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# A CHARACTERISATION OF MORGAN LATTICES

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# A CHARACTERISATION OF MORGAN LATTICES

by Ricardo Maronna

Morgan lattices are defined as distributive lattices possessing a monary operation (-) which obbeys the involutory law (- - a = a) and de Morgan laws. This structure has been studied by C. Moisil[3] (i), J. Kalman[2], and A. Monteiro[4]; an important particular case has also been considered by Byalinicki-Birula and Rasiowa[1] under the name of quasi-boolean algebras.

The purpose of this article is to characterise Morgan lattices by means of operations of infimum (∧) and negation (-). (ii). This work is made easy by the use of Sholander's characterisation[5] of a distributive lattice as a non empty set with the binary operations (∧)

(i) See the bibliographical references at the end of this article.
(ii) It is evident that a dual characterisation can be done using supremum (∨) instead of infimum.

and ($\vee$), which fulfill the following axioms:

$S_1$) $a = a \wedge (a \vee b)$

$S_2$) $a \wedge (b \vee c) = (c \wedge a) \vee (b \wedge a)$

THEOREM 1: <u>Let A be a non empty set, with the operations ($\wedge$) and ($-$). Let us define:</u>

1) $a \vee b = -(-a \wedge -b)$

<u>The system</u> ($A, \wedge, -$) <u>is a Morgan lattice if and only if it obbeys the following axioms:</u>

$M_1$) $a = a \wedge -(-a \wedge -b)$

$M_2$) $a \wedge -(-b \wedge -c) = -(-(c \wedge a) \wedge -(b \wedge a))$ .

DEMONSTRATION: It is evident that $M_1$) and $M_2$) are necessary. To prove that they are sufficient, we shall first prove that A is a distributive lattice; i.e., that it obbeys $S_1$) and $S_2$), which are readily verified.

$S_1$) $a \wedge (a \vee b) = a \wedge -(-a \wedge -b) = a$

$S_2$) $(c \wedge a) \vee (b \wedge a) = -(-(c \wedge a) \wedge -(b \wedge a)) =$

$= a \wedge -(-b \wedge -c) = a \wedge (b \vee c)$

We must still prove:

2) $- -a = a$

3) $-(a \vee b) = -a \wedge -b$

As A is a distributive lattice, we have:

4) $a \wedge a = a$

5) $a \wedge b = b \wedge a$

From 5) and $M_2$) we obtain:

6) $a \wedge -(-b \wedge -c) = -(-(b \wedge a) \wedge -(c \wedge a))$

and replacing b and c by a it results, by 4)

7) $a \wedge -(-a \wedge -a) = -(-(a \wedge a) \wedge -(a \wedge a)) =$

$= -(-a \wedge -a) = - -a$

and applying $M_1$) to the first member of 7), we get 2).

As to 3), it is immediately obtained from 1) and 2).

THEOREM 2: Postulates $M_1$) and $M_2$) are independent.

DEMONSTRATION: Let A be the set $\{0,1\}$. Let us define $-a = a$, and let us define $a \wedge b$ in two different ways:

i) $a \wedge b = a$

ii) $a \wedge b = a \cdot b$

In the first case, $(A, \wedge, -)$ verifies $M_1$, but not $M_2$. With the second definition, $M_2$ is satisfied, but not $M_1$.

# BIBLIOGRAPHY


1 BYALINICKI-BIRULA (A) and RASIOWA (H): On the representation of quasi-boolean algebras; Bulletin de l'Académie Polonnaise des Sciences, Classe III, 5 (1957), pp. 259-261.

2 KALMAN (J.A.): Lattices with involution. Trans. of the American Mathematical Society, 87 (1958), pp. 485-491.

3 MOISIL (Gr.C.): Recherches sur l'algèbre de la logique. Annales Scientifiques de l'Université de Jassy, 22 (1935), pp.1-117.

4 MONTEIRO (Antonio): Matrices de Morgan caractéristiques pour le calcul propositionnel classique. Anais da Academia Brasileira de Ciencias. 32 (1960), pp. 1-7.

5 SHOLANDER (Marlow): Postulates for distributive lattices. Canadian Journal of Mathematics. 3 (1951), pp. 28-30.