\pi}\int_1^\infty\frac{k(-t)}{t^s}dt=\frac{-\operatorname{sen}\pi s}{\pi}\int_1^\infty\frac{h(t)}{t^s}dt=\frac{-\operatorname{sen}\pi s}{\pi}\int_1^\infty\frac{a\log t+C+bt^{-1/2}}{t^s}dt+\frac{-\operatorname{sen}\pi s}{\pi}\int_1^\infty\frac{q(t)}{t^s}dt.$$

The last term defines a function $g_0(s)$ that has a holomorphic extension to $\operatorname{Re}z>0$. Thus,

(11) $$\sum_1^\infty\frac{1}{\lambda_n^s}=\frac{\operatorname{sen}\pi s}{\pi}\int_1^\infty\frac{a\log t+bt^{-1/2}+C}{t^s}dt+g_1(s)=M(s)+g_1(s),\quad s>1,$$

where $g_1(z)=-f_0(z)-g_0(z)$ is holomorphic in $\operatorname{Re}z>0$. The integral is equal to $\frac{a}{(s-1)^2}+\frac{b}{s-1/2}+\frac{C}{s-1}$. Then $M(s)=-\frac{a}{s-1}+\frac{b/\pi}{s-1/2}+g_2(s)$ on $s>1$ where g_2 is holomorphically continuable to $\operatorname{Re}s>0$. Thus, on $\operatorname{Re}z>1$ it holds that

(12) $$\sum_1^\infty\frac{1}{\lambda_n^z}=-\frac{a}{z-1}+\frac{b/\pi}{z-1/2}+g(z),\ g\text{ holomorphic on }\operatorname{Re}z>0,$$

and (6) follows, QED.

If $N(\lambda)$ had a second term in its asymptotic expansion, because of Th. 1 §1 one should have $N(\lambda)=-a\lambda+2b\lambda^{1/2}/\pi+o(\lambda^{1/2})=(|D|/4\pi)\lambda-(\langle J\rangle/4\pi)\sqrt{\lambda}+o(\lambda^{1/2})$.

3. II. LOCAL COORDINATES. To deal with regular regions it is convenient to introduce local coordinates around the boundary in the following way where we follow Å. Pleijel, (cf. [Pl], [B]). Let s be the parameter arc legth on J starting at the origin O, (see figs. 3,4).

The points in J will be denoted by $y=y(s)=(y_1(s),y_2(s))$, $0\le s<\langle J\rangle$. We assume that $y_i(s)\in C^2([0,\langle J\rangle]), y_i(0)=y_i(\langle J\rangle)$. Let $n_i=\vec{n}=\vec{n}(s)=(\vec{n}_1,\vec{n}_2)$ be the interior normal versor at y. Suppose $\delta>0$ sufficiently small and let I be an interval such that $\langle I\rangle\le\langle J\rangle$. *Define* the map T: $(s,t)\to x:=y(s)+t\,\vec{n}$ on the rectangle $C(I):=I\times(-\delta,\delta)$ to the strip $J_\delta:=\{x:dist(x,J)<\delta\}$. We denote with $\xi:=(\xi_1,\xi_2)$ a point of the rectangle $C(I)$ and with $x=(x_1,x_2)$ its image in the strip J_δ. Then, T is written as $T:\xi=(\xi_1,\xi_2)\to x=(x_1,x_2)=(y_1(\xi_1),y_2(\xi_1))+\xi_2(\vec{n}_1(\xi_1),\vec{n}_2(\xi_1))$, $0\le\xi_1<\langle I\rangle$, $|\xi_2|<\delta$. Thus, if $\xi_2=0$ then $x\in J$. Given $\eta=(\eta_1,0)$ *its image will be represented by* $y=y(\eta_1)$ to underline that it is in J. One can get, taking δ sufficiently small, that the map T of the rectangle $\{\xi=(\xi_1,\xi_2):0\le\xi_1\le\langle J\rangle,|\xi_2|<\delta\}$ with vertical sides identified, be a homeomorphism onto J_δ. In fact, since by hypothesis J is C^2, T is a C^1 map and can be written as (note that $n_i=(-\dot{y}_2(\xi_1),\dot{y}_1(\xi_1))$, $|n_i|=1$ because of $\xi_1=s$):

$$(13)\ T(\xi)=\begin{cases}x_1(\xi)=y_1(\xi_1)-\xi_2\dot{y}_2(\xi_1)\\ x_2(\xi)=y_2(\xi_1)+\xi_2\dot{y}_1(\xi_1)\end{cases}.$$

Its jacobian B is the modulus of the determinant of the following matrix,

$$(14)\ \frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)}=\begin{vmatrix}\dot{y}_1(\xi_1)-\xi_2\ddot{y}_2(\xi_1) & -\dot{y}_2(\xi_1)\\ \dot{y}_2(\xi_1)+\xi_2\ddot{y}_1(\xi_1) & \dot{y}_1(\xi_1)\end{vmatrix}=1-\xi_2c(\xi_1).$$

$c(\xi_1)$ is the curvature of J at the point $T(\xi_1,0)\in J$. For δ sufficiently small $1-\xi_2c(\xi_1)>0$ whenever $|\xi_2|<\delta$ because of the continuity of c. In this case $0<B=1-\xi_2c(\xi_1)$ and T is

locally a homeomorphism. Let us see, perhaps by using a smaller δ, that our T is *globally* a homeomorphism. Let $y(s)\in J$ and ε such that T is a homeomorphism from $(s-2\varepsilon,s+2\varepsilon)\times(-\delta,\delta)$ onto $U:=T((s-2\varepsilon,s+2\varepsilon)\times(-\delta,\delta))$, a neighborhood of $y(s)$. Let $V:=\{x\in U: dist(x,J\setminus U)>2dist(x,J)\}$. Then, V is also a neighborhood of $y(s)$ and there exists $0<\tilde{\delta}\leq\delta$ such that $\tilde{U}:=T((s-\varepsilon,s+\varepsilon)\times(-\tilde{\delta},\tilde{\delta}))\subset V$. Since J is compact it is possible to cover it with a finite number of such neighborhoods, $\{\tilde{U}_h : h=1,\cdots,N\}$. Let $\delta_1=\min_h \tilde{\delta}_h$. We claim that T is a homeomorphism from

$\{\xi=(\xi_1,\xi_2): 0\leq\xi_1\leq\langle J\rangle, |\xi_2|<\delta_1\}$ onto J_{δ_1}. In fact, let $x\in J_{\delta_1}$, to prove that there is a unique $y\in J$ such that $dist(x,J)=|x-y|$ assume that y_1,y_2 verify $dist(x,J)=|x-y_k|$, $k=1$, 2. If $y_1\in\tilde{U}\cap J$ then, by construction, $y_2\in U\cap J$. Writing $y_h=y(s_h)$ and defining $\xi_2:=\pm dist(x,J)$, (with the sign + whenever $x\in D$), we deduce that $x=T(s_1,\xi_2)=T(s_2,\xi_2)$. Since T is a homeomorphism on U, we get $s_1=s_2$, *i.e.* $y_1=y_2$. Thus we have proved that T is one to one and onto J_{δ_1}. Given $x=y(\xi_1)+\xi_2 n_i(\xi_1)\in F_\delta$, $\hat{x}$ denotes the *symmetric point with respect to J*: $\hat{x}:=y(\xi_1)-\xi_2 n_i(\xi_1)$.

THEOREM 1. *If* $J\in C^2$ *then on* $\operatorname{Re} z>1$ *it holds that*

(6') $$\int_{1+}^{\infty}\frac{dN(\lambda)}{\lambda^z}=\frac{|D|}{4\pi}\frac{1}{z-1}-\frac{\langle J\rangle}{8\pi}\frac{1}{z-1/2}+G(z),\qquad G(z)\text{ holomorphic on }\operatorname{Re} z>0.$$

PROOF. It only remains to prove formula (4): $\int_D H(p,p;-\chi^2)dp=\frac{\langle J\rangle}{8\chi}+O\left(\frac{\log\chi}{\chi^2}\right)$. If $F_h=D\cap J_h$, because of (3) we have

(15) $$I=\int_D H(x,x;-\chi^2)dx=\int_{F_h}H(x,x;-\chi^2)dx+\int_{D\setminus F_h}H(x,x;-\chi^2)dx=$$

$$=\int_0^h d\xi_2\int_0^{\langle J\rangle}H(x(\xi),x(\xi);-\chi^2)[1-c(\xi_1)\xi_2]d\xi_1+O(e^{-\chi h/2}),$$

where we suppose $h<\delta_1$. We assumed that $\forall x\in D$, $0<\xi_2\le h$ there exists $\hat{x}\notin \overline{D}$ and this holds if h is small enough. Also $B=|B|=1-c(\xi_1)\xi_2>0$ on $\overline{F_h}$. If the boundary were the real axis and D the superior half plane we would have (cf. (1')), for $b=(b_1,0)\in J$, that $H(b,q;-\chi^2)=K_0(\chi|b-\hat{q}|)/2\pi$. Then, from the χ-harmonicity of both sides: $H(\xi,q;-\chi^2)=K_0(\chi|\xi-\hat{q}|)/2\pi$ on $D=\{\mathrm{Im}\xi>0\}$. We shall see that even for D a Jordan region, $R(x,p):=H(x,p;-\chi^2)-\dfrac{K_0(\chi|x-\hat{p}|)}{2\pi}$ is small when $p\in F_h$, $x\in\overline{D}$. In fact, $(\Delta_x-\chi^2)R=0$ on D and R is χ-harmonic. Besides, if $x\in J$,

(16) $|R(x,p)|=\left|K_0(\chi|p-x|)-K_0(\chi|\hat{p}-x|)\right|/2\pi$.

We shall deduce from (16) that for $p\in F_h$ it holds that

(17) $|R(x,p)|\le M\inf(1/\chi,e^{-\chi\, dist(p,J)/4})$ for $x\in J$, with M independent of p, x.

Assuming (17) for a moment, it follows from the maximum principle (i.e. χ-harmonic functions take their positive maximum and their negative minimum on the boundary) that (17) holds also for $x\in D$. Thus, if $R(p):=H(p,p;-\chi^2)-K_0(\chi|p-\hat{p}|)/2\pi$, then

(18) $|R(p)|\le M\inf(1/\chi,e^{-\chi\, dist(p,J)/4})$ for $p\in F_h$.

Therefore, $\left|\int_{F_h}R(p)dp\right|\le\int_{F_h}|R(p)|dp\le\int_{F_h}|R(p)|I_{\{dist(p,J)\ge\varepsilon\}}dp+\int_{F_h}|R(p)|I_{\{dist(p,J)<\varepsilon\}}dp=\mathrm{I}+\mathrm{II}$,

where (cf. (18)): $\mathrm{I}\le M|D|e^{-\chi\varepsilon/4}$, $\mathrm{II}\le M'\varepsilon/\chi$. Taking $\varepsilon=\dfrac{4}{\chi}\log\chi^2$ we get,

$\int_{F_h}|R(p)|dp\le C'/\chi^2+C''(\log\chi)/\chi^2$. Then, for χ great enough we obtain,

(19) $\int_{D\cap J_h}|R(p)|dp=O(1)\dfrac{\log\chi}{\chi^2}$.

Let us prove (17). Given $p\in F_h$, let V be a neighborhood of $O\in J$, O the middle point of the segment $p\hat{p}$ (see fig. 5). After a change of cartesian coordinates such that $p=(0,x_2)$, $\hat{p}=(0,-x_2)$, $J\cap V=\{x=(x_1,f(x_1)):|x_1|<\delta\}$, we can write $f(x_1)=ax_1^2+o(x_1^2)$. Choosing

$\delta > 0$ sufficiently small we can achieve that $|f(x_1)| < C|x_1|^2$ for $|x_1| < \delta$ with C independent of $p \in F_h$. In fact, in the new coordinates the equations of J are:

$$\begin{cases} x_1(s) = (y_1(s) - y_1(0))\dot{y}_1(0) + (y_2(s) - y_2(0))\dot{y}_2(0) \\ x_2(s) = -(y_1(s) - y_1(0))\dot{y}_2(0) + (y_2(s) - y_2(0))\dot{y}_1(0) \end{cases}.$$ Then,

$\dot{x}_1(s) = \dot{y}_1(s)\dot{y}_1(0) + \dot{y}_2(s)\dot{y}_2(0) = 1 + (\dot{y}_1(s) - \dot{y}_1(0))\dot{y}_1(0) + (\dot{y}_2(s) - \dot{y}_2(0))\dot{y}_2(0)$, and there exists $\varepsilon > 0$, depending only upon J, such that $\dot{x}_1(s) > 1/2$ for $|s| < \varepsilon$. We can choose $\varepsilon \le 2h$. Then, $x_1 = x_1(s)$ has an inverse function $s = s(x_1) < 2x_1$ defined at least on $|x_1| < \delta := \varepsilon/2 \le h$. Thus $f(x_1) = x_2(s(x_1))$. From the identities $f'(x_1(s))\dot{x}_1(s) = \dot{x}_2(s)$, $f''(x_1(s))\dot{x}_1^2(s) + f'(x_1(s))\ddot{x}_1(s) = \ddot{x}_2(s)$ we conclude that $f'(0) = 0$ and $|f''(x_1)| < K$ on $|x_1| < \delta$ with K independent of O. Moreover, $f(x_1) = \int_0^{x_1} f''(t)(x_1 - t)dt$ implies that $|f(x_1)| < (K/2)x_1^2$. Now we can replace h in formula (15) by $\delta = \varepsilon/2$ without changing the formal expressions of this formula or those that follow. *Notation:* $\rho := \inf(|x-p|, |x-\hat{p}|)$, $r := \sqrt{x_1^2 + x_2^2}$, (see fig.5). Then,

$\rho^2 = x_1^2 + (x_2 - |f(x_1)|)^2 \ge x_1^2 + \frac{x_2^2}{2} - f^2(x_1)$ and it follows that $\rho^2 \ge x_1^2 + x_2^2/2 - C^2 x_1^4$. Therefore, $\rho^2 \ge (x_1^2 + x_2^2)/2$ if δ is small enough. That is, $\rho > r/\sqrt{2}$. If $\hat{x} = (x_1, -f(x_1))$ represents here the symmetric point to x with respect to $x_2 = 0$, we have: $|x - \hat{p}| = |\hat{x} - p|$. Thus, $||x-p| - |x-\hat{p}|| \le |x - \hat{x}| = 2|f(x_1)|$. If $x \in J \cap V$,

(20) $d := |K_0(\chi|p-x|) - K_0(\chi|p-\hat{x}|)| \le \chi 2|f(x_1)||K_0'(\chi\tilde{\rho})|$,

where $\tilde{\rho}$ is a number between $|x-p|$ and $|x-\hat{p}|$. Because of $\rho > r/\sqrt{2}$ we have $\tilde{\rho} > r/\sqrt{2}$ $(\ge |x_2|/\sqrt{2} = dist(p,J)/\sqrt{2})$. The right hand side of (20) is equal to,

(21) $\frac{2|f(x_1)|}{\tilde{\rho}}|\chi\tilde{\rho}K_0'(\chi\tilde{\rho})| \le c\frac{2|f(x_1)|}{\tilde{\rho}}e^{-\chi\tilde{\rho}/2} \le c2\sqrt{2}\frac{|f(x_1)|}{\sqrt{x_1^2 + x_2^2}}e^{-\chi\tilde{\rho}/2} \le$

$$\leq c'\frac{x_1^2}{\sqrt{x_1^2+x_2^2}}e^{-\chi r/2\sqrt{2}} \leq c'|x_1|e^{-\chi r/2\sqrt{2}} \leq c're^{-\chi r/4} \leq c'' \inf(\frac{1}{\chi},(diamD)e^{-\chi dist(p,J)/4}).$$

If $x \in J \setminus V$ then $|p-x|,|\hat{p}-x| \geq \delta/2 = h/2$ and

(22) $d' := K_0(\chi|p-x|)+K_0(\chi|\hat{p}-x|) \leq c''' e^{-\chi h/4} \leq c^{iv} \inf(1/\chi, e^{-\chi dist(p,J)/4})$.

From (21) and (22) we get $|R(x,p)| \leq \sup(d,d')/2\pi \leq M \inf(1/\chi, e^{-\frac{\chi}{4}dist(p,J)})$, that is (17).

Finally, we evaluate what should be the first approximation Θ of $\int_{F_h} H(p,p;-\chi^2)dp$.

$$\Theta := \int_{F_h}\frac{K_0(\chi|p-\hat{p}|)}{2\pi}dp = \frac{1}{2\pi}\int_0^{\langle J\rangle} d\xi_1 \int_0^h K_0(\chi 2\xi_2)[1-c(\xi_1)\xi_2]d\xi_2 =$$

$$=\frac{1}{2\pi}\int_0^{\langle J\rangle}\{\frac{1}{2\chi}\int_0^{2h\chi} K_0(t)dt - \frac{c(\xi_1)}{(2\chi)^2}\int_0^{2h\chi} K_0(t)tdt\}d\xi_1 =$$

$$=\frac{1}{2\pi}\int_0^{\langle J\rangle}\{\frac{1}{2\chi}\int_0^{\infty} K_0(t)dt - \frac{1}{2\chi}\int_{2\chi h}^{\infty} K_0(t)dt\}d\xi_1 + O(1/\chi^2) =$$

$$=\frac{1}{2\pi}\int_0^{\langle J\rangle}(\frac{1}{2\chi}\int_0^{\infty} K_0(t)dt)d\xi_1 + O(e^{-\chi h}/\chi) + O(1/\chi^2) = (\chi >> 1) = \int_0^{\langle J\rangle}\frac{1}{8\chi}d\xi_1 + O\left(\frac{1}{\chi^2}\right) =$$

$=\frac{\langle J\rangle}{8\chi} + O\left(\frac{1}{\chi^2}\right)$ since $\int_0^{\infty} K_0(r)dr = \frac{\pi}{2}$. Therefore we obtain,

(23) $\frac{1}{2\pi}\int_{F_h} K_0(\chi|p-\hat{p}|)dp = \frac{\langle J\rangle}{8\chi} + O\left(\frac{1}{\chi^2}\right)$.

(23) together with (19) yield (4) and the theorem follows, QED.

4. WEYL, BERRY AND LAPIDUS CONJECTURES. In this section we assume again that D is a Jordan region. We know that if J is sufficiently regular then $N(\lambda) = A\lambda^{\dim D/2} + B\lambda^{(\dim D-1)/2} + o(\lambda^{(\dim D-1)/2})$, $A = |D|(4\pi)^{-1}$, $B = -\langle J\rangle(4\pi)^{-1}$. Here dim represents the Hausdorff dimension. But $\dim D - 1 = 1 = \dim J$. Then we have, $N(\lambda) = A\lambda^{\dim D/2} + B\lambda^{\dim J/2} + o(\lambda^{\dim J/2})$. Berry's conjecture says that an asymptotic expression like the last one must hold when J has no additional regularity. Unfortunately this is not true with all generality. Lapidus conjectures that dim$J$ should be replaced by the Minkowski dimension of J. A reasonable objective is to determine η in an expression like $N(\lambda)=A\lambda+o(\lambda^{\eta})$ if such an exponent exists. This amounts to estimate the discrepancy. Let be $A(t):=\{z\in D: dist(z,J)\leq t\}$. We write $|B|=m(B)=$ exterior plane Lebesgue measure of B. $k=k(\partial D)$ denotes the *Minkowski's interior dimension* of the boundary, $k(\partial D):=\dim_I \partial D = 2-\lim_{t\to 0}\frac{\log|A(t)|}{\log t}$, if the limit exists. A parameter that always exists is $l(\partial D):=\inf\{\alpha\geq 0: |A(t)|=O(t^{2-\alpha}), t\to 0+\}$. It holds that if k can be defined then $k(\partial D)=l(\partial D)$. If the limit does not exist we can still define the *superior interior dimension* as $k^*=k^*(\partial D):=\overline{\dim}_I\partial D=2-\underline{\lim}\frac{\log|A(t)|}{\log t}, t\to 0+$. It is not difficult to prove that $k^*(\partial D)=l(\partial D)$. It is harder to prove the following result.

THEOREM 1. $l(\partial D)=k^*(\partial D)\in[1,2]$.

Instead of using the dimension theory sometimes it is more convenient to deal with simpler analytic or measure theoretic concepts. We shall say that D is ε*-semiregular*, $\varepsilon\in(0,1]$, whenever $|A(t)|=O(t^{\varepsilon})$. If D is ε-semiregular then $m(\partial D)=0$. If J is rectifiable then $A(t)\approx t$. In this case D is 1-semiregular. But note that there are Jordan regions that are not ε-semiregular for any $\varepsilon\in(0,1]$. Let us observe that
$l(\partial D)=2-\sup\{\eta: 0\leq\eta\leq 2, |A(t)|=O(t^{\eta})\}$. Now it is easy to prove next theorem.

THEOREM 2. *i) If* $k^*\in[1,2)$ *then* $2-l(D)=2-k^*\in(0,1]$.

ii) If $0<\varepsilon\in(0,2-k^*)$ *then D is* ε*-semiregular.*

Assume that $2>\sigma>s=k^*(J)$. Then, it can be proved that the discrepancy verifies the inequality $\delta(\lambda)=N(\lambda)-(|D|/4\pi)\lambda\geq-\lambda^{\sigma/2}$ for $\lambda>\lambda_1(\sigma)$, (cf. [F]). But in this case D is an ε-semiregular region for $\varepsilon\in(0,2-s)$. Therefore, there exists an ε in that interval such that $\lambda^{\sigma/2}>M\lambda^{1-\frac{\varepsilon}{2}}$, M a constant, whenever $\lambda>\lambda_2$. In fact, because of Theorems 1 and 2 we can pick up an ε such that $1 > \sigma/2 > 1-\varepsilon/2 > s/2 \geq 1/2$. Next theorem improves slightly the inequality about δ mentioned above.

THEOREM 3. *If D is ε-semiregular then there exists a constant M such that the discrepancy verifies $\delta(\lambda) \geq -M\lambda^{1-\varepsilon/2}$ whenever $\lambda > \lambda_0(\varepsilon)$.*

PROOF. We follow the argument in [F]. Let W be the Whitney's decomposition of D in half open binary squares. Let us denote with $S(i)$ the union of those squares of W of side $2^{-i}, i = 1,2,\cdots$. We have, $S(k+1) \subset D \setminus \bigcup_1^k S(i)$, $D = \bigcup_1^\infty S(k)$, (see Fig. 1). If $k \geq 2$ then $S(k) \subset A(2^{1-k}\sqrt{2})$. Therefore, because of the hypothesis, if $n(k)$=#squares in $S(k)$, we get $n(k)2^{-2k} \leq \left|A(2^{1-k}\sqrt{2})\right| = O(2^{-k\varepsilon})$. Let be $D(k)$=union of the interiors of the squares in $S(j)$, j=1,...,k and $N(\lambda,k)$=#eigenvalues of $D(k)$ not greater than λ. Because of $D(k) \subset D$, it holds that $N(\lambda,k) \leq N(\lambda)$. (We shall prove this inequality in Theorem 7 of next paragraph.) Since $D(k)$ is the union of disjoint open squares we can estimate $N(\lambda,k)$ just counting the eigenvalues of each component square. If D is a square we have $N(\lambda) = (|D|/4\pi)\lambda - \psi(\lambda)$, $0 \leq \psi(\lambda) \leq \langle \partial D \rangle \sqrt{\lambda}/2\pi$. (In fact, the classical eigenfunctions of the Dirichlet problem for a square of side a are $(\operatorname{sen} m\pi x/a)(\operatorname{sen} n\pi y/a)$ except for a non null factor. The corresponding eigenvalues are $\lambda_{m,n} = \pi^2(m^2+n^2)/a^2$, $m,n > 0$ with multiplicity $\#\{(m,n): 0 < m,n;\ m^2+n^2 = a^2\lambda_{m,n}/\pi^2\}$. In consequence, $N(\lambda) = \#F$, $F := \{(m,n): 0 < m,n;\ |(m,n)| \leq a\sqrt{\lambda}/\pi\}$. The union of squares of side 1 with superior right vertices in F has area $N(\lambda)$ and is contained in $C = \{(x,y): 0 \leq x,y; |(x,y)| \leq r = a\sqrt{\lambda}/\pi\}$. Hence $N(\lambda) \leq |D|\lambda/4\pi$=area of C. Similarly $\widetilde{C} = \{(x,y): 1 \leq x,y; |(x,y)| \leq r = a\sqrt{\lambda}/\pi\}$ is contained in the union of squares of side 1 with inferior left vertices in F. Therefore, $N(\lambda) \geq \textit{area of } \widetilde{C} \geq (\pi r^2/4) - 2r = |D|\lambda/4\pi - \langle \partial D \rangle \sqrt{\lambda}/2\pi$.) Thus,

$$N(\lambda) \geq N(\lambda,k) \geq \sum_1^k \frac{2^{-2i}}{4\pi}\lambda\, n(i) - \sum_1^k \frac{2.2^{-i}}{\pi}\sqrt{\lambda}\, n(i) =$$

$$= \frac{|D|}{4\pi}\lambda - \frac{\lambda}{4\pi}\sum_{k+1}^{\infty} 2^{-2i} n(i) - \frac{2}{\pi}\lambda^{1/2}\sum_1^k 2^{-i} n(i) \geq \geq \frac{|D|}{4\pi}\lambda - \frac{\lambda}{4\pi}\sum_{k+1}^{\infty} O(2^{-i\varepsilon}) - \frac{2}{\pi}\lambda^{1/2}\sum_1^k O(2^{i(1-\varepsilon)}) \geq$$

$$\frac{|D|}{4\pi}\lambda - c\lambda 2^{-k\varepsilon} - c'\lambda^{1/2} 2^{k(1-\varepsilon)}.$$

Choosing $2^{k-1} < \lambda^{1/2} \leq 2^k$, it follows that $N(\lambda) \geq (|D|/4\pi)\lambda - M\lambda^{1-\varepsilon/2}$, QED.

5. WEAK SOLUTIONS. We say that the function u defined in the region U (not necessarily a Jordan region) is a *weak solution of* $(-\Delta + \lambda)u = f, u = 0$ on ∂U where $f \in L^2(U)$, if $u \in H_0^1(U)$ and for any $v \in H_0^1(U)$, $\int_U \nabla u \times \nabla v\, dx + \lambda \int_U uv\, dx = \int_U fv\, dx$ holds.

Recall that $H_0^1(U)$ is the closure of $C_0^\infty(U)$ in $H^1(U)$. One can prove that if $f \in C^\infty(U)$ then $u \in C^\infty(U)$.

THEOREM 1 (Lax-Milgram). *Let H be a real Hilbert space and B a bilinear functional* $B: H \times H \to R$ *continuous* ($|B(u,v)| \leq \alpha \|u\|_H \|v\|_H$) *and coercive* (*i.e.,* $\exists \beta > 0$ *such that* $\beta \|u\|_H^2 \leq |B(u,u)|$). *Given F, a real continuous linear functional on H, there exists a unique* $u \in H$ *such that* $B(u,v) = F(v)\ \forall v \in H$.

We shall apply this result with $H = H_0^1(U)$ and $B = B_\gamma(u,v) = I(u,v) + \gamma\langle u,v\rangle$ where $\gamma \geq 0$, $I(u,v) := \int_U \nabla u \times \nabla v dx$. The functional B is *continuous and coercive*:

$B_\gamma(u,u) = \int |\nabla u|^2 dx + \gamma \|u\|_2^2 \geq \beta \|u\|_H^2$. Most of the results of this section will only be stated.

THEOREM 2. *i) Let* $\gamma \geq 0$. *For any* $f \in L^2(U)$ *there exists exactly one weak solution of* $(-\Delta + \gamma)u = f, u = 0$ *on* ∂U.

ii) The application $L_\gamma^{-1}: f \in L^2 \to u_f \in H_0^1$ *is bounded.*

iii) $L_\gamma^{-1}: L^2 \to L^2$ *and is completely continuous.*

PROOF. i) There exists a unique $u_f \in H_0^1$ such that $\forall v \in H_0^1(U)$, $B_\gamma(u_f, v) = \langle f, v \rangle$. That is, $\int_U \nabla u_f \times \nabla v dx + \gamma \int_U u_f v dx = \int_U f v dx$.

ii) If $f_n \to f, u_{f_n} \to u$ then $B_\gamma(u_{f_n}, v) \to B_\gamma(u, v)$, $\langle f_n, v \rangle \to \langle f, v \rangle$. Therefore, L_γ^{-1} is closed with domain L^2. Thus, $\|u_f\|_{H_0^1} = \|L_\gamma^{-1} f\|_{H_0^1} \le K \|f\|_2$.

iii) The immersion of $H_0^1(U)$ in $L^2(U)$ is compact, $H_0^1 \subset\subset L^2$, since U is a bounded region. Then, $K = L_\gamma^{-1}$ is completely continuous of $L^2(U)$ in $L^2(U)$, QED.

THEOREM 3. *The boundary problem:* $(-\Delta + \lambda)u = f, u = 0$ *on* ∂U *has a unique weak solution for any* $f \in L^2$ *or there exists a non trivial solution of the homogeneous problem:* $(-\Delta + \lambda)u = 0,\ u = 0$ *on* ∂U. *The corresponding null space* N_λ *has a finite dimension. The non homogeneous problem admits a weak solution if and only if* $f \perp N_\lambda$ *in* L^2.

THEOREM 4. *The positive infimum* $\lambda_1 := \inf\{I(u,u) : u \in H_0^1, \|u\|_2 = 1\}$ *is taken on a function* $w_1 \in H_0^1(U)$, *a weak solution of* $(-\Delta - \lambda_1)w_1 = 0, w_1 = 0$ *on* ∂U. *It also verifies* $I(w_1, v) = \lambda_1 \langle w_1, v \rangle$ *for any* $v \in H_0^1(U)$. *Moreover, there exist* $w_2, w_3, \dots \in H_0^1$, $\|w_n\|_2 = 1$, *such that, for* $n > 1$, $\lambda_n := \inf\{I(v,v) : v \in H_0^1, \|v\|_2 = 1, \langle v, w_i \rangle = 0,\ i = 1,\dots,n-1\}$, $0 < \lambda_i \le \lambda_n$, *is taken in* w_n. *Besides,* $I(w_n, v) = \lambda_n \langle w_n, v \rangle$ *for any* $v \in H_0^1(U)$.

THEOREM 5. $0 < \lambda_n \le \lambda_{n+1} \to \infty$ *and* $\{w_n\}$ *is a complete ortonormal system in* L^2. *It holds for* $u, w_j, v_k \in H_0^1 \supset M$ *that* $\min_{0 \ne u \perp w_1,\dots,w_{n-1}} \frac{I(u,u)}{\|u\|_2^2} = \lambda_n$, $\min_{\dim M = n} \max_{0 \ne u \in M.} \frac{I(u,u)}{\|u\|_2^2} = \lambda_n$,

$\max_{\{v_1,\dots,v_{n-1}\} lin.indep.} \min_{0 \ne u \perp v_1,\dots,v_{n-1}} \frac{I(u,u)}{\|u\|_2^2} = \lambda_n$, *where* $\perp$ *means orthogonal in* L^2.

THEOREM 6. *i) If* $\partial U \in C^2$, $f \in L^2(U)$ *and* $(-\Delta + \mu)u = f, u = 0$ *on* ∂U *then* $\|u\|_{H^2} \le C(U)\left[\|f\|_2 + \|u\|_2\right]$. *If* $f = 0$ *then* $\|u\|_{H^2} \le C(U)\|u\|_2$.

ii) If $\partial U \in C^2$ *then* $w_i \in H^2 \cap H_0^1 \cap C^\infty$.

Let σ_U be the spectrum of the operator $-\Delta$, $(-\Delta-\lambda)u=0, u=0$ on ∂U, each λ repeated according to its multiplicity.

THEOREM 7. *A) Let $D\subset E$. Then, $\forall j\ \lambda_j(E)\le\lambda_j(D)$ and $N_D(\lambda)\le N_E(\lambda)$.*

B) Let D, E, U regions such that $D\cap U=\varnothing, D\cup U=E$. If $\sigma_D=\{\lambda_j\}$, $\sigma_U=\{\lambda'_j\}$, $\sigma_E=\{\mu_j\}$ then $\mu_{j+k}\le\sup\{\lambda_j,\lambda'_k\}$ and $N_D(\lambda)+N_U(\lambda)\le N_E(\lambda)$.

C) If $\{D_i:i=1,\dots,m\}$ is a disjoint family of regions such that for any i, $D_i\subset E$, E a (bounded) region then $\sum_1^m N_{D_i}(\lambda)\le N_E(\lambda)$.

PROOF. A) If M is a subespace of $H_0^1(D)$ then it is a subspace of $H_0^1(E)\supset H_0^1(D)$, (cf. auxiliary theorem, §6.) Using the minmax formula of Theorem 6 we obtain $\lambda_j(E)\le\lambda_j(D)$.

B) Let $M=M_1+M_2$, $M_1=[v_1,\dots,v_j]\subset H_0^1(D)$ and $M_2=[v'_1,\dots,v'_k]\subset H_0^1(U)$. Let $u=v+v', v\in M_1, v'\in M_2$. Then, $I(u,u)=I(v,v)+I(v',v')$. Because of the minmax formula we have $\mu_{j+k}\le\min_{M=M_1+M_2}\max_{0\ne u\in M}\dfrac{I(v,v)+I(v',v')}{\|v\|^2+\|v'\|^2}$

$$\le\min_{M=M_1+M_2}\frac{\max_{v\in M_1}(I(v,v)/\|v\|^2)\|v\|^2+\max_{v'\in M_2}(I(v',v')/\|v'\|^2)\|v'\|^2}{\|v\|^2+\|v'\|^2}.$$

If $M_1=\{w_1,\dots,w_j\}, M_2=\{w_1',\dots,w_k'\}$, the last expression is not greater than $\dfrac{a\lambda_j+b\lambda'_k}{a+b}$.

Therefore, $\mu_{j+k}\le\sup\{\lambda_j,\lambda_k'\}$,... QED.

6. CLASSICAL AND VARIATIONAL EIGENFUNCTIONS. We shall prove next that the (weak) eigenfunctions and eigenvalues of the Dirichlet problem in a Jordan region $D\subset R^2$ found by means of the variational method are equal to the classical ones, corroborating that we have made a legitimate use in section 4 of the variational results of the preceding theorem 7. For this we consider the Dirichlet problem in the unit circle $U=\{z:|z|<1\}$. Let $\varphi\in C_0^\infty(U)$. We write: $\xi=\rho e^{i\phi}, x=re^{i\theta}, \hat{x}=r^{-1}e^{i\theta}=x/r^2=1/\bar{x}$.

Green's kernel for this region is $G(x,\xi)=\frac{-1}{2\pi}\log\frac{|x-\xi|}{|1-\bar{\xi}x|}=\frac{-1}{2\pi}\left[\log|x-\xi|-\log\left|\hat{\xi}-x\right|-\log|\xi|\right]$

whenever $x\neq\xi\neq 0$. The function $u(x):=\int_U G(x,\xi)\varphi(\xi)d\xi$ is the classical solution of

$-\Delta u=\varphi$: $u\in C^2(U)\cap C(\bar{U})$, $u(z)=0$ if $z\in\partial U=\{z:|z|=1\}$. We show that u belongs to $H_0^1(U)$. In this case the classical solution is also a weak solution such that $u\in H_0^1(U)\cap H^2(U)\cap C^\infty(U)\cap C(\bar{U})$, (cf. section 5).

THEOREM 1. *a) If $K(x,\xi)$ is equal to the Green's kernel of Dirichlet problem in U,*

$G(x,\xi)=\frac{-1}{2\pi}\log\frac{|x-\xi|}{|1-\bar{\xi}x|}$, *or to some of its derivatives,*

$\frac{\partial G}{\partial x_i}(x,\xi)=\frac{-1}{2\pi}\left\{\frac{x_i-\xi_i}{|x-\xi|^2}-\frac{(x_i|\xi|-\xi_i/|\xi|)|\xi|}{\left|x|\xi|-\xi/|\xi|\right|^2}\right\}$, *then it verifies*

(1) $\int_U |K(x,\xi)|d\xi\leq C<\infty$, $\int_U |K(x,\xi)|dx\leq C<\infty$ *for any x,ξ in U, respectively.*

b) If $\varphi\in C_0^\infty(U)$ then $u(x)=\int_U G(x,\xi)\varphi(\xi)d\xi$ belongs to $H_0^1(U)$.

To prove this theorem we need the following result.

THEOREM 2. *If $K(x,\xi)$, $x,\xi\in U$, satisfies the inequalities in (1) then $w(x):=\int_U K(x,y)\varphi(y)dy$ verifies $\int_U |w(x)|^2dx\leq C^2\|\varphi\|_2^2$.*

PROOF. In fact,

$$|w(x)|^2=\left|\int_U K(x,y)\varphi(y)dy\right|^2\leq\left(\int|K(x,y|dy\right)\left(\int|K(x,y)||\varphi(y)|^2dy\right)\leq C\int|K(x,y)||\varphi(y)|^2dy.$$

Thus, $\int_U |w(x)|^2dx\leq C\int dx\int|K(x,y)||\varphi(y)|^2dy\leq C^2\|\varphi\|_2^2$, QED.

PROOF OF THEOREM 1. We have $G(x,\xi)=O(\log|x-\xi|)$ and

$$\frac{\partial G}{\partial x_1}=-\frac{\partial}{4\pi\,\partial x_1}\left[\log|x-\xi|^2-\log|\xi|^2-\log\left|x-\xi/|\xi|^2\right|^2\right]=\frac{-1}{2\pi}\left\{\frac{x_1-\xi_1}{|x-\xi|^2}-\frac{(x_1|\xi|-\xi_1/|\xi|)|\xi|}{\left|x|\xi|-\xi/|\xi|\right|^2}\right\}=$$

$=O(\frac{1}{|x-\xi|})$. Therefore *a*) follows. From *a*) we obtain that $u\in H^1(U)$. Because of B) of next auxiliary theorem we obtain $u\in H_0^1(U)$, QED.

AUXILIARY THEOREM. *A) Let D and E be regions such that* $D\subset E$. *We define, from the functions u in* $H_0^1(D)$ *functions* $\tilde{u}$, $\tilde{u}(x):=u(x), x\in D; \tilde{u}(x):=0, x\in E\setminus D$. *Then, in the sense of distributions it holds that* $(\frac{\partial u}{\partial x_i})^{\sim}=\frac{\partial\tilde{u}}{\partial x_i}$ *and the extended functions define a subspace of* $H_0^1(E)$.

B) Let D be a Jordan region with boundary C^1 *and* $u\in C(\overline{D})$, *null at the boundary. If* $u\in H^1(D)$ *then* $u\in H_0^1(D)$.

For a proof of this theorem see the references in § 10.

THEOREM 3. *Let D be a Jordan region with boundary J.*

i) The classical solution $v\in C(\overline{D})\cap C^2(D)$ *of* $-\Delta v=\psi$, $\psi\in C_0^\infty(D)$, $v\equiv 0$ *en* ∂D *belongs to* $H^1(D)$.

ii) If $J\in C^1$ *then* $v\in H_0^1(D)$ *and is equal a.e. to the weak solution of* $-\Delta v=\psi$, $\psi\in C_0^\infty(D)$, $v=0$ *on* ∂D.

PROOF. i) Let g be the map that applies D conformally onto U and applies $\overline{D}$ topologically onto $\overline{U}$, (Riemann-Carathéodory mapping theorem). Let $\varphi\in C_0^\infty(U)$ and $\Phi(z):=\varphi(g(z))$. Then $\Phi\in C_0^\infty(D)$. Let u be as in b) of Theorem and y $v(z)=u(g(z))$. *Let us define,* $\varsigma=\xi+i\eta=g(z)=g(x+iy)=g_1(x,y)+ig_2(x,y)$. From $-\Delta_{\xi,\eta}u=\varphi(\varsigma)\in C_0^\infty(U)$ it follows that $-\Delta_{x,y}v=\varphi(g(z))\left|\frac{dg}{dz}\right|^2$. The function $\frac{dg}{dz}\ (\neq 0)$ is holomorphic on *D*. Thus, the following function belongs to $C_0^\infty(D)$:

(2) $\psi(x,y) := \varphi(g(z)).\left\{\left(\frac{\partial g_1}{\partial x}\right)^2 + \left(\frac{\partial g_2}{\partial x}\right)^2\right\}$.

Therefore, $-\Delta v(x,y) = \Phi(x,y).\left\{\left(\frac{\partial g_1}{\partial x}\right)^2 + \left(\frac{\partial g_2}{\partial x}\right)^2\right\} = \psi(x,y) \in C_0^\infty(D)$. v is then the classical solution of $-\Delta v = \psi(x,y), v|_{\partial D} \equiv 0$. On the other hand we have $|\nabla_{x,y} v|^2 = |(\nabla_{\xi,\eta} u)(g(z))|^2 \left|\frac{dg}{dz}\right|^2 = |(\nabla_{\xi,\eta} u)(g(z))|^2 \frac{\partial(\xi,\eta)}{\partial(x,y)}$. Then, using Theorem 2 we arrive at $\|\nabla_{x,y} v\|_{L^2(D)} = \|\nabla_{\xi,\eta} u\|_{L^2(U)} \le M\|\varphi\|_{L^2(U)}$. In consequence, the classical solution of $-\Delta v = \psi(x,y)$, $v = 0$ on J, is in $H^1(D)$.

ii) follows easily from the auxiliary theorem, QED.

THEOREM 4. *Let D be a Jordan region with* $J \in C^1$. *Then* $\boldsymbol{G} \equiv L_0^{-1}$.

PROOF. If w is the weak solution of $-\Delta w = \psi(x,y)$, w=0 on J and v is the classical solution of the problem then v=w because of Theorem 3. Thus, we have $v = \boldsymbol{G}\psi = L_0^{-1}\psi$ for any ψ of the form (2), what amounts to say that the equality holds for any $\psi \in C_0^\infty(D)$. If $C_0^\infty(D) \ni \psi_n \to f$ in L^2 then $\boldsymbol{G}(\psi_n) = L_0^{-1}(\psi_n) \to L_0^{-1}(f)$ in H_0^1 and to $\boldsymbol{G}(f)$ in L^2. In consequence, $\boldsymbol{G} \equiv L_0^{-1}$, QED.

Let D be a Jordan region with boundary J and $\psi \in C_0^\infty(D)$. There exist Jordan regions D_n with boundaries $J_n \in C^\infty$, $J_n \to J$, such that $\mathrm{supp}\psi \subset D_1 \subset \overline{D}_n \subset D_{n+1} \subset \overline{D}_{n+1} \subset \subset D = D_0$, $n = 1, 2, \ldots$. (For example, $D_n := \{(x,y) : G(x_0,y_0;x,y) > \varepsilon_n\}$, $\varepsilon_n \downarrow 0$, where $G(x_0,y_0;x,y)$ is the Green's kernel for D). If v_n is the classical solution on D_n from $-\Delta v_n = \psi$, $v_n = 0$ on J_n, we get that $v - \widetilde{v}_n$ is a harmonic function on $D \setminus J_n$, continuous on $\overline{D}$. Therefore, $\|v - \widetilde{v}_n\|_\infty \le \sup_{z \in J_n} |v(z)| \xrightarrow[n\to\infty]{} 0$. That is, v is the límit in L^∞, and then in L^2, of $\widetilde{v}_n$. In particular we have $\forall n \ \|v_n\|_2 \le M < \infty$.

Let us replace by u's the v's to denote weak solutions. Thus, $u_n = v_n$. We shall see that ii) of Theorem 3 holds with greater generality.

TEOREMA 5. *Let D be a Jordan region with boundary J. The weak solution u of* $-\Delta u = \psi$, $\psi \in C_0^\infty(D)$, $u = 0$ *on J, coincides with the classical solution v.*

PROOF. $\iint\limits_D |\nabla \widetilde{u}_n|^2 dxdy = \int \nabla \widetilde{u}_n . \nabla \widetilde{u}_n = \iint\limits_{D_n} \nabla u_n . \nabla u_n = \int \psi u_n \le \|\psi\|_2 \|u_n\|_2 \le M \|\psi\|_2 = M'$.

Then, $\|\widetilde{u}_n\|_{H_0^1} \le K < \infty$. We use next, in $H_0^1(D)$, the equivalent norm $|\cdot|_{H_0^1}$ associated to the scalar product $I(u,v) = \int\limits_D \nabla u \times \nabla v \, dx$. Define $W_1 := \widetilde{u}_1$, $W_n := \widetilde{u}_n - \widetilde{u}_{n-1}$ in such a way that $\sum_1^N W_j = \widetilde{u}_N$. It holds that $W_n \in H_0^1(D_n) \square H_0^1(D_{n-1})$. In fact, let $\varphi \in C_0^\infty(D_{n-1})$. Since u_m is a weak solution in D_m we have $I(W_n, \varphi) = \int\limits_D \psi(\varphi - \varphi) dx = 0$. Then, $\sum_1^N |W_j|_{H_0^1}^2 = |\widetilde{u}_N|_{H_0^1}^2 \le K^2$ holds in $H_0^1(D)$. Because of this, $\sum_1^N W_j = \widetilde{u}_N$ converges in $H_0^1(D)$ to a function $u \in H_0^1(D)$, a weak solution. Because of $\widetilde{u}_N = \widetilde{v}_N$, the sequence $\{\widetilde{u}_N\}$ converges in L^2 to the classical solution v. Thus, $u = v$, QED.

THEOREM 6. *i) If D is a Jordan region then* $\boldsymbol{G} \equiv L_0^{-1}$.

ii) Let $\{w_i\}$ *be the family of normalized eigenfunctions of the Dirichlet problem obtained via the variational method and* $\sigma_V = \{\mu_i\}$ *the associated spectrum. Let* $\{\varphi_i\}$ *be the family of classical normalized eigenfunctions and* $\sigma = \{\lambda_i\}$ *its spectrum. Then* $\sigma_V = \sigma$ *and for any i the eigenspaces (of finite dimension) coincide:* $N(\mu_j) = N(\lambda_j)$. $\{\varphi_i\} \approx \{w_i\}$: *each variational eigenfunction is a linear combination of classical eigenfunctions contained in its eigenspace:* $w_i = \sum_{\lambda_k = \mu_i} c_k \varphi_k$ *a.e., and conversely.*

PROOF. Using Theorem 5 we can repeat the proof of Theorem 4 showing so that i) holds. ii) is an immediate consequence of i), QED.

7. FINAL REMARKS. In this last paragraph we collect three bidimensional examples to foster a non optimistic view of the conjectures. Assume that dim is Hausdorff or Minkowski dimension. For the examples (a), (b), next formula (*) does not hold:

(*) $N(\lambda) = A\lambda + B\lambda^{d/2} + o(\lambda^{d/2})$, $d = \dim \partial D$, $A = |D|/4\pi$ and B a non null constant.

(a) Assume that D is a Jordan plane region with a boundary of positive area, i.e., d=2. If (*) holds then we would have $N(\lambda) \sim (A+B)\lambda$, contradicting Weyl's asymptotic formula (§2, Th. 5).

(b) Gromes considers a biangular region D on the unit sphere with vertices at the poles. Here d=1. Let $\{x\} := x - [x] - 1/2$. If the angle β between its (meridian) sides is a rational multiple of π, $\beta = \pi(a/b)$, then $N(\lambda) = \frac{|D|}{4\pi}\lambda - \frac{\langle \partial D\rangle + \delta(\lambda)}{4\pi}\sqrt{\lambda} + O(1)$ where $\delta(\lambda) = (1/a)\{b\sqrt{\lambda + 1/4} - b/2\}$. Thus, instead of a constant B we have an oscillating function $B = B(\lambda)$.

(c) Let $D = \bigcup_{i,j=0}^{1} S_{ij}$, $S_{ij} = \{i\pi/2 < x < (i+1)\pi/2\} \times \{j\pi/2 < x < (j+1)\pi/2\}$. Then, D is not a Jordan region but d=1. For each $n, m = 1, 2, \ldots$, the functions $sin(2nx)sin(2my)I_{S_{ij}}(x,y)$ are four linearly independent eigenfunctions corresponding to the eigenvalue $4(n^2 + m^2)$, (I_S stands for the characteristic function of S). If $\Omega := \{x, y : 0 < x < \pi, 0 < y < \pi\}$ then $sin(nx)sin(my)$ $n, m = 1, 2, \ldots$ are the eigenfunctions for the eigenvalues $n^2 + m^2$. Therefore we have the relation: $N_D(\lambda) = 4\#\{n, m : n^2 + m^2 < \lambda/4\} = 4N_\Omega(\lambda/4)$. For Ω it holds

(**) $$N_\Omega(\lambda) = \frac{|\Omega|}{4\pi}\lambda - \sqrt{\lambda}\left(\frac{\langle \partial\Omega\rangle}{4\pi} + o(1)\right).$$

Then, $N_D(\lambda) = \frac{|D|}{4\pi}\lambda - \sqrt{\lambda}\left(\frac{2\langle \partial\Omega\rangle}{4\pi} + o(1)\right)$. But $\langle \partial D\rangle < 2\langle \partial\Omega\rangle$ and (**) does not hold for D.

However, if D had the squares $\overline{S_{ij}}$ without points in common we would have

$$N_D(\lambda) = \frac{|D|}{4\pi}\lambda - \sqrt{\lambda}\left(\frac{\langle \partial D\rangle}{4\pi} + o(1)\right).$$

8. FIGURES.

Fig.1

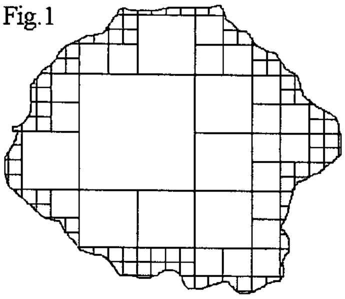


Fig.2

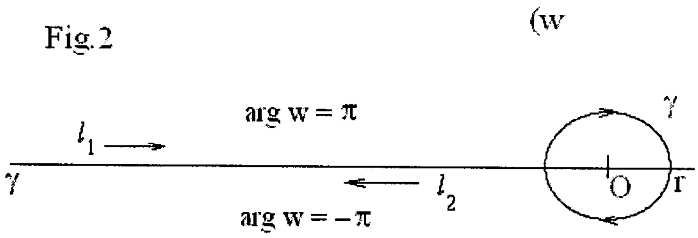
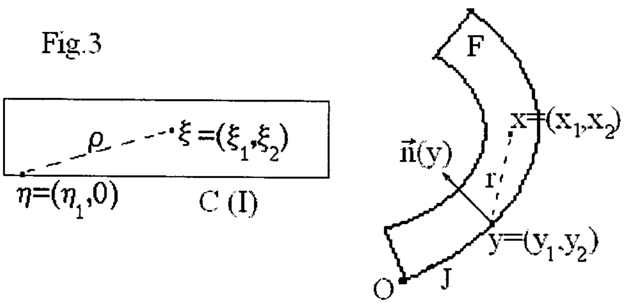


Fig.3



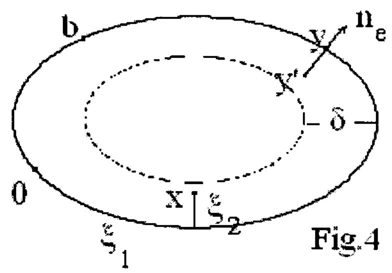
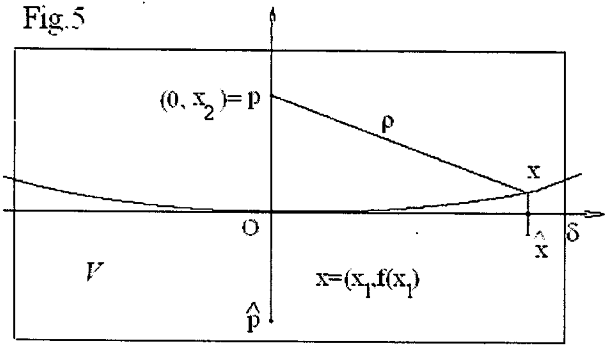


Fig.4

Fig.5



9. APPENDIX. Next we quote a *tauberian* proposition where *we assume* that $\varphi(0)=0$, (cf. [T] and [W]).

AUXILIARY LEMMA. *i) Let* $\varphi(t)\geq 0, \sigma>0, x\to 0, T\to\infty$. *Then,*

$$\int_0^\infty e^{-xt}\varphi(t)dt \sim \frac{A}{x^\sigma} \Rightarrow \int_0^T \varphi(t)dt \sim A\frac{T^\sigma}{\Gamma(\sigma+1)}.$$

ii) If $\varphi(x)$ *is non decreasing and non negative and* $p\geq 1$ *then*

$$\int_0^T \varphi(y)dy \sim AT^p \Rightarrow \varphi(T)\sim ApT^{p-1}$$

iii) Let $\varphi(x)$ *be as in ii). If* $\gamma>-2$ *then*

$$\int_0^\infty e^{-xt}d\varphi(t)\sim A/x \Rightarrow \int_0^T t^\gamma\varphi(t)dt \sim AT^{\gamma+2}/(\gamma+2).$$

The non negative jump function $N(\lambda)$, $0\leq\lambda<\infty$, $N(\lambda)=\sum_{\lambda_n\leq\lambda}1$, is right continuous and vanishes on $[0,\lambda_1)$. Because of $A\lambda_n \sim n$, $A=\frac{|D|}{4\pi}$, the series $\sum_{n=1}^\infty e^{-\lambda_n x}$ converges and defines the function $Z(x)=\int_0^\infty e^{-xt}dN(t)=\lim_{T\to\infty}\int_0^T e^{-xt}dN(t)$ on $(0,\infty)$.

LEMMA 1. $N(\lambda)\sim\frac{|D|}{4\pi}\lambda$, $\lambda\to\infty \Rightarrow Z(x)=\sum_{n=1}^\infty e^{-\lambda_n x}\sim\frac{|D|}{4\pi x}$, $x\to 0+$.

In fact, integrating by parts we get $Z(x)=x\int_0^\infty e^{-xt}N(t)dt$ with $N(t)\sim At$. An abelian theorem allows us to assert that $Z(x)\sim x(A/x^2)$, ([W], Ch. 8), QED.

On the other hand, for $x>0$ there exists $Z(x):=\int_0^\infty e^{-xt}dN(t)$. Thus, if $Z(x)\sim A/x$ then $\int_0^T N(t)dt\sim\frac{AT^2}{2}$, (cf. *iii*), for $\gamma=0$, auxiliary lemma). Since $N(t)\uparrow$ we get, using *ii*) of the same lemma, that $N(t) \sim At$. Therefore, for $\lambda \to \infty$, $x \to 0$, the following tauberian proposition holds,

LEMMA 2. $Z(x) = \sum_{n=1}^{\infty} e^{-\lambda_n x} = \int_0^{\infty} e^{-xt} dN(t) \sim A/x \quad \Rightarrow \quad N(\lambda) \sim A\lambda$.

LEMMA 3. *Let* $N(\lambda) = A\lambda + B\sqrt{\lambda} + o(\sqrt{\lambda}), \lambda \to \infty$. *Then,* $Z(x) = \frac{a}{x} + \frac{b}{\sqrt{x}} + o(x^{-1/2})$, $x \downarrow 0$, *with* $a = A,\ b = \frac{B\sqrt{\pi}}{2}$.

PROOF. $\int_0^{\infty} e^{-xt} dN(t) = x\int_0^{\infty} e^{-xt} N(t)dt = x\int_0^{\infty} e^{-xt}(At + B\sqrt{t} + o(\sqrt{t}))dt = \frac{A\Gamma(2)}{x} + \frac{B\Gamma(3/2)}{\sqrt{x}} + x\int_0^{\infty} e^{-xt} o(\sqrt{t})dt = \frac{a}{x} + \frac{b}{\sqrt{x}} + R(x)$. But $|R(x)| \le x\left(\int_0^M Ke^{-xt}\sqrt{t}\,dt + \int_M^{\infty} \varepsilon e^{-xt}\sqrt{t}\,dt\right)$, $\varepsilon > 0$, $M = M(\varepsilon)$. If $G(u) = K\int_0^u \sqrt{v}\,e^{-v}dv$ then $|R(x)| \le \frac{G(xM) + \Gamma(3/2)\varepsilon}{\sqrt{x}}$ For x small enough one obtains, $|R(x)| \le \varepsilon(1 + \sqrt{\pi}/2)/\sqrt{x}$. Then, $\sqrt{x}\,R(x) = o(1)$, QED.

10. REFERENCES

For §1, [I], [Ku], [T], [L]; for §2, [BP] parte I, [C], [G], [MO]; §3, [Pl], [B]; §4, [F], [L], [S]; §5, [E], [CH]; §6, [A], [Tr], [H], [Ke], [Tj], [BP] parte III, is, in part, a preprint of this paper; §7, [Gr.], [BC]; §9, [T], [W].

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AGRADECIMIENTOS: A la Dra. María Inés Platzeck por la buena voluntad mostrada en la publicación de este trabajo. Al referee, cuyos comentarios matemáticos nos hicieron ver la necesidad de incorporar el apéndice (§9) y modificar el título.