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par

Luiz Monteiro

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146. *Boolean Elements in Lukasiewicz Algebras. I*

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1965)

0. *INTRODUCTION.* In the theory of the (three-valued) Lukasiewicz algebras founded by Gr. Moisil, the possibility operator plays an important role. Moisil denotes the operator by M and we shall denote by ∇ it defined on a distributive lattice A and it is uniquely determined by the set K of all elements $k\in A$ such that $\nabla k=k$.

The purpose of this note is to establish characteristic properties of the family K. In § 1 we summarize some theorems on closure operators defined on lattices. In § 2, we study these operators in the case of Kleene algebras, and in § 3 we apply these results to the problem suggested by A. Monteiro.*)

1. *CLOSURE LATTICES.* Let $(L, 0, 1, \wedge, \vee)$ be a lattice with first and last elements. If a unary operator ∇ is defined on L such that:

$C\,1)$ $\nabla 0=0$, $C\,2)$ $x\leq \nabla x$,

$C\,3)$ $\nabla(x\vee y)=\nabla x\vee\nabla y$, $C\,4)$ $\nabla\nabla x=\nabla x$,

we shall say that the system $(L, 0, 1, \wedge, \vee, \nabla)$ is a *closure lattice*, and the operator ∇ is a *closure operator*. This notion is a generalization of closure operators on topological spaces and was studied by N. Nakamura [17] (see also [16] and [18]).

It is easy to prove that:

$C\,5)$ If $x\leq y$, then $\nabla x\leq\nabla y$, or equivalently,

$C\,6)$ $\nabla(x\wedge y)\leq\nabla x\wedge\nabla y$.

In [18] it was proved that

1.1. *The family K of all invariant elements of a closure operator has the following properties*:

$K\,1)$ *K is a sub-lattice of L containing* 0 *and* 1.

$K\,2)$ *K is lower relatively complete*: *that is, for all $x\in L$, the set $\{k\in K : x\leq k\}$ has an infimum belonging to K.*

Moreover we have

$$(1)\qquad \nabla x=\wedge\{k\in K : x\leq k\}.$$

Conversely, if K is a subset of L with the properties $K\,1)$ and $K\,2)$, (1) defines a closure operator ∇ on L, and K is the set of all invariant elements by ∇.

*) The results of this paper were presented to the "Unión Matemática Argentina" in October 1964.

We shall say that a unary operator Δ defined on L satisfying

$I\,1)$ $\Delta 1=1$, $I\,2)$ $\Delta x \leq x$,

$I\,3)$ $\Delta(x \wedge y)=\Delta x \wedge \Delta y$, $I\,4)$ $\Delta\Delta x=\Delta x$

is an *interior operator*.

In [18] the dual form of 1.1. was also proved:

1.2. *The family H of all invariant elements of an interior operator Δ has the following properties*:

$H\,1)$ *H is a sub-lattice of L containing* 0 *and* 1.

$H\,2)$ *H is upper relatively complete: that is, for all $x \in L$ the set $\{h \in H : h \leq x\}$ has a supremum belonging to H.*

Moreover we have

$$\Delta x=\vee\{h \in H : h \leq x\}. \tag{2}$$

Conversely, If H is a subset of L with the properties H 1) and H 2), (2) defines an interior operator Δ on L, and H is the set of all invariant elements by Δ.

2. *KLEENE ALGEBRAS*. Let $(A, \wedge, \vee)$ be a distributive lattice. If a unary operation $\sim$ is defined on A such that:

$M\,1)$ $\sim\sim x=x$, $M\,2)$ $\sim(x \vee y)=\sim x \wedge \sim y$,

we shall say that the system $(A, \wedge, \vee, \sim)$ is a *de Morgan lattice*. This notion has been introduced by Gr. Moisil ([11], p. 91) and studied by J. Kalman [7] under the name of *distributive i-lattice*. It is easy to prove that $\sim$ is an *involution* ([4], p. 4), that is, it satisfies $M1$) and

$M\,3)$ $x \leq y$ if and only if $\sim y \leq \sim x$.

As $\sim$ is an involution, we have that if $\{x_i\}_{i \in I}$ is a family of elements of A such that $\bigvee_{i \in I} x_i$ exists, then $\bigwedge_{i \in I} \sim x_i$ also exists and we have

$M\,4)$ $$\sim \bigvee_{i \in I} x_i=\bigwedge_{i \in I} \sim x_i.$$

Analogously, if $\bigwedge_{i \in I} x_i$ exists, then $\bigvee_{i \in I} \sim x_i$ also exists and

$M\,5)$ $$\sim \bigwedge_{i \in I} x_i=\bigvee_{i \in I} \sim x_i.$$

If A has the last element 1, we shall say that A is a *de Morgan algebra*. This notion has been studied by A. Bialynicki-Birula and H. Rasiowa ([3], [2]) under the name of *quasi-Boolean algebras*. In this case, $0=\sim 1$ is the first element of A.

If the operation $\sim$ also verifies the condition

$K)$ $$x \wedge \sim x \leq y \vee \sim y,$$

we shall say that A is a *Kleene lattice* (*algebra*). A three-element algebra of this kind was used by S. C. Kleene as a characteristic matrix of a propositional calculus ([8], [9], p. 334). These lattices were studied by J. Kalman [7] with the name of *normal distributive i-lattices*. An important example of Kleene algebras are the N-lattices of H. Rasiowa [19]. We have used the terminology introduced in [15] and [5].

Let A be a de Morgan algebra. We shall say that a *sub-algebra B of A is lower (upper) relatively complete*, if B has property K 2) of 1.1 (property H 2) of 1.2.).

From M 3), M 4), and M 5) we can easily prove the following

2.1. *LEMMA. A sub-algebra B of a de Morgan algebra A is lower relatively complete if and only if it is upper relatively complete. In this case the operators ∇ and Δ respectively defined by* (1) *of* 1.1. *and* (2) *of* 1.2. *are related by the following formulae:*

(1) $\Delta x = \sim \nabla \sim x$, (2) $\nabla x = \sim \Delta \sim x$.

We shall say that $x \in A$ is a *Boolean element* if there exists an element $-x \in A$ such that $x \wedge -x = 0$ and $x \vee -x = 1$. We know that if it exists, $-x$ is unique, and will be called the *Boolean complement* of x. Let B be the set of all Boolean elements of A. Clearly B is a Boolean algebra.

We shall use the following result[*)] by A. Monteiro. For completeness, we give the proof.

2.2. *LEMMA. Let A be a Kleene algebra. If $z \in A$ has a Boolean complement $-z$, then $-z = \sim z$.*

PROOF: By hypothesis we have

(1) $z \vee -z = 1$, (2) $z \wedge -z = 0$,

therefore by M 2)

(3) $\sim z \wedge \sim -z = 0$, (4) $\sim z \vee \sim -z = 1$,

this means

(5) $-\sim z = \sim -z$, (6) $-\sim -z = \sim z$,

and so $\sim z$ and $\sim -z$ are also Boolean elements. By K) we can write.

(7) $$z \wedge \sim z \leq -z \vee \sim -z.$$

As z, $-z$, $\sim z$, $\sim -z$ are Boolean elements, so are $(z \wedge \sim z)$ and $(-z \vee \sim -z)$. Then by (7) we have $-(-z \vee \sim -z) \leq -(z \wedge \sim z)$, that is $z \wedge - \sim -z \leq -z \vee - \sim z$, hence, by (5) $z \wedge \sim z \leq -z \vee - \sim z$.

From this relation we deduce

$$z \wedge \sim z \leq (-z \vee - \sim z) \wedge \sim z = (-z \wedge \sim z) \vee (\sim z \wedge - \sim z),$$

hence, by (3) and (5), $z \wedge \sim z \leq -z \wedge \sim z$ and then

$$z \wedge \sim z \leq z \wedge -z \wedge \sim z = 0.$$

So, $z \wedge \sim z = 0$ and by M2), $z \vee \sim z = 1$, which proves $\sim z = -z$.

Q.E.D.

2.3. *COROLLARY. The set B of all Boolean elements of a Kleene algebra A is a subalgebra of A.*

3. (*THREE-VALUED*) *LUKASIEWICZ ALGEBRAS*. The notion of (three-valued) Lukasiewicz algebra was introduced and developed by Gr. Moisil ([12], [13], [14]) to study the three-valued logic of J. Lukasiewicz [10]. Its role is similar to Boolean algebras in

[*)] Unpublished.

classical logic. We shall use the following A. Monteiro's definition [6] that is equivalent to Gr. Moisil's:

3.1. *DEFINITION. A (three-valued) Lukasiewicz algebra is a system* $(A, 1, \wedge, \vee, \sim, \nabla)$ *such that* $(A, 1, \wedge, \vee, \sim)$ *is a Kleene algebra and* ∇ *is a unary operator defined on* A *that satisfies the following axioms:**)

$L\,1)$ $\nabla(x \wedge y) \leq \nabla x \wedge \nabla y$, $L\,2)$ $\sim x \vee \nabla x = 1$,

$L\,3)$ $x \wedge \sim x = \sim x \wedge \nabla x$.

Let us recall some properties ([12]-[14]):

L 4) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, $L\,5)$ $\nabla(x \vee y) = \nabla x \vee \nabla y$,

$L\,6)$ $x \leq \nabla x$, $L\,7)$ $\nabla\nabla x = \nabla x$,

$L\,8)$ $\nabla x = x$ if and only if x is a Boolean element of A, .

$L\,9)$ $\nabla 0 = 0$.

We must notice that the (three-valued) Lukasiewicz algebras are examples of Kleene algebras where a non-trivial operator satisfying $L\,4)$, $L\,6)$, $L\,7)$, and $L\,9)$ is defined, unlike Boolean algebras, where G. Bergman [1] proved that the identity operator is the only one that satisfies such conditions.

Properties $L\,5)$, $L\,6)$, $L\,7)$, and $L9)$ show that ∇ is a closure operator on A, hence according to 1.1 and $L8)$, it follows that the subalgebra B of all Boolean elements of A is lower relatively complete, and for all $x \in A$ we have

$L\,10)$ $\nabla x = \wedge\{b \in B : x \leq b\}$.

According to 2.1 and 2.3 we can define the operator Δ, (that is interpreted as the *necessity operator* and noted as ν by Moisil) dual of ∇, by the formula:

$L\,11)$ $\Delta x = \vee\{b \in B : b \leq x\}$

and we have the relations (1) and (2) of 2.1.

Moisil proved the following *determination principle* [12]:

$L\,12)$ $x \leq y$ if and only if $\Delta x \leq \Delta y$ and $\nabla x \leq \nabla y$.

From $L10)$, $L11)$, and $L12)$ we easily see that the subalgebra B is *separating*, that is, if $y \not\leq x$ for $x, y \in A$ then, there exists $b \in B$ such that $x \leq b$ and $y \not\leq b$ or there exists $b' \in B$ such that $b' \leq y$ and $b' \not\leq x$.

In short, we can assert that *the family of invariant elements of the operator* ∇ *coincides with the subalgebra of all Boolean elements of* A, *that is lower relatively complete and separating.* The next theorem shows that these properties characterize the set of invariant elements of ∇.

3.2. *THEOREM. Let* A *be a Kleene algebra such that the family* B *of its Boolean elements is lower relatively complete and separating. Then one and only one (three-valued) Lukasiewicz*

*) The operation $\sim$ was noted as N by Moisil.

algebra structure can be defined on A.

PROOF: As B is lower relatively complete, by the formula L 10), we can define the operator ∇ and according to 1.1, ∇ will have the properties C 1)–C 6) and B will be the family of all invariant elements of ∇. To prove the theorem it is sufficient to show that ∇ also satisfies axioms L 2) and L 3).

Let us prove L 2). By C 2), $x\leq\nabla x$, by M 3) it follows that $\sim\nabla x\leq\sim x$. As ∇x is a Boolean element, from 2.2 it follows that $\sim\nabla x$ is the Boolean complement of ∇x. Hence we have

$$1=\sim\nabla x\vee\nabla x\leq\sim x\vee\nabla x\leq 1.$$

Let us prove L 3). First of all by 1.2, 2.1, and 2.3 the operator Δ can be defined, and will have properties I 1)–I 4), moreover it satisfies

(1) $$\Delta x=\sim\nabla\sim x.$$

As $x\leq\nabla x$, it is clear that $\sim x\wedge x\leq\sim x\wedge\nabla x$, then, to prove $L3$) we need to show

(2) $$\nabla x\wedge\sim x\leq\sim x\wedge x.$$

This last proof will be done in two steps:

I. Let us prove the following property:

(P) If x, y of A satisfy

(P 1) $y\leq\sim x$,

(P 2) for all $b\in B$ such that $x\leq b$ we have $y\leq b$,

then $y\leq x$.

For, let us suppose that $x, y\in A$, x, y satisfy P 1) and P 2) and $y\not\leq x$. As there cannot exist $b\in B$ such that $x\leq b$ and $y\not\leq b$ by P 2), from the separation property of B it follows that there exists $b'\in B$ such that $b'\leq y$ and $b'\not\leq x$. By $b'\not\leq x$, we have in particular

(3) $$b'\neq 0.$$

Moreover

(4) $$b'\leq y\leq\sim x.$$

$\nabla x\in B$ and C 2) imply $x\leq\nabla x$. By P 2), we have $y\leq\nabla x$ and $b'\leq y$, hence

(5) $$b'\leq\nabla x.$$

From (4) and (5)

(6) $$b'\leq\sim x\wedge\nabla x.$$

Applying Δ to both sides of (6) and recalling 1.2 and the formula (1), we have by 2.2

$$\Delta b'=b'\leq\Delta(\sim x\wedge\nabla x)=\Delta\sim x\wedge\Delta\nabla x=\Delta\sim x\wedge\nabla x$$
$$=\sim\nabla x\wedge\nabla x=0.$$

Then $b'=0$, which contradicts (3). Therefore we have $y\leq x$, and (P) is proved.

II. (P) and the lower relatively completness of B imply (2). For, making $y=\sim x\wedge\nabla x$ we have

(7) $$y\leq\sim x.$$

Therefore x, y satisfy P 1). They also satisfy P 2). For, if $b\in B$ and $x\leq b$, then $\nabla x\leq\nabla b=b$ so $y\leq\nabla x\leq b$. Then by (P), we have

(8) $$y=\sim x\wedge\nabla x\leq x.$$

From (7) and (8) we have (2). Q.E.D.

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147. *Boolean Elements in Lukasiewicz Algebras. II*

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1965)

0. *INTRODUCTION.* R. Cignoli has proved the following result:

0.1. *THEOREM: Let A be a Kleene algebra. It is possible to define on A a structure of Lukasiewicz algebra if and only if the family B of all Boolean elements of A has the following properties:*

$B\,1)$ *B is separating.*

$B\,2)$ *B is lower relatively complete.*

The purpose of this note is to show that if, instead of a Kleene algebra, A is a distributive lattice with first (0) and last element (1), then we can define on A a structure of Lukasiewicz algebra if and only if the family B has the properties $B\,1)$, $B\,2)$, and

$B\,3)$ *B is upper relatively complete.*

We shall use the notations and definitions of [1].

In § 1 we introduce an alternative definition of Lukasiewicz algebra which is useful for the purpose of this paper.

1. *DEFINITION OF LUKASIEWICZ ALGEBRAS.* We can define the notion of (three-valued) Lukasiewicz algebra introduced and developed by Gr. Moisil [3], [4], [5] in the following way [6], [7]:

1.1. *DEFINITION: A (three-valued) Lukasiewicz algebra is a system $(A, 1, \wedge, \vee, \sim, \nabla)$ where $(A, 1, \wedge, \vee, \sim)$ is a de Morgan lattice and ∇ is a unary operator defined on A satisfying the following axioms:*

$L\,1)$ $\sim x \vee \nabla x = 1,$ $\qquad$ $L\,2)$ $x \wedge \sim x = \sim x \wedge \nabla x,$

$L\,3)$ $\nabla(x \wedge y) = \nabla x \wedge \nabla y.$

In [6] (Theorem 4.3) it was proved that in a (three-valued) Lukasiewicz algebra the operation $\sim$ also satisfies the condition

$K)$ $\qquad x \wedge \sim x \leq y \vee \sim y,$

that is, the system $(A, 1, \wedge, \vee, \sim)$ is not only a de Morgan algebra but a Kleene algebra.

A. Monteiro has proved that if we postulate the condition $K)$, then we can replace axiom $L\,3)$ of definition 1.1 by the weaker

$L'3)$ $\qquad \nabla(x \wedge y) \leq \nabla x \wedge \nabla y.$

More exactly:

1.2. *THEOREM: Let $(A, 1, \wedge, \vee, \sim, \nabla)$ be a system such that $(A, 1, \wedge, \vee, \sim)$ is a Kleene algebra and ∇ is a unary operator defined on A satisfying axioms $L\,1)$, $L\,2)$, and $L'3)$. Then* $(A, 1, \wedge,$

$\vee, \sim, \nabla)$ *is a (three-valued) Lukasiewicz algebra.*

PROOF: As Kleene algebras are special kind of de Morgan algebras, to prove the theorem we need show that

(1) $$\nabla x \wedge \nabla y \leq \nabla(x \wedge y).$$

We will prove (1) in the following steps:

a) $x \leq \nabla x$.

By L 1) we have

$$x \wedge (\sim x \vee \nabla x) = x \wedge 1 = x,$$

then

$$(x \wedge \sim x) \vee (x \wedge \nabla x) = x$$

and, recalling L 2), we can write:

$$(\sim x \wedge \nabla x) \vee (x \wedge \nabla x) = x.$$

Therefore

$$x = (\sim x \vee x) \wedge \nabla x \leq \nabla x.$$

b) If $\sim x \wedge z \leq x$, then $z \leq \nabla x$.

Suppose that $\sim x \wedge z \leq x$, we have

$$(\sim x \wedge z) \vee \nabla x \leq x \vee \nabla x$$

and then, by a), we can write:

$$(\sim x \vee \nabla x) \wedge (z \vee \nabla x) \leq \nabla x$$

and recalling L 1)

$$z \vee \nabla x \leq \nabla x,$$

therefore

$$z \leq \nabla x.$$

c) $\sim x \wedge \nabla x \wedge \nabla y \leq x$.

Using L 2) we can write:

$$\sim x \wedge \nabla x \wedge \nabla y = \sim x \wedge x \wedge \nabla y \leq x.$$

d) $\sim x \wedge \nabla x \wedge \nabla y \leq y$.

By L 2), K), and a) we have

$$\sim x \wedge \nabla x \wedge \nabla y = \sim x \wedge x \wedge \nabla y \leq (\sim y \vee y) \wedge \nabla y = (\sim y \wedge \nabla y) \vee (y \wedge \nabla y)$$
$$= (\sim y \wedge \nabla y) \vee y = (y \wedge \sim y) \vee y = y.$$

From c) and d) we have

e) $\sim x \wedge \nabla x \wedge \nabla y \leq x \wedge y$.

From e), interchanging x by y, we have

f) $\sim y \wedge \nabla x \wedge \nabla y \leq x \wedge y$.

From e) and f), taking acount of M 2) it follows that

g) $\sim(x \wedge y) \wedge \nabla x \wedge \nabla y \leq x \wedge y$.

Finally, from b) and g) we have (1).

2. *CHARACTERISTIC PROPERTIES OF BOOLEAN ELEMENTS OF LUKASIEWICZ ALGEBRAS.* **Let $(A, 0, 1, \wedge, \vee)$ be a distributive lattice with first and last element. If $x \in A$ has a Boolean complement, we shall denote it by $-x$. It is convenient to recall the following property:**

2.1. *If z is a Boolean element of A, then for all $x\in A$*

$$x\wedge z=0 \text{ is equivalent to } x\leq -z.$$

2.2. LEMMA: *Let $(A, 0, 1, \wedge, \vee)$ be a distributive lattice and let B be the sublattice of all Boolean elements of A.*

a) *If B is lower relatively complete, then the operator ∇ defined on A by the formula:*

$$\nabla x=\wedge\{b\in B : x\leq b\}$$

has the following properties:

C 1) $\nabla 0=0$, C 2) $x\leq \nabla x$, C 3) $\nabla(x\vee y)=\nabla x\vee\nabla y$,
C 4) $\nabla\nabla x=\nabla x$. C 5) If $x\leq y$, then $\nabla x\leq\nabla y$,
C 6) $\nabla x=x$ if and only if $x\in B$, C 7) $\nabla(x\wedge\nabla y)=\nabla x\wedge\nabla y$.

b) *If B is upper relatively complete, then the operator Δ defined on A by the formula:*

$$\Delta x=\vee\{b\in B : b\leq x\}$$

has the following properties:

I 1) $\Delta 1=1$, I 2) $\Delta x\leq x$, I 3) $\Delta(x\wedge y)=\Delta x\wedge\Delta y$,
I 4) $\Delta\Delta x=\Delta x$, I 5) If $x\leq y$, then $x\Delta\leq\Delta y$,
I 6) $\Delta x=x$ if and only if $x\in B$, I 7) $\Delta(x\vee\Delta y)=\Delta x\vee\Delta y$.

PROOF: a) The properties C 1)–C 6) are a consequence of the fact that B is a sublattice of A containing 0 and 1 and lower relatively complete (see [1]).

Let us prove C 7):

As $x\wedge\nabla y\leq x$, it follows from C 5) that

(1) $$\nabla(x\wedge\nabla y)\leq\nabla x$$

and since $x\wedge\nabla y\leq\nabla y$, from C 5) and C 4) we have

(2) $$\nabla(x\wedge\nabla y)\leq\nabla y.$$

On the other hand, by C 2) we can write:

$$x\wedge\nabla x\wedge\nabla y\wedge -\nabla(x\wedge\nabla y)=(x\wedge\nabla y)\wedge -\nabla(x\wedge\nabla y)$$
$$\leq \nabla(x\wedge\nabla y)\wedge -\nabla(x\wedge\nabla y)=0.$$

Since $\nabla x\wedge\nabla y\wedge -\nabla(x\wedge\nabla y)\in B$, we have (by 2.1)

(3) $$x\leq -\nabla x\vee -\nabla y\vee\nabla(x\wedge\nabla y).$$

Meeting both sides of (3) with ∇x and using C 2) and (1) we have

$$x=x\wedge\nabla x\leq(\nabla x\wedge -\nabla y)\vee\nabla(x\wedge\nabla y)\leq\nabla x$$

Hence, by C 5), C 3), and C 6) it follows that

$$\nabla x=(\nabla x\wedge -\nabla y)\vee\nabla(x\wedge\nabla y)$$

and then, by (2)

$$\nabla x\wedge\nabla y=\nabla(x\wedge\nabla y).$$

It is not necessary to prove b), for it is the dual form of a).

Q.E.D.

(Compare this result with [2]).

We shall say that a sublattice B of a lattice A is *relatively complete* if it is both lower and upper relatively complete.

2.3. *THEOREM*: *Let* $(A, 0, 1, \wedge, \vee)$ *be a distributive lattice with first and last element such that the sublattice* B *of all its Boolean elements is relatively complete and separating. Then, defining the operators* ∇, Δ, *and* $\sim$ *by the formulae*:

$$\nabla x = \wedge\{b \in B : x \leq b\}, \qquad \Delta x = \vee\{b \in B : b \leq x\},$$
$$\sim x = (-\Delta x \wedge x) \vee -\nabla x,$$

the system $(A, 1, \wedge, \vee, \sim, \nabla)$ *is a (three-valued) Lukasiewicz algebra.*

PROOF: We shall use all properties shown in 2.2 without reference. The theorem will be proved in the following steps:

a) $\nabla(x \wedge y) \leq \nabla x \wedge \nabla y$.

It follows immediately from $C\,5)$.

b) $\sim x \vee \nabla x = 1$.

It easily follows from the definition of $\sim x$.

c) $x \wedge \sim x = \sim x \wedge \nabla x$.

Taking account of 2.1, we have $x \wedge -\nabla x = 0$, then

$$x \wedge \sim x = x \wedge ((-\Delta x \wedge x) \vee -\nabla x) = -\Delta x \wedge x.$$

But we also have

$$\sim x \wedge \nabla x = ((-\Delta x \wedge x) \vee -\nabla x) \wedge \nabla x = -\Delta x \wedge x.$$

d) If $z \in B$, then $\sim z = -z$.

By $z \in B$, we have $\Delta z = z = \nabla z$, then

$$\sim z = (-\Delta z \wedge z) \vee -\nabla z = (-z \wedge z) \vee -z = -z.$$

e) $-\Delta x = \sim \Delta x$ and $-\nabla x = \sim \nabla x$.

It is an immediate consequence of d).

f) $\Delta x = \sim \nabla \sim x$.

First of all, we have $-\Delta x = \nabla -\Delta x$, $-\nabla x = \nabla -\nabla x$, hence we can write

$$\begin{aligned}\nabla \sim x &= \nabla((-\Delta x \wedge x) \vee -\nabla x) = \nabla(-\Delta x \wedge x) \vee \nabla -\nabla x \\ &= \nabla(\nabla -\Delta x \wedge x) \vee -\nabla x = (\nabla -\Delta x \wedge \nabla x) \vee -\nabla x \\ &= (-\Delta x \wedge \nabla x) \vee -\nabla x = -\Delta x \vee -\nabla x = -\Delta x\end{aligned}$$

and then f) follows from e).

g) $\nabla x = \sim \Delta \sim x$.

The proof of g) is analogous to that of f).

h) $\sim \sim x = x$.

By e), f), and g) we have

$$\begin{aligned}\sim \sim x &= (-\Delta \sim x \wedge \sim x) \vee -\nabla \sim x = (\nabla x \wedge \sim x) \vee \Delta x \\ &= (\nabla x \wedge ((-\Delta x \wedge x) \vee -\nabla x))) \vee \Delta x = (-\Delta x \wedge x) \vee \Delta x = x.\end{aligned}$$

i) $x \leq y$ if and only if $\Delta x \leq \Delta y$ and $\nabla x \leq \nabla y$.

If $x \leq y$, then $\Delta x \leq \Delta y$ and $\nabla x \leq \nabla y$.

Conversely, if $\Delta x \leq \Delta y$, then for all $z' \in B$ we have:

$$z' \leq x \text{ implies } z' \leq y$$

and if $\nabla x \leq \nabla y$, then for all $z \in B$ we have:

$$y \leq z \text{ implies } x \leq z,$$

therefore, by the separating property of B, we must have $x \leq y$.

j) If $x \leq y$, then $\sim y \leq \sim x$.

According to i), it is sufficient to prove that $\Delta \sim x \leq \Delta \sim y$ and $\nabla \sim x \leq \nabla \sim y$.

But by e), g), and h) we have $\Delta \sim y = -\nabla y$ and $\Delta \sim x = -\nabla x$, hence, if $x \leq y$, it follows that $\Delta \sim y \leq \Delta \sim x$. Analogously we can prove $\nabla \sim y \leq \nabla \sim x$.

k) $\sim (x \wedge y) = \sim x \vee \sim y$.

It easily follows from h) and k).

l) $x \wedge \sim x \leq y \vee \sim y$.

As we have shown in the proof of c), $x \wedge \sim x = -\Delta x \wedge x$, thus $\Delta(x \wedge \sim x) = 0$ and a fortiori

(1) $$\Delta(x \wedge \sim x) \leq \Delta(y \vee \sim y).$$

On the other hand, $y \vee \sim y = y \vee (-\Delta y \wedge y) \vee -\nabla y = y \vee -\nabla y$, therefore $\nabla(y \vee \sim y) = 1$, and then we have

(2) $$\nabla(x \wedge \sim x) \leq \nabla(y \vee \sim y)$$

and $x \wedge \sim x \leq y \vee \sim y$ follows from i), (1), and (2).

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NOTAS DE LOGICA MATEMATICA

N° 25

SUR LES ALGEBRES DE LUKASIEWICZ INJECTIVES

par

Luiz Monteiro

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INSTITUTO DE MATEMATICA

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Este número es una separata del artículo "Sur les Algebres de Lukasiewicz injectives", por Luiz Monteiro, publicado en Proceedings of the Japan Academy, Vol.41,N°7 (1965), pag.578-581, en la cual estan corregidos los errores de impresión. El mismo contiene resultados obtenidos en el Instituto de Matemática de la Universidad Nacional del Sur. Un prepint de este artículo fué preparado en el año 1964, pero por diversas razones no pudo publicarse.

Cet numero est un tirage-à-part de l'article "Sur les Algebres de Lukasiewicz Injectives", par Luiz Monteiro, publié dans les Proceedings of the Japan Academy, Vol.41, N°7 (1965), pag.578-581, dans lequel les erreurs d'impresión on été corrigées. Il contient des résultats obtenus dans l'Instituto de Matemática de la Universidad Nacional del Sur. Un préprint de cet article devrait être publié dans cette collection dans l'année 1964, mais par plusieures raisons il n'a pas été possible de le faire paraître à cette époque.

126. *Sur les Algèbres de Lukasiewicz Injectives*

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(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 13, 1965)

A. Monteiro a proposé[1] de déterminer les algèbres de Lukasiewicz (trivalentes) qui sont injectives et il a conjecturé qu'elles doivent être les algèbres complètes et centrées. Nous avons montré non seulement qu'il en est ainsi, mais aussi que ce résultat peut être considéré comme une conséquence d'un important théorème de Roman Sikorski [8].

Nous supposerons connues certaines notions sur les algèbres de Boole et sur les algèbres de Lukasiewicz (voir [3]-[7]).

Les opérateurs de possibilité (M), de nécessité (ν) et de négation (N) seront representés dans cette note respectivement par ∇, Δ, $\sim$.

Un centre d'une algèbre de Lukasiewicz A est un élément $c\in A$ tel que $\sim c=c$.[2] D'après Moisil A ne peut avoir qu'un seul centre.

Si A a un centre c, Moisil [5] a montré que pour tout $x\in A$: $x=\Delta x\vee(c\wedge\nabla x\wedge\nabla\sim x)$.

Remarquons que

$$x=(\Delta x\vee c)\wedge(\Delta x\vee\nabla x)\wedge(\Delta x\vee\nabla\sim x)$$

et comme $\Delta x\vee\nabla x=\nabla x$; $\Delta x\vee\nabla\sim x=\Delta x\vee\sim\Delta x=1$, alors on peut écrire plus simplement:

$$x=(\Delta x\vee c)\wedge\nabla x$$

ou ce qui est équivalent

$$x=(\nabla x\wedge c)\vee\Delta x.$$

Un élément $k\in A$ sera dit booléen si k a un complément, c'est-à-dire s'il existe $k'\in A$ tel que $k\vee k'=1$, $k\wedge k'=0$. S'il en est ainsi on a $k'=\sim k$. D'après Moisil [3] pour qu'un élément x soit booléen il faut et il suffit que $\nabla x=x$, ou ce qui revient au même $\Delta x=x$.

Représentons par $K(A)$ l'algèbre de Boole des éléments booléens de l'algèbre de Lukasiewicz A.

1.1. *LEMME*: *Si C est une algèbre de Lukasiewicz complète,*[3] *alors l'algèbre de Boole $K(C)$ est complète.*

DEMONSTRATION. Soit $\{k_i\}_{i\in I}$ une famille d'éléments de $K(C)$, et soit $k=\bigvee_{i\in I} k_i$. Montrons que $k\in K(C)$. En effet comme $k_i\le k$ pour tout $i\in I$, alors, Δ étant monotone, nous pouvons écrire:

1) Dans un cour sur les Algèbres de Lukasiewicz réalisé pendant le premier semestre de 1963 à l'Universidad Nacional del Sur.

2) Voir [3] p. 446.

3) C'est-à-dire le réticulé C est complet.

$$k_i = \Delta k_i \leq \Delta k \text{ pour tout } i \in I$$

d'oú $k = \bigvee_{i \in I} k_i \leq \Delta k \leq k$, donc $\Delta k = k$, et par conséquent $k \in K(C)$.

1.2. *DEFINITION*: *Une algèbre de Lukasiewicz C sera dite injective si quel que soit l'algèbre de Lukasiewicz A et une sous algèbre B de A, alors pour tout homomorphisme f de B dans C, il existe un homomorphisme h de A dans C qui est une extension de f (c'est-à-dire $h(b)=f(b)$ pour tout $b \in B$).*

Il est bien connu que:

1.3. *THEOREME*: *Toute algèbre de Lukasiewicz est sousalgèbre d'une algèbre de Lukasiewicz complète et centrée.*

1.4. *THEOREME*: *Une algèbre de Lukasiewicz C est injective si et seulement si C est complète et centrèe.*

DEMONSTRATION. Nécessaire: Soit C une algèbre de Lukasiewicz injective, alors, par le théorème 1.3. nous pouvons affirmer que C peut être étendue à une algèbre de Lukasiewicz A complète et centrée. Soit f la transformation de la sous-algèbre C de A dans C, donnée par $f(x)=x$ pour tout $x \in C$. Il est évident que f est un isomorphisme de C sur C.

Comme C est injective, alors il existe un homomorphisme h de A dans C qui prolonge f.

Soit $\{c_i\}_{i \in I}$ une famille d'éléments de C. Montrons que cette famille a une borne supérieure dans C. En effet comme $c_i \in A$ pour tout $i \in I$ alors il existe l'élément $\bigvee_{i \in I} c_i = c_0 \in A$, donc $c_i \wedge c_0 = c_i$ pour tout $i \in I$, et par conséquent $h(c_i \wedge c_0) = h(c_i)$ c'est-à-dire $h(c_i) \wedge h(c_0) = h(c_i)$. Comme $c_i \in C$ et h est une extension de l'homomorphisme f, nous aurons $f(c_i) \wedge h(c_0) = f(c_i)$ pour tout $i \in I$, c'est-à-dire $c_i \wedge h(c_0) = c_i$ pour tout $i \in I$, alors: $h(c_0) \in C$ est un majorant (dans C) de la famille $\{c_i\}_{i \in I}$.

Supposons maintenant qu'il existe un élément $c' \in C$ tel que $c_i \leq c'$ pour tout $i \in I$, alors $c_0 \leq c'$ et par conséquent $h(c_0) \leq h(c') = c'$. Nous venons de montrer que $h(c_0) \in C$ est la borne supérieure (dans C) de la famille $\{c_i\}_{i \in I}$, donc C est complète.

Voyons que C est centrée. En effet comme A a un centre c, alors nous aurons $h(c) = h(\sim c) = \sim h(c)$, donc $h(c)$ est un centre de C.

Suffisante: Soit C une algèbre de Lukasiewicz complète et centrée, A une algèbre de Lukasiewicz, B une sous-algèbre de A et f un homomorphisme de B dans C.

$K(A)$, $K(C)$, et $K(B)$ sont des algèbres de Boole, et par le lemme 1.1., $K(C)$ est complète. Soit f' la restriction de f à $K(B)$, alors f' est un homomorphisme booléen de $K(B)$ dans $K(C)$, donc par un théorème de Sikorski[1] il existe un homomorphisme booléen h' de

1) Voir [8] et [2].

$K(A)$ dans $K(C)$ qui est une extension de f'.

Posons par définition: $h(x)=(h'(\Delta x)\vee c)\wedge h'(\nabla x)$ pour tout x de A.

Voyons maintenant que h est un homomorphisme de A dans C. Pour cela on doit démontrer que:

$H1)$ $h(x\wedge y)=h(x)\wedge h(y)$.

En effet

$$\begin{aligned}h(x\wedge y)&=(h'(\Delta(x\wedge y))\vee c)\wedge h'(\nabla(x\wedge y))\\&=(h'(\Delta x\wedge\Delta y)\vee c)\wedge h'(\nabla x\wedge\nabla y)\\&=((h'(\Delta x)\wedge h'(\Delta y))\vee c)\wedge h'(\nabla x)\wedge h'(\nabla y)\\&=((h'(\Delta x)\vee c)\wedge(h'(\Delta y)\vee c))\wedge h'(\nabla x)\wedge h'(\nabla y)\\&=h(x)\wedge h(y)\ .\end{aligned}$$

$H2)$ $\sim h(x)=h(\sim x)$.

$$\begin{aligned}\sim h(x)&=\sim((h'(\Delta x)\vee c)\wedge h'(\nabla x))\\&=(\sim h'(\Delta x)\wedge\sim c)\vee\sim h'(\nabla x)\\&=((h'(\sim\Delta x)\wedge c)\vee h'(\sim\nabla x)\\&=(h'(\nabla\sim x)\vee h'(\Delta\sim x))\wedge(c\vee h'(\Delta\sim x))\\&=(h'(\Delta\sim x)\vee c)\wedge(h'(\nabla\sim x\vee\Delta\sim x))\\&=(h'(\Delta\sim x)\vee c)\wedge h'(\nabla\sim x)\\&=h(\sim x)\ .\end{aligned}$$

$H3)$ $\nabla h(x)=h(\nabla x)$.

$$\begin{aligned}\nabla h(x)&=\nabla(h'(\nabla x)\wedge(h'(\Delta x)\vee c))\\&=\nabla h'(\nabla x)\wedge(\nabla h'(\Delta x)\vee\nabla c)\\&=\nabla h'(\nabla x)\wedge(\nabla h'(\Delta x)\vee 1)\\&=\nabla h'(\nabla x)=h'(\nabla x)\\&=h'(\nabla x)\wedge(h'(\nabla x)\vee c)\\&=h'(\nabla\nabla x)\wedge(h'(\Delta\nabla x)\vee c)=h(\nabla x).\end{aligned}$$

Voyons finalement que h est une extension de f.

Soit $b\in B$, alors comme B est une sous-algèbre de A, nous aurons $\Delta b,\nabla b\in B$, plus précisément $\Delta b,\nabla b\in K(B)$, donc:

$$\text{(i)}\qquad h'(\Delta b)=f'(\Delta b)=f(\Delta b),$$
$$\text{(ii)}\qquad h'(\nabla b)=f'(\nabla b)=f(\nabla b)$$

et par ailleurs nous pouvons écrire, en utilisant (i) et (ii):

$$\begin{aligned}h(b)&=h'(\nabla b)\wedge(h'(\Delta b)\vee c)=f(\nabla b)\wedge(f(\Delta b)\vee c)\\&=\nabla f(b)\wedge(\Delta f(b)\vee c)=f(b)\end{aligned}$$

et la démostration est terminée.

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