NOTAS DE LOGICA MATEMATICA

16

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Two sets of axioms for boolean algebras

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TWO SETS OF AXIOMS FOR BOOLEAN ALGEBRAS

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In (1), B.A. Bernstein has given a set of axioms for Boolean algebras in terms of implication and of the constant 0 (first element of the algebra), based on the following two identities:

I) $(a \to b) \to a = a$

II) $(d \to d) \to ((a \to b) \to c) = (((c \to 0) \to a) \to ((b \to c) \to 0)) \to 0$

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(1) See the bibliographical references at the end of this article.

In A) we give a simpler axiomatic of the same kind; and, in B) another, in terms of implication and negation, which is a slight modification of the former.

Our characterizations can be useful, for instance, in order to simplify the verification that Lindenbaum algebra of the classic propositional calculus, for certain of its usual formulations, such as those of Wajsberg and of Frege-Lukasiewicz (see Church, (iii)), is a Boolean algebra.

A) AXIOMS IN TERMS OF IMPLICATION AND 0 .

Let $\mathcal{A} = (X, \to, 0)$ be a system where 0 denotes a fixed element of X and $\to$ a binary operation defined on X, such that, for every $a, b, c \in X$, the following equalities are verified:

1) $(a \to a) \to b = b$

2) $a \to ((c \to 0) \to (b \to 0)) = (a \to b) \to (a \to c)$

The following equalities hold in $\mathcal{A}$:

3) $(c \to 0) \to (b \to 0) = b \to c$

is obtained substituting $a \to a$ for a in 2) and using 1).

4) $(c \to 0) \to 0 = c$

Making $b = 0 \to 0$ in 3) and applying 1)

5) $a \to (b \to c) = (a \to b) \to (a \to c)$

By 2) and 3)

6) $a \to a = 0 \to 0$

Using 4) and 1)

7) $a \to (0 \to 0) = 0 \to 0$

By 5) and 6)

8) $(a \to b) \to b = (b \to 0) \to a = (a \to 0) \to b = (b \to a) \to a$

Applying successively 3) , 4) , 5) , 4) :

$$(a \to b) \to b = ((b \to 0) \to (a \to 0)) \to ((b \to 0) \to 0) =$$
$$= (b \to 0) \to ((a \to 0) \to 0) = (b \to 0) \to a$$

Permuting a and b we have: $(b \to a) \to a = (a \to 0) \to b$

8) is obtained observing that, by 3) and 4), $(a \to 0) \to b = (b \to 0) \to a$

9) <u>DEFINITION: $1 = 0 \to 0$</u>

10) DEFINITION: $-a = a \to 0$

Taking in account the above definitions, it follows:

1') $1 \to b = b$

3') $-c \to -b = b \to c$

4') $--c = c$

6') $a \to a = 1$

7') $a \to 1 = 1$

8') $(a \to b) \to b = -b \to a = -a \to b = (b \to a) \to a$

9') $-0 = 1$

11) DEFINITION: $a \leq b$ if and only if $a \to b = 1$

From 5) and 7) it is immediately seen:

12) If $a \leq b$, then $c \to a \leq c \to b$

13) DEFINITION: $a \vee b = -a \to b$

14) DEFINITION: $a \wedge b = -(-a \vee -b)$

THEOREM: The system $\mathcal{B} = (X, \vee, \wedge, -, 0, 1)$ is a Boolean algebra in which $a \to b = -a \vee b$.

PROOF: 1º) X is a partially ordered set through the relation $\leq$.

The reflexivity follows immediately from 6').

From $a \leq b$ and $b \leq a$ we obtain $a = b$. In fact, by 8), $(a \to b) \to b = (b \to a) \to a$ and, by hypothesis, $a \to b = b \to a = 1$, hence, by 1'), $b = a$.

From $a \leq b$ and $b \leq c$, i.e. $a \to b = b \to c = 1$, substituting in 5) we have $a \to 1 = 1 \to (a \to c)$. By 7') and 1') we obtain $a \to c = 1$, i.e. $a \leq c$.

2º) The mapping which to every $x \in X$ assigns the value $-x$ is a dual order automorphism of period two, i.e.

i) $x \leq y$ is equivalent to $-y \leq -x$

ii) $--x = x$

i) and ii) follow immediately from 3') and 4').

3º) 0 and 1 are, respectively, the first and the last elements of X.

It is sufficient to prove, by 2º) and 9'), that 1 is the last element, but this is immediat from 7').

4º) X is a lattice, with supremum and infimum given by definitions 13) and 14).

Let $a,b \in X$; we shall prove now that $-a \to b$ is the supremum of a and b, with respect to the order defined in X.

a) $a \leq -a \rightarrow b$, $b \leq -a \rightarrow b$

Using successively 5), 6') and 7') we have:
$b \rightarrow (-a \rightarrow b) = (b \rightarrow -a) \rightarrow (b \rightarrow b) = 1$, i.e. $b \leq -a \rightarrow b$.
Taking in account that, by 8'), $-a \rightarrow b = -b \rightarrow a$, we also have $a \leq -a \rightarrow b$.

b) If $a \leq c$, $b \leq c$, then $-a \rightarrow b \leq c$.

Applying 12) we obtain: $-a \rightarrow b \leq -a \rightarrow c$ and $-c \rightarrow a \leq$ $\leq -c \rightarrow c$. By 8') again, $-c \rightarrow c = (c \rightarrow c) \rightarrow c = c$, hence $-a \rightarrow b \leq c$.

From 2º) it follows that there exists the infimum of each pair a,b of elements of X, and that this is just $a \wedge b = -(-a \vee -b)$.

5º) $-a$ is the unique solution of the equations

α) $a \vee x = 1$

β) $a \wedge x = 0$

α) can be written as $-a \rightarrow x = 1$, or, which is equivalent, (α') $-a \leq x$.

β) is equivalent to $-a \vee -x = 1$, which can be written as $x \rightarrow -a = 1$, i.e. (β') $x \leq -a$.

It is now evident that $-a$ is the unique solution of (α') and (β').

From a theorem of Birkhoff (see (11), p. 171) it follows that $\mathcal{B}$ is a Boolean algebra.

Finally, $a \rightarrow b = -a \vee b$ is derived immediately from

13) and 4').

INDEPENDENCE: Axioms 1) and 2) are independent. Axiom 1) is verified by $X = \{0,1\}$ with the implication defined by table 1, but axiom 2) does not hold. The same set X, with the implication defined by table 2, is an example in which axiom 2) holds, but axiom 1) does not.

$\to$	0	1
0	0	1
1	1	0

Table 1

$\to$	0	1
0	0	1
1	1	1

Table 2

B) AXIOMS IN TERMS OF IMPLICATION AND NEGATION

Consider the system $\mathcal{B} = (X, \Rightarrow, \sim)$, where $\Rightarrow$ is a binary operation and $\sim$ a unary operation defined on X, such that the following equalities are identically verified:

a) $(a \Rightarrow a) \Rightarrow \sim\sim b = b$

b) $a \Rightarrow \sim\sim(\sim c \Rightarrow \sim b) = (a \Rightarrow \sim\sim b) \Rightarrow (a \Rightarrow \sim\sim c)$

The following equalities are derived from a) and b):

c) $\sim c \supset \sim b = b \supset c$

Substituting $a \supset a$ for a in b) and using a)

d) $\sim\sim b \supset \sim\sim c = b \supset c$

e) $a \supset (b \supset c) = (a \supset b) \supset (a \supset c)$

Using successively d), c), b), d), we have

$$\begin{aligned} a \supset (b \supset c) &= \sim\sim a \supset \sim\sim (b \supset c) = \\ &= \sim\sim a \supset \sim\sim (\sim c \supset \sim b) = \\ &= (\sim\sim a \supset \sim\sim b) \supset (\sim\sim a \supset \sim\sim c) = \\ &= (a \supset b) \supset (a \supset c) \end{aligned}$$

f) $\sim\sim b = b$

Using successively a), c), b), e), d), and a) we have

$$\begin{aligned} \sim\sim b &= (a \supset a) \supset \sim\sim((a \supset a) \supset \sim\sim\sim\sim b) = \\ &= (a \supset a) \supset \sim\sim(\sim\sim\sim\sim\sim b \supset \sim(a \supset a)) = \\ &= ((a \supset a) \supset \sim\sim(a \supset a)) \supset ((a \supset a) \supset \sim\sim\sim\sim\sim\sim b) = \\ &= (a \supset a) \supset (\sim\sim(a \supset a) \supset \sim\sim\sim\sim\sim\sim b) = \\ &= (a \supset a) \supset ((a \supset a) \supset \sim\sim\sim\sim b) = \\ &= (a \supset a) \supset \sim\sim b = \\ &= b \end{aligned}$$

g) $(a \supset a) \supset b = b$

By a) and f)

h) $a \supset a = b \supset b$

Using successively g), c), e), c) and g)

$$a \supset a = ((b \supset b) \supset a) \supset ((b \supset b) \supset a) =$$
$$= (\sim a \supset \sim (b \supset b)) \supset (\sim a \supset \sim (b \supset b)) =$$
$$= \sim a \supset (\sim (b \supset b) \supset \sim (b \supset b)) =$$
$$= \sim a \supset ((b \supset b) \supset (b \supset b)) =$$
$$= \sim a \supset (b \supset b)$$

Substituting b for a we have:

$$b \supset b = \sim b \supset (a \supset a)$$

h) follows observing that from e), c), e) and f) we obtain:

$$\sim a \supset (b \supset b) = (\sim a \supset b) \supset (\sim a \supset b) =$$
$$= (\sim b \supset \sim \sim a) \supset (\sim b \supset \sim \sim a) =$$
$$= \sim b \supset (\sim \sim a \supset \sim \sim a) =$$
$$= \sim b \supset (a \supset a)$$

We shall denote with Λ the element $\sim (a \supset a)$, which by h) is independent from the defining element a:

i) <u>DEFINITION: $\Lambda = \sim(a \supset a)$</u>.

j) $a \supset \Lambda = \sim a$

By c), f) and g):

$$a \supset \Lambda = a \supset \sim(a \supset a) = \sim\sim(a \supset a) \supset \sim a =$$
$$= (a \supset a) \supset \sim a = \sim a$$

Given $\mathcal{A} = (X, \rightarrow, 0)$ verifying the axioms 1) and 2), we define $\mathcal{A}^{\sim} = (X, \supset, \sim)$ by the relations:

(i) $x \supset y = x \rightarrow y$

(ii) $\sim x = x \rightarrow 0 = -x$

and given $\mathcal{B} = (X, \supset, \sim)$ verifying the axioms a) and b), we define $\mathcal{B}^{\circ} = (X, \rightarrow, 0)$ by the relations:

(i') $x \rightarrow y = x \supset y$

(ii') $0 = \sim(a \supset a) = \Lambda$

<u>THEOREM: The systems $\mathcal{A} = (X, \rightarrow, 0)$ and $\mathcal{B} = (X, \supset, \sim)$ are equivalents, i.e.</u>

1º) $\mathcal{A}^{\sim}$ verifies a) and b)

2º) $\mathcal{B}^{\circ}$ verifies 1) and 2)

3º) $\mathcal{A}^{\sim\circ} = \mathcal{A}$

4º) $\mathcal{B}^{\circ\sim} = \mathcal{B}$

PROOF: 1º) The equality a) is deduced from 1) and 4') using the translations (i) and (ii). The equality b) is obtained in the same way from 2) and 4').

2º) The equality 1) is the translation of g). The equality 2) is obtained from e) and c) using definition i).

3º) In $\mathcal{A}$, $0 = -1 = -(a \rightarrow a)$, then, this element coincides with $\Lambda = \sim(a \supset a)$ in $\mathcal{A}$ and, therefore, with the element 0 of $\mathcal{A}^{\sim\circ}$.

4º) For every $x \in X$, the negation $\sim x$ in $\mathcal{B}$ is translated, by (ii), in $-x = x \rightarrow 0$ in $\mathcal{B}^{\circ}$, element which coincides with $\sim x$, by j).

Tables 1 and 2 indicated in A) permit to verify the independence of axioms a) and b) putting $\sim x = x \rightarrow 0$.

We note finally that the equalities

1') $(a \supset a) \supset b = b$

2') $a \supset (\sim c \supset \sim b) = (a \supset b) \supset (a \supset c)$

which are faithful translations of equalities 1) and 2), do not permit to characterize Boolean algebras. In fact, the system $(X, \supset, \sim)$, where $X = \{1,2,3\}$ and $\supset$, $\sim$ are defined by the tables:

$\supset$	1	2	3
1	1	2	3
2	1	1	1
3	1	1	1

x	$\sim x$
1	2
2	1
3	1

verifies 1') and 2'), and it is not, evidently, a Boolean algebra.

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