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AGNES ILONA BENEDEK AND RAFAEL PANZONE

NULL SERIES:

Two Applications

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May 1979.

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ON THE MATHEMATICAL MODEL OF A VIBRATING MECHANICAL DEVICE

by

Agnes Benedek and Rafael Panzone

ABSTRACT. The main objective of this paper is to exhibit as in [6] the null series of an ordinary differential irregular problem and the role played by them in the solution of the original partial differential problem, which involves, in this case, the one-dimensional wave equation.

1. Assume we have a circular shaft of constant diameter with disks at the extremes. Torsional vibrations will be produced if opposite twisting couples at the ends of the shaft are suddenly removed. Then the disks will rotate in opposite directions, (cf. [5], pp.9-13). If the circular cross sections of the shaft during torsional vibrations remain plane and their radii remain straight then the angle of twist $\theta = \theta(x,t)$ at the cross section at x will satisfy the wave equation under the following initial and boundary conditions (cf. [5], pp. 318-323):

$$(1)\quad \begin{cases} \dfrac{\partial^2\theta}{\partial t^2} = a^2\dfrac{\partial^2\theta}{\partial x^2} & 0 \leqslant x \leqslant \ell \;,\quad 0 < t \\ \dfrac{\partial^2\theta}{\partial t^2} = b\dfrac{\partial\theta}{\partial x} & x = 0 \\ \dfrac{\partial^2\theta}{\partial t^2} = -c\dfrac{\partial\theta}{\partial x} & x = \ell \\ \theta(x,0) = f(x) & \\ \theta_t(x,0) = g(x) & \end{cases}$$

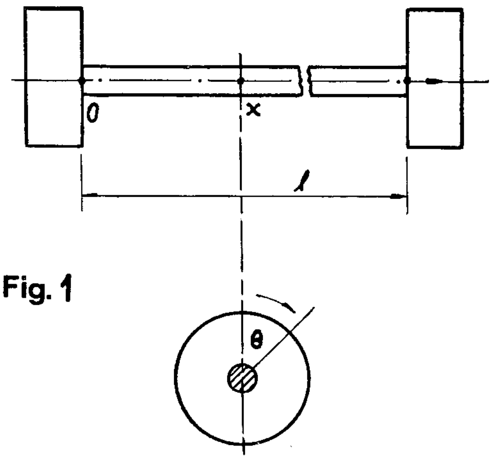


Fig. 1

We shall study the mathematical model described by (1) assuming $a,b,c > 0$ and f *and* g *of bounded variation on* [0,1]. After separation of variables: $\theta = u(x).T(t)$, and we arrive at the following differential problem:

$$(2) \qquad \begin{cases} u'' - \lambda u = 0 \\ \ell m.\lambda u(0) = u'(0) \\ \ell n.\lambda u(\ell) = -u'(\ell) \end{cases} , \quad m = a^2/b\ell \quad , \quad n = a^2/c\ell \quad 0 \leqslant x \leqslant \ell ,$$

$$(3) \qquad T'' - a^2\lambda T = 0 \quad , \quad 0 < t$$

For $\lambda = -\beta^2/\ell^2$ we have the eigenfunctions:

$$(4) \qquad u_\beta(x) = \cos\frac{\beta x}{\ell} - m\beta \sin\frac{\beta x}{\ell} \quad , \quad \Phi(\beta) = 0 \quad , \quad \beta \geqslant 0 ,$$

$$(5) \qquad \Phi(\beta) \equiv (m+n)\beta - (mn\,\beta^2 - 1)\mathrm{tg}\,\beta =$$

$$= \frac{-1}{\cos\beta} \begin{vmatrix} m\beta & 1 \\ n\beta\cos\beta + \sin\beta & n\beta\sin\beta - \cos\beta \end{vmatrix}$$

Correspondingly $\lambda \sim -n^2\pi^2/\ell^2$ and

$$(6) \qquad T_\beta(t) = \begin{cases} A_\beta \cos\frac{a\beta t}{\ell} + B_\beta \sin\frac{a\beta t}{\ell} & \text{for} \quad \beta > 0 \\ A_0 + tB_0 & \text{for} \quad \beta = 0 \end{cases}$$

We can expect to express a solution of (1) in the form:

$$(7) \qquad \theta(x,t) = \sum_\beta T_\beta(t)\, u_\beta(x)$$

Then, the coefficients A_β , B_β must verify:

$$(8)' \quad \begin{cases} \theta\ (x,0) = \sum A_\beta\, u_\beta(x) = f(x) \\ \theta_t(x,0) = \sum_{\beta\neq 0} B_\beta \frac{a\ \beta}{\ell} u_\beta(x) + B_0 u_0(x) = g(x) \end{cases}$$

For a detailed treatment of expansions (8), cf. [3], Ch.VI; there it is shown that

$$(9) \quad \begin{cases} \|u_0\|_2^2 = \ell \\ \|u_\beta\|_2^2 = \frac{\ell}{4\beta} [\, 2\beta(1+m^2\beta^2)+(1-m^2\beta^2)\ \sin 2\beta + 2m\beta(\cos 2\beta-1)] . \end{cases}$$

2. Let us write $P(\lambda) = \ell m.\lambda$, $Q(\lambda) = -1$, $\widetilde{P}(\lambda) = \ell n.\lambda$, $\widetilde{Q}(\lambda) = 1$. Then the differential problem (2) becomes:

$$(2') \quad \begin{cases} y'' - \lambda y = 0 \\ P(\lambda)\ y(0) + Q(\lambda)\ y'(0) = 0 \\ \widetilde{P}(\lambda)\ y(\ell) + \widetilde{Q}(\lambda)\ y'(\ell) = 0 \end{cases}$$

where $1 = \deg P = p > q = \deg Q = 0$, $1 = \deg \widetilde{P} = \widetilde{p} > \widetilde{q} = \deg \widetilde{Q} = 0$. From [3],Th. 4, Ch. III, it follows that *the Orr series*[1)] *of any function in* $L^2([0,\ell])$ *converges to* 0 *at* $x=0$ *and at* $x=\ell$. If $z_1(x) = x/\ell$, $z_2(x) = (\ell-x)/\ell$, the expansion in eigenfunctions of (2') given by:

$$z_i(x) = \sum_\beta \frac{B(z_i,u_\beta)}{B(u_\beta,u_\beta)} u_\beta(x) = \sum_\beta D_\beta(z_i)\ u_\beta(x) \quad ,$$

$$B(u,v) = \int_0^\ell u(x)\ v(x)\ dx + \ell m.u(0)\ v(0) + \ell n.u(\ell)\ v(\ell) \quad ,$$

---

1) By the Orr series of a function we understand a certain L^2-expansion in eigenfunctions obtained by the method of residues, (cf. Appendix , [6]).

converges to $z_i(x)$ *uniformly in* $[0,\ell]$, ([2]). (Observe that $u_\beta(x)$ is not, in general, a periodic function of period ℓ:

$$u_\beta(x+\ell) = u_\beta(x) \cos \beta + u'_\beta(x)(\ell/\beta) \sin \beta) .$$

Because of [3], Th.3, Ch. III, the Orr series of $z_i(x)$:

$$z_i(x) = \sum_\beta c_\beta(z_i)\, u_\beta(x) .$$

converges *uniformly to* $z_i(x)$ *on compact sets of* $(0,\ell)$. Then

$$(10) \quad \sum_\beta (D_\beta(z_i)-c_\beta(z_i))u_\beta(x) = \sum_\beta \eta_\beta(z_i)v_\beta(x) \quad , \quad v_\beta = u_\beta/\|u_\beta\|_2 \quad ,$$

is a null series i.e., $\{\eta_\beta(z_i)\} \in \ell^2$ and (10) converges to 0 in L^2 (in this case it also converges to 0 uniformly in compact sets of $(0,\ell)$). Since for i=1, (10) converges to 0 at x=0 and to 1 at $x=\ell$, and for i=2 it converges to 1 at x=0 and to 0 at $x=\ell$, it follows that $\{\eta_\beta(z_1)\}$ and $\{\eta_\beta(z_2)\}$ are linearly independent elements of ℓ^2. From [3], Ch. V, we know that the dimension of the subspace of null series in ℓ^2 is equal to:

$$g = \text{degrees of freedom} = \left[\frac{2p \vee (2q+1)}{2}\right] + \left[\frac{2\tilde{p} \vee (2\tilde{q}+1)}{2}\right] = 2 .$$

Therefore, that subspace is generated by $\{\{\eta(z_i) : i = 1,2\}$. This implies that given f and g we have two degrees of freedom in the choice of the coefficients of each L^2- expansion in (8).

Then, there exist 4 *parameters* whose selection determines the coefficients in (8). In this case the parameters are *the values of* f *and* g *at* 0 *and* ℓ. In fact, the sets of coefficients of the expansions of an L^2- function f are of the form:

$$\{c_\beta(f)\} + r.\{\eta_\beta(z_1)\} + s.\{\eta_\beta(z_2)\} = \{\gamma_\beta(f)\} ,$$

and $\sum_\beta \gamma_\beta(f)\, u_\beta(x) = f(x)$ in L^2, $= r$ at $x=\ell$, $= s$ at $x=0$,

$= \frac{f(x+0) + f(x-0)}{2}$ for $x \in (0,\ell)$ if f is of bounded variation in a neighbourhood of x.

One of our purposes is to see how the indetermination in the expansions of the initial conditions (8) appears in the solutions of problem (1). But first let us see the following proposition.

PROPOSITION 1. i) If $\sum_\beta A_\beta u_\beta$ is a null series and $x \pm at \in (0,\ell)$ then $\sum_\beta A_\beta \cos \frac{a\beta t}{\ell} u_\beta(x) = 0$.

ii) If $B_0 u_0 + \sum_{\beta \neq 0} B_\beta \frac{a\beta}{\ell} u_\beta$ is a null series and $x \pm at \in (0,\ell)$ then $B_0 t.u_0(x) + \sum_{\beta\neq 0} B_\beta \sin \frac{a\beta t}{\ell} u_\beta(x) = 0$.

PROOF. From (4) it follows that $\cos \frac{a\beta t}{\ell} . u_\beta(x) = \frac{1}{2}[u_\beta(x+at) + u_\beta(x-at)]$ and this implies i). On the other hand,

$$2u_\beta(x) \sin \frac{a\beta t}{\ell} = \frac{\beta}{\ell}\int_{x-at}^{x+at} u_\beta(y)\, dy.$$

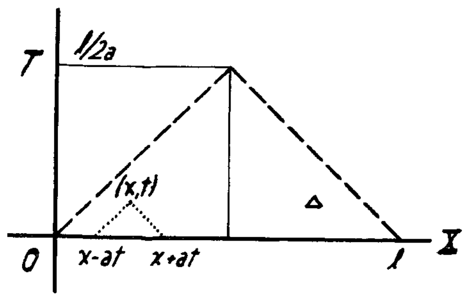


Fig. 2

Then, $a\sum_{\beta\neq 0} B_\beta \sin \frac{a\beta t}{\ell} u_\beta(x) =$

$$= \sum_{\beta\neq 0} \frac{aB_\beta}{2}\frac{\beta}{\ell}\int_{x-at}^{x+at} u_\beta(y)\, dy = \frac{1}{2}\int_{x-at}^{x+at} [\sum_{\beta\neq 0} B_\beta \frac{a\beta}{\ell} u_\beta]\, dy .$$ QED.

In consequence, if $\Delta = \{(x,t);\ t > 0,\ x \pm at \in (0,\ell)\}$ then $\theta(x,t)$ for $(x,t) \in \Delta$ will not depend upon the choice of the parameters f(0), f(1), g(0), g(1) in the expansions (8). This is explained by the fact that if $x \in (0,\ell)$, $t > 0$, $(x,t) \in \Delta$, then the perturbations at 0 and ℓ did not arrive at the point x at the instant t.

3. Next lemma is a result from the folklore of distribution theory. We call D(A) the space of test functions in the open set A and D'(A) the space of distributions in A.

LEMMA 1. Let $\mathbb{R}$ be a region in the plane such that its intersection with any straight line parallel to one of the axes ξ, η, is an interval. Let u be a distribution, $u \in D'(\mathbb{R})$, such that $\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$. Then, there exist distributions S and T such that $S \in D'(E)$, $T \in D'(F)$, $E = \text{proj}_\xi \mathbb{R}$, $F = \text{proj}_\eta \mathbb{R}$, and

(11) $$u = S_\xi \otimes 1_\eta + 1_\xi \otimes T_\eta \; .$$

PROOF. Let R be an open rectangle, $R \subset \mathbb{R}$, $R = I \times J$, and $\alpha \in D(I)$, $\beta \in D(J)$, both of integral one. Assume that $\varphi(\xi,\eta) \in D(R)$. Using the nomenclature of [4] we define:

$$\psi(\xi,\eta) = \varphi(\xi,\eta) - \alpha(\xi) \, . \, 1_\xi(\varphi) -$$
$$-\beta(\eta) . 1_\eta(\varphi) + \alpha(\xi)\, \beta(\eta) (1_\xi \otimes 1_\eta)(\varphi) .$$

It holds: $\int_{-\infty}^{\infty} \psi(\xi,y)\, dy = 0 =$

$= \int_{-\infty}^{+\infty} \psi(x,\eta)\, dx$, that is ,

$$\psi(\xi,\eta) = \int_{-\infty}^{\eta} dy \int_{-\infty}^{\xi} \psi(x,y) dx \in D(R),$$

$$\psi \in \frac{\partial^2}{\partial\xi\partial\eta} D(R) \; .$$

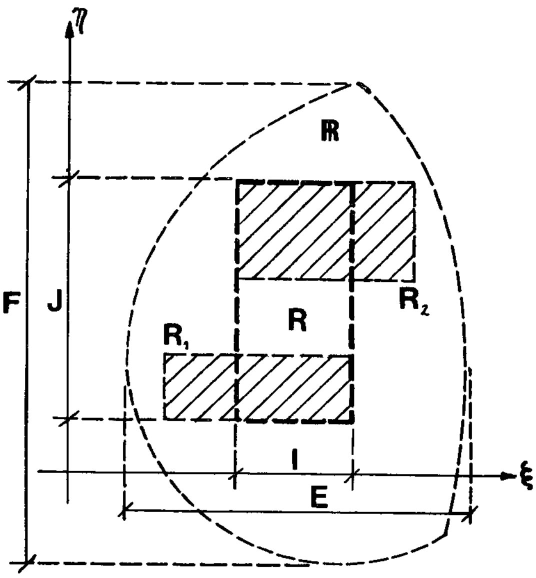


Fig. 3

Now define in I the distribution S and in J the distribution T by:

$$S(\chi) = u(\chi(\xi) \, . \, \beta(\eta)) \quad ; \quad T(\kappa) = u(\alpha(\xi) \, . \, \kappa(\eta)) .$$

Taking into account that $u(\psi) = u(\partial^2\psi/\partial\xi\partial\eta) = 0$ we get

(12) $$u(\varphi) = (1_\xi \otimes T_\eta)(\varphi) + (S_\xi \otimes 1_\eta)(\varphi) + c(1_\xi \otimes 1_\eta)(\varphi), \quad c = -u(\alpha . \beta) .$$

In consequence, the lemma is "locally" true if we take care of incorporating the last summand into any of the others. S and T depend on α and β but if these functions are replaced by α' and β' respectively, the new S' and T' differ from the former ones only in additive constants. In fact, from

$$S'_\xi \otimes 1_\eta + 1_\xi \otimes T'_\eta + c' \, 1_\xi \otimes 1_\eta = S_\xi \otimes 1_\eta + 1_\xi \otimes T_\eta + c \, 1_\xi \otimes 1_\eta \, .$$

one gets, if $\alpha = \alpha'$ and therefore $T = T'$, that $S' = S + (c-c')1_\xi$.

Now, let us call S_i and T_i the distributions associated to $R_i = I_i \times J_i$, $i = 1,2$, $S_i \in D'(I_i)$, $T_i \in D'(J_i)$. If $I = I_i \cap I_2 \neq \emptyset$ then there exists a rectangle $R = I \times J$, $J \supset J_1 \cup J_2$. Since T_1 and T_2 can be related to a T obtained with a function α with support in I, one sees that $T_1 - T_2$ is equal to a constant wherever it is defined.

If we cover F with a family of intervals J_r such that in each of them $\frac{\partial T_r}{\partial y}$ is defined and in the intersection of two of these intervals J_r , J_s , $\frac{\partial T_r}{\partial y} = \frac{\partial T_s}{\partial y}$ holds, then there exists a unique distribution of $D'(F)$ equal to $\frac{\partial T_r}{\partial y}$ in J_r, $\forall$ r. Let T be one of its primitives. In an analogous way we obtain a distribution $S \in D'(E)$.

Since $u - S_\xi \otimes 1_\eta - 1_\xi \otimes T_\eta$ is locally constant, it is a constant, and then equal to $c(1_\xi \otimes 1_\eta)$. QED.

REMARK. Now it is easy to see that

$$S_\xi \otimes 1_\eta + 1_\xi \otimes T_\eta = \tilde{S}_\xi \otimes 1_\eta + 1_\xi \otimes \tilde{T}_\eta$$

if and only if there exists a constant C such that

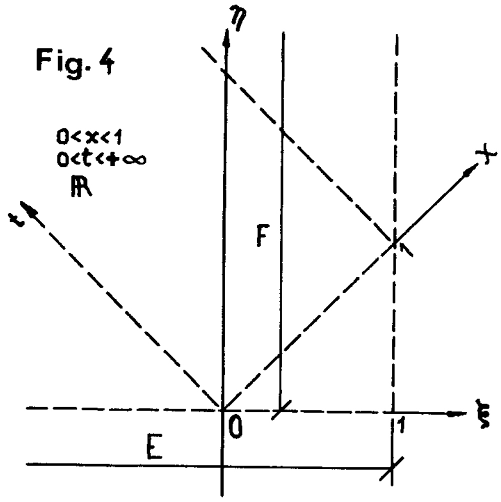
$$\tilde{S}_\xi = S_\xi + C \qquad , \qquad T_\eta = \tilde{T}_\eta - C \, .$$

We shall apply the preceding proposition to a region $R = R_{\xi\eta} \subset E \times F$, $E = (-\infty,1)$, $F = (0,\infty)$; $R_{\xi\eta}$ is the image of $R_{xy} = \{(x,t): 0 < x < 1, 0 < t < \infty\}$ by the mapping:

$$\xi = x - t \quad , \quad \eta = x + t$$

where the jacobian $\frac{\partial(\xi,\eta)}{\partial(x,t)} = 2.$

Fig. 4



Let us call τ the operator defined by

$$(\tau\ \psi)(x,t) = \psi(x-t,x+t) \quad \text{if} \quad \psi = \psi(\xi,\eta) \in D(R_{\xi\eta}).$$

If $P \in D'(R_{xy})$ then we define:

$$(13) \qquad (\tau^{-1} P)(\psi) := P(2.\tau\ \psi).$$

Then: $(\tau^{-1}(\frac{\partial P}{\partial t}))(\psi) = 2\ \frac{\partial P}{\partial t}\ (\tau\ \psi) = -2P(\frac{\partial(\tau\ \psi)}{\partial t}) = -2P(\tau(\frac{\partial\psi}{\partial\eta} - \frac{\partial\psi}{\partial\xi})) =$

$$= -(\tau^{-1}\ P)(\frac{\partial\psi}{\partial\eta} - \frac{\partial\psi}{\partial\xi}) = [(\frac{\partial}{\partial\eta} - \frac{\partial}{\partial\xi})(\tau^{-1}\ P)](\psi)$$

and therefore,

$$(14) \qquad \tau^{-1}\ \frac{\partial^2 P}{\partial t^2} = (\frac{\partial}{\partial\eta} - \frac{\partial}{\partial\xi})^2 \tau^{-1} P \ ; \quad \tau^{-1}\ \frac{\partial^2 P}{\partial x^2} = (\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\xi})^2\ \tau^{-1} P.$$

(14) implies that

$$\tau^{-1}\ (\frac{\partial^2 P}{\partial t^2} - \frac{\partial^2 P}{\partial x^2}) = -4\ \frac{\partial^2}{\partial\xi\partial\eta}\ (\tau^{-1}\ P).$$

If $(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})\theta = 0$ then from the preceding lemma:

$\tau^{-1}\theta = S_\xi \otimes 1_\eta + 1_\xi \otimes T_\eta$ with $S \in D'(E)$, $T \in D'(F)$. If $\varphi \in D(R_{xt})$ we have:

(15) $\theta_{x,t}(\varphi) = S_\xi(\int_0^\infty \varphi(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2}) \frac{d\eta}{2}) + T_\eta(\int_{-\infty}^1 \varphi(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2}) \frac{d\xi}{2})$.

If T were a function we would have (cf. (13)):

$$T_\eta(1_\xi(\tau^{-1} \frac{\varphi}{2})) = \int_0^\infty T(\eta)d\eta \int_{-\infty}^1 \varphi(\frac{\xi+\eta}{2},\frac{\eta-\xi}{2}) \frac{d\xi}{2} = \iint_{R_{xt}} T(x+t)\varphi(x,t)\,dx\,dt.$$

In consequence, if S and T are functions, from (15) we see that θ is of the form $\theta_{x,t} = S(x-t) + T(x+t)$.

In the general case: $S \in D'(E)$, $T \in D'(F)$, we shall write

(16) $$\theta_{x,t} = S_{x-t} + T_{x+t} .$$

4. By $\theta_{x,0}$ we shall understand the distribution S+T, i.e., *the restriction of* $\theta_{x,t}$ *to* $0 < x < 1$. If $\varphi(x,y) = \phi(\xi,\eta)$ then

$$(\frac{\partial}{\partial t} S_{x-t})(\varphi) = -S_{x-t}(\frac{\partial\varphi}{\partial t}) = -S_\xi(\int_0^\infty \frac{\partial\varphi}{\partial t} (\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2}) \frac{d\eta}{2}) =$$

$$= -S_\xi(\int_0^\infty \frac{\partial\varphi}{\partial t} (x,t) \frac{d\eta}{2}) = -S_\xi(\int_0^\infty (\frac{\partial\phi}{\partial\eta} - \frac{\partial\phi}{\partial\xi}) (\xi,\eta) \frac{d\eta}{2}) =$$

$$= S_\xi(\int_0^\infty \frac{\partial\phi}{\partial\xi} \frac{d\eta}{2}) = S_\xi(\int_0^\infty \frac{\partial\varphi}{\partial\xi} (x,y) \frac{d\eta}{2}) =$$

$$= S_\xi(\int_0^\infty \frac{\partial\varphi}{\partial\xi} (\frac{\eta+\xi}{2}, \frac{\eta-\xi}{2}) \frac{d\eta}{2}) = -S'_\xi(\int_0^{+\infty} \varphi\frac{d\eta}{2}) = -S'_{x-t}(\varphi),$$

where $S'_{x-t} := (S')_{x-t}$. Therefore:

(17) $$\frac{\partial S_{x-t}}{\partial t} = -S'_{x-t}, \quad \frac{\partial S_{x-t}}{\partial x} = S'_{x-t}, \quad \frac{\partial T_{x+t}}{\partial t} = T'_{x+t}, \quad \frac{\partial T_{x+t}}{\partial x} = T'_{x+t}.$$

In consequence we formulate *the initial conditions* of the problem on the interval (0,1) in the following way:

(18) $\theta_{x,0} = S+T = f(x)$ in $D'((0,1))$; $(\frac{\partial\theta}{\partial t})_{x,0} = T'-S' = g(x)$ in $D'((0,1))$.

(18) determines S and T in (0,1) in the sense that if $\tilde{S}$, $\tilde{T}$, also verify (18) then $\tilde{S} = S + C$, $\tilde{T} = T - C$, C a constant. In fact, this follows from $2T' = f' + g = 2\tilde{T}'$, $2S' = f' - g = 2\tilde{S}'$. (And also from the remark to Lemma 1).

Then a solution θ_{xt} , if it exists, must be found among the distributions of the form (16) which satisfy (18) in (0,1).

From (17) we obtain

$$(19)\quad \begin{cases} \dfrac{\partial^2\theta}{\partial t^2} - b\,\dfrac{\partial\theta}{\partial x} = S''_{x-t} + T''_{x+t} - b(S'_{x-t} + T'_{x+t}) \\[2ex] \dfrac{\partial^2\theta}{\partial t^2} + c\,\dfrac{\partial\theta}{\partial x} = S''_{x-t} + T''_{x+t} + c(S'_{x-t} + T'_{x+t}). \end{cases}$$

Since our main objective is to analyze qualitatively problem (1), *in what follows we shall restrict ourselves to the case* $a = b = c = \ell = 1$.

By definition, the restrictions to x=0 of the first distribution in (19) and to x=1 of the second one, are:

$$(20)\quad \begin{cases} S''_{-t} + T''_{t} - S'_{-t} - T'_{t} \in D'((0 < t < \infty)) \\[1ex] S''_{1-t} + S'_{1-t} + T''_{1+t} + T'_{1+t} \in D'((0 < t < \infty)) \end{cases},$$

where $P_{-t}(\varphi) := P(\varphi(-t))$ and $P_t := P$. Analogously $P_{1+t}(\varphi) := P(\varphi(t-1))$, $P_{1-t}(\varphi) := P(\varphi(-t+1))$.

So we shall formulate *the boundary conditions* of the problem as

$$(21)\quad \begin{cases} S''_{-t} + T''_{t} - (S'_{-t} + T'_{t}) = 0 & \text{in } D'((0,\infty)) \\ S''_{1-t} + S'_{1-t} + T''_{1+t} + T'_{1+t} = 0 & \text{in } D'((0,\infty)). \end{cases}$$

and by a *solution of problem* (1) *in the distribution sense we mean a distribution* $\theta_{x,t}$ *as given in* (16): $\theta_{x,t} = S_{x-t} + T_{x+t}$, *with* S *and* T *verifying* (18) *and* (21).

5. EXISTENCE OF SOLUTION. We know that each summad in (7) satisfies the wave equation and that we can choose coefficients A_β and B_β in such a way that (8) is verified in $L^2(0,1)$. This implies $\{A_\beta \|u_\beta\|\} \in \ell^2$, $\{\beta\ B_\beta\ \|u_\beta\|\} \in \ell^2$, (cf.[1], §1, formula (8)).

Then, from $T_\beta(t)\ u_\beta(x) = (A_\beta \|u_\beta\| \cos \beta t + B_\beta \|u_\beta\| \sin \beta t)\ v_\beta(x)$ if $\beta \neq 0$, we see that $\sum T_\beta\ u_\beta$ converges in $L^2((0,1) \times (0,T))$ to a solution of the wave equation. That is, there exists in $D'((0,1) \times (0,\infty))$ a function, represented by the series, solution of the wave equation.

Let us call

$$(22)\quad \begin{cases} S = \frac{1}{2} \sum A_\beta\ u_\beta(x) - \frac{1}{2} [\sum B_\beta\ \beta \int_0^x u_\beta\ dy + B_0 \int_0^x u_0\ dy] \\ T = \frac{1}{2} \sum A_\beta\ u_\beta(x) + \frac{1}{2} [\sum B_\beta\ \beta \int_0^x u_\beta\ dy + B_0 \int_0^x u_0\ dy] \end{cases}$$

We shall show that these expresions define a distribution $S \in D'((-\infty,1))$ and a distribution $T \in D'((0,\infty))$ such that $\theta_{x,t} = S_{x-t} + T_{x+t}$ is a solution of problem (1) in the distribution sense. Let us prove these facts.

First observe that for any real z and β great enough , $\beta > (\int_z^{z+1} |u_\beta(y)|^2\ dy)^{1/2} > \beta/2$, as it follows from (4). If

$\varphi \in D((-\infty,+\infty))$ then $\int_{-\infty}^{+\infty} \sin \beta y\, \varphi(y)\, dy$ is rapidly decaying at infinity. Therefore, from $\{A_\beta\, \beta\} \in \ell^2$ and $A_\beta\, u_\beta = (A_\beta\, \beta)(\frac{\cos \beta x}{\beta} - \sin \beta x)$ we see that $\langle \sum_{|\beta|<N} A_\beta\, u_\beta, \varphi \rangle$ converges as $N \longrightarrow \infty$ and this implies that $\sum_\beta A_\beta\, u_\beta(x)$ defines a distribution belonging to $D'(-\infty,+\infty)$.

Let $\sum T_n$ be a convergent series of distributions and τ_n a primitive of T_n. If τ is a primitive of $T = \sum T_n$ then $\tau = \sum \tau_n$ holds whenever for a function $\varphi_0 \in D$ of non null integral: $\tau(\varphi_0) = \sum \tau_n(\varphi_0)$, (cf. [4], II, §4).

The series $\sum_\beta B_\beta\, \beta\, u_\beta(x) + B_0\, u_0(x)$ converges in $L^2(0,1)$ to g and therefore $B_0 \int_0^x u_0\, dy + \sum_\beta B_\beta\, \beta \int_0^x u_\beta(y) dy$ converges uniformly to $\int_0^x g(y)\, dy$. Taking $\varphi_0 \in D(0,1)$, $\int_0^1 \varphi_0(y)\, dy \neq 0$, and applying the proposition mentioned above, one sees that $B_0 \int_0^x u_0\, dy +$ $+ \sum_\beta B_\beta\, \beta \int_0^x u_\beta\, dy$ defines a distribution in $D'(-\infty,+\infty)$.

From the results of the preceding paragraph we get, if $\theta_{x,t} =$ $= S_{x-t} + T_{x+t}$, and $S \in D'(-\infty,1)$, $T \in D'(0,\infty)$ are the distributions that we have just found, that

$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})\theta_{x,t} = 0$. Besides $\theta_{x,0} = S + T = f$, $(\frac{\partial \theta}{\partial t})_{x,0} = T'-S' = g$.

Since

$$\theta_{x,t} = \frac{1}{2} \sum A_\beta (u_\beta(x-t)+u_\beta(x+t)) + \frac{1}{2}\left[\sum B_\beta \beta \int_{x-t}^{x+t} u_\beta(y) dy + B_0 \int_{x-t}^{x+t} u_0(y) dy\right],$$

from (4), as in the proof of proposition 1, it follows that $\theta_{x,t} = \sum T_\beta(t)\, u_\beta(x)$, and therefore that $\theta_{x,t}$ is a function.

On the other we have

$$S'_{-t} = \frac{1}{2} \sum A_\beta\, u'_\beta(-t) - \frac{1}{2} \sum B_\beta \beta . u_\beta(-t) - \frac{1}{2} B_0\, u_0(-t)$$

$$S''_{-t} = \frac{1}{2} \sum A_\beta\, u''_\beta(-t) - \frac{1}{2} \sum B_\beta \beta . u'_\beta(-t) .$$

Analogous expressions are obtained for T'_t and T''_t. Taking into account that $u''_\beta = -\beta^2 . u_\beta$, after replacing u_β and u'_β by their expressions obtained from (4) we see that S and T satisfy the first equation in (21). Recalling that *the eigenvalue equation* is $2\beta - (\beta^2-1)\, \mathrm{tg}\, \beta = 0$, it follows in a similar fashion that S and T satisfy the second equation in (21). Thus, it is proved that $\theta_{x,t} = \sum T_\beta\, u_\beta$ *is a function, solution of* (1) *in the distribution sense.*

REMARKS. 1. Observe that the set that really entered into consiration until now is

$$\overline{R} = \{(x,t) : 0 \leqslant x \leqslant 1,\ 0 \leqslant t < \infty\} \setminus \{(0,0),(1,0)\}.$$

2. If f and g are also continuous on (0,1) then the series (8) and (22) converge uniformly on closed subintervals, (cf.(10), §2 and [3], Th. 3, Ch. III).

6. UNIQUENESS. If θ and $\tilde{\theta}$ are two solutions of problem (1) in the distribution sense then we can assume $S - \tilde{S} = 0 = T - \tilde{T}$ on the unit interval. Therefore, to see *how many solutions exist in the distribution sense,* it is necessary and sufficient to find *how many pairs* S, T, *null on* (0,1) *verifying* (21) *exist.* Let S, T be such a pair. If T is known on (n,n+1), then from the first equa-

tion in (21) it follows that S is known on $(-(n+1),-n)$ except for a summand of the form $a_n + b_n e^t$ (cf. [4], Ch. V, §6). From the second equation in (21) it is seen that if S is known on $(-(n+1),-n)$ then T is known on $(n+2,n+3)$ except for a summand of the form $A_n + B_n e^{-t}$. Therefore, on intervals $(k,k+1)$, $k \in Z$, contained in $(-\infty,1)$ for S and in $(0,\infty)$ for T, these distributions coincide with functions [S] and [T] respectively which together with their derivatives have limits at the end points k and k+1.

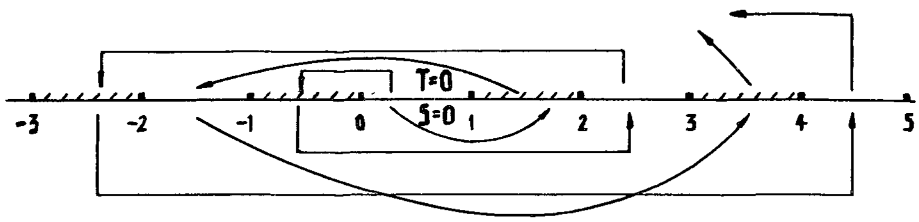


Fig. 5

In consequence S and T are determined uniquely on $(-\infty,1)\setminus Z$ and $(0,\infty)\setminus Z$ respectively, if their jumps and those of their first derivatives are known at the integers in $(-\infty,1)$ and $(0,\infty)$.

Therefore:

$$(23)\quad \begin{cases} T = [T] + \sum_{n=1}^{\infty} q_n , & q_n = \sum_{j=0}^{N_n} \gamma_{nj}\, \delta_n^{(j)} , \quad T = 0 \text{ on } (0,1), \\ S = [S] + \sum_{n=0}^{\infty} p_{-n} , & p_{-n} = \sum_{j=0}^{M_n} \beta_{nj}\, \delta_{-n}^{(j)} , \quad S = 0 \text{ on } (0,1) \end{cases}$$

If c_n (C_n) is the jump of [T] ([S]) at $x=n$ ($x=-n$), we have:

$$(24)\qquad [T]' = [T'] + \sum_{1}^{\infty} c_n\, \delta_n \quad ; \quad [S]' = [S'] + \sum_{0}^{\infty} C_n\, \delta_{-n} ,$$

since $[[T]'] = [T']$, (cf. [4], Ch. II, §2).

And, if d_n (D_n) is the jump of [T'] ([S']) at $x=n$ ($x=-n$), it follows that:

$$(25)\quad [T]'' = [T''] + \sum_1^\infty c_n \delta'_n + \sum_1^\infty d_n \delta_n \ ; \quad [S]'' = [S''] + \sum_0^\infty C_n \delta'_{-n} + \sum_0^\infty D_n \delta_{-n} .$$

It is easy to verify that $(\delta_{-n}{}^{(j)})_{-t} = (-1)^j . \delta_n{}^{(j)}$,

$$(\delta_m{}^{(j)})_{1-t} = (-1)^j \delta_{1-m}^{(j)} \quad ; \quad (\delta_m{}^{(j)})_{1+t} = \delta_{m-1}^{(j)}$$

since $R_{-t}(\varphi) = R(\psi)$, $\psi(x) = \varphi(-x)$; $R_{1\pm t}(\varphi) = R(\eta)$, $\eta(x) = \varphi(\pm(x-1))$.

Making use of these formulae and of (21), (24) and (25) we obtain:

$$\sum_1^\infty C_n(\delta'_{-n})_{-t} + \sum_1^\infty D_n(\delta_{-n})_{-t} + \sum_1^\infty c_n \delta'_n + \sum_1^\infty d_n \delta_n -$$

$$-\{\sum_1^\infty C_n(\delta_{-n})_{-t} + \sum_1^\infty c_n \delta_n\} + \sum_{n=1}^\infty \{ \sum_{j=0}^{M_n} \beta_{nj} (\delta_{-n}^{(j+2)})_{-t} +$$

$$+ \sum_{j=0}^{N_n} \gamma_{nj} \delta_n^{(j+2)}\} - \sum_{n=1}^\infty \{ \sum_0^{M_n} \beta_{nj} (\delta_{-n}^{(j+1)})_{-t} + \sum_0^{N_n} \gamma_{nj} \delta_n^{(j+1)}\} = 0,$$

$0 < t < \infty$.

Then, necessarily $M_n = N_n$, $n = 1,2,3,\ldots$, and

$$(26)\quad \sum_1^\infty (D_n + d_n - C_n - c_n)\delta_n + \sum_1^\infty (c_n - C_n + \beta_{n0} - \gamma_{n0})\,\delta'_n +$$

$$+ \sum_1^\infty (\beta_{n0} - \beta_{n1} + \gamma_{n0} - \gamma_{n1})\delta''_n + \sum_1^\infty(-\beta_{n1} + \beta_{n2} + \gamma_{n1} - \gamma_{n2})\delta'''_n +$$

$$+ \ldots + \sum_1^\infty (\beta_{n,M_n} (-1)^{M_n} + \gamma_{n,M_n})\, \delta_n^{(M_n+2)} = 0$$

Repeating this process but using the second relation in (21) we get

$$\sum_0^\infty (C_n + D_n)(\delta_{-n})_{1-t} + \sum_0^\infty C_n(\delta'_{-n})_{1-t} + \sum_2^\infty (c_n + d_n)(\delta_n)_{1+t} +$$

$$+ \sum_2^\infty c_n\ (\delta'_n)_{1+t} + \sum_0^\infty [\sum_{j=0}^{M_n} \beta_{nj}(\delta_{-n}^{(j+2)})_{1-t} + \sum_{j=0}^{M_n} \beta_{nj}(\delta_{-n}^{(j+1)})_{1-t}] +$$

$$+ \sum_2^\infty [\sum_{j=0}^{N_n} \gamma_{nj}(\delta_n^{(j+2)})_{1+t} + \sum_{j=0}^{N_n} \gamma_{nj}(\delta_n^{(j+1)})_{1+t}] = 0,\ 0 < t < \infty .$$

Therefore

$$\sum_0^\infty (C_n + D_n)\delta_{1+n} + \sum_2^\infty (c_n + d_n)\delta_{n-1} - \sum_0^\infty C_n\ \delta'_{1+n} + \sum_2^\infty c_n\ \delta'_{n-1} +$$

$$+ \sum_{n=0}^\infty \sum_{j=0}^{M_n} \beta_{nj}\ (-1)^{j+1}(\delta_{n+1}^{(j+1)} - \delta_{n+1}^{(j+2)}) +$$

$$+ \sum_{n=2}^\infty \sum_{j=0}^{N_n} \gamma_{nj}\ (\delta_{n-1}^{(j+1)} + \delta_{n-1}^{(j+2)}) = 0\ .$$

That is, always in the interval $(0,+\infty)$:

$$(27)\quad \sum_{n=1}^\infty [(C_{n-1} + D_{n-1})\delta_n - C_{n-1}\ \delta'_n] + \sum_1^\infty (c_{n+1} + d_{n+1})\delta_n + c_{n+1}\delta'_n] +$$

$$+ \sum_{n=1}^\infty \{\sum_{j=0}^{M_{n-1}} \beta_{n-1,j}\ (-1)^{j+1}\ (\delta_n^{(j+1)} - \delta_n^{(j+2)}) +$$

$$+ \sum_{j=0}^{N_{n+1}} \gamma_{n+1,j}\ (\delta_n^{(j+1)} + \delta_n^{(j+2)})\} = 0.$$

From here we conclude that $M_{n-1} = N_{n+1}$, $n = 1,2,3,\ldots$; then

$$(28)\qquad M_0 = M_2 = M_4 = \ldots = N_2 = N_4 = \ldots ;$$
$$N_1 = N_3 = \ldots = M_1 = M_3 = \ldots ..$$

From (26) and (27) we obtain the following equations for

$n = 1,2,3,\ldots$:

$$(29)\quad \begin{cases} D_n - C_n + d_n - c_n = 0 & C_{n-1} + D_{n-1} + c_{n+1} + d_{n+1} = 0 \\ \beta_{n,0} - \gamma_{n,0} - C_n + c_n = 0 & -\beta_{n-1,0} + \gamma_{n+1,0} - C_{n-1} + c_{n+1} = 0 \\ \beta_{n,0} - \beta_{n,1} + \gamma_{n,0} - \gamma_{n,1} = 0 & \beta_{n-1,0} + \beta_{n-1,1} + \gamma_{n+1,0} + \gamma_{n+1,1} = 0 \\ -\beta_{n,1} + \beta_{n,2} + \gamma_{n,1} - \gamma_{n,2} = 0 & -\beta_{n-1,1} - \beta_{n-1,2} + \gamma_{n+1,1} + \gamma_{n+1,2} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \beta_{n,M_n} (-1)^{M_n} + \gamma_{n,M_n} = 0 & \beta_{n-1,M_{n-1}} (-1)^{M_{n-1}} + \gamma_{n+1,N_{n+1}} = 0 \end{cases}$$

Taking into account (28), it follows that if one knows

$$(30)\quad C_0,\ D_0,\ M_0,\ \beta_{0,0},\ldots,\beta_{0,M_0}\ ;\ c_1,\ d_1,\ N_1,\ \gamma_{1,0},\ldots,\ \gamma_{1,N_1}$$

then from (29) it is possible to deduce *all* other constants that enter into the definitions of S and T. Besides, *if* one already knows that S *and* T *are functions, then it suffices to know the four parameters*

$$(31)\qquad C_0\ ,\ D_0\ ;\ c_1\ ,\ d_1\ ,$$

to determine completely S *and* T. They correspond to the jump at 0 of S (C_0) and that of its derivative (D_0), and the jump at 1 of T (c_1) and of its derivative (d_1).

LEMMA 2. Assume $\{A_\beta\}$ is the set of coefficients of a null series such that: $\sum A_\beta\, u_\beta(0) = \gamma_0$, $\sum A_\beta\, u_\beta(1) = \gamma_1$. Then

i) $\sum A_\beta\, u_\beta(x)$ converges in $D'(-\infty,+\infty)$ to a function R,

ii) $R(x) + R(-x)$ is a continuous function on $(-\infty,+\infty)$,

iii) $R(1+x) + R(1-x)$ is also a continuous function on $(-\infty,+\infty)$,

iv) $R(-0) = 2\,\gamma_0$, $R(1+0) = 2\,\gamma_1$,

v) $\sum A_\beta \int_0^x u_\beta(t)\,dt$ converges absolutely and uniformly in compact sets of $(-\infty,+\infty)$ to an absolutely continuous function $P(x)$ such that $P'(x) \equiv R(x)$ on $(-1,0) \cup (1,2)$ and is a continuous function there,

vi) $P'(0-) = 2\gamma_0$, $P'(1+0) = 2\gamma_1$, $P(x) = 0$ for $x \in (0,1)$.

PROOF. Let us recall that $\sum A_\beta u_\beta(x) = 0$ in $L^2(0,1)$ and converges uniformly to 0 in compact sets of $(0,1)$, (cf. §2). For $\varphi \in D$, $\langle u_\beta,\varphi\rangle = \int_{-\infty}^{\infty} u_\beta(x)\,\varphi(x)\,dx$ is a rapidly decaying function of β at infinity. Since $\{\beta A_\beta\} \in \ell^2$, $\{A_\beta\} \in \ell^1$ and $\sum \langle A_\beta u_\beta,\varphi\rangle$ converges $\forall \varphi \in D(-\infty,+\infty)$, and therefore defines a distribution R.

From $u_\beta(x) + u_\beta(-x) = 2\cos\beta x = 0(1)$ it follows that $\sum A_\beta (u_\beta(x) + u_\beta(-x)) = 2\sum A_\beta \cdot \cos\beta x$ converges uniformly to a continuous function.

Analogously, $u_\beta(1-x) + u_\beta(1+x) = 2(\cos\beta - \beta\sin\beta)\cos\beta x = 2u_\beta(1)\cos\beta x = 0(1)$. In fact, from the eigenvalue equation we have for $\beta\neq 0$: $u_\beta(1) = \cos\beta - \beta\sin\beta = -(\cos\beta + \frac{\sin\beta}{\beta})$.

Then $\sum A_\beta (u_\beta(1+x) + u_\beta(1-x)) = 2\sum A_\beta u_\beta(1)\cos\beta x$ also converges uniformly to a continuous function. In consequence, $\sum A_\beta u_\beta(x)$ converges in L^2 on any finite subinterval of $(-\infty,+\infty)$, and the distribution R is a function. Since $R(x) \equiv 0$ on $(0,1)$

$$\lim_{x\to -0} R(x) = \lim_{x\to -0} (R(x) + R(-x)) = 2\sum A_\beta u_\beta(0) = 2\gamma_0 ,$$

$$\lim_{x\to +0} R(1+x) = \lim_{x\to +0} (R(1+x) + R(1-x)) = 2\sum A_\beta u_\beta(1) = 2\gamma_1 .$$

Thus, i) - iv) are proved. Let us see v) and vi). The L^2-convergence of $\sum A_\beta u_\beta(x)$ on any interval implies that $\sum A_\beta \int_0^x u_\beta(s)\,ds$ converges uniformly to a an absolutely continuous function $P(x)$. Therefore $\sum A_\beta u_\beta(x) = P'(x)$ in the sense of distributions and necessarily $P'(x) = R(x)$ a.e. .

Since $\sum A_\beta u_\beta(x)$ converges uniformly on compact sets of $(0,1)$, $R(x)$ is continuous on $(-1,0)$ and on $(1,2)$.

In consequence, $P'(x) \equiv R(x)$ on these intervals and vi) follows from iv). QED.

THEOREM 1. Assume f and g of bounded variation on $[0,1]$. The series (7) with coefficients verifying (8) in $L^2(0,1)$ define the only solutions θ of (1) in the distribution sense which are functions.

To prove Theorem 1 we shall make use of next Lemma 3.

LEMMA 3. A distribution $\theta_{x,t} = T_{x+t} + S_{x-t}$ on $R_{x,t} = \{(x,t);\ 0 < x < 1,\ t > 0\} \subset E\times F = (-\infty < \xi < 1)\times(0 < \eta < \infty)$ is a function iff $T \in D'(0,\infty)$ and $S \in D'(-\infty,1)$ are functions.

PROOF. Assume $U \subset E$, $V \subset F$ such that $U\times V \subset R_{\xi,\eta}$. Let $\chi(\xi) \in D(U)$, $\psi(\eta) \in D(V)$. Then

$\varphi(x,t) = \chi(x-t).\psi(x+t) \in D(R)$ and

$$\theta(\varphi) = c_1\ S(\chi) + c_2 \int \chi(\xi)\ d\xi\ ,$$

$$c_1 = \frac{1}{2}\int \psi(\eta)\ d\eta\ , \quad c_2 = T(\psi)/2\ ,$$

as follows from (15). If θ is a function on R and ψ is kept fixed with $c_1 \neq 0$, we have

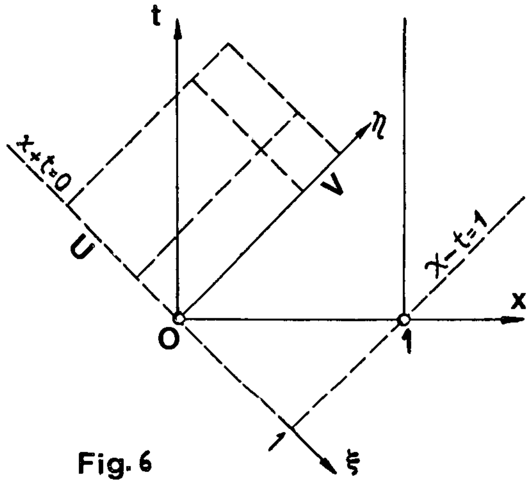


Fig. 6

$$S(\chi) = \int\left(\int (2c_1)^{-1}\theta\left(\frac{\xi+\eta}{2}\ ,\ \frac{\eta-\xi}{2}\right)\psi(\eta)d\eta\right)\chi(\xi)d\xi\ -\ (c_2/c_1)\int \chi(\xi)d\xi\ =$$

$$= \int \sigma(\xi)\ \chi(\xi)\ d\xi\ ,$$

with σ a locally integrable function on U. Then S is a function. Analogously it follows that T is a function. QED.

PROOF OF THEOREM 1. We call *admissible solutions* those functions θ solutions of (1) in the distribution sense given by series (7) with coefficients satisfying (8) in $L^2(0,1)$ such that

$S = \frac{1}{2}(f - \int_0^x g\, dy)$, $T = \frac{1}{2}(f + \int_0^x g\, dy)$ on $(0,1)$. The proof of the theorem will be accomplished if we prove the following *proposition:* the admissible solutions are all the solutions with S and T functions.

Because of the existence of at least one admissible solution for each pair f, g, it is sufficient to prove the proposition in the case $f \equiv 0 \equiv g$.

Then assume $f = g = 0$. If θ is an admissible solution necessarily $S = T = 0$ on $(0,1)$. In this situation the series (7) is determined by the four numbers $f(0)$, $f(1)$, $g(0)$ and $g(1)$. We have already shown that there is a one-to-one linear correspondence between 4-tuples $\{f(0),\dots\}$ and pairs of sequences of coefficients of null series $(\{A_\beta\},\{A'_\beta\})$, $A'_\beta = \beta B_\beta$ if $\beta \neq 0$, $A'_\beta = B_0$ if $\beta = 0$.
Given $(\{A_\beta\}, \{A'_\beta\})$ there is a pair $(S,T) = \alpha(\{A_\beta\},\{A'_\beta\})$ defined by (22). The application α is linear. These pairs (S,T) are in a one-to-one linear correspondence with a set of 4-tuples (C_0, D_0, c_1, d_1).

If α were one-to-one, the map $\{f(0),\dots\} \longrightarrow \{C_0,\dots\}$ would be onto and according to what we have proved above this would show that the pairs (S,T) defined by (22) are the only solutions which are functions, thus proving the proposition. Assume that $\alpha(\{A_\beta\} , \{A'_\beta\}) = (S,T) = (0,0)$ and that $\sum A_\beta u_\beta(0) = \gamma_0$, $\sum A_\beta u_\beta(1) = \gamma_1$, $\sum A'_\beta u_\beta(0) = \gamma'_0$, $\sum A'_\beta u_\beta(1) = \gamma'_1$.
Let us call R the function $R(x)$ of lemma 2 defined with the set $\{A_\beta\}$ and P the function $P(x)$, as in lemma 2, but obtained with the set of coefficients $\{A'_\beta\}$.
Then $S = \frac{1}{2}(R(x)-P(x))$ on $(-\infty,1)$ and $T = \frac{1}{2}(R(x)+P(x))$ on $(0,\infty)$.

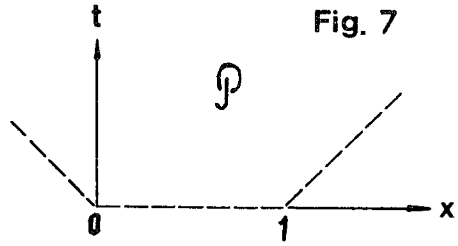
Since $P(x) \equiv 0$ on $[0,1]$ and is continuous function, we obtain from lemma 2 that $S=0$ implies $\gamma_0 = 0$. Also that $T=0$ implies $\gamma_1 = 0$. Therefore $\{A_\beta\} = 0$ and then $R=0$. In consequence, $2T = 0 = = P(x)$ on $(0,\infty)$ and $-2S = 0 = P(x)$ on $(-\infty,1)$. We get from this that $\gamma'_1 = 0 = \gamma'_0$, and $\{A'_\beta\} = 0$. Therefore, α is 1-1. QED.

We finish this paragraph with a lemma on the behaviour of those solutions that arise from null series.

Let us call $\mathcal{P}$ *the set of points* (x,t) *such that:* $t > 0$, $-t < x < t+1$.

By *a solution of problem* (1) we shall understand a solution in the distribution sense that verifies the initial conditions in a way to be described.

Fig. 7



LEMMA 4. Let $\{A_\beta\}$ be the set of coefficients of a null series that converges to φ_0 at $x=0$ and to φ_1 at $x=1$, and $\{A'_\beta\}$ the set of coefficients of a null series that converges to γ_0 at $x=0$ and to γ_1 at $x=1$.

Then if $A'_\beta = \beta B_\beta$ for $\beta \neq 0$, $A'_0 = B_0$, $\theta(x,t) = \sum A_\beta \cos \beta t . u_\beta(x) + + \sum B_\beta \sin \beta t . u_\beta(x) + B_0 t . u_0(x)$ is a solution of problem (1) for $f = g = 0$ on $(0,1)$ such that

i) $\theta(x,t)$ and $\theta_t(x,t)$ tend to 0 uniformly in compact sets of $(0,1)$ when $t \longrightarrow 0$;

ii) $\theta(0,t) \longrightarrow \varphi_0$, $\theta(1,t) \longrightarrow \varphi_1$;

iii) $\theta_t(0,t) \longrightarrow \gamma_0 - \varphi_0$, $\theta_t(1,t) \longrightarrow \gamma_1 - \varphi_1$;

iv) $\varphi_0 = 0 = \varphi_1$ implies that $\theta(x,t)$ is continuous on $\overline{\mathcal{P}}$, and therefore that $\theta(x,t) \xrightarrow{\cdot} 0$ on $[0,1]$ when $t \longrightarrow 0$.

PROOF. $\theta(x,t) = 0$ if $0 < x-t < x+t < 1$ (cf. proposition 1) and i) follows. On the other hand we have:

$\theta(x,t) = T(x+t) + S(x-t)$, where $T(x) = \frac{1}{2}(R(x) + P(x))$ on $0 < x$, $S(x) = \frac{1}{2}(R(x) - P(x))$ on $x < 1$, $R(x) = \sum A_\beta u_\beta(x)$,

$P(x) = \sum B_\beta \beta \int_0^x u_\beta(t)\,dt + B_0 x$.

From lemma 2 it follows that $\lim_{t\to 0+} \theta(0,t) = \lim_{t\to 0+} (T(t)+S(-t)) =$

$= \lim_{t\to 0+} T(t) + \lim_{t\to 0+} \frac{1}{2}(R(-t) - P(-t)) = 0 + \lim_{t\to 0+} \frac{R(-t)}{2} = \frac{R(-0)}{2} = \varphi_0$.

Also that: $\lim_{t\to 0+} \theta(1,t) = \lim_{t\to 0+} T(1+t) = \lim_{t\to 0+} \frac{R(1+t)}{2} = \varphi_1$. Thus ii) is proved. Let us see iii). We have

$\theta_t(x,t) = T'_{x+t} - S'_{x-t}$ and $\theta_t(0,t) = T'_t - S'_{-t} = T'(t) - S'(-t)$ on $\{0 < t\}\setminus Z$.

Then: $\lim_{t\to 0+} \theta_t(0,t) = \lim_{t\to 0+} -S'(-t) = \lim_{t\to 0+} \frac{P'(-t)-R'(-t)}{2}$.

The series $\sum A_\beta \beta \sin \beta x$ converges in $L^2(-1,0)$ since $u_\beta(x) = \cos \beta x - \beta \sin \beta x$ and $\{A_\beta\} \in \ell^1$. Therefore, if $-1 < x < 0$, $R(x) = 2 \sum A_\beta \cos \beta x$ and also $R'(x) = -2 \sum A_\beta \beta \sin \beta x$ a.e. On the other hand S, T, S' and T' converge when the argument tends to an integer. In consequence, from v), vi) lemma 2, it follows that $R'(y) = 2 S'(y) + P'(y)$ is continuous on $(-1,0)$ and has a limit for $y \longrightarrow 0-$.

Then, from $2 \sum A_\beta \cos \beta(-x) - R'(x) = 2 \sum A_\beta u_\beta(-x) = 0$, $-1 < x < 0$, we obtain: $R'(-0) = 2 \sum A_\beta = 2 \varphi_0$, and therefore $\lim_{t\to 0+} \theta_t(0,t) = \gamma_0 - \varphi_0$.

As before we have: $\lim_{t\to 0+} \theta_t(1,t) = \lim_{t\to 0+} T'(1+t) =$

$$= \lim_{t\to 0+} \frac{P'(1+t) + R'(1+t)}{2} .$$

For $0 < t < 1$: $R(1+t) = R(1+t) + R(1-t) = 2 \sum A_\beta \cos \beta t . u_\beta(1)$ (the series converges uniformly since $u_\beta(1) = 0(1)$) and $R'(1+t) =$

$= -2 \sum A_\beta \, \beta \sin \beta t . u_\beta(1)$ a.e. (the series converges in $L^2(0,1)$ since $\sum A_\beta \, u_\beta(t) \, u_\beta(1)$ converges in $L^2(0,1)$ and this because $\{A_\beta \, u_\beta(1)\} \in \ell^2$). Taking into account that $\beta \cos \beta + \sin \beta =$ $= -\beta \, (\cos \beta - \beta \sin \beta)$ it follows easily that $u_\beta(t) \, u_\beta(1) =$ $= u_\beta(1-t)$ and then $R(1-t) = \sum A_\beta \, u_\beta(t) \, u_\beta(1) = 0$ on $0 < t < 1$. In consequence: $2 \sum A_\beta \cos \beta t . u_\beta(1) + R'(1+t) = 2 \, R(1-t) = 0$. Since again $\lim_{t \to 0+} R'(1+t)$ exists, we obtain $R'(1+) =$ $= -2 \sum A_\beta \, u_\beta(1) = -2 \, \varphi_1$, and also that : $\lim_{t \to 0+} \theta_t(1,t) = \gamma_1 - \varphi_1$.

iv) The hypothesis implies that $\{A_\beta\} = 0$. Then, $S = -P/2$, $T = P/2$.

QED.

7. The solution of problem (1) with $a = b = c = \ell = 1$ that we are seeking must satisfy:

$\theta_{tt} - \theta_{xx} = 0$ in R; $\theta_{tt} - \theta_x = 0$ for $x=0$, $t > 0$; $\theta_{tt} + \theta_x = 0$ for $x=1$, $t > 0$.

Let us assume that $\theta(0,x) = f(x) \in C^2([0,1])$ and $\theta_t(0,x) = g(x) \in$ $\in C^1([0,1])$. In this case there exists a function, solution in the distribution sense (§5): $\theta(x,t) = T(x+t) + S(x-t)$ with S and T functions (cf. Lemma 3) such that $T = \frac{f}{2} + \frac{h}{2}$, $S = \frac{f}{2} - \frac{h}{2}$, $h(x) = \int_0^x g(y) \, dy$, for $0 < x < 1$.

The boundary conditions show as in §6 that the knowledge of T(S) on an open interval I_i determines S(T) on another interval I_k, except for summands of the form: $A_k \, e^{\pm t} + B_k$, since T and S must satisfy the following differential equations:

$$(32) \quad \begin{cases} T''(t) - T'(t) = -[S''(-t) - S'(-t)] \quad , \quad 0 < t < \infty \\ T''(1+t) + T'(1+t) = -[S''(1-t) + S'(1-t)] \, , \; 0 < t < \infty . \end{cases}$$

With a convenient choice of the A_k's and B_k's it is possible to obtain $T \in C^1([0,\infty))$, $S \in C^1((-\infty,1])$. Let us assume this regularity for S and T. *Under which conditions it is possible to assure that* $T \in C^2([0,\infty))$ *and* $S \in C^2((-\infty,1])$?

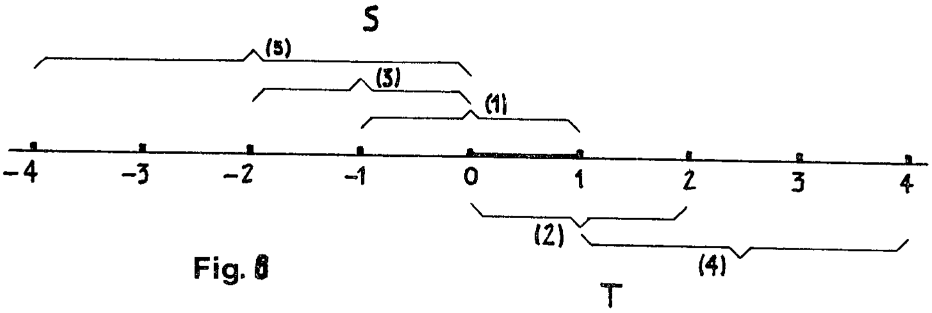


Fig. 8

First, let us consider S and the interval $-1 < s < 1$. If $s \in (-1,0)$: $S''(s) - S'(s) = -\frac{1}{2}[f''(-s) + g'(-s) - f'(-s) - g(-s)]$ implies $S''(0-) = f'(0) - \frac{g'(0)}{2} - \frac{f''(0)}{2}$ since $S'(0) = (\frac{f'-h'}{2})(0)$.

On the other hand from the definition of S we have:
$S''(0+) = \frac{f''(0) - g'(0)}{2}$.

That is, $f''(0) = f'(0)$ *is a necessary and sufficient condition for* $S''(0-) = S''(0+)$. The plausibility of the condition follows easily: if for $x=0=t$, $\theta_{tt} = \theta_{xx}$ and $\theta_{tt} = \theta_x$ then $\theta_{xx}(0,0) = \theta_x(0,0)$.

Next, let us consider T and the interval $0 < s < 2$. In an analogous way it can be seen that $f''(1) = -f'(1)$ *is a necessary and sufficient condition for* $T''(1-) = T''(1+)$, and an explanation for this can be given as before.

Then,

(33) $$f''(0) = f'(0) \quad , \quad f''(1) = -f'(1)$$

are requirements on the initial conditions imposed by the boundary conditions to get a certain regularity of the solution.

If $-2 < t < 0$ then $S''(t) - S'(t) = -[T''(-t) - T'(-t)] = d(t) \in$

$\in C((-2,0))$.

Since $S(0)$, $S'(0)$ are known, S is determined in that interval and coincides in $(-1,0)$ with the function already obtained. In consequence, in this step we have $S(x)$ well defined on $(-2,1)$ and twice continuously differentiable there. By reflection at $x=1$ we can extend T from $(0,2)$ to $(0,4)$ and in such a way that $T \in C^2$, (see Fig. 8). Repeating these steps we conclude that the first point of next lemma holds.

LEMMA 5. i) Assume $f \in C^2([0,1])$, $g \in C^1([0,1])$. If (33) holds then there exist S and T verifying (32) such that $S = \frac{f-h}{2}$ $T = \frac{(f+h)}{2}$, $h = \int_0^x g(s)\,ds$ on $x \in (0,1)$ and $S \in C^2((-\infty,1])$, $T \in C^2([0,\infty))$. This pair S, T is unique.

ii) Assume $f \in C([0,1])$, $g \in L^2(0,1)$. Then there exist S and T verifying (32) such that $S = \frac{f-h}{2}$, $T = \frac{f+h}{2}$, $h = \int_0^x g(s)\,ds$ on $x \in (0,1)$ and $T \in C([0,\infty))$, $S \in C((-\infty,1])$. This pair S, T is not unique.

iii) If $f \in C^1([0,1])$, $g \in C([0,1])$, then there exist S and T verifying (32) such that $S = \frac{f-h}{2}$, $T = \frac{f+h}{2}$, $h = \int_0^x g(s)\,ds$ on $x \in (0,1)$, and $S \in C^1((-\infty,1])$, $T \in C^1([0,\infty))$. This pair is unique.

PROOF. In the proof of ii) and iii) we make use of the following *proposition:*

a) Let F be a distribution on the open set $\Omega \subset (-\infty,\infty)$ such that $F'' + a F' = G''$ where a is a constant and G a continuous function. Then F is a continuous function.

b) If $F'' + a F' = G'$ then $F \in C^1(\Omega)$, (cfr. [4], p.131).

Now ii) follows from a) observing that S and T constructed by the "ping-pong method" given by relations (32), are uniformly continuous in the intervals I_k. A convenient choice of the constants A_k and B_k in each step yields continuity at the integers.

iii) follows from b) of preceding proposition.

QED.

LEMMA 6. Let f, f' and g be uniformly continuous in (0,1), $h = \int_0^x g\, ds$. Let S and T be functions such that on (0,1): $S = (f-h)/2$, $T = (f+h)/2$, and they verify (32). Then, i) $S \in C^1$ on $\{(-\infty,1)\setminus Z\}$ and $T \in C^1$ on $\{(0,\infty)\setminus Z\}$, being uniformly continuous together with their first derivatives on any finite subinterval.

The limits
$$C_0 = C_0(S) = \lim_{t\to 0+} S(t) - S(-t)$$
$$D_0 = D_0(S) = \lim_{t\to 0+} S'(t) - S'(-t)$$
and
$$c_1 = c_1(T) = \lim_{t\to 0+} T(1+t) - T(1-t)$$
$$d_1 = d_1(T) = \lim_{t\to 0+} T'(1+t) - T'(1-t)$$

determine uniquely S and T.

ii) $C_0 = c_1 = 0$ implies the continuity of S and T on their respective domains.

PROOF. Let $\tilde{S}$, $\tilde{T}$ as in iii) lemma 5. Let us call $\hat{S} = S - \tilde{S}$, $\hat{T} = T - \tilde{T}$. Then $\hat{S}$ and $\hat{T}$ are null in (0,1) and are determined uniquely by $C_0(\hat{S})$, $D_0(\hat{S})$ and $c_1(\hat{T})$, $d_1(\hat{T})$ (cfr. § 6).

Since $C_0(\hat{S}) = C_0(S)$, $D_0(\hat{S}) = D_0(S)$, $c_1(\hat{T}) = c_1(T)$, $d_1(\hat{T}) = d_1(T)$, i) follows.

ii) follows from the analogous property for $\hat{S}$ and $\hat{T}$.

QED.

8. In this section we study the behaviour of the solution $\theta(x,t)$ and of its derivative $\theta_t(x,t)$ for $t \to 0$ and $x \in [0,1]$.

We shall restrict ourselves to cases where concrete applications may be at hand. Suppose to begin with that f is continuous and g of bounded variation there.

For $0 < x-t < x+t < 1$, we have

$$\theta(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)ds + \frac{f(x+t)+f(x-t)}{2} . \tag{34}$$

Its derivative in the distribution sense is

$$\frac{\partial}{\partial t}\theta(x,t) = (g(x+t)+g(x-t))/2 + (f'_{x+t} - f'_{x-t})/2 , \tag{35}$$

and tends to g(x) in D'(0,1) if $t \to 0$.

For convergence in a finer topology as that of D'-more suitable for applications-f must be more regular. Let f, f', g be uniformly continuous in (0,1).

For such f, g we describe more precisely the correspondence

$$(\{A_\beta\} , \{A'_\beta\}) \longleftrightarrow (S,T) \longleftrightarrow (C_0, D_0, c_1, d_1) \tag{36}$$

giving the following complement to Lemma 6:

THEOREM 2. Let f, f' and g be uniformly continuous in (0,1). Then

i) $C_0 = f(0+) - \varphi_0$, $D_0 = \gamma_0 - g(0+) + C_0$,

ii) $c_1 = \varphi_1 - f(1-0)$, $d_1 = \gamma_1 - g(1-0) - c_1$,

where $f = \sum A_\beta u_\beta$ and $g = \sum A'_\beta u_\beta$ in $L^2(0,1)$, $\sum A_\beta u_\beta(0) = \varphi_0$, $\sum A'_\beta u_\beta(0) = \gamma_0$, $\sum A_\beta u_\beta(1) = \varphi_1$, $\sum A'_\beta u_\beta(1) = \gamma_1$.

PROOF. Let $S = \frac{1}{2}(R-P)$, $T = \frac{1}{2}(R+P)$, $R(y) = \sum A_\beta u_\beta(y)$,

$P(x) = \sum A'_\beta \int_0^x u_\beta(s)\, ds$.

Then $C_0 = \lim_{t\to 0+} [S(t)-S(-t)] = \frac{1}{2}\lim_{t\to 0+} [(R(t)-R(-t))-(P(t)-P(-t))]$.

But, leaving to the reader some a.e. arguments, we have

$$(37)\quad \Delta R = R(t) - R(-t) = \sum A_\beta \Delta u_\beta = -2 \sum A_\beta \beta \sin \beta t =$$
$$= 2 \sum A_\beta (u_\beta(t) - \cos \beta t) = 2(f(t) - \sum A_\beta \cos \beta t) \longrightarrow$$
$$\longrightarrow 2 (f(0+) - \varphi_0) \text{ for } t \longrightarrow 0+ , \text{ and}$$

$$(38)\quad \Delta P = P(t)-P(-t) = \sum A'_\beta \int_{-t}^{t} u_\beta(s)ds = \int_{-t}^{t} (\sum A'_\beta u_\beta(s))ds \underset{t\to 0+}{\longrightarrow} 0.$$

Also $D_0 = \lim_{t\to 0+} [S'(t)-S'(-t)] = \lim_{t\to 0+} \frac{1}{2} (\Delta R' - \Delta P')$.

But

$$(39)\quad \Delta R' = \sum A_\beta \frac{d}{dt}(u_\beta(t)+u_\beta(-t)) = -2 \sum A_\beta \beta \sin \beta t \longrightarrow 2 C_0 .$$

$$(40)\quad \Delta P' = \sum A'_\beta(u_\beta(t)-u_\beta(-t)) = 2 \sum A'_\beta (u_\beta(t) - \cos \beta t) =$$
$$= 2 (g(t) - \sum A'_\beta \cos \beta t) \longrightarrow 2 (g(0+) - \gamma_0) .$$

Therefore $D_0 = C_0 - (g(0+) - \gamma_0)$, and i) is proved.

We have

$$(41)\quad \begin{cases} u_\beta(x+t) = u_\beta(x)\, u_\beta(t) - (\beta^2+1) \sin \beta x \sin \beta t \\ u_\beta(x+t) + u_\beta(x-t) = 2 u_\beta (x) \cos \beta t \\ u_\beta(1-t) = u_\beta (t)\, u_\beta (1) \end{cases}$$

where to get the last identity the eigen-value equation must be used. Then, from (41) we have for $t > 0$.

$$(42)\quad R(1+t)-R(1-t) = \sum A_\beta (u_\beta(1+t) - u_\beta(1-t)) =$$
$$= 2 \sum A_\beta u_\beta(1) \beta \sin \beta t =$$
$$= \sum A_\beta (u_\beta(1+t) + u_\beta(1-t)) - 2 \sum A_\beta u_\beta(1-t) =$$
$$= \sum 2 A_\beta u_\beta(1) \cos \beta t - 2 f(1-t) \underset{t\to 0+}{\longrightarrow} 2(\varphi_1 - f(1-0)).$$

and

(43) $P(1+t)-P(1-t) = \sum A'_\beta \int_{1-t}^{1+t} u_\beta(x)dx = \int_{1-t}^{1+t} \sum A'_\beta u_\beta(x)dx \xrightarrow[t\to 0+]{} 0$

In consequence

$$T(1+t) - T(1-t) = \frac{1}{2} \Delta R + \frac{1}{2} \Delta P \longrightarrow \varphi_1 - f(1-0) = c_1 \text{ for } t \longrightarrow +0.$$

Also

(44) $R'(1+t)-R'(1-t) = \sum A_\beta \ (u_\beta(1+t) + u_\beta(1-t))' =$

$= -2 \sum A_\beta \ u_\beta(1) \ \beta \sin \beta t = - \Delta R \longrightarrow -2 c_1 \text{ for } t \longrightarrow 0+$,

and

(45) $P'(1+t)-P'(1-t) = \sum A'_\beta \ (u_\beta(1+t) - u_\beta(1-t)) =$

$= \sum A'_\beta \ (u_\beta(1+t) + u_\beta(1-t)) - 2 \sum A'_\beta \ u_\beta(1-t) =$

$= 2 \sum A'_\beta \ u_\beta(1) \cos \beta t - 2 g(1-t) \xrightarrow[t\to 0+]{} 2(\gamma_1 - g(1-0)).$

In consequence

$$\Delta T' \longrightarrow -c_1 + (\gamma_1 - g(1-0))$$

QED.

Finally we want to study the behaviour of θ and θ_t fot $t \longrightarrow 0$, under the same hypothesis of Theorem 2 when the set $\{A_\beta\}$ is chosen in such a way that $\theta(0,t) \longrightarrow f(0+)$, $\theta(1,t) \longrightarrow f(1-)$.

THEOREM 3. Let f, f', g be uniformly continuous in (0,1) and A_β such that $\varphi_0 = f(0+)$, $\varphi_1 = f(1-)$.

Then

i) $\theta(x,t) \longrightarrow f$ uniformly in (0,1) , $\theta(0,t) \longrightarrow f(0+)$
$\theta(1,t) \longrightarrow f(1-)$,

ii) $\frac{\partial \theta}{\partial t}(x,t) \longrightarrow g(x)$ uniformly in compact sets of (0,1) ,

iii) $\frac{\partial \theta}{\partial t}(0,t) \longrightarrow \gamma_0$, $\frac{\partial \theta}{\partial t}(1,t) \longrightarrow \gamma_1$

PROOF. From the hypothesis and Theorem 2 it follows that $C_0 = c_1 = 0$ Then by lemma 6, ii), S and T are continuous and uniformly continuous in (0,1). Therefore $\theta(x,t)$ is continuously extendable from P to $\overline{P}$, and i) follows.

ii) Given $\varepsilon > 0$, from (35) we obtain for $t \longrightarrow 0$, $0 < t < 1$,

$\frac{\partial\theta}{\partial t} = g(x) + o(1)$, $o(1)$ being uniform in $[\varepsilon, 1-\varepsilon]$.

iii) $\theta_t(0,t) = T'_t - S'_{-t} = T'_t - S'_t + S'_t - S'_{-t} =$

$= g(t) + S'_t - S'_{-t} \longrightarrow g(0+) + D_0(S) = \text{(by Th. 2)} =$

$= \gamma_0 + c_0 = \gamma_0$

$\theta_t(1,t) = T'_{1+t} - S'_{1-t} = T'_{1+t} - T'_{1-t} + T'_{1-t} - S'_{1-t} =$

$= T'_{1+t} - T'_{1-t} + g(1-t) \underset{t\to 0+}{\longrightarrow} d_1(T) + g(1-) =$

$= \text{(by Th. 2)} = \gamma_1 - c_1 = \gamma_1$. QED.

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AN EXAMPLE OF BESSEL EXPANSION ARISING FROM AN ANOMALOUS BOUNDARY DIFFERENTIAL PROBLEM

by

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ABSTRACT. The example introduced at the beginning of paper [1] is discussed and thus, this paper becomes an application of that one.

1. INTRODUCTION. a) Expansions of the form:

$$(1)\quad u = u(r,\theta,\varphi) = u(r,\theta) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} C_{n,i} \frac{J_{n+1/2}(s_{n,i}\, r)}{\sqrt{s_{n,i}\, r}} P_n(\cos\theta) e^{-s_{n,i}^2 \alpha^2 t}$$

where $0 \leqslant \theta < \pi$, $0 \leqslant \varphi < 2\pi$, $0 < r < \infty$, are solutions with axial symmetry of the heat equation $\Delta u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$, ([7], p.289).

If the problem is to find the temperature in a solid sphere of radius one whose surface is kept at temperature zero and its initial temperature is $f(r,\theta)$ then we must have $J_{n+1/2}(s_{n,i}) = 0$, $i = 1,2,\ldots$, $n = 1,2,\ldots$, and the $C_{n,i}$ must be chosen so that

$$(2)\quad f(r,\theta) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} C_{n,i} \frac{J_{n+1/2}(s_{n,i}\, r)}{\sqrt{s_{n,i}\, r}} P_n(\cos\theta).$$

Since $\sqrt{r}\, J_{n+1/2}(s_{n,i}\, r)\, P_n(\cos\theta)$ is an o.n. system, the coefficients $C_{n,i}$ are determined by the values (a.e) of $f(r,\theta)$.

In the particular case where $f(r,\theta) = f_o(r).\cos\theta$, since $P_1(\cos\theta) = \cos\theta$, $C_{n,i} = 0$ for $n \neq 1$. If we call $g(r) = r.f_o(r)$ and $B_j = C_{1,j}/\sqrt{s_j}$, $s_j = s_{1,j}$, we get

$$(3) \qquad g(r) = \sum_1^\infty B_j \{\sqrt{r}\, J_{3/2}(s_j.r)\} \quad , \quad J_{3/2}(s_j) = 0 \ .$$

b) If radiation of heat occurs from the sphere into an infinite medium of constant temperature zero, we have the following boundary condition ([3], p.312):

$$\frac{\partial u}{\partial n} + \sigma u = 0 \quad , \quad \sigma > 0 \ .$$

If we look for a solution of the diffusion equation satisfying this boundary condition, the separation of variables provides the following axially symmetric elementary solutions

$$\frac{1}{\sqrt{r}} J_{n+1/2}(s.r)\, P_n(\cos\theta)\, e^{-s^2 \alpha^2 t}$$

where s verifies

$$(4) \qquad s.J'_{n+1/2}(s) + (\sigma - 1/2)\, J_{n+1/2}(s) = 0 \ .$$

If the initial distribution of temperature f is independent of φ then an expansion like (2) holds. In the special case for which $f = f_o(r).\cos\theta$ we must again have (3) but where s satisfies the following relation at $r = 1$:

$$(5) \qquad \frac{d}{dr}\{\sqrt{r}\, J_{3/2}(s_j.r)\} + (\sigma - 1)\{\sqrt{r}\, J_{3/2}(s_j.r)\} = 0 \ .$$

In fact, this follows from (4) taking into account that

$$(6) \qquad s.J'_{3/2}(s.r) = \frac{1}{\sqrt{r}}\left[\frac{d}{dr}(\sqrt{r}\, J_{3/2}(s.r)) - \frac{1}{2\sqrt{r}} J_{3/2}(s.r)\right] \ .$$

c) The boundary condition (5) and that which appears in (3) are *ordinary boundary conditions* for Bessel's equation. Let us see an example of an *irregular boundary condition* for this differential equation.

Assume as before a solid sphere of radius one with an initial distribution of temperature axially symmetric with respect to the z-axis, cooled in a mass of liquid kept at uniform temperature v(t) at each instant t. The liquid receives heat also from its surroundings at constant temperature T_1.

If $u = u(r,\theta,t)$ denotes the temperature at the point (r,θ,φ) of the sphere at time t then $v(t) = u(1,\theta,t)$, $t > 0$, and

$$\Delta u = \frac{1}{r^2 \operatorname{sen}\theta} \{\frac{\partial}{\partial r}(r^2 u_r \operatorname{sen}\theta) + \frac{\partial}{\partial\theta}(u_\theta \operatorname{sen}\theta)\} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} .$$

The rate of accumulation of heat in the liquid is proportional to $\frac{dv}{dt}$ and equal to a linear combination of $-\left.\frac{\partial u}{\partial r}\right|_{r=1-}$ and of $(T_1 - v(t))$. Then, it holds with a $\nu > 0$, that ([6], pp. 262-3):

$$(7) \qquad \left.\frac{\sigma}{\alpha^2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} + \nu(u - T_1)\right|_{r=1-} = 0$$

If $w = u - T_1$ we obtain

$$(8) \qquad \Delta w = \frac{1}{\alpha^2}\frac{\partial w}{\partial t} \quad , \quad \left.\frac{\sigma}{\alpha^2}\frac{\partial w}{\partial t} + \frac{\partial w}{\partial r} + \nu w\right|_{r=1-} = 0 .$$

For a suitable choice of the units we can assume that $\alpha = 1$ and study the following differential system:

$$(9) \qquad \begin{cases} \Delta w = \frac{\partial w}{\partial t} \quad , \quad x = (x_1, x_2, x_3) \quad , \quad r = |x| < 1 \quad , \quad t > 0 . \\ \left.\sigma\frac{\partial w}{\partial t} + \frac{\partial w}{\partial r} + \nu w\right|_{r=1-} = 0 \quad , \sigma > 0 \quad , \quad \nu > 0 \text{ for } t > 0 . \end{cases}$$

$$(10)\quad \begin{cases} \lim\limits_{t\to 0+} w(x,t) = f(x) = f_o(r)\cos\theta \quad , \quad r<1 \ , \\ \lim\limits_{t\to 0+} w(x,t) = v_o \quad , \quad |x| = 1 \quad v_o = v(0) \ . \end{cases}$$

Separating variables in this problem, the solutions of the heat equation of the form: $w = R(r)\,\Theta(\theta)\,\Psi(t)$ verify (cf.[7], p.288):

$\frac{1}{\text{sen}\,\theta}\,(\Theta'(\theta)\,\text{sen}\,\theta)' + (n+1)\,n\,\Theta(\theta) = 0, \quad n = 0,1,2,\ldots ,$

$\Psi'(t) = -s^2\Psi(t)$

$$(11)\qquad R_{rr} + \frac{2\,R_r}{r} + \left(s^2 - \frac{n(n+1)}{r^2}\right) R = 0$$

In consequence

$$R = r^{-1/2}\left(c_1 s^{-(n+1/2)} J_{n+1/2}(s.r) + c_2 s^{n+1/2} J_{-(n+1/2)}(s.r)\right)$$

($s = 0$ is not an exceptional value in this formula). For the boundedness of R at the origin it is necessary that $c_2 = 0$. So we may take $c_1 = 1$ and call $R = R_{n,s} = r^{-1/2}\, s^{-(n+1/2)}\, J_{n+1/2}(s.r)$.

The boundary condition in (9) reduces to

$\sigma\,\Psi'\,R + \frac{dR}{dr}\,\Psi + \nu\,R\,\Psi \Big|_{r=1-} = 0$, therefore

$$(12)\qquad (\nu - \sigma\,s^2)\,R + \frac{dR}{dr} = 0 \quad , \quad r = 1 \quad , \quad R = R_{n,s} \ .$$

This is the equation of the eigen-frequencies s, which for each n, form a countable set Λ_n.

Then, we try to find a solution of (9), (10) of the form:

$$w = w(r,\theta,t) = \sum_{n,s} c_{n,s}\, e^{-s^2 t}\, R_{n,s}(r)\, P_n(\cos\theta).$$

Because of the initial conditions (10), we try expansions with $n = 0,1$. That is

(13) $w(r,\theta,t) = \sum_{s\varepsilon\Lambda_o} c_{o,s} e^{-s^2 t} R_{o,s}(r) + \sum_{s\varepsilon\Lambda_1} c_{1,s} e^{-ts^2} R_{1,s}(r) \cos\theta$.

Let us call $y = y_{n,s}(r) = r\, R_{n,s}(r) = \sqrt{r}.J_{n+1/2}(s.r)/s^{n+1/2}$.

Then y satisfies, in view of (11),

(14) $$y'' + (s^2 + \frac{1/4 - (n+1/2)^2}{r^2})\, y = 0 ,$$

and in view of (12) the boundary condition

(15) $$(\nu - \sigma s^2 - 1)\, y(1) + y'(1) = 0 .$$

That is, a boundary condition of the form

$$P(\lambda)\, y(1) + Q(\lambda)\, y'(1) = 0$$

where P and Q are polynomials in $\lambda = s^2$, not both constants and without common zeroes.

Besides, the boundedness of R at the origin implies the boundary condition:

(16) $$y(0) = 0.$$

From (13), in order to satisfy the initial condition (10),

(16 a) $$0 = \sum c_{o,s} \frac{y_{o,s}(r)}{r} , \quad r < 1 \quad ; \quad \sum c_{o,s}\, y_{o,s}(1) = v_o$$

(16 b) $$f_o(r) = \sum c_{1,s} \frac{y_{1,s}(r)}{r} , \quad r < 1 \quad ; \quad \sum c_{1,s}\, y_{1,s}(1) = 0$$

should hold. Expansions in series of Bessel functions of order n, $n \geqslant 0$, n an integer, arising from an irregular boundary problem with P and Q linear functions of λ, were already considered by P. Zecca in his paper [8].

2. NULL SERIES. Let us consider (16 a). $y_{o,s}(r)$ satisfies, in view of (14), (15) and (16), the boundary problem

$$(17)\quad \begin{cases} y'' + s^2 y = 0 \\ y(0) = 0 \\ y(1)\ P(s^2) + y'(1) = 0 \quad \text{where} \quad P(\lambda) = \nu-\sigma\lambda-1 \quad , \quad \nu,\sigma > 0. \end{cases}$$

We suppose that the eigen-frequencies of this problem are *simple* (cfr. Appendix II).

For the boundary problem (17) there exists a null series with sum equal to one at $r = 1$ and such that any other null series is a multiple of this one (the factor being the sum of the series at $r = 1$).

[This was proved, for example, in a previous joint paper by the authors. Alternative proof: according to [5], Ch.III, Th.4, in this situation the expansion of a $\Phi \in L^2(0,1)$ obtained through the calculus of residues-which was called in another paper Orr's expansion - converges to 0 at $r = 1$. Now take an eigenfunction $y(r)$ solution of (17) for a value s_o such that $P(s_o^2) \neq 0$. Then $y(1) \neq 0$, and the difference between $y(r)$ and its Orr series is a null series that converges to 0 in square mean and at each point in $[0,1)$, and converges to $y(1)$ at $r = 1$.

Since g (the degrees of freedom) is equal to one, any other null series is a multiple of this one].

So there exist coefficients $c_{o,s}$ such that $\sum c_{o,s}\ y_{o,s}(r)$ converges to 0 in $L^2(0,1)$, $\{c_{o,s}\ \|y_{o,s}\|_2\} \in \ell^2$ and $\sum c_{o,s} y_{o,s}(1) = v_o$.

This last series converges absolutely. In fact, $y_{o,s}(r) = \sqrt{\frac{2}{\pi}}\ \frac{\sin sr}{s}$ and then $y_{o,s}/\|y_{o,s}\| = 0(\sin s)$. From the boundary condition it follows that $s.\sin s = 0(1)$. Thus, $y_{o,s}/\|y_{o,s}\| = 0(1/s) \in \ell^2$. Therefore

(18) $$\sum c_{o,s} \frac{y_{o,s}(r)}{r} = 0 \quad \text{in} \quad L^2(|x| < 1) \ ,$$

and (16 a) is true in the mean.

Now we turn to (16 b). $y_{1,s}$ satisfies, in view of (14), (15) and (16), the boundary problem

(19) $$\begin{cases} y'' + (s^2 - \frac{2}{r^2})\, y = 0 \\ y(1)\, P(s^2) + y'(1) = 0 \quad , \quad P(\lambda) \text{ the same as in (17)} . \\ y(0) = 0 \end{cases}$$

It is easy to see that in (19) the boundary condition $y(0) = 0$ is equivalent to $y(r) \in L^2(0,1)$. The system of eigenfunctions verifying (19) is a particular case of those studied in [1].

There, in § 3, it is proved that the eigen-frequencies s_n verify: $s_n = n\pi + 0(1)$.

Besides, we suppose again that they are *all simple* (cfr. Appendix II).

In [1], § 7, it is proved that if $r\, f_o(r) \in L^2(0,1)$ (what is just to say that $f(x) \in L^2(|x| < 1)$), then there exists $\{c_{1,s}\}$ such that

(20) $\sum c_{1,s} y_{1,s}(r) = r\, f_o(r)$ in $L^2(0,1)$ and $\{c_{1,s} \| y_{1,s} \|_2\} \in \ell^2$.

The series in (20) just being the Orr series. We have also

(21) $$\sum c_{1,s}\, y_{1,s}(1) = 0$$

(cfr. Appendix I), and this series converges absolutely since

$\left\{ \frac{y_{1,s}(1)}{\| y_{1,s} \|_2} \right\} \in \ell^2$. In fact, $y_{1,s}(1)/\| y_{1,s} \|_2$ = (Prop. 3, [1]) =

$= \sqrt{s}.J_{3/2}(s)\ 0(1)$ = (boundary cond.) $= \sqrt{s}.J'_{3/2}(s)\ 0(1/s) =$

$= 0(1/s)$.

3. EXISTENCE. THEOREM 1. Suppose $f(x) = f_o(r)\cos\theta \in L^2(|x| < 1)$. Then, there exists a function $w(x,t)$, C^∞ on $|x| \leqslant 1$, $t > 0$, that verifies (9) and such that $\lim_{t\to 0+} w(x,t) = f(x)$ in $L^2(|x| < 1)$, $\lim_{t\to 0+} w(x,t) = v_o$ uniformly on $|x| = 1$.

PROOF. Having chosen the coefficients $c_{o,s}$ and $c_{1,s}$ we show that (13) is a solution.

Since

$$\frac{y_{n,s}(r)}{\|y_{n,s}\|} = N_s \sqrt{r.s}\ .\ J_{n+1/2}(r\ s) \tag{22}$$

where N_s is bounded, (Prop. 3, [1]), we have

$$\frac{d^j}{dr^j}\ \frac{y_{n,s}(r)/r}{\|y_{n,s}\|} = 0(s^{j+1}) \tag{23}$$

Then term by term differentiation is valid in (13) for $t > 0$, $r > 0$, and

$$w(r,\theta,t) = \sum_{s\varepsilon\Lambda_o} c_{o,s} e^{-s^2 t} \frac{y_{o,s}(r)}{r} + \cos\theta . \sum_{s\varepsilon\Lambda_1} c_{1,s} e^{-s^2 t} \frac{y_{1,s}(r)}{r} \tag{24}$$

defines a C^∞ function in $t > 0$, $0 < r$, that verifies the boundary condition (9), since each summand does. Since the summands are solutions of the heat equation in, say, $|x| < 2$, $t > 0$, and the equation is hypoelliptic, the continuous function w belongs also to $C^\infty(x,t)$ in that region.

For $r = 1$ we have, with absolute convergence,

$$\sum_{\Lambda_o} c_{o,s}\, y_{o,s}(1) + \cos\theta . \sum_{\Lambda_1} c_{1,s}\, y_{1,s}(1) = v_o\ ,$$

therefore, uniformly on $|x| = 1$, $\lim_{t\to 0+} w(x,t) = v_o$.

Also by construction

$$r(w(r,\theta,t) - f(x)) = \sum_{\Lambda_o} c_{o,s}(e^{-s^2t} - 1)\, y_{o,s}(r) \; +$$

$$+ \sum_{\Lambda_1} c_{1,s}(e^{-s^2t} - 1)\, y_{1,s}(r) \cos\theta \; =$$

$$= S_o(r,t) + S_1(r,t)\cos\theta$$

$$\int_{|x|<1} |w(r,\theta,t)-f(x)|^2\, dx = 2\pi \int_0^1 r^2\, dr \int_0^\pi \operatorname{sen}\theta\, d\theta\, |w - f|^2 =$$

$$= 4\pi \int_0^1 |S_o(r,t)|^2 dr + \frac{4}{3}\pi \int_0^1 |S_1(r,t)|^2 dt$$

From [1], proof of theorem 1, jj), it follows that

$$\int_0^1 |S_i(r,t)|^2\, dr \leqslant K.\sum_{s\varepsilon\Lambda_i} |c_{i,s}|^2\, \|y_{i,s}\|^2\, |e^{-s^2\,t} - 1|^2$$

which tends to 0 with $t \longrightarrow 0+$ by Lebesgue's dominated convergence theorem.

QED.

4. UNIQUENESS. Assume $w(x,t)$ is a function, C^∞ $(t > 0,\ |x| \leqslant 1)$ that verifies (9) and (10) in the sense of Theorem 1. We have,

THEOREM 2. If $G(\tau) = \int_{|x|<1} w^2(x,\tau)dx + \sigma\int_{|x|=1} w^2(x,\tau)d\Sigma$ then

$0 \leqslant G(\tau) \leqslant \|f\|_2^2 + 4\,\pi\,\sigma\, v_o^2$ for $\tau > 0$.

PROOF.

(25) $$\frac{1}{2}\frac{d}{d\tau} G(\tau) = \int_{|x|<1} \frac{\partial w}{\partial \tau}\, w\, dx + \sigma \int_{|x|=1} \frac{\partial w}{\partial \tau}\, w\, d\Sigma\ .$$

But

(26) $$\int_{|x|<1} \frac{\partial w}{\partial \tau}\, w\, dx = \int_{|x|<1} \Delta\, w.w\, dx = \text{by Green's formula} =$$

$$= \int_{|x|=1} \frac{\partial w}{\partial r} w \, d\Sigma - \int_{|x|<1} |\mathrm{grad}\ w|^2 \, dx .$$

Substituting (26) in (25) we obtain

$$\frac{1}{2} \frac{d}{d\tau} G(\tau) = - \int_{|x|<1} |\mathrm{grad}\ w|^2 \, dx + \int_{|x|=1} w . \left(\frac{\partial w}{\partial r} + \sigma \frac{\partial w}{\partial \tau}\right) d\Sigma =$$

$$= (\mathrm{cfr.}(9)) = - \int_{|x|<1} |\mathrm{grad}\ w|^2 \, dx - \int_{|x|=1} \nu . w^2 \, d\Sigma \leqslant 0.$$

Therefore $G(\tau) \leqslant \lim_{t \to 0+} G(\tau) = \int_{|x|<1} |f|^2 \, dx + 4 \pi \sigma v_o^2$. QED.

Theorem 1 shows not only the uniqueness of the solution but also the well-posedness of the problem.

APPENDIX I

We used in the preceding discussion the following complement to paper [1]. To prove it we assume familiarity with that paper and use its nomenclature. We want to show that the expansion of $f \in L^2$ obtained through the calculus of residues - which was called Orr's expansion - *converges to* 0 *at* x=1 *if* $p > q$. Orr's coefficients are exactly those chosen in (20)-(21) and denoted there by $\{c_{1,s} \|y_{1,s}\|_2\}$. If $G(x,y,\lambda)$ denotes Green's kernel and $\{C_n\}$ a set of adequate contours of integration, our desired result is equivalent to:

PROPOSITION. If $p > q$ then $\int_{C_n} G_\lambda(f)(1)\, d\lambda =$

$$= \int_{C_n} d\lambda \int_0^1 G(1,y,\lambda)\, f(y)\, dy = o(1) .$$

PROOF. We shall only sketch a proof. References of formulae are always to paper [1]. Then (cf. (15) and (55)):

$$G(1,y,s^2) = -\Phi(1)\Psi(y) \;;\; \Phi(1) = -Q(s^2) \;;\; \Psi(y) = \frac{e^{(y-1)|\sigma|}\; 0(1)}{s^{2p}}$$

and for $f \in L^1(0,1)$ we have

$$\int_{C_n} G_\lambda(f)(1)\, d\lambda = \int_{C_n} d\lambda \int_0^1 G(1,y,\lambda)\, f(y)\, dy =$$

$$= - \int_{C_n} Q(\lambda)\, d\lambda \int_0^1 \Psi(y)\, f(y)\, dy$$

where C_n is a contour adequately chosen. Let ψ be absolutely continuous, null in a neighborhood of x=1, such that $\int_0^1 |f-\psi|\, dx < \varepsilon$.

Call $A(\lambda) = \int_0^1 \Psi(y)\, \psi(y)\, dy$ and $B(\lambda) = \int_0^1 \Psi(y)\, (f-\psi)\, dy$.

Then $B(\lambda) = o(1)/s^{2p}$ and

(*) $$\int_{C_n} Q(\lambda)\ B(\lambda)\ d\lambda = o(1).$$

Now we consider the integral: $\int_{C_n} Q(\lambda)\ A(\lambda)\ d\lambda =$

$$= 2\int_{D_n} Q(s^2)\ A(s^2)\ s\ ds$$

where D_n is the semi-contour " $\sqrt{C_n}$ ". If $\hat{\Psi}$ denotes the primitive of Ψ null at x=0 we obtain by integration by parts:

(**) $$A(\lambda) = \int_0^1 \Psi(y)\ \psi(y)\ dy = -\int_0^1 \hat{\Psi}(y)\ \psi'(y)\ dy\ .$$

Next we estimate

$$\hat{\Psi}(y) = \int_0^y -\sqrt{u}\ \frac{J_\nu(u\ s)}{s^\nu\ w(s^2)}\,du = \frac{1}{s^{\nu+3/2}w(s^2)}\int_0^{ys} \sqrt{u}\ J_\nu(u)\ du$$

(cf. (33)). We have:

$$\int_0^t \sqrt{z}\ J_\nu(z)\ dz = \int_0^t z^{1/2-\nu-1}(z^{\nu+1}\ J_{\nu+1})'\ dz =$$

$$= t^{1/2}\ J_{\nu+1}(t) + (\nu+1/2)\int_0^t z^{-1/2}\ J_{\nu+1}(z)\ dz =$$

$$= 0(1)+0(1)\int_0^t z^{-1/2}J_{\nu+1}(z)\ dz =$$

$$= 0(1)+0(1)\int_0^t z^{-3/2}\ J_{\nu+2}(z)\ dz = 0(1)\ ,$$

and then: $\hat{\Psi}(y) = 0(1)\ /\ [\,s^{\nu+3/2}.w(s^2)]\,.$

From (33) and (55), it follows that $\frac{1}{s^{\nu}.w(s^2)} = \frac{0(1)}{s^{2p-1/2}.e^{|\text{Im } s|}}$.

Thus $\Psi(y) = \frac{0(1)}{s^{2p+1}}$.

From this estimation and (**), it follows that

$A(\lambda) = 0(1) \,/\, s^{2p+1}$, and in consequence $\int_{C_n} Q(\lambda)\, A(\lambda)\, d\lambda =$

$= \int_{D_n} \frac{0(1)\ s^{2q}}{s^{2p+1}}\, s\, ds = 0(1/s).$ QED.

APPENDIX II

Here we show that the pairs of values σ,ν such that the *eigenfrequencies of boundary problems* (17) *and* (19) *that are not simple form a set of first category* in the (σ,ν)-plane. Therefore the assumption that these eigenfrequencies are simple is reasonable.

We treat both, problems (17) and (19) simultaneously. Suppose $\sigma,\nu > 0$.

a) $s = 0$ is never an eigenfrequency,
$y = r^{n+1}$, $n = 0$ or 1, is the solution verifying the differential equation and $y(0) = 0$ in (17) and (19) respectively. Therefore $y(1)\ P(0) + y'(1) = (\nu-1) + (n+1) = \nu+n$ never vanishes for this solution.

b) If $s\neq 0$, $\sqrt{r}\ J_{n+1/2}(r.s)$, $n=0$ or 1, is, except for a constant factor, the solution of the differential equation verifying $y(0) = 0$ in (17) and (19) respectively. So the equation of the eigenfrequencies $s\neq 0$ is

$$J_{n+1/2}(s)\ P(s^2) + \frac{d}{dr}\left(\sqrt{r}\ J_{n+1/2}(r.s)\right)\Big|_{r=1} = 0 \quad , \quad \text{that is}$$

(†) $$(\nu - \sigma\ s^2 - 1/2)\ J_{n+1/2}(s) + s\ J'_{n+1/2}(s) = 0$$

If we equal to 0 its derivative with respect to s we obtain

$$\left(-(2\sigma+1)s + \frac{(n+1/2)^2}{s}\right) J_{n+1/2}(s) + (\nu - \sigma\ s^2 - 1/2)\ J'_{n+1/2}(s) = 0.$$

Therefore s is a multiple zero of (†) only if

$$0 = \begin{vmatrix} \nu - \sigma\ s^2 - 1/2 & s \\ \frac{(n+1/2)^2}{s} - s(2\sigma+1) & \nu-\sigma s^2-1/2 \end{vmatrix} = \sigma^2 s^4 + [3\sigma - 2\nu\sigma + 1]\ s^2 + (\nu+n)(\nu-n-1)$$

These values of s, $s = s(\sigma,\nu)$, satisfy the following formula:

$$2\sigma^2.s^2 = (2\nu-3)\sigma-1 \pm\sqrt{[(2\nu-3)\sigma - 1]^2 - 4\sigma^2(\nu+n)(\nu-n-1)} \; .$$

Such an s is actually a multiple eigenfrequency if it verifies (†). That is, if s^2 is a zero of the entire function of s^2,

$$F(s^2) = (\nu-\sigma s^2-1/2)\,\frac{J_{n+1/2}(s)}{s^{n+1/2}} + \frac{s\,J'_{n+1/2}(s)}{s^{n+1/2}} \; .$$

This yields: $\{\sigma,\nu\colon \sigma > 0,\ \nu > 0,$ (†) has a multiple zero$\}$ =

$= \{\sigma,\nu\colon \sigma > 0,\ \nu > 0,$

$F([(2\nu-3)\sigma-1 + \sqrt{[(2\nu-3)\sigma-1]^2-4\sigma^2(\nu+n)(\nu-n-1)}]/2\sigma^2) = 0$

or $F([(2\nu-3)\sigma-1 -\sqrt{\qquad}]/2\sigma^2) = 0\}$.

Since $s.J'_{n+1/2}(s)/J_{n+1/2}(s)$ is a meromorphic function but not algebraic, $\{(\sigma,\nu)\colon \sigma > 0,\ \nu > 0,$ (†) has a multiple zero$\}$ is nowhere dense. QED.

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