NOTAS DE ALGEBRA Y ANALISIS

4

AGNES BENEDEK, EDGARDO GÜICHAL AND RAFAEL PANZONE

On Certain Non Harmonic Fourier Expansions As Eigenfunction Expansions Of Non Regular Sturm-Liouville Sistems

1974

INSTITUTO DE MATEMATICA

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ON CERTAIN NON HARMONIC FOURIER EXPANSIONS AS EIGENFUNCTION EXPANSIONS OF NON REGULAR STURM-LIOUVILLE SYSTEMS

by

A.Benedek, E.Güichal and R.Panzone

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Dedicado a la memoria de

nuestros amigos y colegas

JUAN CARLOS MERLO

y

EVELIO TOMAS OKLANDER

RÉSUMÉ

Nous étudions ici un cas particulier du système:

$$(*)\quad \begin{cases} y''(x) + (\nu - q(x))y(x) = 0 \ , \\ \sum_{j=0}^{m} \beta_j \ y^{(j)}(0) = 0 \quad , \quad \sum_{j=0}^{n} \alpha_j \ y^{(j)}(1) = 0 \ , \\ \alpha_j, \beta_j \text{ nombres réels, } \alpha_n \neq 0 \ , \ n \geqslant 2. \end{cases}$$

Plus exactement, nous considérons le cas $q \equiv 0$, $m = 2$, $|\beta_0| + |\beta_1| \neq 0$, dont leurs solutions sont données par des exponentielles, sauf pour $\nu = 0$.

Nous trouvons des développements qui ressemblent ceux de la théorie de Fourier non harmonique, et d'après ceci le nom du travail

Nous nous bornerons ici à étudier ce cas particulier, parce-que celùi-ci contient les applications les plus connues, et d'ailleurs, présente les moindres difficultés techniques.

Dans un autre travail nous étudierons le cas général (*), au moins, avec une q suffisamment régulier.

ON CERTAIN NON HARMONIC FOURIER EXPANSIONS AS EIGENFUNCTION EXPANSIONS OF NON REGULAR STURM-LIOUVILLE SYSTEMS

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I. INTRODUCTION.

1. J.Prescott considers in his book, [17], the following mechanical system: a rod moving axially with constant velocity and bearing at one of its ends a finite mass. If the other end is suddenly stopped severe dynamic stresses are generated in the rod and oscilations in the axial direction are produced. Mathematically, the problem is reduced to the study of the one-dimension wave equation. Using the classical method of separation of variables the following second-order ordinary differential equation comes under consideration:

$$(1) \qquad \begin{cases} X''(x) + \lambda^2 X(x) = 0 \;, & 0 < x < b \;, \\ X(0) = 0 \;, & X'(b) = \lambda^2 X(b) \;. \end{cases}$$

Timoshenko and Young in their book study another similar mechanical system. A bar fixed at one end with a load in the other one. If a force is applied at the lower end and suddenly released then vibrations occur in the bar as in the preceding mechanical device. Mathematically, the problem of the displacements of the bar is solved as before changing only the initial conditions. Separation of variables leads again to a non regular Sturm-Liouville system like (1).
P.A.A.Laura called our attention to the eigenfunction expansions associated to this system. The eigenfunctions form a

complete set in $L^2(0,b)$ but not orthogonal with respect to Lebesgue measure as in the case when the second boundary condition is replaced by an ordinary one. In [12], [13], he and his coworkers study a cable-like system with an energy absorbing device placed at the upper end in order to decrease the severity of the dynamic stresses caused by the sudden loading. The second order differential system is now:

$$(2)\qquad \begin{cases} X'' + \lambda^2 X = 0 \ , \ 0 < x < b \ , \\ X(0) = q.X'(0) \ , \ X'(b) = \lambda^2 X(b), \ q \text{ a constant.} \end{cases}$$

Analogous differential systems occur in certain problems of flow of heat or diffusion. In relation with this, cf. the papers by Bauer, [2], and Langer, [11].

2. In what follows we shall study a generalization of system (2). Observing that the second boundary condition is a linear relation between X' and X'', one is immediately led to consider the following system:

$$(3)\qquad \begin{cases} y'' + \lambda^2 y = 0 \ , \ 0 \leqslant x \leqslant 1 \ , \\ \beta_0 \, y(0) + \beta_1 \, y'(0) = 0 \ , \\ \alpha_0 y(1) + \alpha_1 y'(1) + \ldots + \alpha_n y^{(n)}(1) = 0 \ , \end{cases}$$

$\sum|\beta_j| \neq 0$, $\alpha_n \neq 0$, $n \geqslant 2$, α_i and β_j real numbers.

The solutions of this system are, in general, linear combinations of exponentials. Then, the expansions associated could be considered as belonging to the theory of non harmonic Fourier series. However, they do not fall inside the usual theory. For example, Paley and Wiener in their famous book study non harmonic expansions which are unique in the sense that only one set of coefficients is associated to each function. This uniqueness is consequence of the biorthogonality of the system they consider. This situation does not occur in our case. In relation with the topic of non harmonic expansions, cf. [4], [7], [14], [15], [17].

Approximation theory suggests several problems. We have restricted ourselves to the analogues of Plancherel theorem, Riesz-Fischer result and Dirichlet-Jordan theorem of the Fourier analysis.

To prove these results, it is convenient to show firstly that the set of eigenfunctions is complete in $L^2(0,1)$. We do this following a paper of W.M.F. Orr published at the beginning of this century. This paper uses ideas that seem to be due to H.S.Carslaw.

3. The case n=2 has been studied by several authors, cf. [5], [2], [8], [11].

If $\alpha_1 = \beta_1 = 1$, $\alpha_2 > 0$, n=2, Churchill proves as a particular result of his work the following proposition:

Let f *be a continuous function with a derivative bounded everywhere by a constant. If* $\{y_i\}$ *is the set of eigenfunctions of the system then the expansion*

$$(4) \qquad f(x) = \sum_0^\infty (B(f,y_i)/B(y_i,y_i))y_i(x) \ ,$$

holds uniformly in [0,1]. *Here,*

$$(5)\begin{cases} B(u,v) = \int_0^1 uv\ dx + \alpha_2\ u(1)v(1) = \int_0^1 uv\ d\sigma \ , \\ d\sigma = dx + \alpha_2 . \delta_1 . \end{cases}$$

This result implies that $\{y_i/\sqrt{B(y_i,y_i)}\}$ is an orthonormal basis in $L^2(d\sigma)$.

Since $\alpha_2 > 0$, the bilinear form (5) defines an inner product and the system is called *normal*. In the four papers just mentioned the idea of orthogonality of the eigenfunctions plays an important role.

Our aim is to consider simultaneously normal and non normal systems. To do this we free ourselves of the idea of orthogonality of the eigenfunctions. In this sense we shall exhibit a bilinear form B such that $B(y_i, y_j) = 0$ if $i \neq j$, which is not, in general, an inner product.

4. The case n=3 appears also in the literature, (cf. [6]), as a particular case of a more general setting. It is proved under the situation that Dück calls "*eigentlich definite*" that certain eigenfunction expansions hold.

II. COMPLETENESS.

1. Let us assume that $F_i(s)$, $i=1,2,3,4$, is a polynomial $\not\equiv 0$ such that:

$$(1) \qquad \deg F_1 = \deg F_2 \quad ; \quad \deg F_3 = \deg F_4 \ .$$

For a fixed s, let us consider a function:

$$(2) \qquad A\,e^{sx} + B\,e^{-sx} \quad , \quad -\infty < a \leqslant x \leqslant b < \infty \ ,$$

satisfying

$$(3) \qquad \begin{aligned} &A\,e^{sa}\,F_1(s) + B\,e^{-sa}\,F_2(s) = 0 \\ &A\,e^{sb}\,F_3(s) + B\,e^{-sb}\,F_4(s) = 0 \quad , \end{aligned}$$

$$(4) \qquad \begin{aligned} \Delta(s) \equiv F_2(s)\,F_3(s)\,\exp s(b-a)\, - \\ - F_1(s)\,F_4(s)\,\exp s(a-b) = 0. \end{aligned}$$

We *assume* also that:

$$(5) \qquad \Delta(s) = 0 \ , \ s\neq 0 \quad \Rightarrow \quad \Delta'(s) \neq 0 \ .$$

That is, every non null zero of Δ is of first order. The following result holds:

THEOREM 1. i) *Let* $\phi(x)$ *be of finite variation on* $[a,b]$, *and* $x \in (a,b)$. *Assume that* $\Delta(s) = s^p.h(s)$, $h(0) \neq 0$. *Then, there exists a series of functions* (2) *verifying* (3) *and* (4) *such that:*

$$(6) \ \phi(x+0) + \phi(x-0) = P(x) + \sum(A_s e^{sx} + B_s e^{-sx}) \ ,$$

where $P(x)$ *is a polynomial of degree less than* p.

ii) *If* $\phi \in C([a,b])$ *and is null on neighborhoods of* a *and* b, *the convergence is uniform on* (a,b).

Part i) of this theorem is due to Orr, [14]. His method of proof proves also part ii). For the sake of completeness we shall derive it. In the middle of the proof the following proposition is used, (cf.[21], II):

Let be $\int_a^b |g(t)|dt < \infty$, $\int_a^b |f(t)|dt < \infty$, $f \equiv g$ on a neighborhood of $x \in (a,b)$.

Then, there exists the limit:

$$\lim_{k\to+\infty} \frac{1}{\pi} \int_a^b f(y) \frac{\sin k(x-y)}{x-y} dy = L$$

if and only if $(S_{a,b}g)(x) = L$, where $(S_{a,b}g)(x)$ is the sum of the Fourier series of g relative to (a,b),(i.e., the expansion of g with respect to the system $\{\exp i(2\pi n(x-a)/(b-a))\}$). Besides, if $f \in C^1([a,b])$ and is zero on a neighborhood of a and b and $k \to +\infty$, then:

$$(7) \qquad \frac{1}{\pi} \int_a^b f(y) \frac{\sin k(x-y)}{x-y} dy \longrightarrow f(x)$$

holds uniformly on (a,b).

Proof of ii). We can assume without loss of generality that $\phi(x)$ is a real function.

The function $\Delta(z)$ is entire and therefore the set of its zeroes is at most denumerable. Let be $F = F_1F_4/F_2F_3$ and $p = \lim_{z\to\infty} F^{-1}(z)$. An application of Rouché's theorem shows that functions: $\exp(-2z(b-a)) - \frac{F_3 \cdot F_2}{F_1 \cdot F_4}$ and $\exp(-2z(b-a))-p$ have the same number of zeroes in anular regiones great enough. Moreover, it proves that the zeroes of Δ are of the form:

(8) $\quad S_n = \sigma + n\pi i/(b-a) + o(1)$

where σ is a fixed determination of $-\dfrac{\lg p}{2(b-a)}$

Each term of the series in (6) is a multiple of

(9) $\quad e^{s(x-a)}F_2(s) - e^{-s(x-a)}F_1(s)$

and of

(10) $\quad e^{s(x-b)}\,F_4(s) - e^{-s(x-b)}\,F_3(s).$

This suggests the consideration of the following contour integral:

(11) $$\int_C dz \int_a^b [\{e^{z(u-a)}F_2(z) - e^{-z(u-a)}F_1(z)\ .$$

$$.\{e^{z(x-b)}F_4(z) - e^{-z(x-b)}F_3(z)\}\ \Delta^{-1}(z)]\ \phi(u)du.$$

C is a circumference with center at the origin of radius h such that $\Delta \neq 0$ on C and oriented in the usual positive sense. We call $C_+ = C \cap \{\mathrm{Re}\ z > 0\}$, $C' = C_+ \cap \{|\arg z| \leqslant \leqslant \pi/2 - \delta\}$, $C'' = (C_+ - C') \cap \{\mathrm{Im}\ z > 0\}$, $C''' = C_+ - (C' \cup C'')$.

The bracket in (11) can be written as

(12) $$I = e^{z(u-x)} . \frac{-1-F\ e^{2z(a-b+x-u)}+G\ e^{2z(a-u)}+H\ e^{z(2x-2b)}}{1-F\ e^{-2z(b-a)}}$$

where F is defined as above, $G = F_1/F_2$, $H = F_4/F_3$.

ϕ is zero outside of $(a+\varepsilon, b-\varepsilon)$ and therefore the inner integral in (11) must be restricted to this interval. For $u \in (a+\varepsilon, b-\varepsilon)$, we have:

$$(13)\quad |e^{z(2a-2b+x-u)}| \vee |e^{z(2a-u-x)}| \vee |e^{z(u+x-2b}| \vee$$

$$\vee\ |e^{-2z(b-a)}| \leqslant e^{-\varepsilon|\mathrm{Re}\ z|}$$

For a certain constant M we obtain:

$$(14)\quad \left|\int_{C'} dz \int_a^b [\,I+e^{z(u-x)}]\phi du\right| \leqslant M \int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}-\delta} e^{-h\varepsilon\cos\varphi}\, hd\varphi \ ,$$

since F,G and H are asymptotically constants. This implies that the left member of (14) tends to zero if $h \to \infty$. If h tends to infinity along a certain set of numbers adequately chosen (for example $h_n = [\,n+(1+\mathrm{sgn}\ p)/4]\pi/(b-a))$,

it is possible to say something analogous for the integrals over C" and C"'. In fact, if h is so chosen and $z \in C''$, for n great enough we get: $|1-F(z).\exp(-2z(b-a))| \geqslant \eta > 0$.

Since ϕ is of bounded variation, it is the difference of two positive decreasing continuous functions. Let ψ be one of them. Then

$$\int_{C''} dz \int_a^b (I + e^{z(u-x)})\psi(u)du$$

has an absolute value bounded by the sum of the absolute values of the expressions:

$$(15)\quad \begin{aligned} &\int_{C''} dz\ [\,\psi(a) \int_a^{\xi} \mathrm{Re}(I+e^{z(u-x)})du] \ ; \\ &\int_{C''} dz\ [\,\psi(a) \int_a^{\xi} \mathrm{Im}(I+e^{z(u-x)})du] \ . \end{aligned}$$

In fact, this is a consequence of the second mean value theorem.

From (12), it follows easily that:

$$(16) \qquad \int_a^{\xi} (I+e^{z(u-x)})du = O(1/z) ,$$

with O independent of $x \in (a,b)$.

(16) implies that the brackets in (15) are $O(1/h)$ and that the integrals are $O(\delta)$. The same estimations occur when one considers the integral over C"'.

Therefore, the integral (11) restricted to C_+ is equiconvergent with:

$$(17) \qquad - \int_a^b \phi(u)du \int_{C_+} e^{z(u-x)} dz ,$$

uniformly on $a < x < b$, when $h_n \rightarrow +\infty$ (h_n = radius of $C = C(n)$).

If $C_- = C \cap \{\mathrm{Re}\ z < 0\}$, from the change of variable $w = -z$ we get:

$$(18) \qquad \int_{C_-} dz \int_a^b I(z,u)\phi(u)du = \int_{C_+} dw \int_a^b K(w,u)\phi(u)du ,$$

where $K(z,u)$ is obtained from $I(z,u)$ replacing $F_1(z), F_2(z), F_3(z), F_4(z)$ by $F_2(-z), F_1(-z), F_4(-z), F_3(-z)$ respectively. In consequence, the second member of (18) is obtained from (11) with another family of polynomials.

Therefore, it equiconverges (uniformly on (a,b)) with (17).

From this, we finally arrive to the conclusion:

$\int_C dz \int_a^b I(z,u)\phi(u)du$ is uniformly equiconvergent with

$$-2 \int_a^b \phi(u)du \int_{C_+} e^{z(u-x)} dz = -4i \int_a^b \frac{\sin(u-x)h_n}{u-x} \phi(u)du ,$$

which converges uniformly to $-4\pi i \phi(x)$ for $h_n \rightarrow +\infty$.

On the other hand, an application of the theorem of residues shows that (11) is equal to a sum of the form

$$(19) \qquad \sum_{0<|s|<h} \alpha_s \cdot \{e^{s(x-b)} F_4(s) - F_3(s)\, e^{-s(x-b)}\}$$

plus the residue at the origin. Let us call

$$f(x,z) = [e^{z(x-b)} F_4(z) - F_3(z) e^{-z(x-b)}] .$$
$$\cdot \int_a^b [e^{z(u-a)} F_2(z) - F_1(z) e^{-(u-a)z}] \phi(u) du.$$

The residue at z=0 of

$$f/\Delta = f(x,z)/z^{\omega} h(z) = \{A_0(x) + A_1(x) z + \dots +$$
$$+ A_{\omega-1}(x)\ z^{\omega-1} + \dots\}/z^{\omega}$$

is $A_{\omega-1}$, where

$$A_{\omega-1}(x) = \frac{1}{(\omega-1)!} \frac{d^{\omega-1}}{dz^{\omega-1}} \frac{f(x,z)}{h(z)} \Bigg|_{z=0} = \sum_{k=0}^{\omega-1} r_k\, x^k ,$$

is a polynomial of degree less than ω. QED.

The method of proof used by Orr like that used by Langer [11], are inspired in the techniques of the calculus of residues applied to certain integrales involving a Green function of a differential system.

III. EIGENFUNCTIONS AND EIGENVALUES

1. Let us call $F_1(z)$ the real polynomial $\beta_0+\beta_1 z \not\equiv 0$ and F_3 the polynomial $\sum_{j=0}^{n} \alpha_j z^j$, also real with $n \geqslant 2$ and $\alpha_n \neq 0$.
By definition: $F_2(z) = F_1(-z)$ and $F_4(z) = F_3(-z)$.
The solutions $Ae^{sx} + Be^{-sx}$ of the differential system:

$$(1) \quad \begin{cases} u'' - s^2 u = 0 \ , \ 0 \leqslant x \leqslant 1, \\ F_1(d/dx)u(0) = 0 \ ; \ F_3(d/dx)u(1) = 0 \ , \end{cases}$$

have coefficients that satisfy:

$$(2) \quad AF_1(s) + BF_2(s) = 0 \ ; \ AF_3(s)e^s + BF_4(s)e^{-s} = 0.$$

The values of s for which there exist non trivial solutions of (1) are the roots of the equation:

$$(3) \quad \Delta(s) \equiv e^s F_2 F_3 - e^{-s} F_1 F_4 = 0.$$

We shall make two hypotheses on the differential system, which will be assumed from now on, except in few occasions where we explicitly mention that some of them does not hold. The second one is justifiable only in a first approach to the problem.

<u>1st hypotesis</u>: $F_3(z)$ is not divisible by an even real polynomial of positive degree.
If F_3 had a divisor $f(z^2)$ and $F_3(z) = F_3^0(z)f(z^2)$ then the boundary condition at x=1 would be satisfied by <u>any</u> solution $u_t = Ae^{tx} + Be^{-tx}$ with $f(t^2) = 0$ or else if $f(t^2) \neq 0$, then

$F_3(d/dx)u_t(1) = 0$ is equivalent to $F_3^0(d/dx)u_t(1) = 0$. Then the study of system (1) is practically reduced to that obtained replacing F_3 by F_3^0. So the first hypothesis minimizes the set of "eigenfunctions". If such an f exists, it divides the g.c.d.(F_3,F_4). This g.c.d. is real and either ever or odd, since if s is a common zero of F_3 and F_4, $-s$ is also one. So the *first hypothesis is equivalent* to: g.c.d$(F_3,F_4) = 1$ or z.

2nd hypothesis: every non null zero of Δ is of first order and null is a zero of Δ of at most third order. This hypothesis restricts oneself to "eigenvalues" of "multiplicity one". One word of explanation deserves the case $\Delta(0) = 0$, since this always happens. Call $\delta(z) = \Delta(z)/z$. The solution $u(x)$ such that $u(0) = -2\beta_1$, $u'(0) = 2\beta_0$, is an entire function of s^2 ([20],Th. 1.5) and $F_3(d/dx)u(1) = \delta(s)$. In consequence, if null is a zero of δ it is at least of order two and therefore at least of third order for Δ. So the *second hypothesis is equivalent* to: the even entire function $\delta(s)$ has simple nonnull zeroes and null is a zero of at most second order.

DEFINITION. *An eigenvalue is a zero of δ and an eigenfunction is a non trivial solution satisfying the boundary conditions.*

To avoid uncertainty in the determination of the general solution we introduce a new *definition:* $u_s(x) = u(x,s) = F_2(s)e^{sx} - F_1(s)e^{-sx}$ if $s \neq 0$, $u_0 = 2(-\beta_1 + \beta_0 x)$ if $s=0$. It follows inmediately that $u(x,0) = \lim u(x,s)/s$ for

$s \to 0$. Since $F_1(s) = F_2(s) = 0$ cannot be verified for $s \neq 0$, then $u \not\equiv 0$, and therefore u is, for every s, a non trivial solution of the differential equation and the first boundary condition.

2. THEOREM 1. *If s is an eigenvalue then -s and $\bar{s}$ also are. Outside of a circle great enough the eigenvalues of* (1) *are purely imaginary, of the form:*

$i\ t_m$ *where* $t_m = \pm(2m+1)\pi/2 + 0(1)$ *or* $t_m = \pm m\pi + 0(1)$,

and are simple roots of Δ *even if the second hypothesis does not hold.*

PROOF. The first part of the theorem follows from the power series expansion of Δ around the origin, which is real, and odd. The roots of this function must satisfy:

(4) $e^{2s} = P(s)/P(-s)$

where P is a polynomial of degree n or n+1. In consequence, the distance of one of them to the nearest tends to π for $s \to \infty$. If $s = it$, t real then $e^{it} = \pm P(it)/|P(it)|$. It follows that

(5) $\operatorname{tg} t = \operatorname{Im} P(it)/\operatorname{Re} P(it) = N(t)/D(t)$.

This equation determines the imaginary roots of Δ. If the degree of N is greater (lower) than that of D, (5) has infinite positive solutions which approach indefinitely to odd multiples of $\pi/2$ (multiples of π). This, with the previous observation about the distance of two roots imply the thesis. If s were a root of multiplicity greater than one it would satisfy (4) and an analogous relation obtained after replacing P(s) and P(-s) by polynomials Q and R.

Therefore, s would be a root of the polynomial:
$P(s)R(s)-P(-s)Q(s)$, which has only a finite number of roots

QED.

3. Λ will denote the set of eigenvalues $\neq 0$, i.e., it coincides with the spectrum when 0 is not a root of δ. A(s) will denote an auxiliary function with domain Λ :

$A(s) = e^s F_2(s)/F_4(s) = e^{-s} F_1(s)/F_3(s)$.

Since F_3 and F_4 have no common zero in Λ, A is well defined. Next we prove a basic technical result. A more complete statement is given as Th.2 in Ch.VI.

THEOREM 2. *If* s *and* t *belong to* Λ *and* $s^2 \neq t^2$, *we have:*

$$\int_0^1 (F_2(s)e^{sx}-F_1(s)e^{-sx})(F_2(t)e^{tx}-F_1(t)e^{-tx})dx =$$

$$= A(s)\ A(t).st.V(s^2,t^2)\ ,$$

$$V(s^2,t^2) = \sum c_{pq}\ s^{2p}\ t^{2q}\quad ,\qquad 0 \leqslant p,q \leqslant [n/2]-1\ ,$$

$$c_{pq} = -4 \sum{}' \{\alpha_{2k}\ \alpha_{2j+1} :\ k > j\ ,\ k+j-1 = p+q\} +$$

$$+4 \sum{}' \{\alpha_{2k}\ \alpha_{2j+1} :\ k < j\ ,\ k+j-1 = p+q\}.$$

The prime means that all the summands are not necesarily present.
Precisely, $p,q \in [j,k)$ *in the first sum and* $p,q \in [k,j)$ *in the second one.*

PROOF. If $R(s,t) = (d/dx)u_s(1).u_t(1)$, we have:

$$(6)\quad (s^2-t^2) \int_0^1 u_s u_t\ dx = u'_s u_t - u_s u'_t \Big|_0^1 = R(s,t)-R(t,s).$$

Using that $\Delta(s) = \Delta(t) = 0$, we get:

$$(7)\quad R(s,t) = s(F_2(s)e^{s} + F_1(s)e^{-s})(F_2(t)e^{t} - F_1(t)e^{-t}) =$$

$$= A(s)A(t)s.[(F_4(s) + F_3(s))(F_4(t)-F_3(t))]$$

Calling $S(s,t)$ to s times the bracket in (7) we obtain:

$$(8)\quad R(s,t) - R(t,s) = A(s)A(t)\{S(s,t)-S(t,s)\} =$$

$$= A(s)A(t)\{P(s)Q(t)-P(t)Q(s)\},$$

where $P(s) = s\{F_3(-s) + F_3(s)\}$, $Q(s) = F_3(-s) - F_3(s)$, are odd polynomials of degree less than or equal to $n+1$ and n, respectively. On the other hand we have:

$$(9)\quad P(s)Q(t)-P(t)Q(s)=-4\sum{}'\alpha_{2k}\alpha_{2j+1}(s^{2k+1}t^{2j+1}-s^{2j+1}t^{2k+1}),$$

where the prime means that the sum is over the set of j and k such that $2k \leqslant n$, $2j+1 \leqslant n$, $j \neq k$. The parenthesis in (9) is equal to :

$$(10)\qquad st(s^{2k}\,t^{2j} - s^{2j}\,t^{2k}).$$

If $k > j$, the parenthesis in (10) equals to:

$$(11)\quad (s^2-t^2)(s^{2k-2}t^{2j} + \dots + s^{2j}t^{2k-2});\ 2k \leqslant n;\ 2j \leqslant n-2$$

In fact, the parenthesis of (10) is divisible by s^2-t^2 and equals (11) for $2k \leqslant n$, $2j \leqslant n-1$. If n is even then $2j < n-1$, and if it is odd, it follows from $2j < 2k < n$. The right member in (9) restricted to $k > j$ equals to:

$$(12)\quad st(s^2-t^2).\sum_{p,q} s^{2p}t^{2q} \sum_{k+j-1=p+q} \varepsilon_{kj}\,\alpha_{2k}\,\alpha_{2j+1},$$

where $2p+2, 2q+2 \leqslant n$ and ε_{kj} takes the value 0 or -4.

If $j > k$, the opposite of the parenthesis in (9) equals to:

$$(13)\quad st(s^2-t^2)(s^{2j-2}t^{2k}+\ldots+s^{2k}t^{2j-2}),\ 2j \leqslant n-1, 2k \leqslant n-3$$

From (13) we obtain an expression like (12) but where $2p+2$ and $2q+2$ are less than n and where the ε's take the value 0 or 4.

QED.

REMARK. For future references we observe that:

$$V(s^2,t^2) = (s^2-t^2)^{-1}[(F_4(s)+F_3(s))(F_4(t)-F_3(t))/t - \\ - (F_4(t)+F_3(t))(F_4(s)-F_3(s))/s].$$

This is an immediate consequence of (8) and the definitions of P and Q.

4. Let us call $v_s(x) = v(x;s) = u(x;s)/\|u(.;s)\|_2$ the *normalization* in $L^2(0,1)$ of the function u_s defined in §1.

THEOREM 3. *If* s *and* t *are eigenvalues such that* $s^2 \neq t^2$ *then*

$$(v_s,v_t) = \int_0^1 v(x;s)\overline{v(x;t)}\,dx = \begin{cases} 0(1)/st, & \text{for } s,t \in \Lambda \\ 0(1)/s, & \textit{for } t=0, s \in \Lambda \end{cases}$$

PROOF. Assume that s and $t \in \Lambda$ and s is not a root of F_4. If s is an imaginary number, we have: $u_s = 2i\ \mathrm{Im}(F_2(s)e^{sx})$ and

$$(14)\quad \|u_s\|^2 = (u_s,u_s) = 4\int_0^1 \mathrm{Im}\ (F_2(s)e^{sx})dx = -\int_0^1 u_s^2\ dx\ .$$

The last integral is equal to:

$$(15)\quad 0(1)e^{2s}\ s^{-1}\ F_2^2(s) + 2\beta_1 s^2 + 2\beta_0(\beta_1-\beta_0)\ .$$

In fact, from $\Delta(s) = 0$, it follows that

(16) $\int_0^1 u_s^2 dx = e^{2s} F_2^2(s)(1-F_3^2/F_4^2)/2s+(F_1^2-F_2^2)/2s-2F_1F_2$

which is equal to (15). In consequence, from theorem (2) and (15),

$$\int_0^1 v_s v_t \, dx = A(s)A(t) \; st \; V(s^2,t^2)/\|u_s\|.\|u_t\| =$$

$$= O(1)(st)^{1-n} \; V(s^2,t^2) = O(1)/st \; .$$

Observing that the last integral coincides with $(v_s, v_{\bar{t}})$ and that $\bar{t}^2 \neq s^2$, the theorem follows.

When t=0, from (6) we obtain for any $s \in \Lambda$:

$$s^2(u_s,u_0) = (u_s' \, u_0 - u_s \, u_0')(1) =$$

$$= 2e^s.\{s(\beta_0-\beta_1)(F_2(s)+F_1(s)e^{-2s}) - \beta_0(F_2(s)-F_1(s)e^{-2s})\} =$$

$$= 2A(s) \; \{s(\beta_0-\beta_1)(F_4+F_3) - \beta_0(F_4-F_3)\} \; .$$

Therefore,

(17) $(v_s,v_0)=A(s).\{2(\beta_0-\beta_1)(F_4+F_3)-2\beta_0(F_4-F_3)/s\}/s\|u_0\|.\|u_s\|.$

(17) and (15) imply that $(v_s,v_0) = O(1)/s$.

QED.

Making use of *real inner* product $\langle .,. \rangle$ theorem 3 can be written as: $\langle v_s,v_t \rangle = O(1)/|st|$ whenever s and t are purely imaginary numbers and $|s| \neq |t|$.

IV. THE CASE n=2.

1. This is a simple and important case. Precisely, $F_1 \equiv 1-qs$, $F_3 \equiv ps+s^2$, $q \geqslant 0$, $p > 0$, describes the mechanical applications mentioned in I, (cf. [12], [13], [17] [19]). Besides, the case n=2 is useful to explore the general theory with examples and counterexamples.
We shall assume in all this chapter that

(1) $F_1(s) = \beta + s$, $F_3(s) = \alpha s - s^2$,α and β real,

and that the first and second hypothesis *do not* necessarily hold.
This chapter is not used in what follows and can be skipped.

2. THEOREM 1. i) *If* $\alpha\beta \leqslant 0$, Δ *has no zero outside the real and imaginary axes.*

ii) *If* $\beta=0$ *then* $\alpha\neq 1$ *iff the second hypothesis holds.*

iii) *There exists a number* $\varepsilon > 0$ *such that* $|s| < \varepsilon$ *implies the existence of* α *and* β *such that* $\Delta(s)=0$, *whenever* s *is of the form* $\gamma + i\delta$, $\gamma > 0$, $\delta > 0$. *There exists* s *of this form for which no set of polynomials* (1) *verifies* $\Delta(s)=0$.

iv) *In* iii) *the non ordered set* $\{\alpha,\beta\}$ *is uniquely determined and is composed of positive numbers.*

v) *If* $s > 0$, *it is possible to find positive, not identical numbers* α,β, *such that* s *is a root of order greater than one for* Δ.

vi) *If* $\alpha < 0$, $\beta < 0$, *there is no root of* Δ *outside the imaginary axis, and the second hypothesis holds.*

PROOF. i) It is enough to prove that if $s = \gamma + i\delta$, $\gamma, \delta > 0$, then $\Delta(s) \neq 0$.
If this were not so, we would have:

$$(2)\quad \operatorname{cotgh} s = (F_1F_4+F_2F_3)/(F_1F_4-F_2F_3) = (\alpha+\beta)s/(s^2+\alpha\beta) =$$

$$= (\alpha+\beta)\{\gamma(\gamma^2+\delta^2+\alpha\beta) - i\delta(\gamma^2+\delta^2-\alpha\beta)\}/|s^2+\alpha\beta|^2 =$$

$$= (\sinh 2\gamma - i \sin 2\delta)/2(\sin^2 \delta + \sinh^2\gamma) ,$$

since $s^2+\alpha\beta \neq 0$.

But then:

$$(3)\quad (\sinh 2\gamma)/2\gamma(\gamma^2+\delta^2+\alpha\beta) = (\sin 2\delta)/2\delta(\gamma^2+\delta^2-\alpha\beta).$$

$\alpha.\beta \leqslant 0$, implies that the modulus of the left hand side of (3) is larger than that of the right hand side, contradiction. Then $\Delta(s) \neq 0$.

iii) From (3) we obtain:

$$(4)\quad \alpha\beta = \frac{\{(\sinh 2\gamma)/2\gamma - (\sin 2\delta)/2\delta\}|s|^2}{\{(\sinh 2\gamma)/2\gamma + (\sin 2\delta)/2\delta\}} .$$

(4) yields:

$$(s^2+\alpha\beta)/s = (\sinh 2\gamma + i \sin 2\delta)/\{(\sinh 2\gamma)/2\gamma+(\sin 2\delta)/2\delta\}.$$

From the third and last member of equality (2) we get:

$$(5)\quad \alpha+\beta = \frac{\sinh^2 2\gamma + \sin^2 2\delta}{2(\sinh^2\gamma+\sin^2\delta)\{(\sinh 2\gamma)/2\gamma+(\sin 2\delta)/2\delta)\}} .$$

But:

$$(6) \quad \frac{\sinh^2 2\gamma + \sin^2 2\delta}{4(\sinh^2\gamma + \sin^2\delta)} = \sinh^2\gamma + \cos^2\delta .$$

Therefore:

$$(7) \quad \alpha+\beta = 2(\sinh^2\gamma + \cos^2\delta)/\{(\sinh 2\gamma)/2\gamma + (\sin 2\delta)/2\delta\}.$$

$D = (\alpha+\beta)^2 - 4\alpha\beta$ is the discriminant of the second order equation with α and β as roots. Then $D > 0$ iff

$$(8) \quad (\sinh^2\gamma + \cos^2\delta)^2 > [(\frac{\sinh 2\gamma}{2\gamma})^2 - (\frac{\sin 2\delta}{2\delta})^2](\gamma^2+\delta^2).$$

If $\gamma^2 + \delta^2 < \varepsilon^2$, (8) holds and therefore there exist α and β real satisfying the thesis. If $\delta = \pi(2k+1)/2$, k an integer, the inequality in (8) is verified in the opposite sense and therefore $D < 0$. iii) follows.

iv) The preceding proof also shows that $\alpha\beta = f(\gamma,\delta) > 0$, $\alpha+\beta = g(\gamma,\delta) > 0$. From this follows that α and β are positive whenever they are real and that together with (1), the polynomials $F_1^0 = \alpha+s$, $F_3^0 = \beta s - s^2$ have an equation of eigenvalues for which s is a root. These *two* sets of polynomials are the *only* ones with this property.

v) Let us call $M = \cosh s$, $N = \sinh s$. Then $M^2 - N^2 = 1$. From $\delta(s) = \delta'(s) = 0$ we obtain:

$$(9) \quad -(\alpha+\beta)sN + \alpha\beta M = -s^2 M \; ; \; -(\alpha+\beta)(sM+N) + \alpha\beta N = -s^2 N - 2sM.$$

If $J = s + MN$,

$$(10) \quad \alpha+\beta = 2sM^2/J \quad ; \quad \alpha\beta = (-s^3 + MNs^2)/J.$$

Then, $D = (s^2(s^2+M^2))/J^2 > 0$ if s is real and non null. On the other hand if s is real or purely imaginary the right hand sides in (10) are greater or equal to 0. Therefore whenever (10) has real solutions they are non-negative and v) follows.

ii) From i) it follows that there is no zero outside the real and imaginary axes. If $s\neq 0$ and $\delta(s) = \delta'(s) = 0$, the second equation in (10) is equivalent to $2s = \sinh 2s$. But this equation has no real or purely imaginary root different of null. Since $\Delta/s^2 \equiv e^{-s}(s+\alpha) + e^{s}(s-\alpha)$, the second hypothesis does not hold iff $s-\alpha \operatorname{tgh} s = 0$ has 0 as a root of order greater than one, i.e., iff $\alpha=1$.

vi) From Th.2, Ch.III, we obtain for $\bar{s} = t$ neither real nor imaginary:

$$(11) \quad \int_0^1 |u(x;s)|^2 dx = |F_2(s)|^2 . e^{2\mathrm{Re}\ s} . |s|^2 . 4\alpha / |F_4(s)|^2 ,$$

since $V = 4\alpha$. But this is impossible since the rigth hand member is negative. $\delta(0) \neq 0$ follows immediately and if s is a positive number, $\delta(0) = 0$ would imply:

$$s^2 \cosh^2 s - (\alpha+\beta)s \sinh s + \alpha\beta \cosh s = 0 ,$$

which is impossible.

If $s=ik$, $k > 0$, were a zero of order greater than one of δ, then from (10) we would have:

$$\alpha+\beta = (2k \cos^2 k)/(k + \sin k.\cos k) .$$

The denominator is positive since $kx + 2^{-1} \sin 2kx > 0$ and a contradiction follows. This proves vi). QED.

3. We have restricted ourselves to a set of polynomials depending on two parameters α,β. In some situations (cf.

for example [11]) where more parameters are involved, it is also possible to prove that the second hypothesis holds.

COROLLARY. *If* $\alpha,\beta < 0$,

$$\int_{-\infty}^{\infty} x^{-2}\ \lg|\delta(ix)/2\alpha\beta|\,dx = -\pi\ .$$

PROOF. $\delta(iz)$ is an even entire function of order one with real zeroes and satisfying $\delta(0) = 2\alpha\beta$. The number of zeroes in $(0,r)$ is asymptotically equal to r/π. Then, it is possible to apply Th.XXIII of [15].

QED.

V. L^2-EXPANSIONS.

1. In this chapter we show that each function f of $L^2(0,1)$ admits an expansion in eigenfunctions. We call S the part of the spectrum contained in the upper half-plane: $S = \{s;\ \delta(s) = 0,\ 0 \leqslant \operatorname{Arg} s < \pi\}$. One of the reasons for this choice is that $\{v_s(x);\ s \in S\}$ is a *linearly independent* set. The function J defined by

$$(1)\quad J(f;z;x) = \\ = \int_0^1 u(t;z)[\, e^{z(x-1)}F_4(z) - F_3(z)e^{-z(x-1)}]\, f(t)\Delta^{-1}(z)dt$$

has a residue at $s \in S \cap \Lambda$:

$$(2)\quad [\, e^{s(x-1)}F_4(s) - F_3(s)e^{-s(x-1)}]\,(\Delta'(s))^{-1}\langle f,u_s\rangle = \\ = (A(s)\Delta'(s))^{-1}\langle f,u_s\rangle . u_s(x)\ .$$

If 0 is a zero of order one for Δ, there is no residue of J at z=0. But, if 0 is of order three, then

$$\text{Res J at } z = 0 = \int_0^1 f(t)(u/z)([\ldots]/z)(\Delta/z^3)^{-1}\ dt\ \Big|_{z=0} = \\ = 4(\Delta/z^3)^{-1}(0)\int_0^1 f(t)(\beta_0 t-\beta_1)(\alpha_0 x-(\alpha_0+\alpha_1))dt.$$

From $\Delta(0) = \Delta'(0) = 0$ we obtain $\alpha_0/\beta_0 = (\alpha_0+\alpha_1)/\beta_1$ and therefore the residue at z=0 is equal to

$$(3)\qquad H_0\ u_0(x)\ \langle f,u_0\rangle$$

where H_0 is a real constant.

u_s is defined in III,1, in such a way that it satisfies the first boundary condition. If $s \in \Lambda$, then it also satisfies the second one. If $s=0$, $u_0(x)$ satisfies the second boundary condition iff $\alpha_0\beta_1 = \beta_0(\alpha_0+\alpha_1)$, and therefore, the u_0 that appears in (3) *is* an eigenfunction. (Observe that from the first hypothesis follows: $\alpha_0=0 \Rightarrow \alpha_1\neq 0$).

We have proved the following *proposition:* for every $s \in S$, the residue of J at s is equal to

$$(4) \qquad H(s)\, \langle f,u_s\rangle .u_s(x)$$

where $H(0) = H_0$, $H(s) = (A(s)\Delta'(s))^{-1}$ if $s\neq 0$.

The function H is defined on all the spectrum and verifies $H(\bar{\mu}) = \overline{H(\mu)}$, i.e. it behaves like the eigenfunctions: $u_{\bar{\mu}} = \bar{u}_\mu$. If $\mu \in \Lambda$, we have $u_{-\mu} = -u_\mu$. But $A(\mu) = A(-\mu)$, and therefore, because of the evenness of Δ' we obtain $H(\mu) = H(-\mu)$. If besides μ is purely imaginary: $\bar{u}_\mu = -u_\mu$, $H(\mu) = \overline{H(\mu)}$. In consequence, for f real and $s \in S \cap \Lambda$:

$$(5) \qquad H(-\bar{s})\; u(x;-\bar{s})\, \langle f,u_{-\bar{s}}\rangle = \overline{H(s)\; u(x;s)\, \langle f,u_s\rangle}.$$

2. The main objective of this chapter is to prove the following analogues of Plancherel's theorem and Dirichlet and Jordan's theorem.

THEOREM 1. i) *Every* $f \in L^2(0,1)$ *admits an* L^2*-expansion in eigenfunctions:*

$$(6) \qquad f = \sum_{s\in S} (nb)_s\, v_s \quad , \qquad b_s = b(f)_s = (f,v_s) \; ,$$

and where n *is an hermitian matrix such that* $n_{ss} = -H(s)\|u_s\|^2$

if s *is real* $\neq 0$, $n_{00} = -H(0)\|u_0\|^2/2$, $n_{ss} = +H(s)\|u_s\|^2$ *if* s *is imaginary*, $n_{s,-\bar{s}} = +H(s)\|u_s\|^2$ *if* s *is neither real nor imaginary*, $n_{st} = 0$ *elsewhere.*

ii) $n_{ss} \longrightarrow 1$ *for* $|s| \to \infty$.

iii) *If* f *is of bounded variation and* $x \in (0,1)$, *then*

$$\sum_{s\in S} (nb)_s \, v_s = (f(x+0) + f(x-0))/2 \; .$$

Let us call A the matrix defined by $\bar{A}_{st} = (v_s, v_t) = A_{ts}$, $s,t \in S$. Then A is the Gramian of the set of eigenfunctions v_s with index in S. b=b(f) will designate the vector (column) associated to $f \in L^2$ with components:

$$b_s = \langle f, v_s \rangle = (f, v_s) \; .$$

That is, the components of b are the *Fourier products*. If in L^2, $f = \sum c_s v_s$, then $\{c_s\}$, $s \in S$, will be called a set of *Fourier coefficients*. c will designate the vector with components c_s. When for each s, $c_s = (nb)_s$, $\{c_s\}$ will be called the set of *Orr coefficients*. In this case, $c_s \sim b_s$. Next we prove a very useful result, whose part i) resembles Riesz-Fischer theorem.

THEOREM 2. *Let* f *be a complex valued function in* L^2 *and* $s \in S$.

i) $f = \sum c_s v_s$ *in* L^2 *iff* $\sum |c_s|^2 < \infty$.

ii) $f = \sum c_s v_s$ *in* L^2 *implies* $A.c = b(f) \in l_2$.

iii) $f \in L^2$ *implies* $b(f) \in l_2$.

iv) A: $l_2 \longrightarrow l_2$.

v) *Let be* $N > M \longrightarrow \infty$. *Then*

$$(7) \quad \|\textstyle\sum_M^N c_s v_s\|_2^2 = (\sum_M^N |c_s|^2)(1 + o(1))$$

where $o(1) = 0(1)/M$. *If* $\sum_M^\infty |c_s|^2 < \infty$, *then* (7) *holds with* $N=\infty$. *The summation in* (7) *means* $M < |s| < N$.

PROOF OF THEOREM 2. From III, Th.3, we obtain:

$$(8) \quad \begin{cases} \|u_s\|_2^2 \sim -2\beta_1^2 s^2 = 2\beta_1^2|s|^2 & \text{if } \beta_1 \neq 0 , \\ \|u_s\|_2^2 \sim 2\beta_0^2 & \text{if } \beta_1 = 0 , \end{cases}$$

and also that

$$(9) \quad A_{st} = 0(1)/|st| , \quad \text{for } s\neq t , \ s,t \in S \cap \Lambda.$$

i) follows from v), and (7) is a consequence of

$$\int_0^1 |\textstyle\sum c_s v_s|^2 dt = \sum |c_s|^2 + \sum_{s\neq t} c_s \bar{c}_t A_{ts} ,$$

$$|\sum_{s\neq t} (c_s/s)(\bar{c}_t/t)| \leqslant (\textstyle\sum |c_s/s|)^2 \leqslant (\sum|c_s|^2)(\sum|s|^{-2})0(1) =$$

$$= 0(1)\|c\|_2^2 M^{-1} ,$$

where all the summations are between M and N.

ii) follows from

$$b_k = (f,v_k) = \textstyle\sum c_s(v_s,v_k) = \sum c_s A_{ks} =$$

$$= c_k + 0(1) \sum_{0\neq s\neq k} |c_s|/|sk| + 0(1)|c_0|k^{-1} = c_k + 0(1)\|c\|/k.$$

iv) is consequence of i) and ii). Let us see iii). Observe that if $s=iK$, $K > 0$,

(10) $u_s = 2i[\beta_0 \sin Kx - \beta_1 K \cos Kx]$.

Then,

$b_s(f) = 2i\beta_0(f, \sin Kx)/\|u_s\| - 2i\beta_1 (f, \cos Kx)(K/\|u_s\|)$.

If $\beta_1=0$ or not, we obtain

(11) $b_s(f) = 0(1)(f,t(Kx))$, $t(z)=\sin z$ or $\cos z$.

Moreover, we already know that K is asymptotically like a multiple of π or an odd multiple of $\pi/2$. Precisely, its difference with such a multiple is $d=d(K)$ and

(12) $d = 0(1)/K$

In consequence, for $r=m$ or $=(2m+1)/2$

(13) $t(Kx) = t(r\pi x) + 0(1)/K$, $K \sim \pi r$,

(14) $b_s(f) = 0(1)(f,t(r\pi x)) + 0(1)\|f\|_2/K$

But $\{t(r\pi x)\}$ is, except for a factor, an orthonormal system on (0,2) which implies that $(f,t(r\pi x)) \in l_2$. Then, $b \in l_2$.

QED.

3. PROOF OF THEOREM 1. Let C be a circumference of radius h not passing through eigenvalues. From II, Th.1, we know that if $f \in C^1(0,1)$ and is null in neighborhoods of 0 and 1, it holds uniformly:

(15) $(-2\pi i)^{-1}\int_C J(f,z,x)dz = -\sum_{|s|<h} \langle f,u_s\rangle H(s)u_s(x) \xrightarrow[h\to\infty]{} 2f$.

Here the summation is over all the eigenvalues. Recalling that $H(-a) = H(a)$ and that $a \in S \cap \Lambda$ implies $-a$ is eigenvalue not belonging to S, we obtain:

$$(16)\quad \sum_{0\neq|s|<h} \langle f,u_s\rangle H(s)u_s = 2 \sum_{\substack{0\neq|s|<h \\ s\in S}} \langle f,u_s\rangle H(s)u_s \ .$$

Then, for a set dense in L , it holds

$$(17)\quad f(x) = \lim_{h\to\infty} \sum_{\substack{|s|<h \\ s\in S}} (nb)_s\, v_s \quad , \quad b_s=(f,v_s).$$

Using that $\Delta(s)=0$ when $s \in S$, we obtain

$$(18)\quad \Delta'(s) = 2e^sF_2(s)F_3(s)+e^s(F_2F_3)'-e^{-s}(F_1F_4)' \sim 2e^sF_2F_3.$$

$$(19)\quad H(s) = e^sF_3(s)/\Delta'(s)F_1(s) \sim (2F_1F_2)^{-1}.$$

From (8), it follows that

$$(20)\quad \|u_s\|_2^2 = - \int_0^1 u_s^2 \ dx \sim 2F_1(s)F_2(s) \ .$$

Therefore, $n_{ss} \longrightarrow 1$ for $s \longrightarrow \infty$. This proves ii) and to complete the proof of i) assume that a sequence $\{f_m\}$ verifying (17) converges to f in L^2. (14) implies that $\|b(f)\|_2 = 0(1)\|f\|_2$ and therefore, $b(f_m) \longrightarrow b(f)$ in l_2. In consequence:

$$c^{(m)} = nb(f_m) \longrightarrow c = nb(f) \quad \text{in} \quad l_2.$$

Since:

$$\Big\| \sum_{n=N}^{\infty} (c^{(n)}-c)_s v_s\Big\|_2^2 = [\sum_N^{\infty} |(c^{(n)}-c)_s|^2][1 + 0(1)/N] \ ,$$

we finally obtain: $\|\sum_s (c_s^{(n)}-c_s)v_s\|_2 = 0(1)\|c^{(n)}-c\|_2$. This implies that $f_m = \sum c_s^{(m)}v_s \longrightarrow \sum c_s v_s$ in L^2. Then, the last series defines a.e. a function that must coincide with f, proving so that every $f \in L^2$ admits an expansion with Orr coefficients. iii) follows from II, Th.1, i).

QED.

REMARK. From Theorem 2 it is clear that domain of A = l_2 = the set of *all Fourier coefficients* and range of A = the set of *all Fourier products*.

VI. THE GRAMIAN.

1. This chapter is devoted to the study of the Gramian A of the system defined by (1) of ch.III.
Recall that $n > 1$ and so the ordinary Sturm-Liouville case is excluded.

THEOREM 1. i) $A=I+T$, *where* I *is the unit matrix and* T *is a matrix of finite Hilbert-Schmidt norm.*

ii) A *is not a diagonal matrix: except eventually for a finite number of* s, $A_{st}\neq 0$ *for all* t *with* $|t| > M(s)$.

iii) *No* $f \in L^2$ *has a unique expansion in eigenfunctions in* $L^2(0,1)$.

PROOF. i) $T_{ij} = 0$ if $i=j$, $T_{ij}=A_{ij}$ if $i\neq j$. From III, Th.3, it follows that $\sum_{i,j} |T_{ij}|^2 < \infty$.

ii) To prove ii) it is enough to show that the polynomial $S(s,t)-S(t,s)$ in Th.2, ch.III, is not identically zero. If it were so, $S(s,t) \equiv S(t,s)$ which is equivalent to $s(F_4(s)+F_3(s)) \equiv C(F_3(s)-F_4(s))$ where C is a constant.Then, $(C+s)F_4 \equiv (C-s)F_3$. In consequence $F_3(-C) = 0$ and $F_3(s) = (C+s)\,\emptyset(s)$, so $F_4(s) = (C-s)\,\emptyset(s)$. From this and $F_3(s) = F_4(-s)$ it follows that $\emptyset(s) = \emptyset(-s)$. Since $n \geqslant 2$, F_3 and F_4 have an even common factor of positive degree, contradiction.

iii) Assume that $f = \sum c_j v_j$ and write the Gramian as $A = T-(-1)I$. Then, f has a unique expansion iff for $c \in l_2$,

$Ac=0$ is equivalent to $c=0$. Since T defines a completely continuous operator, Fredholm theorems imply that uniqueness is equivalent to $A\ l_2 = l_2$. But the range of A is $\{b(f);\ f \in L^2\}$. On the other hand from the expansion theorem we know that $\sum A_{ij}(nb)_j(f) = b_i(f)\ \forall\, i$. If range of A were l_2, we could choose $b_i(f) = \delta_{ik}$, which implies $A_{ik}n_{kk} = \delta_{ik}$ if $|k|$ is great enough. Taking k such that $n_{kk}\neq 0$ (Th.1, ch.V) and $i\neq k$ such that $A_{ik}\neq 0$, a contradiction is obtained.

QED.

2. The first part of the preceding theorem admits a more precise statement. This is next theorem 3, but to prove it we shall need an extension of Th.2, III. If 0 is an eigenvalue we *define:* $A(0) = \beta_0/\alpha_0 = \beta_1/(\alpha_0+\alpha_1)$, (cf.§1, V). Therefore, $A(.)$ is defined on all the spectrum. $u_s/s \longrightarrow u_0$ and therefore $(|s|/s)v_s \longrightarrow v_0$ for $s \longrightarrow 0$ in the upper half-plane. This makes plausible a generalization of Th.2, III, which is the content of next theorem. We define now the *auxiliary function* $\Omega(z)$, equal to 0 if $z=0$, $=1$ elsewhere.

THEOREM 2. *If* $t \in S$, $s \in S-(0)$, *and* $-s\neq\bar{t}$, *then*

$$(v_t,v_s) = [A(\bar{s})A(t)/\|u_s\|\|u_t\|].\sum c_{pq}\bar{s}^{2p+1}\ t^{2q+\Omega(t)}$$

PROOF. If $t\neq 0$, this is nothing but Th.2, III, because of $c_{pq}=c_{qp}$. Assume $t=0 \in S$. Then, formula (17) of III can be written as:

$$(v_s,v_o) = [A(s)/\|u_s\|s][A(0)/\|u_0\|]\ .$$

$$.\{-2\alpha_0(F_4(s)-F_3(s))/s - 2\alpha_1(F_4(s)+F_3(s))\} .$$

Since $2\alpha_0 = (F_4+F_3)(0)$, $-2\alpha_1 = ((F_4-F_3)/t)(0)$, we obtain:

$$(v_s,v_0) = [\,.\,][\,.\,]\; s^2\, V(s^2,0).$$

In fact, the last formula is a consequence of the remark to Th.2, III. From it and $(v_0,v_s) = (v_{\bar{s}},v_0)$ the theorem follows.

QED.

3. Let us *denote* with k_p, $0 \leqslant p \leqslant [n/2] - 1$, the elements of l_2, which are defined by (3) in the proof of next theorem. k_p^t will designate the transpose of k_p, i.e., the row vector with the same components as the column vector. So, if $\bar{k}_p$ is that element of l_2 whose components are the complex conjugates to those of k_p, $Y_{pq} = \bar{k}_p.k_q^t$ will denote the infinite matrix, a 1-term dyad, such that $(\bar{k}_p.k_q^t)_{ij} = (\bar{k}_p)_i\,(k_q)_j$.

If c_{pq} are the real coefficients introduced in Th.2, ch.III, we define:

$$w_p = \sum_q c_{pq}\, k_q \quad , \quad 0 \leqslant p,q \leqslant [n/2] - 1.$$

A finite range operator will be associated with these vectors:

$$L\,h = \sum_p \langle w_p , h \rangle\, \bar{k}_p$$

where $\langle\, ,\, \rangle$ denotes the usual "real" inner product in l_2. The matrix form of this operator is

$$\sum c_{pq}\, Y_{pq} \;.$$

THEOREM 3. i) $A=D+L$ *where* L *is a real linear combination of 1-term dyads* $L = \sum_{p,q} c_{pq}\, \bar{k}_p . k_q^t = \sum c_{pq}\, Y_{pq}$,

$0 \leqslant p,q \leqslant [n/2] - 1$ *and the* c*'s are the same as given in Th.2, III.*

D *is a matrix such that* $D_{ii} = 1 - L_{ii}$ *real whenever* i *is real or imaginary,* $D_{s,-\bar{s}} = -\langle v_{-\bar{s}}, v_{-\bar{s}} \rangle - L_{s,-\bar{s}}$ *if* s *is neither real nor imaginary,* $D_{ij} = 0$ *elsewhere.*

ii) *The set* $\{k_p\} \subset l_2$ *is linearly independent.*

iii) $Dn = I$.

iv) $\forall f \in L^2$, $\forall p$, $nb \perp \bar{w}_p$.

If we assume for example that there are only two eigenvalues $s, -\bar{s}$ *in* S *outside the real and imaginary axes, the Gramian* A *looks like:*

	s	$-\bar{s}$	i	j	.	.	.	.
s	L_{ss}	$-\langle v_{\bar{s}}, v_{\bar{s}} \rangle$	$L_{s,i}$	$L_{s,j}$				
$-\bar{s}$	$-\langle v_s, v_s \rangle$	$L_{-\bar{s},-\bar{s}}$	$L_{-\bar{s},i}$	$L_{-\bar{s},j}$				$L_{v,u}$
i	$L_{i,s}$	$L_{i,-\bar{s}}$	1	$L_{i,j}$				
j	$L_{j,s}$	$L_{j,-\bar{s}}$	$L_{j,i}$	1				
.								
.						1		
.								
		$L_{u,v}$						1

and the corresponding matrix D *is equal to:*

	s	$-\bar{s}$	i	j	...
s	0	$-\langle v_{-\bar{s}}, v_{-\bar{s}} \rangle - L_{s,-\bar{s}}$	0	0	...
$-\bar{s}$	$-\langle v_s, v_s \rangle - L_{-\bar{s},s}$	0	0	0	...
i	0	0	$1-L_{ii}$	0	...
j	0	0	0	$1-L_{jj}$	..
⋮	⋮	⋮	⋮	⋮	

PROOF OF THEOREM 3. It is convenient to "normalize" the function A(.). We *define*

(1) $g(s) = A(s)/\|u_s\|$, $s \in S$.

From Th.2, we obtain:

(2) $A_{st} = (v_t, v_s) = g(\bar{s})g(t)\sum c_{pq}\bar{s}^{2p+1}\, t^{2q+\Omega(t)}$, $-\bar{s} \neq t$,

for $s \in S-(0)$, $t \in S$, and where Ω is the function defined in §2.
Let us call:

(3) $k_p(s) = (k_p)(s) = g(s)s^{2p+\Omega(s)}$, $s \in S$, $0 \leqslant p \leqslant [n/2]-1$.

Then,

(4) $A_{st} = \sum c_{pq}(Y_{pq})_{st} = L_{st}$ if $s \neq 0$, $s \neq -\bar{t}$, and $c_{pq} = c_{qp}$, real.

This implies that A=D+L, D as described, since $(v_s, v_{-\bar{s}}) = -\langle v_s, v_s \rangle$. From the definition of A(.) (cf.III, §3 and VI, §2), we see that $g(\bar{s}) = \bar{g}(s)$. Therefore, L_{ss} is real $\forall$ s on the coordinate axes and i) follows.

Because of (20), V, we have

$|g(s)|^2 = |A(s)|^2/\|u_s\|^2 \sim |F_2(s)|^2/|2F_1(s)F_2(s)||F_4(s)|^2$.

Therefore, $|g|$ is asymptotically like a constant times $|s|^{-n}$. Since $2p+1 \leqslant n-1$, $k_p(s) = 0(1)\ s^{-1}$ and belongs to l_2. From the hypothesis on the polynomials F_i, we see that $F_2(s) = 0$, $s \in S-(0)$, implies $F_4(s) = 0$ and $F_3(s) \neq 0$, $F_1(s) \neq 0$. *Therefore*, $A(s) \neq 0$. That $A(0) \neq 0$ if 0 is an eigenvalue, follows directly from the definition given in §2. *In consequence*, $g(s) \neq 0\ \ \forall s \in S$.

If for a set of a's: $\sum a_r k_r = 0$, we would have $g(s)(\sum a_r s^{2r+1}) = 0$ for infinitely many s. Since g is different from 0, the polynomial inside the parenthesis would have an infinite number of roots, contradiction.

This proves ii). To finish the proof only remains to determine the matrix Dn. Observe that $Al_2 = \{b;\ Ac=b,\ c \in l_2\} = l_2 \ominus (T^*-(-1)I)^{-1}(0)$.

Then, third Fredholm theorem implies that the range of A is of finite codimension. If c=c(f) denotes the Orr coefficients of $f \in L^2$ we have from i)

(5) $\quad (D + \sum c_{pq} Y_{pq})\ c(f) = b$.

Since c=nb, from (5) we get

(6) $(I-Dn)b = \sum \langle w_p, nb \rangle \overline{k}_p = L\,nb = Lc$.

The last formula shows that (I-Dn)b runs in a space of finite dimension while b varies in one of finite codimension. Therefore, if h runs in l_2, (I-Dn)h varies in a subspace of l_2 of finite dimension, since Dn is a diagonal matrix and $(Dn)_{ss} = D_{ss}\, n_{ss} = 0(1)$ for $|s|$ great enough.
In consequence, the matrix I-Dn must have all its elements equal to zero with the exception of a finite number of them:

(7) $D_{jj} . n_{jj} = 1$ for $|j| > K$.

From (6) and (7) we get:

$$[\sum \langle w_p, nb \rangle \overline{s}^{2p+1}]\ \overline{g}(s) = 0 \quad \text{for} \quad |s| > K.$$

Therefore, the expression inside the brackets must be zero for infinite values of s and then, $\overline{w}_p \perp nb$, $\forall p\ \forall b(f)$ proving iv). Using (6) again we get:

$(I-Dn)b = 0\ \forall b(f)$ and iii) follows.

QED.

REMARK 1. iv) says that Orr coefficients verify Lc=0. Conversely, all the elements in the null space of L are Orr coefficients. In fact, assume that Lc=0. Then, for the function f such that $f = \sum c_j v_j$ we have, Ac = Dc = b. Since $D = n^{-1}$, it follows that $c = D^{-1}b = nb$. Then, COROLLARY. c is an Orr coefficient iff Lc=0.

REMARK 2. The preceding Th.3, together with Th.2 of next chapter, show that the study of the differential systems under consideration can be reduced to the study of certain finite-range operators on l_2. For the ordinary Sturm-Liouville theory: A=I and the finite-range operator is 0.

VII. DEGREES OF FREEDOM OF THE SYSTEM.

1. We already know (Th.1,VI) that the eigenfunction expansions we are studying are not unique. There is a certain freedom of choice of coefficients in an L^2-expansion. In this chapter we try to clarify this notion.

In §3, VI, we introduced the matrix $L = \sum c_{pq} Y_{pq}$, that we can write

(1) $$L = \sum \overline{k}_p . w_p^t \quad , \quad 0 \leqslant p \leqslant [n/2]-1 \ .$$

Let us *call* $\beta_p(c) = \langle w_p, c \rangle$. Then for $c \in l_2$.

(2) $$Lc = \sum \beta_p(c)\ \overline{k}_p \ .$$

From Th.3, VI, we get:

(3) $$c = D^{-1}b - \sum_p (D^{-1}\overline{k}_p)\ \beta_p(c) \ ,$$

where b is as always the Fourier product vector b(f) associated to the L^2-function f with Fourier coefficient vector c. In consequence,

(4) $$\beta_r(c) = \beta_r(D^{-1}b) - \sum_p \beta_r(D^{-1}\ \overline{k}_p)\ \beta_p(c) \ .$$

If K is the $[n/2] \times [n/2]$-matrix such that

(5) $$K_{ij} = \beta_i(D^{-1}\ \overline{k}_j) = \langle w_i, D^{-1}\ \overline{k}_j \rangle \quad ,$$

from (4) we obtain:

(6) $$(I+K)\ \beta(c) = \beta(D^{-1}b) \ ,$$

where $\beta(c)$ is the $[n/2]$-dimensional column vector $\{\beta_p(c)\}$.

THEOREM 1. $\beta(D^{-1}b) = 0$ *and* $\forall c \in l_2$; $(I+K)\beta(c) = 0$.

PROOF. (3) can be written:

(7) $$Dc + \sum_p \beta_p(c)\, \overline{k}_p = b \ .$$

If $c = nb$, from iv), Th.3, ch.VI, we see that c is orthogonal to the w's. That is, $\beta_p(nb) = \beta_p(D^{-1}b) = 0$. The second part of the theorem follows from (6).

QED.

2. DEFINITION. *The set of eigenfunctions* $\{v_s\}$ *(or the differential system) is said to have* g *degrees of freedom if the set of* c*'s such that* $\sum c_j v_j = 0$ *in* L^2 *is a subspace* σ *of* l_2 *of dimension* g. *Clearly, the set of* c*'s such that* $\sum c_j v_j = f$ *for a fixed* $f \in L^2$ *is the traslation of a subspace of* l_2 *with dimension* g.

THEOREM 2. i) σ *is the null space of the operator defined by matrix* A, *i.e., the eigenspace of the completely continuous operator* T *(defined in* VI*) corresponding to the eigen value* -1.

ii) g = *dimension of* σ = *dimension of the subspace of solutions* β *of the equation:* $(I+K)\beta = 0$.

iii) $1 \leqslant g \leqslant [n/2]$. *In particular, if* $n=2$ *or* $n=3$, $g=1$.

iv) $\det(I+K) = 0$.

PROOF. Since $f=0$ is equivalent to $b(f) = 0$, we see that $c \in \sigma$ iff $Ac = 0$. The mapping $\sigma \ni c \longrightarrow \beta(c)$ is an injection. In fact, $\beta(c) = 0$ implies $Lc=0$. From $Ac = Dc+Lc =$ $= Dc$ and $Ac = 0$, it follows $c=0$. Then, $\dim \sigma \leqslant$ $\leqslant \dim(I+K)^{-1}(0)$ is a consequence of the preceding theorem. Assume that β is in the null space of $(I+K)$. Defining $c = -\sum \beta_p D^{-1}\overline{k}_p$ we see that $\langle w_q, c \rangle = -(K\beta)_q = \beta_q$. Therefore, $c = -D^{-1}Lc$ and $c \in \sigma$. This proves ii). iii) follows immediately from $g \neq 0$ (iii), Th.1, VI) and that β is an $[n/2]$-dimensional vector. $g \geqslant 1$ implies iv).

QED.

3. We shall write $v_s^{(h)}$ instead of $(d/dx)^h\, v_s(x)$ and prove that

(8) $$k_p(s) = \sum_h a_{ph}\, v_s^{(h)}(1)$$

Analogously,

(9) $$w_p(s) = \sum_h b_{ph}\, v_s^{(h)}(1)$$

Associated to the sets of a's and b's there is a bilinear form B:

(10) $$B(f,g) = (f,g) - \sum_{h,l} [\sum_p b_{ph}\overline{a}_{pl}]\,(f^{(h)}\,\overline{g}^{(l)})(1) \;,$$

defined at least on C^∞-functions $f(x), g(x)$.

THEOREM 3. *There exist a set* $\{a_{ph}\}$ *and a set* $\{b_{ph}\}$, $0 \leqslant h < n$, *of real numbers such that* (8),(9) *and*

(11) $$B(v_s, v_t) = 0 \quad , \quad \forall s \forall t \in S \quad s \neq -\overline{t},$$ *hold.*

PROOF. We know that (cf.(3), VI):

(12) $$k_p(s) = \frac{A(s)}{\|u_s\|} \cdot s^{2p+\Omega(s)}$$

On the other hand, at least for $s\neq 0$,

(13) $$\sum_h a_{ph}\, v_s^{(h)}(1) = \frac{1}{\|u_s\|}\,[\, I_p(s).F_2(s).e^{s} - I_p(-s).F_1(s).e^{-s}]$$

where $I_p(s) = \sum_h a_{ph}\, s^h$.

By the definition of A(s) (III, §3), we have for $s \in S$

$$F_2(s).e^{s} = F_4(s).A(s) \quad , \quad F_1(s).e^{-s} = F_3(s).A(s).$$

Substituting in the righ member of (13), we get:

$$\sum_h a_{ph}\, v_s^{(h)}(1) = \frac{A(s)}{\|u_s\|} \cdot [\, I_p(s).F_4(s) - I_p(-s).F_3(s)] .$$

So, for $s \in S-(0)$

(14) $$k_p(s) = \sum_h a_{ph}\, v_s^{(h)}(1)$$

if the polynomial $I_p(s)$ satisfies:

(15) $$I_p(s).F_4(s) - I_p(-s).F_3(s) = s^{2p+1} .$$

Since g.c.d.(F_3,F_4) = 1 or s, there exist real polynomials C(s), D(s) such that

(16) $$C(s).F_4(s) + D(s).F_3(s) = s^{2p+1}$$

Moreover, C and D can be chosen to have degree less than n. In fact, if $C = C_1+F_3.Q$ and $D = D_1+F_4.R$, with deg C_1, deg $D_1 < n$, then from (16) we get

$$C_1.F_4 + D_1.F_3 - s^{2p+1} = -F_3.F_4.(Q+R)$$

Since the left member has degree < 2n, (Q+R) must be identically 0 and C_1, D_1 also verify (16).

Subtracting from (16) the expression obtained from it by changing s into -s, we get

$$F_4.(C(s)-D(-s))/2 - F_3.(C(-s)-D(s))/2 = s^{2p+1} .$$

Thus, the polynomial $I_p(s) = (C(s)-D(-s))/2$ verifies (15), having degree less than n. Therefore, (14) holds.

If 0 is an eigenvalue, we have

$$I_p(d/dx)\ v_0(1) = 2[\beta_0.a_{p0} - \beta_1.a_{p0} + \beta_0.a_{p1}]/\|u_0\| =$$

$$= A(0).2[\alpha_0 a_{p1} - \alpha_1 a_{p0}]/\|u_0\| =$$

$$= \frac{A(0)}{\|u_0\|}\left[\frac{F_4(s).I_p(s)-F_3(s).I_p(-s)}{s}\right]_{s=0}$$

From (15) and (12) we then get

(17) $\qquad I_p(d/dx)\ v_0(1) = k_p(0)$

That is, (14) holds for any $s \in S$.

From the definition of w_p and (14) we also get for $s \in S$:

(18) $\quad w_p(s) = G_p(d/dx)v_s(1)$, where $G_p(s) = \sum_q c_{pq} I_q(s) = \sum b_{ph} s^h$.

The bilinear form B, when $f=v_s$, $g=v_t$ is equal to

(19) $\qquad A_{ts} - \sum_p G_p(d/dx)v_s(1).\overline{I_p(d/dx)v_t(1)} =$

$$= A_{ts} - \sum_p w_p(s)\bar{k}_p(t) = A_{ts} - L_{ts} = D_{ts}$$

Then, (19) = 0 whenever $s \neq -\bar{t}$.

QED.

The meaning of (19) is that the form $\{B(v_s,v_t)\}$ *coincides* with the matrix D. We have obtained also a *representation of the matrix* L in terms of the values of at most n-1 derivatives of the eigenfunctions taken at the extreme x=1.

VIII. AN APPLICATION.

1. In this chapter we shall study a particular case of the preceding theory, precisely, that described at the very beginning of this work. To avoid the unnecessary repetition of the imaginary unity we write the equation as:

(1) $u'' + k^2u \equiv u'' - (ik)^2u = 0 \ , \quad 0 \leqslant x \leqslant 1$

The boundary conditions are now:

(2) $u(0) = 0 \ , \quad a.u''(1) + u'(1) = 0 \ , \quad a \neq 0 \ .$

Then, $F_1(s) \equiv 1$, $F_3 \equiv as^2 + s$. We shall *assume that* $a > 0$. In this case any eigenvalue is purely imaginary and therefore k is always real different from zero. (If $a < 0$, there are two real eigenvalues). In this chapter we call *eigenvalue* any value of k such that ik is eigenvalue in the sense of Ch.III. Now, k is an eigenvalue iff

(3) $\operatorname{tg} k = 1/ak \ , \quad k - \dot{\pi} = o(1) \ .$

Writing u_k instead of u_{ik}/i we have: $u_k = 2 \sin kx$, and

(4) $\int_0^1 u_m \, u_n \, dx = -4a \sin n \sin m \ , \text{ if } m \neq n \ ,$

(5) $\int_0^1 u_k^2 \, dx = 2(1 - a \sin^2 k) \ .$

For k and 1 positive eigenvalues, we have

(6) $A_{k1} = 1 \text{ if } k=1 \ , \quad A_{k1} = \dfrac{(-2a \sin k \sin 1)}{((1-a \sin^2 k)(1-a \sin^2 1))^{1/2}}$

We know that the Gramian A can be written as D+L, where D is now a diagonal matrix

$$(7)\quad D_k = D_{kk} = 1 + \frac{2a \sin^2 k}{1 - a \sin^2 k} = \frac{1 + a \sin^2 k}{1 - a \sin^2 k} .$$

In this case is immediately verified that for k eigenvalue $\sin k = 0(1/k)$, and that $0 \neq D_k \sim 1$.
Let us *define* $R = \sum \sin^2 k/(1 + a \sin^2 k)$, where k runs in the set of positive eigenvalues. Next theorem is formula (4.26) of [9], p.207. An alternative proof is given later.

THEOREM 1. $2a R = 1$.

From ch.VII, we see that g=1 and there is only one vector $k_0(s)$. It holds:

$$(8)\qquad k_0(s) = \{-i \sin k / \sqrt{2(1 - a \sin^2 k)}\}.$$

Then $w_0 = c_{00} k_0$. But $c_{00} = -4\, \alpha_2 \alpha_1$ as it follows from Ch.III. *Then,* $c_{00} = -4a$, and we have:

$$(9)\qquad w_0 = -4a\ k_0.$$

Then, *the matrix* L *is equal to* $-4a\ \bar{k}_0 . k_0^t$, and the matrix K of Ch.VII has only one entry equal to

$$(10)\qquad K_{00} = \langle w_0, D^{-1}\ \bar{k}_0 \rangle .$$

But $K_{00} = -2a\ R$ as it follows from (7), (8) and (9). From Th.1, Ch.VII, we get finally: $K_{00} = -1$. And this proves Theorem 1.

2. THEOREM 2. *If* $v_k = u_k/\|u_k\|$,

i) $\sum v_k(1)v_k(x)/D_k = 0$ *in* L^2 ;

ii) $\sum v_k(1)v_k(x)/D_k = 0$ *if* $0 \leqslant x < 1$,

$= 1/a$ *if* $x=1$;

iii) $\forall f \in L^2$: $\sum (b_k/D_k)\ v_k(1) = 0$;

iv) *if* $f \in L$ *real and* $\phi \in (-\infty,\infty)$, *there exist a vector* $C \in l_2$ *such that*

$$(11) \qquad \sum C_j v_j = f \ (L^2) \quad , \quad \sum C_j v_j(1) = \phi ,$$

holds. This vector is unique.

PROOF. i) Let C be a vector in l_2 non-orthogonal to $\overline{k}_0$. If AC=b and $c = D^{-1}b$, we have: $C + D^{-1}LC = c$, and therefore

$$f = \sum C_j v_j = \sum (b_j/D_j)v_j + 4a(\sum k_0(j)C_j)(\sum \overline{k}_0(j)v_j/D_j) =$$

$$= f + i2a\ (C,D^{-1}\overline{k}_0).\sum v_j(1)v_j(x)/D_j, \text{ in } l_2 ,$$

since

$$(12) \qquad \overline{k}_0(j) = (i/2)\ v_j(1)$$

ii) Let G(x) be the function defined by

$$(13) \qquad G(x) = \sum v_j(1)v_j(x)/D_j = 2 \sum \frac{\sin jx \sin j}{1 + a \sin^2 j} ,$$

whenever the series converges.

From (3) we obtain :

(14) $$aG(x)/2 = \sum \frac{\cos j \sin jx}{(1 + a \sin^2 j)j} =$$

$$\sum \left[\frac{\cos j \sin jx}{(1 + a \sin^2 j)j} - \frac{\cos \pi J \sin \pi Jx}{(1 + a \sin^2 \pi J)\pi J}\right] + \sum \frac{(-1)^J \sin \pi J}{\pi J}$$

where J is the integer such that $j - \pi J = o(1)$.

Using the mean value theorem, we see that the first summation is equal to

$$\sum (j - \pi J) \frac{d}{dt} \frac{\cos t \sin tx}{t(1 + a \sin^2 t)} ,$$

where the derivatives are taken at values of t in the intervals $(j, \pi J)$.

Since $|j - \pi J| = 0(1/J)$ and the derivative is $0(1/J)$, the last series converges uniformly to a continuous function $H(x)$ for $0 \leqslant x \leqslant 1$. Therefore

(15) $$aG(x)/2 = H(x) + \sum_{J=1}^{\infty} (-1)^J (\sin \pi Jx)/\pi J .$$

The series in (15) converges to $-x/2$ on $[0,1)$. Then $G(x)$ is defined and continuous on $[0,1)$, and equal to zero there (cf. i)). Therefore

$H(x) = x/2$ for $0 \leqslant x \leqslant 1$. From (15) we get:

(16) $$G(1) = 1/a.$$

By the way, since $G(1) = 2 \sum \sin^2 jx/(1 + a.\sin^2 jx) = 2.R$, from (16) *it follows again that* $2R.a = 1$.

iii) and iv). Let us define a vector $C \in l_2$:

(17) $$C_j = b_j/D_j + \phi \, . \, av_j(1)/2D_j$$

and $\tau = \sum b_j v_j(1)/D_j$.

According to Ch.VII, §1, Corollary and (8) and (9):

$0 = \beta(D^{-1}b) = \langle w_0, D^{-1}b \rangle = \sum w_0(j)b_j/D_j = -4a \sum k_0(j)b_j/D_j =$

$= 4ai\tau$.

Then the function defined by $\sum C_j v_j$ coincides with f. In fact, because of i) this series coincides in L^2 with $\sum (b_j/D_j)v_j$ which converges in L^2 to f. But

$\sum C_j v_j(1) = \tau + \phi(a/2) \sum v_j^2(1)/D_j = 0 + \phi$,

because of ii).

QED.

iv) of the preceding theorem shows that the indetermination in the expansion of f due to g=1 is reduced to the indetermination of ϕ which is equal to the value of the expansion at x=1.

The method of calculation of coefficients given in [17], [19], [11], [12], [13] consists in using those coefficients for which the expansion converges at x=1 to f(1).

3. THEOREM 3. *There exists a bilinear form*

$$B(f,g) = \int_0^1 fg\, dx + a\, f(1)g(1) ,$$

such that $B(v_s,v_t) = 0$ *for* $s \neq t$, s *and* t *eigenvalues*.

PROOF. According to (15), ch.VII, we must have:

$$s = I(s)(as^2-s) - I(-s)(as^2+s).$$

Choosing I(s) constant and equal to -1/2, we get the polynomial I necessary to define the bilinear form of Th.3, Ch.VII. Then

$$k_0(s) = -(1/2)v_s(1) \;; \quad w_0(s) = -c_{00}\, v_s(1)/2 \;.$$

Since (cf.(9)) $c_{00} = -4a$, we have, from (19), Ch.VII.,

$$B(f,g) = (f,g) - (c_{00}/4)\, f(1)g(1) = (f,g) + a\, f(1)g(1).$$

QED.

This result coincides with the inner product used by Churchill in [5] , (cf. ch. I, (5)).

INSTITUTO DE MATEMATICA
UNIVERSIDAD NACIONAL DEL SUR
BAHIA BLANCA - ARGENTINA

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