NOTAS DE ALGEBRA y ANALISIS

9

AGNES ILONA BENEDEK AND RAFAEL PANZONE

ON INVERSE EIGENVALUE PROBLEMS FOR A SECOND-ORDER DIFFERENTIAL EQUATION WITH PARAMETER CONTAINED IN THE BOUNDARY CONDITIONS

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ON INVERSE EIGENVALUE PROBLEMS FOR A SECOND-ORDER DIFFERENTIAL EQUATION WITH PARAMETER CONTAINED IN THE BOUNDARY CONDITIONS (*)

by A.Benedek and R.Panzone

SUMMARY. We prove some theorems for a two-point boundary value problem with boundary conditions linearly dependent on the eigenvalue parameter which are extensions of well-known results due to Borg, Levinson and Hochstadt-Lieberman.

1. INTRODUCTION. We consider the differential equation on $0 \leqslant x \leqslant \pi$,

(Q) $\quad u'' + (\lambda - Q)u = 0 \quad , \quad Q \in L^1(0,\pi)$ and real,

with the boundary conditions

(α) $\quad y(0)\cos\alpha + y'(0)\sin\alpha = 0 \quad , \quad 0 \leqslant \alpha < \pi$,

[β] $\quad -(\beta_1 y(\pi) - \beta_2 y'(\pi)) = \lambda(\beta_1' y(\pi) - \beta_2' y'(\pi)) \quad , \quad \rho = \beta_1'\beta_2 - \beta_1\beta_2' > 0.$

Let us denote with $\Lambda(Q,(\alpha),[\beta])$ the set of eigenvalues of the problem (Q),(α),[β].

THEOREM 1, a) *If* $\Lambda(Q,(\alpha),[\beta]) \cap \Lambda(Q,(\alpha),[\gamma]) = \emptyset$ *then* Q *is the only function in* $L^1(0,\pi)$ *such that* (Q) *has these spectrums with the indicated boundary conditions.*

b) *If* $\Lambda(Q,(\alpha),[\beta]) \cap \Lambda(Q,(\omega),[\beta]) = \emptyset$ *then the same result holds,* (cf. [B],[L]).

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THEOREM 2. *Assume now that* $Q(x) = Q(\pi - x)$. *If* $[\tilde{\beta}]$ *denotes the boundary condition at* $x=0$ *"symmetric" to* $[\beta]$:

$$[\tilde{\beta}] \qquad -(\beta_1 y(0)+\beta_2 y'(0)) = \lambda(\beta_1' y(0)+\beta_2' y'(0))$$

then Q *is the only function in* $L^1(0,\pi)$ *such that* $(Q),[\tilde{\beta}],[\beta]$ *has the spectrum* $\Lambda(Q,[\tilde{\beta}],[\beta])$, (cf. [B],[L]).

To prove these theorems we shall make use of results of Titchmarsh's book [T] which are stated there for Q continuous but do hold also for Q summable. Also we shall borrow many results proved in Fulton's paper [F] which hold for the two-point boundary value problem (Q), $(\alpha),[\beta]$. In [F] Q is assumed continuous but if τ denotes the operator $\tau u := -u''+Qu$ and the differential equation (Q) $\tau u = \lambda u$ is understood as usual to be verified almost everywhere, u and u' absolutely continuous, then all the results in [F] are still valid. J.Walter in [W] has given an operator-theoretic formulation of that irregular boundary problem which is used by Fulton.

It assumes the form

$$A(F) := \begin{pmatrix} -F_1''(x)+Q(x)F_1(x) \\ -R(F_1) \end{pmatrix} , \qquad F = \begin{pmatrix} F_1(x) \\ F_2 \end{pmatrix} ,$$

with domain $D(A) \subset H$:

$$D(A) = \{F \in H \mid F_1(x), F_1'(x) \text{ absolutely continuous in } [0,\pi],\ \tau F \in L^2(0,\pi),\ F_1(a)\cos\alpha + F_1'(a)\sin\alpha = 0,\ F_2 = R'(F_1)\},$$

where $H = L^2(0,\pi) \oplus C$ (cf.[F], (2.1)-(2.5)),

$$\|F\|_H^2 = \int_0^\pi |F_1|^2 dx + |F_2|^2/\rho \text{ , and}$$

$$(1) \qquad \begin{cases} R'(u) = R_\beta'(u) := \beta_1'.u(\pi) - \beta_2'.u'(\pi) , \\ R\,(u) = R_\beta(u) := \beta_1.u(\pi) - \beta_2.u'(\pi) . \end{cases}$$

Except for the proof that A is densely defined in H we leave to the reader the verification that even without changing the definition of

D(A) the results of [F] hold for $Q \in L^1$ with the obvious changes of "a.e." instead of "everywhere".

The classical boundary problem: $\tau y = \mu y, (\alpha), R(y) = 0$ has, for certain μ, a nontrivial solution y_0 which necessarily verifies $R'(y_0) \neq 0$. Then $Y = \begin{pmatrix} y_0 \\ R'(y_0) \end{pmatrix} \in D(A)$. Consider the problem $\tau z = \nu z$, $(\alpha), R'(z) = 0$. Let us denote with z_0 a linear combination of its eigenfunctions such that $\| z_0 - F_1 + cy_0 \|_2 < \varepsilon$ where $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \in H$, is given and $c = F_2/R'(y_0)$. Therefore if $Z = \begin{pmatrix} z_0 \\ 0 \end{pmatrix}$ then $Z \in D(A)$ and $\| F-(cY+Z) \|_H = \| F_1 - cy_0 - z_0 \|_2 < \varepsilon$. In consequence, $\overline{D(A)} = H$.

We shall also use results of Güichal's thesis [G]. Here too the theorems were proved under the hypothesis of Q continuous but they hold for Q summable with the obvious modifications.

The following result will be proved:

THEOREM 3. *Assume* $\tilde{Q}$ *and* Q *in* $L^1(0,\pi)$ *and* $Q = \tilde{Q}$ *a.e. in* $(\frac{\pi}{2},\pi)$. *If* $\Lambda(Q,(\alpha),[\beta]) = \Lambda(\tilde{Q},(\alpha),[\beta])$ *then* $Q = \tilde{Q}$ *a.e. in* $(0,\pi)$, (cf.[HL]).

We observe that the hypothesis $\Lambda(Q,(\alpha),[\beta]) \cap \Lambda(Q,(\alpha),[\gamma]) = \emptyset$ when $[\beta]$ is replaced by (β) and $[\gamma]$ by (γ), $0 \leqslant \alpha,\beta,\gamma < \pi$, is equivalent to $\beta \neq \gamma$, or what is the same, to $\sin(\beta-\gamma) \neq 0$. In fact, if y_β is a nontrivial solution of $(Q),(\alpha),(\beta)$ and y_γ one of $(Q),(\alpha),(\gamma)$, same λ, then $y_\beta = cy_\gamma$, $c \neq 0$. Therefore y_β satisfies (β) and (γ), which implies $\sin(\beta-\gamma) = 0$.

2. PROOF OF THEOREM 1. a) Let $\phi(x,\lambda)$, $\chi(x,\lambda)$ be two solutions of (Q) defined by $\phi(0,\lambda) = \sin\alpha$, $\phi'(0,\lambda) = -\cos\alpha$, $\chi(\pi,\lambda) = \beta_2'\lambda + \beta_2$, $\chi'(\pi,\lambda) = \beta_1'\lambda + \beta_1$.

The wronskian $w(\lambda) = W(\phi,\chi)$:

$$(2) \qquad w(\lambda) = (\beta_1'\lambda+\beta_1)\phi(\pi,\lambda)-(\beta_2'\lambda+\beta_2)\phi'(\pi,\lambda) = \lambda R_\beta'(\phi) + R_\beta(\phi) ,$$

will be called *the characteristic function* of the boundary value problem. Next we adapt the proof in [L] to our situation.

LEMMA 1. *The characteristic function is uniquely determined by the spectrum and the boundary conditions* (α),[β].

First we recall that (2) is an entire function with simple real zeroes which define the spectrum, ([F], p.296). To fix ideas we shall restrict ourselves to case 1 of [F]: $\alpha\neq 0$, $\beta_2'\neq 0$. In this case:
$w(\lambda) = \beta_2' \sin\alpha\ s^3 \sin\pi s + O(|s|^2 e^{|t|\pi})$, where $s = \sqrt{\lambda} = \sigma+it$, ([F], p.299). Therefore $w(\lambda)$ is an entire function of order 1/2 and since $n-3/2 < s_n = \sqrt{\lambda_n} < n-1/2$ for sufficiently large n, ([F], p.300), Hadamard's factorization theorem asserts that $w(\lambda) = C\,P(\lambda)$ where $P(\lambda)$, the canonical product, is of genus 0 and C is a constant. For $s = it$ and $t \longrightarrow \infty$ we have:

$$\beta_2' \sin\alpha \frac{t^3 \operatorname{sh}\pi t}{P(-t^2)} \longrightarrow C.$$

Since P is known, $w(\lambda)$ is determined. QED.

If $\tilde{Q} \in L^1$ is used instead of Q we shall write $\tilde{\phi}$ and $\tilde{\chi}$ instead of ϕ and χ. If [γ] denotes the boundary condition:

$$[\gamma] \qquad -(\gamma_1 y(\pi)-\gamma_2 y'(\pi)) = \lambda(\gamma_1' y(\pi)-\gamma_2' y'(\pi)) \quad , \quad \gamma_1'\gamma_2-\gamma_1\gamma_2' > 0 \quad ,$$

we have:

LEMMA 2. $\Lambda(Q,(\alpha),[\beta]) \cap \Lambda(Q,(\alpha),[\gamma])=\emptyset$, $\Lambda(Q,(\alpha),[\beta]) = \Lambda(\tilde{Q},(\alpha),[\beta])$ *and* $\Lambda(Q,(\alpha),[\gamma]) = \Lambda(\tilde{Q},(\alpha),[\gamma])$ *imply that for any* $\lambda_n \in \Lambda(Q,(\alpha),[\beta])$,

$$(3) \qquad C_n = \frac{\phi(x,\lambda_n)}{\chi(x,\lambda_n)} = \frac{\tilde{\phi}(x,\lambda_n)}{\tilde{\chi}(x,\lambda_n)} .$$

In fact, assume that C_n ($\neq 0$) is defined by the first equality and call $\tilde{C}_n = \tilde{\phi}(x,\lambda_n)/\tilde{\chi}(x,\lambda_n)$. Now observe that because of the hypothesis and lemma 1 the characteristic function for (Q),(α),[γ] and that one for

$(\tilde{Q}),(\alpha),[\gamma]$ coincide.

Let us denote it by $w_\gamma(\lambda)$. $w_\beta(\lambda)$ is defined analogously. If $w_\beta(\lambda_n) = 0$: $w_\gamma(\lambda_n) = \lambda_n R'_\gamma(\tilde{\phi})+R_\gamma(\tilde{\phi}) = \tilde{C}_n[\lambda_n R'_\gamma(\tilde{\chi})+R_\gamma(\tilde{\chi})] =$
$= \tilde{C}_n[\lambda_n R'_\gamma(\chi)+R_\gamma(\chi)] = (\tilde{C}_n/C_n)(\lambda_n R'_\gamma(\phi)+R_\gamma(\phi)) = (\tilde{C}_n/C_n)\; w_\gamma(\lambda_n)$,
where the equality of the brackets is obtained by direct calculation. Since $w_\gamma(\lambda_n) \neq 0$, $\tilde{C}_n/C_n = 1$ follows. QED.

To continue with the proof we collect some results. In case 1 of [F], $w_\beta(\lambda) = \beta'_2\, s^3 \sin\pi s \sin\alpha + O(|s|^2 e^{|t|\pi})$ (p.299) and the eigenfrequencies s_n verify $s_n = (n-1) + O(1/n)$, (p.300). Besides $C_n = 1/k_n$ ((3.9), p.296) and because of formulae (3.13) and (3.14) of [F] :
$w'_\beta(\lambda_n)/k_n = \|\Phi(.,\lambda_n)\|_H^2$ where $\Phi(x,\lambda_n) = \begin{pmatrix}\phi(x,\lambda_n)\\ R'_\beta(\phi(.,\lambda_n))\end{pmatrix}$.

a) will follow from:

PROPOSITION 1. *Assume that* $\tilde{Q}$ *is another function in* $L^1(0,\pi)$ *for which* $\Lambda(\tilde{Q},(\alpha),[\beta]) = \Lambda(Q,(\alpha),[\beta])$ *and* (3) *holds. Then* $Q = \tilde{Q}$ *a.e.*.

Firstly we have: $w'_\beta(\lambda_n)/k_n = \|\tilde{\Phi}(.,\lambda_n)\|_H^2$ where
$\tilde{\Phi}(x,\lambda_n) = \begin{pmatrix}\tilde{\phi}(x,\lambda_n)\\ R'_\beta(\tilde{\phi}(.,\lambda_n))\end{pmatrix}$ since w_β does not depend on Q.

Besides from [T], §1.7, we know that $\phi(x,\lambda) = \text{sen}\alpha \cos sx + O\left(\frac{e^{|t|x}}{|s|}\right)$ uniformly for $x \in [0,\pi]$ and from [F], p.304, that $\chi(x,\lambda) =$
$= \beta'_2\, s^2 \cos s(\pi-x) + O(|s|\; e^{|t|(\pi-x)})$, also uniformly for $x \in [0,\pi]$.
The same estimates hold for $\tilde{\phi}$ and $\tilde{\chi}$.

Assume now that $\begin{pmatrix}f\\ R'_\beta(f)\end{pmatrix} \in D(A)$, and define for $w(\lambda) = w_\beta(\lambda)$:

$$H(x,s^2) = H(x,\lambda) = \frac{\chi(x,\lambda)}{w(\lambda)} \int_0^x \tilde{\phi}(\xi,\lambda)\; f(\xi)\; d\xi .$$

Next we assume $|s| = n-1/2$. Using the estimates mentioned above we see that in case 1 of [F] the last integral is equal to

$\frac{\sin\alpha \;\sin sx}{s} f(x) + O\left(\frac{e^{|t|x}}{|s|}\right)(\delta + xe^{-\delta|t|})$, ([L], p.28), for δ a small positive quantity. Therefore:

(4) $$H(x,s^2) = [f(x)\; \frac{\cos\; s(\pi-x)}{s\; \sin\; \pi s} . \frac{\sin\; sx}{s}] . [1 + O(\frac{1}{|s|})] +$$

$$+ O\left(\frac{e^{-|t|x}}{|s|}\right) . O\left(\frac{e^{|t|x}}{|s|}\right) (\delta + xe^{-\delta|t|}).$$

We obtain for Γ, the circle of radius $(n-1/2)^2$, and $\gamma = \sqrt{\Gamma}$, that

$$\int_\Gamma H(x,\lambda)d\lambda = \int_\gamma H(x,s^2)s\; ds = \frac{f(x)}{2} \int_\gamma \frac{\sin\; s\pi + \sin\; s(2x-\pi)}{s.\sin\; \pi s} .$$

$$.(1 + O(\frac{1}{|s|}))\; ds + O(\int_\gamma (x\; e^{-\delta|t|} + \delta)\, |\frac{ds}{s}|)$$

and as in [L], pp.28-29, it follows that

(5) $$\int_\gamma H(x,s^2)sds = \pi i f(x) + O(\frac{1}{n}) + \frac{f(x)}{2} \int_\gamma \frac{\sin\; s(2x-\pi)}{s.\sin\pi s}\; (1+O(\frac{1}{|s|}))\; ds +$$

$$+ O(\delta \vee \frac{1}{\delta(n-1/2)}).$$

If $\delta = \frac{1}{\sqrt{n-1/2}}$ we get:

$$\int_\gamma H(x,s^2)sds - \pi i f(x) = O(\frac{1}{\sqrt{n}}) + O(\int_\gamma e^{|t|2(x-\pi)}|\frac{ds}{s}|) = o(1) ,$$

uniformly on compact sets of $(0,\pi)$ and boundedly in the closed interval $[0,\pi]$. If instead of $H(x,\lambda)$ we consider

(6) $$K(x,\lambda) = \frac{\phi(x,\lambda)}{w(\lambda)} \int_x^\pi \tilde{\chi}(\xi,\lambda) f(\xi) d\xi$$

we obtain, with $o(1)$ as before,

(7) $$\frac{1}{2\pi i}\int_\Gamma (H+K)d\lambda = f(x) + o(1).$$

Then from (3) and (7) it follows (in L^2):

(8) $$f(x) = \sum \frac{\phi(x,\lambda_n)\int_0^\pi \tilde{\phi}(\xi,\lambda_n)f(\xi)d\xi}{C_n . w'(\lambda_n)} = \sum \psi_n(x) . \int_0^\pi \tilde{\psi}_n(\xi)f(\xi)d\xi$$

where $\psi_n(x) = \phi(x,\lambda_n)/\|\Phi(.,\lambda_n)\|_H$, etc..

(8) holds in all other cases which are handled in the same way. For example if $\alpha=0$ and $\beta_2'\neq 0$ we are in case 2 of [F]. In this case $s_n = n - \frac{1}{2} + O(\frac{1}{n})$ and $w(\lambda) = \beta_2' \, s^2 \cos s\pi + O(|s|e^{|t|\pi})$. Therefore on γ, the circle of radius n, the first bracket in (4) is now equal to: $\frac{\cos s(\pi-x)}{\cos s\pi} \, \frac{\cos sx}{s^2} f(x)$, (observe that the hypothesis $\begin{pmatrix} f \\ R'(f) \end{pmatrix} \in D(A)$, $R' = R_\beta'$, implies $f(0) = 0$ if $\sin\alpha = 0$). The ordinary expansion $f = \sum \psi_n \langle \psi_n, f \rangle$ follows from (8) when $Q = \tilde{Q}$.

A particular case of (8) is

(9) $$0 = \psi_n(x) - \sum_m \psi_m(x)\int_0^\pi \tilde{\psi}_m(t)\psi_n(t)dt = -\sum_{m\neq n}\langle \tilde{\psi}_m,\psi_n\rangle\psi_m + $$
$$+ (1 - \langle \tilde{\psi}_n,\psi_n\rangle)\psi_n.$$

It defines a null series.

Since $(A-\lambda_n)\Psi_n = 0$ it follows that $\lambda_n R'(\psi_n)+R(\psi_n) = 0$. Then, if $R'(\psi_n) = 0$ we would have $R(\psi_n) = 0$, and also $\psi_n(\pi) = \psi_n'(\pi) = 0$, contradiction. In consequence, $R'(\psi_n) \neq 0 \; \forall \, n$, and (3.28) of [F] defines a nontrivial null series. The coefficientes of any other null series are proportional to those of this one ([G], ch.V, p.51, or [BP]). This implies that

$$(10) \qquad \langle \tilde\psi_m, \psi_n \rangle = h_n \, R'(\psi_m) \quad , \quad m \neq n \quad ; \quad -1 + \langle \tilde\psi_m, \psi_m \rangle = h_n \, R'(\psi_n) .$$

But from [F], (3.13), $R'(\psi_m) = \rho/(k_m \| \Phi_m \|) = \rho C_m / \| \Phi_m \|_H = \rho \tilde C_m / \| \tilde\Phi_m \|_H$, and in consequence:

$$(11) \qquad R'(\psi_m) = R'(\tilde\psi_m) \quad \forall \ m.$$

Taking into account (10) and (11) we obtain:

$$(12) \qquad \langle \tilde\psi_m, \psi_n \rangle = h_n \, R'(\tilde\psi_m) \ , \ m \neq n \quad ; \quad -1 + \langle \tilde\psi_n, \psi_n \rangle = h_n \, R'(\tilde\psi_n) .$$

The dual expansion of (8) is

$$(13) \qquad f(x) = \sum \tilde\psi_n(x) \int_0^\pi \psi_n(t) f(t) dt$$

The preceding argument applied to this expansion provides the following set of relations:

$$(14) \qquad \langle \psi_m, \tilde\psi_n \rangle = \tilde h_n \, R'(\psi_m) \ , \ m \neq n \quad ; \quad -1 + \langle \psi_n, \tilde\psi_n \rangle = \tilde h_n \, R'(\psi_n) \ ,$$

which together with (12) imply $\forall$ m,n: $h_n / R'(\psi_n) = \tilde h_m / R'(\tilde\psi_m)$.

From this it follows that

$$(15) \qquad h_n = \tilde h_n \quad , \quad \langle \tilde\psi_m, \psi_n \rangle = \langle \psi_m, \tilde\psi_n \rangle \quad , \quad \forall \ m,n.$$

From (8) (or [F] (3.30)) we know that

$$\tilde\psi_n(x) = \sum \psi_m(x) \langle \psi_m, \tilde\psi_n \rangle \ .$$

This together with (9) and (15) imply $\psi_n = \tilde\psi_n \ \forall \ n$. Taking into account that these are eigenfunctions for the same eigenvalue λ_n: $\tilde Q \psi_n = Q \psi_n$ a.e. which implies $\tilde Q = Q$ a.e. QED.

b) Next we show that (3) holds even under the hypothesis of part b). Part b) will then follow from proposition 1. Observe that $0 \leq \alpha, \omega < \pi$, $\alpha \neq \omega$. We shall distinguish the characteristic functions by the indexes α and ω only since they do not depend on Q or $\tilde Q$. If $\tilde\tau(x,\lambda)$ denotes the solution of $(\tilde Q)$ such that $\tilde\tau(0,\lambda) = \sin \omega$, $\tilde\tau'(0,\lambda) =$

$= -\cos\omega$ we have: $w_\omega(\lambda) = \tilde{\tau}\,\tilde{\chi}' - \tilde{\tau}'\,\tilde{\chi}$.

Therefore, recalling that $\tilde{C}_n = \tilde{\phi}(x,\lambda_n)/\tilde{\chi}(x,\lambda_n)$ we get:

(16) $$w_\omega(\lambda_n) = \sin(\alpha-\omega)/\tilde{C}_n.$$

But the same argument applied to (Q) instead of $(\tilde{Q})$ shows that $w_\omega(\lambda_n) = \sin(\alpha-\omega)/C_n$. Then $C_n = \tilde{C}_n \quad \forall\, n$. QED.

3. PROOF OF THEOREM 2. The characteristic function $w(\lambda)$ of problem $(Q),[\tilde{\beta}],[\beta]$ is, by definition, given by $W(\phi,\chi)$, where $\phi(0,\lambda) = \beta_2+\lambda\beta_2'$, $\phi'(0,\lambda) = -(\beta_1+\lambda\beta_1')$, $\chi(\pi,\lambda) = \beta_2+\lambda\beta_2'$, $\chi'(\pi,\lambda) = \beta_1+\lambda\beta_1'$. Since $\phi(\pi-x,\lambda)$ and $\chi(x,\lambda)$ verify the differential equation (Q) and the same initial conditions at $x=\pi$, we have $\phi(\pi-x,\lambda) = \chi(x,\lambda)$. Therefore, $w(\lambda) = -(\phi(x,\lambda).\phi'(\pi-x,\lambda)+\phi'(x,\lambda).\phi(\pi-x,\lambda)) = -2.w_1(\lambda).w_2(\lambda)$, $w_1(\lambda) = \phi(\frac{\pi}{2},\lambda)$, $w_2(\lambda) = \phi'(\frac{\pi}{2},\lambda)$. w_1 and w_2 are, respectively, the characteristic functions of the eigenvalue problems on $\frac{\pi}{2} \leqslant x \leqslant \pi$:

1) $(Q),(0),[\beta]$, $((0) \equiv y(\frac{\pi}{2})=0)$; 2) $(Q),(\frac{\pi}{2}),[\beta]$, $((\frac{\pi}{2}) \equiv y'(\frac{\pi}{2})=0)$.

In consequence, w_1 and w_2 have no zeroes in common and all their zeroes are simple. Because of the above relation $w(\lambda)$ is an entire function of order $\leqslant 1/2$ and of genus 0 with asymptotic expansion not dependent on Q. So, as in Lemma 1, $w(\lambda)$ is uniquely characterized by the spectrum and the boundary conditions. The problem is to determine from $w(\lambda)$ alone which of its zeroes are zeroes of w_1 and which ones are zeroes of w_2. If this is done, Theorem 1 implies that Q is uniquely determined by w on $[\pi/2,\pi]$ and therefore on $[0,\pi]$.

Because of the symmetry of the problem an eigenfunction $\phi(x,\lambda_n)$ satisfies the relation: $\phi(x,\lambda_n) = C_n.\phi(\pi-x,\lambda_n) = C_n.\chi(x,\lambda_n)$, and also : $\phi'(x,\lambda_n) = -C_n\phi'(\pi-x,\lambda_n)$. Then, $C_n = 1$ if $\phi'(\pi/2,\lambda_n) = 0$ and $C_n = -1$ if $\phi(\pi/2,\lambda_n) = 0$. So, the sign of C_n determines whether

λ_n is a zero of w_1 or w_2. Since the function w is real on the real axis and since its zeroes are real and simple, it will be sufficient to prove that

(17) $$C_n = \text{sign } w'(\lambda_n)$$

to have a criterium to separate the zeroes. We have

$$(\lambda_n-\lambda)\cdot\int_0^\pi \phi(x,\lambda_n)\phi(x,\lambda)dx = C_n(W(\chi(\pi,\lambda_n),\phi(\pi,\lambda))-W(\chi(0,\lambda_n),\phi(0,\lambda))) =$$

$$= C_n[(\beta_2'\lambda_n+\beta_2)\phi'(\pi,\lambda)-(\beta_1'\lambda_n+\beta_1)\phi(\pi,\lambda)+\chi(0,\lambda_n)(\beta_1'\lambda+\beta_1) +$$

$$+ \chi'(0,\lambda_n)(\beta_2'\lambda+\beta_2)] = C_n[-w(\lambda)+(\lambda_n-\lambda).\{\beta_2'\phi'(\pi,\lambda)-\beta_1'\phi(\pi,\lambda) -$$

$$- \chi(0,\lambda_n)\beta_1'-\chi'(0,\lambda_n)\beta_2'\}].$$

Therefore, if we call $R'_{\tilde{\beta}}(\psi) = \psi(0)\beta_1'+\psi'(0)\beta_2'$, (cf.(1)), then we obtain:

$$\int_0^\pi \phi(x,\lambda_n)\phi(x,\lambda)dx = C_n[w(\lambda)/(\lambda-\lambda_n) - R'_\beta(\phi(.,\lambda))- R'_{\tilde{\beta}}(\chi(.,\lambda_n))]$$

which, for $\lambda \longrightarrow \lambda_n$ tends to

$$\int_0^\pi |\phi(x,\lambda_n)|^2dx = C_n[w'(\lambda_n) - R'_\beta(\phi(.,\lambda_n))- R'_{\tilde{\beta}}(\chi(.,\lambda_n))].$$

But since $R'_\beta(\phi(.,\lambda_n)) = C_n\rho$, $R'_{\tilde{\beta}}(\chi(.,\lambda_n)) = \rho/C_n$, we obtain:

(18) $$0 < \int_0^\pi |\phi(x,\lambda_n)|^2dx + (C_n^2+1)\rho = C_n.w'(\lambda_n).$$ QED.

The preceding classification of the zeroes of $w(\lambda)$ as zeroes of w_1 or w_2 shows, since $w'(\lambda_n)$ changes sign alternately, that the zeroes of w_1 and w_2 interlace. Since for $w_1(\lambda_n) = 0$, $\phi(x,\lambda_n)$ has an odd number of zeroes in $(0,\pi)$ and for $w_2(\lambda_n) = 0$, $\phi(x,\lambda_n)$ has an even number of zeroes in $(0,\pi)$, we have proved:

PROPOSITION. *Under the hypothesis of Theorem 2 the number of zeroes* *of $\phi(x,\lambda_n)$ in $(0,\pi)$ changes in an odd number when* n *increases in one.*

This proposition - which can be used to classify the zeroes instead of (17) - is consequence of the symmetry of the problem and it does not hold in the general case. In fact, assume $\phi_\lambda(x) = \phi(x,\lambda)$ is defined as in §2 for the problem (Q),(α),[β]. It can be shown that $\phi_\lambda'(\pi)/\phi_\lambda(\pi)$ decreases from $+\infty$ to $-\infty$ when λ increases between to consecutive zeroes of $\phi_\lambda(\pi)$, μ and μ'. λ is an eigenvalue for that problem iff $\phi_\lambda'(\pi)/\phi_\lambda(\pi) = (\lambda\beta_1'+\beta_1)/(\lambda\beta_2'+\beta_2)$. Therefore, if $-\beta_2/\beta_2' \in (\mu,\mu')$ then two eigenvalues of (Q),(α),[β] belong to this interval and the corresponding eigenfunctions have the same number of zeroes on $(0,\pi)$.

4. PROOF OF THEOREM 3. Next we exhibit the main steps of a proof of theorem 3 following the pattern given in [H-L] which the reader should consult for more details. We shall restrict ourselves to case 1 of [F] and use the notation and some results of §2 of this paper.

From the differential equations (Q) and ($\tilde{Q}$) and the boundary condition at x=0 we obtain: $-\tilde{\phi}(\pi,\lambda)\phi'(\pi,\lambda) + \tilde{\phi}'(\pi,\lambda)\phi(\pi,\lambda) = = \int_0^\pi (\tilde{Q}-Q)(x)\phi(x,\lambda)\tilde{\phi}(x,\lambda)dx$. If we call $H(\lambda)$ the left hand side of this equality then from the hypothesis it follows that:

$$\int_0^{\pi/2} (\tilde{Q}-Q)\phi\tilde{\phi}dx = H(\lambda) \quad , \ \forall\ \lambda. \tag{19}$$

$H(\lambda)$ is an entire function, null at the zeroes of $w(\lambda)$ and such that $H(\lambda) = O(e^{\pi|t|})$. Let us also see that the entire function

$$\eta(\lambda) = H(\lambda)/w(\lambda) = O(1/\sqrt{|\lambda|}). \tag{20}$$

(This estimation will imply $\eta(\lambda) \equiv 0$ and therefore $\int_0^{\pi/2} (\tilde{Q}-Q)\phi\tilde{\phi}dx \equiv 0$.

From the last identity $Q=\tilde{Q}$ a.e. follows as can be seen in [H-L], pp.679-680).

Let us denote with F a closed disk with center at the origin of radius R great enough, A the region in the complement of the disk such that $|\mathrm{Im}\ z| > 1/2$ and B_n a square with center at $z = n$ with sides of length 1. Since $|\sin(\sigma+it)|^2 = \sin^2\sigma + \mathrm{sh}^2 t$, on A: $|\eta(\lambda)| =$
$= |H(\lambda)/w(\lambda)| = O(1)[e^{|t|\pi}/|\beta_2' \sin\alpha . s^3 \sin\pi s|] = O(e^{\pi|t|}/|\lambda|^{3/2} . \mathrm{sh}|t|\pi) =$
$= O(|\lambda|^{-3/2})$.

On the other hand, on ∂B_n : $|s^3 . \sin\pi s| \geq \varepsilon . |s|^3$, $\varepsilon > 0$, since the eigenfrequencies satisfy: $s_n = (n+1) + O(1/n)$, and R is adequately great. Therefore, $|\eta(\lambda)| \leq M(\varepsilon) . |s|^{-3}$ on ∂B_n, and in consequence on B_n. Since this estimation holds under translations of the form $s \longrightarrow s+1$, we finally get: $\eta(\lambda) = O(|\lambda|^{-3/2})$. For other cases the less generous result (20) is obtained. QED.

REFERENCES

[B] BORG, G., *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe,* Acta Math., Vol.78, (1945), 1-96.

[B-P] BENEDEK, A. and PANZONE, R., *On the structure of the set of eigenfunctions of certain irregular boundary problems,* to appear.

[F] FULTON, Ch.T., *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions,* Proc. Royal Soc. Edinburgh, 77A, (1977), 293-308.

[G] GÜICHAL, E.N., *Sobre un problema de Sturm-Liouville con condiciones de contorno irregulares,* Notas de Algebra y Análisis n°7, Inst. de Matemática, Univ.Nac. del Sur, R.Argentina (1978), 1-78.

[Ha] HALD, O.H., *On discrete and numerical inverse Sturm-Liouville problems,* Uppsala Univ., Dep. of Computer Sc. Report n°42, (1972), 1-175.

[H-L] HOCHSTADT, H. and LIEBERMAN, B., *An inverse Sturm-Liouville problem with mixed given data,* Siam J.Appl. Math.34, (1978), 676-680.

[L] LEVINSON, N., *The inverse Sturm-Liouville problem,* Mat. Tidsskrift, (1949), 25-30.

[Le] LEVIN, B.I., *Distribution of zeros of entire functions,* New York, (1964).

[T] TITCHMARSH, E.C., *Eigenfunction expansions,* I, Oxford, (1962).

[W] WALTER, J., *Regular eigenvalue problems with eigenvalue parameter in the boundary conditions,*Math.Z. 133 (1973),301-312.