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NOTES ON n-VALUED POST ALGEBRAS

by

Manuel Abad

1. HISTORICAL INTRODUCTION AND SUMMARY OF RESULTS.

In 1920, E.Post [16] defined the most important operations of n-valued logic as a generalization of the usual 2-valued calculus, and discussed some of their properties by means of tables of values. Earlier a special concept of 3-valued logic was introduced by Lukasiewicz [9].

Rosenbloom [18] in 1942, studied Post algebras from an algebraic standpoint, giving the abstract algebraic properties of the systems independently of their interpretation as logics, and he gave the first postulate-set for Post algebras. However, Rosenbloom's system of axioms was a very difficult one and this complexity hindered the development of the theory until Epstein's paper [7], appeared in 1960. Epstein [7] and Traczyk [21] simplified Rosenbloom axiom system by making use of a greater number of operations and of the existence of an underlying Boolean algebra of a given Post algebra, and they proved the equational definability of the class of Post algebras. Since then several papers on Post algebras, as well as their generalizations and applications not only in logic but in other branches of mathematics, have been published (See Traczyk [22], Chang and Horn [5], Dwinger [6], Rouseau [19], Cignoli [4], Rasiowa [17], Monteiro [14]).

On the other hand, Halmos [8] introduced in 1955 the monadic algebras with the aim of developing an algebraic version of the logical operation of quantification, and in 1970 L.Monteiro [13] introduced the notion of three-valued monadic Lukasiewicz algebra which is a generalization of monadic Boolean algebras. If the underlying Lukasiewicz algebra is centered, this theory coincides with that of three-valued monadic Post algebras, which have been developed by L.Monteiro in [14].

As a natural generalization, in this note we are concerned with various aspects of n-valued monadic Post algebras. We give a brief exposition of the basic properties of n-valued monadic Post algebras, the theory of monadic deductive systems and representation theorem. Also we investigate free n-valued monadic Post algebras with a set G of generators, for G finite.

2. n-VALUED MONADIC POST ALGEBRAS.

In this section we first recall some definitions and also summarize those properties of n-valued Post algebras that we will need in the future. In the second place we introduce the notion of n-valued monadic Post algebra and we establish some of its basic properties.

The following definition is due to T.Traczyk [21]:

DEFINITION 1. Let n be a fixed integer $\geqslant 2$. An n-valued Post algebra is a system $\langle P,0,1,\wedge,\vee,e_1,\ldots,e_{n-2}\rangle$ such that $\langle P,0,1,\wedge,\vee\rangle$ is a distributive lattice with zero 0 and unit 1, and $e_1,\ldots,e_{n-2}$ are n-2 elements of P such that

P1) $0 = e_o \leqslant e_1 \leqslant \ldots \leqslant e_{n-2} \leqslant e_{n-1} = 1$

P2) If B(P) denotes the Boolean algebra of all complemented elements of P, for any $x \in P$, there exist elements $b_1,\ldots,b_{n-1}$ belonging to B(P) such that $x = (b_1 \wedge e_1) \vee (b_2 \wedge e_2) \vee \ldots \vee b_{n-1}$

P3) If $b \in B(P)$ and $b \wedge e_j \leqslant e_{j-1}$ for some j $(1 \leqslant j \leqslant n-1)$, then $b = 0$.

The elements $e_o, e_1,\ldots,e_{n-1}$ are distinct and unique [7].

On the other hand, it can be proved by putting $d_i = \bigvee_{j=i}^{n-1} b_j$,

$1 \leqslant i \leqslant n-1$, that every element x of the Post algebra P can be written also in the form

$$x = (d_1 \wedge e_1) \vee (d_2 \wedge e_2) \vee \ldots \vee d_{n-1}$$

with $d_i \in B(P)$, $1 \leqslant i \leqslant n-1$, and $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_{n-1}$.

This representation is unique ([7] , [21]).

If we denote $d_i = D_i(x)$ and if b' is the Boolean complement of an element $b \in B(P)$, then Epstein [7] proved that the operator

$$\sim x = \bigvee_{i=1}^{n-1} (e_i \wedge (D_{n-i}(x))')$$

verifies

M1) $\sim\sim x = x$

M2) $\sim(x \vee y) = \sim x \wedge \sim y$

Also, if we write $s_i(x) = D_{n-i}(x)$ $(1 \leqslant i \leqslant n-1)$ then the following properties are satisfied:

L1) $s_i(x \vee y) = s_i x \vee s_i y$

L2) $s_i x \vee \sim s_i x = 1$

L3) $s_i s_j x = s_j x$

L4) $s_i \sim x = \sim s_{n-i} x$

L5) $s_1 x \leqslant s_2 x \leqslant \ldots \leqslant s_{n-1} x$

L6) The Moisil determination principle:

If $s_i x = s_i y$ for $i = 1,2,\ldots,n-1$, then $x=y$,

and in addition $$s_i e_j = \begin{cases} 0 & \text{if} \quad 1+j < n \\ 1 & \text{if} \quad i+j \geqslant n \end{cases} \qquad (1)$$

Moreover, the two following properties are known to be true in any n-valued Post algebra:

$$\sim x \vee s_{n-1} x = 1 \ , \quad \text{and} \quad s_i x \wedge \sim s_i x = 0.$$

Now we re-write from Cignoli's paper the following theorem which characterizes the n-valued Post algebras.

THEOREM 2. P is an n-valued Post algebra if and only if P is an n-valued Lukasiewicz algebra and P has n-2 elements $e_1,\dots,e_{n-2}$ which satisfy (1).

The proof can be found in [4].

DEFINITION 3. By an n-valued monadic Post algebra is understood a system $\langle P,\exists\rangle$ such that P is an n-valued Post algebra and $\exists$ is a unary operation defined on P and satisfying the following conditions:

$$\begin{array}{ll} \text{E0)} & \exists 0 = 0 \\ \text{E1)} & x \leqslant \exists x \\ \text{E2)} & \exists(x \wedge \exists y) = \exists x \wedge \exists y \\ \text{E3)} & \exists s_i x = s_i \exists x \end{array}$$

(See H.Bass [2], P.Halmos [8], L.Monteiro [13], [15]).

As usual, the operator $\exists$ on an n-valued monadic Post algebra is called an existential quantifier, and via the equation $\forall x = \sim\exists\sim x$ the concept of a universal quantifier is defined.

We know that any n-element chain (e.g. of real numbers) $e_o < e_1 < \dots < e_{n-1}$ is an n-valued Post algebra. We shall denote by P_n the particular n-valued Post algebra of all fractions $e_j = {}^j/_{n-1}$, $0 \leqslant j \leqslant n-1$, considered as a sublattice of the real numbers [4], with

$$\sim({}^j/_{n-1}) = {}^{n-1-j}/_{n-1}$$

$$s_i({}^j/_{n-1}) = \begin{cases} 0 & \text{if } i+j < n \\ 1 & \text{if } i+j \geqslant n \end{cases}$$

The n-valued Post algebra of all sequences $x = (x_1,\dots,x_k)$, with $x_i \in P_n$ and with pointwise defined operations will be denoted by P_n^k, and $P^*_{n,k} = (P_n^k,\exists)$ the n-valued monadic Post algebra where

$$\exists(x_1,x_2,\dots,x_k) = (a,a,\dots,a) \quad , \quad a = \max\{x_1,x_2,\dots,x_k\}.$$

Certain elementary consequences of the definition are contained in the following lemma.

LEMMA 4. In any n-valued monadic Post algebra the following properties hold:

1) $\exists 1 = 1$; 2) $\exists\exists x = \exists x$; 3) If $x \leqslant y$ then $\exists x \leqslant \exists y$;

4) $\sim x \vee s_{n-1} \exists x = 1$; 5) $\exists x \vee s_{n-1} \sim x = 1$.

Proof. For 1), put $x = 1$ in E1) [8]. For 2), put $x = 1$ in E2) and apply E4), $\exists (1 \wedge \exists y) = \exists 1 \wedge \exists y$, then $\exists\exists y = \exists y$.

If $x \leqslant y$ then $\exists x \leqslant \exists y$. Since $y \leqslant \exists y$, we have $x \leqslant \exists y$, then $x \wedge \exists y = x$, and by E2) $\exists(x \wedge \exists y) = \exists x \wedge \exists y = \exists x$. Then 3). From $1 = \sim x \vee s_{n-1} x \leqslant \sim x \vee s_{n-1} \exists x$ it follows 4). Finally, since $x \leqslant \exists x$, then $1 = x \vee s_{n-1} \sim x \leqslant \exists x \vee s_{n-1} \sim x$, and we have 5).

DEFINITION 5. An element k of an n-valued monadic Post algebra P is said to be a constant if $\exists k = k$.

Let K(P) be the set of constants of P. Clearly, K(P) is the range of the quantifier $\exists$, that is, $K(P) = \exists(P)$. Furthermore, 0 and 1 belong to K(P).

Throughout the following statements it is assumed that P is an n-valued monadic Post algebra and $\exists$ is a quantifier on P.

LEMMA 6. If $s_i x \in K(P)$, $1 \leqslant i \leqslant n-1$, then $x \in K(P)$.

Proof. If $s_i x \in K(P)$, $1 \leqslant i \leqslant n-1$, then $\exists s_i x = s_i x$, $1 \leqslant i \leqslant n-1$, and by E3), $s_i \exists x = s_i x$, $1 \leqslant i \leqslant n-1$, then by L6), $\exists x = x$. Therefore $x \in K$.

COROLLARY 7. The elements $e_1, e_2, \ldots, e_{n-2}$ belong to K(P).

Proof. Since $s_i e_j = 0$ or $s_i e_j = 1$ for every i and j, and since $0 \in K(P)$ and $1 \in K(P)$, it follows from the above lemma that $e_j \in K(P)$, $1 \leqslant j \leqslant n-2$.

This corollary together with next theorem shows that K(P) is a Post subalgebra of P.

THEOREM 8. $K(P)$ is closed under the formation of $\wedge$, $\sim$ and s_i.

Proof. If x and y are in $K(P)$, then $x = \exists x$ and $y = \exists y$. Thus $x \wedge y = \exists x \wedge \exists y = \exists(x \wedge \exists y) = \exists(x \wedge y)$. This proves that $K(P)$ is closed under the formation of infima.
If $x \in K(P)$, then $\exists x = x$ and we can write $0 = \exists 0 =$
$= \exists(s_i x \wedge \sim s_i x) = \exists(s_i x \wedge s_{n-i} \sim x) = \exists(s_i \exists x \wedge s_{n-i} \sim x) =$
$= \exists(\exists s_i x \wedge s_{n-i} \sim x) = \exists s_i x \wedge \exists s_{n-i} \sim x = \exists s_i x \wedge s_{n-i} \exists \sim x$.
But in a Post algebra, if $a \wedge b = 0$ then $b \leqslant \sim a$. Then $s_i x =$
$= s_i \exists x = \exists s_i x \leqslant \sim s_{n-i} \exists \sim x$. Therefore

$$s_{n-i} \exists \sim x \leqslant \sim s_i x = s_{n-i} \sim x \leqslant \exists s_{n-i} \sim x = s_{n-i} \exists \sim x.$$

Consequently $s_{n-i} \exists \sim x = s_{n-i} \sim x$ for $i = 1,\ldots,n-1$, and for the Moisil determination principle, we have $\exists \sim x = \sim x$ and therefore $\sim x \in K(P)$. This proves that $K(P)$ is closed under $\sim$.
Finally, if $x \in K(P)$, i.e. $\exists x = x$, then $s_i \exists x = s_i x$ for $i = 1,\ldots,n-1$ and by E3), $\exists s_i x = s_i x$, so that $s_i x \in K(P)$.
This completes the proof of the theorem.

COROLLARY 9. If x and y are in $K(P)$, then $x \vee y$ belongs to $K(P)$.

The following two formulae hold in any n-valued monadic Post algebra and are stated formally for convenience of reference. The proofs can be found in [13].

$$\exists(x \vee y) = \exists x \vee \exists y \quad ; \quad \exists \sim \exists x = \sim \exists x$$

Also, it is well known that if $B(P)$ is the center of P, that is, the set of all complemented elements of P, and if $K_i =$
$= \{x \in P\colon s_i x = x\}$ we have the following result proved by G. Moisil in [10] and R.Cignoli in [4]:

$$K_1 = K_2 = \ldots = K_{n-1} = B(P)$$

Then we have

LEMMA 10. If $x \in B(P)$, then $\exists x \in B(P)$.

Proof. If x is in $B(P)$, then $s_i x = x$ for all i, $1 \leqslant i \leqslant n-1$, and consequently $\exists s_i x = \exists x$. By E3), $s_i \exists x = \exists x$ for all i, $1 \leqslant i \leqslant n-1$, and therefore $\exists x \in B(P)$.

COROLLARY 11. The system $\langle B(P), \exists \rangle$ is a monadic Boolean algebra.

LEMMA 12. If $B(P) \subseteq K(P)$, then $\exists x = x$ for all x of P. (This quantifier is called discrete).

Proof. If x is in P, $s_i x$ is in $B(P)$ for all i, $1 \leqslant i \leqslant n-1$, and since $B(P) \subseteq K(P)$ it follows that $s_i x \in K(P)$ for all i, $1 \leqslant i \leqslant n-1$, and from lemma 6, $x \in K(P)$. Then $\exists x = x$.

As a consequence of this lemma it follows that the discrete quantifier is the only one which can be defined on the Post algebra P_n. Indeed, $B(P_n) = \{0,1\}$ and then $B(P_n) \subseteq K(P_n)$.

3. MONADIC HOMOMORPHISMS.

Let $\langle P, \exists \rangle$ and $\langle P', \exists \rangle$ be two n-valued monadic Post algebras with distinghished elements $e_o, e_1, \ldots, e_{n-1}$ and $e'_o, e'_1, \ldots, e'_{n-1}$ respectively. A mapping $h: P \longrightarrow P'$ is said to be a monadic homomorphism if h is a Post homomorphism preserving the $\exists$ operation. In other words, h is a monadic homomorphism if and only if for all $x, y \in P$

$$h(x \vee y) = h(x) \vee h(y)$$

$$h(\sim x) = \sim h(x)$$

$$h(s_i x) = s_i h(x)$$

$$h(e_i) = e'_i \ , \quad 0 \leqslant i \leqslant n-1$$

$$h(\exists x) = \exists h(x)$$

By a monadic deductive system in an n-valued monadic Post algebra $\langle P, \exists \rangle$ is understood a filter $D \subseteq P$ such that if x belongs to D then $\forall x$ and $s_1 x$ belong to D.

It can be proved in the usual way that all homomorphic images of an n-valued monadic Post algebra P can be found up to isomorphism by the congruences associated to monadic deductive systems of P. In other words, if D is a monadic deductive system of P and we define $x \equiv y \pmod D$ if and only if there exists an element d in D such that $x \wedge d = y \wedge d$, then $\equiv$ is a congruence relation on the algebra P. Conversely, suppose $\equiv$ is a congruence relation in P, then $D = \{x; x \equiv 1\}$ is a monadic deductive system of P. Then the ordered set of all congruences of P is isomorphic to the set of all monadic deductive systems of P. The set P/D of all equivalence classes, algebrized in the natural fashion, becomes an n-valued monadic Post algebra, and the mapping $x \to |x|$ carrying each x in P into its equivalence class in P/D, is an epimorphism from P onto P/D (see [3]). This result can be derived from the fact that n-valued monadic Post algebra is equationally definable.

When P has only trivial congruences, P is called simple. Then it is clear that an n-valued monadic Post algebra P is simple if and only if {1} is the only proper monadic deductive system of P, and D is a maximal monadic deductive system of P if and only if P/D is simple.

In addition we have:

THEOREM 13. If P is an n-valued monadic Post algebra then the following are equivalent:

i) P is simple
ii) K(P) is a simple n-valued Post algebra
iii) B(P) is a simple monadic Boolean algebra
iv) $K(P) \cap B(P)$ is a simple Boolean algebra
v) P is subdirectly irreducible.

Proof. It is not difficult to prove that there is a one-to-one correspondence between monadic deductive systems in P, deductive systems in K(P), monadic filters in B(P) and filters in $K(P) \cap B(P)$. So that, the conditions i), ii), iii) and iv) are equivalent.

Clearly, if P is simple P is subdirectly irreducible.

Now, let us suppose P is not subdirectly irreducible. Then there exists a set of monadic deductive systems D_γ such that $D_\gamma \neq \{1\}$ and $\cap D_\gamma = \{1\}$. Then the corresponding filters $D_\gamma \cap K(P) \cap B(P)$ give a set of filters in the Boolean algebra $K(P) \cap B(P)$ with $D_\gamma \cap K(P) \cap B(P) \neq \{1\}$ and $\cap D_\gamma \cap K(P) \cap B(P) = \{1\}$. So $K(P) \cap B(P)$ is not subdirectly irreducible and then $K(P) \cap B(P)$ is not simple. Consequently P is not simple, a contradiction. This completes the proof of the theorem.

THEOREM 14. Every n-valued monadic Post algebra is a subdirect product of simple algebras.

Proof. This follows from the above Theorem by Birkhoff's Theorem [3].

The following theorem is an important characterization of simple n-valued monadic Post algebras.

THEOREM 15. An n-valued monadic Post algebra P is simple if and only if $K(P) = \{0,e_1,\ldots,e_{n-2},1\}$.

Proof. We always have $\{0,e_1,\ldots,e_{n-2},1\} \subseteq K(P)$.

Suppose $K(P) = \{0,e_1,\ldots,e_{n-2},1\}$. Since none of the elements $e_1,\ldots,e_{n-2}$ is complemented (see [7]) it follows that $K(P) \cap B(P) = \{0,1\}$, hence from Theorem 13, P is simple.

For the converse, suppose P is simple. Then by Theorem 13 K(P) is a simple n-valued Post algebra. But P_n is the only simple n-valued Post algebra (See [4]). Therefore K(P) and P_n are isomorphic algebras, hence $K(P) = \{0,e_1,\ldots,e_{n-2},1\}$.

Some of the above results can be improved in the case P is a finite algebra.

Let us remark that in every n-valued monadic Post algebra P, the principal filter generated by a, F(a), is a monadic deductive system if and only if a belongs to $K(P) \cap B(P)$.

If P is finite, every filter is principal and then it follows that the family of monadic deductive systems of P is the family of principal filters $F(a)$, with $a \in K(P) \cap B(P)$. In addition, it is clear that $F(a)$ is a maximal monadic deductive system if and only if a is an atom of the Boolean algebra $K(P) \cap B(P)$.

If $b_1, b_2, \ldots, b_s$ are the atoms of $K(P) \cap B(P)$ then the algebras $S_i = P/_{F(b_i)}$, $1 \leqslant i \leqslant s$, are simple, and we know that P is isomorphic to a subalgebra of $\prod_{i=1}^{s} S_i$, where the isomorphism ϕ is given by $\phi(x) = (\phi_\gamma(x))_{1\leq\gamma\leq s}$, $\phi_\gamma \colon P \longrightarrow S_\gamma$ being the natural homomorphism.

If P is finite, ϕ is also surjective. Indeed, if $y = (y_\gamma)_{1\leq\gamma\leq s}$ belongs to $\prod_{i=1}^{s} P/_{F(b_i)}$, for each γ let $x_\gamma \in P$ be such that $\phi_\gamma(x_\gamma) = y_\gamma$. Then $x = \bigvee_{j=1}^{s} (x_\gamma \wedge b_\gamma)$ is such that $\phi_\gamma(x) = y_\gamma$, hence $\phi(x) = y$.

Then we have the following theorem:

THEOREM 16. Any finite n-valued monadic Post algebra P (with more than one element) is isomorphic to the direct product $\prod_{\gamma=1}^{s} P/_{F(b_\gamma)}$, where $b_1, \ldots, b_s$ is the set of all atoms of the Boolean algebra $K(P) \cap B(P)$.

To end this section, we are going to point out some properties of finitely generated n-valued monadic Post algebras that we shall need in the following section.

If $G \subseteq P$ we shall note $S(G)$ the subalgebra of P generated by G, and $SP(G)$ the subalgebra of P as n-valued Post algebra generated by G.

LEMMA 17. If P is a simple n-valued monadic Post algebra and G a generating set of P (that is $S(G) = P$) then $SP(G) = P$.

Proof. From Theorem 15 we have that $K(P) = \{0, e_1, \ldots, e_{n-2}, 1\}$ and therefore $K(P) \subseteq SP(G)$, then $SP(G)$ is a subalgebra which contains G. This implies $S(G) \subseteq SP(G)$. Hence $SP(G) = P$.

As a consequence, if P is a simple n-valued monadic Post algebra, $G \subseteq P$, G finite of cardinal $N[G] = r$ and $S(G) = P$, then P is finite. In fact, $n \leqslant N[P] \leqslant n^{n^r}$. This follows from the fact that P is a homomorphic image of the free n-valued Post algebra on r generators which has n^{n^r} elements (See [4]).

If P is an n-valued monadic Post algebra with a finite set G of r generators and M is a maximal monadic deductive system of P, then $P/_M$ is also r-finitely generated and from above remark, $n \leqslant N[P/_M] \leqslant n^{n^r}$. So $P/_M$ is isomorphic as n-valued Post algebra, to P_n^k, $1 \leqslant k \leqslant n^r$. Besides, $P/_M$ is simple and then $K(P/_M) = \{0, e_1, \ldots, e_{n-1}, 1\}$, therefore $P/_M$ is isomorphic, as n-valued monadic Post algebra, to $P^*_{n,k} = \langle P_n^k, \exists\rangle$, $1 \leqslant k \leqslant n^r$. Then from Theorem 14, P is isomorphic to a subalgebra of $\prod_{k=1}^{n^r} (P^*_{n,k})^{\alpha_k}$, where α_k is the number of times which the axis $P^*_{n,k}$ appears in the decomposition of P as a subdirect product of simple algebras.

Let us see that α_k is finite.

If $\mathcal{M}_k$ is the set of maximal monadic deductive systems M of P such that $P/_M$ is isomorphic to $P^*_{n,k}$, it is clear that $\alpha_k = N[\mathcal{M}_k]$.

Let $Epi(P, P^*_{n,k})$ be the set of all epimorphisms from P onto $P^*_{n,k}$, $F(G, P^*_{n,k})$ the set of all functions from G into $P^*_{n,k}$. The mapping $h \longrightarrow Ker\ h$ carrying each $h \in Epi(P, P^*_{n,k})$ into its kernel in $\mathcal{M}_k$ is clearly surjective, and the mapping $h \longrightarrow h|_G$ carrying each $h \in Epi(P, P^*_{n,k})$ into its restriction to G is injective, being that if $h|_G = h'|_G$ then $\{x \in P: h(x) = h'(x)\}$ is a subalgebra of P which contains G.

Then $\alpha_k = N[\mathcal{M}_k] \leqslant N[Epi(P, P^*_{n,k})] \leqslant N[F(G, P^*_{n,k})] < \infty$.

Therefore, $P = \prod_{k=1}^{n^r} (P^*_{n,k})^{\alpha_k}$ and we have proved

THEOREM 18. Every finitely generated n-valued monadic Post algegebra is finite.

We shall use this Theorem in the sequel.

4. FREE n-VALUED MONADIC POST ALGEBRA.

We devote this section to the study of the most general n-valued monadic Post algebra, that is, the free algebra. We shall obtain its algebraic structure when we take for a generating set a finite set G, following for the construction a technique due to L.Monteiro [12]. We recall the definition:

DEFINITION 19. Given a set G of cardinality $c > 0$, an n-valued monadic Post algebra $F_n(c)$ is said to be a free algebra over a class of similar algebras generated by G if the following conditions are satisfied:

i) $G \subseteq F_n(c)$ and the subalgebra generated by G is $F_n(c)$

ii) Every mapping f of G into an arbitrary n-valued monadic Post algebra P can be extended to a homomorphism h from $F_n(c)$ into P.

The homomorphism h of the definition is uniquely determined by f. Moreover, since the n-valued monadic Post algebras are equationally definable, it is known from a Theorem of G. Birkhoff [3] that $F_n(c)$ exists and it is unique up to isomorphism.

From now on, we shall concentrate our study on the free n-valued monadic Post algebra on a finite set G of r generators and we shall denote by $G = \{g_1, g_2, \ldots, g_r\}$ the generating set of $F_n(r)$.

It follows from Theorem 18 that $F_n(r)$ is finite. So $F_n(r)$ can be written

$$F_n(r) = \prod_{M \in \mathcal{M}} F_n(r)/_M$$

where $\mathcal{M}$ is the (finite) family of all maximal monadic deductive systems of $F_n(r)$.

We know that $F_n(r)/_M$ is isomorphic to $P^*_{n,k}$ for some k, $1 \leqslant k \leqslant n^r$, then if we put $M_k = \{M \in \mathcal{M} : {}^{F_n(r)}/_M$ is isomorphic to $P^*_{n,k}\}$, we have $\mathcal{M} = \bigcup_{k=1}^{n^r} M_k$, $M_i \cap M_j = \emptyset$ if $i \neq j$, and by putting $\alpha_k = N[M_k]$, we can write

$$F_n(r) = \prod_{k=1}^{n^r} (P^*_{n,k})^{\alpha_k}$$

So we must to calculate α_k.

First we have the following result:

LEMMA 20. If G is a free generating set of $F_n(r)$ and if $P_n(r) = SP(G)$ is the Post subalgebra of $F_n(r)$ generated by G, then G is a free generating set of the n-valued Post algebra $P_n(r)$.

Proof. If A is any n-valued Post algebra and f is a mapping from G into A, consider the n-valued monadic Post algebra $A^* = \langle A, \exists \rangle$ where $\exists$ is the discrete quantifier on A. Since $F_n(r)$ is the free n-valued monadic Post algebra there exists a homomorphism h from $F_n(r)$ into A^* extending f. The restriction h' of h to $P_n(r)$ is a Post homomorphism from $P_n(r)$ into A extending f.

Now we are going to determine the numbers α_k. For this purpose we denote, for $1 \leqslant k \leqslant n^r$, $Epi(F_n(r), P^*_{n,k})$ the set of all epimorphisms from $F_n(r)$ onto $P^*_{n,k}$, $Epi(P_n(r), P_n^k)$ the set of all (Post) epimorphisms from $P_n(r)$ onto P_n^k, $Aut(P^*_{n,k})$ the set of all automorphisms of $P^*_{n,k}$. Then we have

LEMMA 21. $$\alpha_k = \frac{N[Epi(F_n(r), P^*_{n,k})]}{N[Aut(P^*_{n,k})]}$$

Proof. We know that the mapping $h \longrightarrow s(h) = Ker\ h$ carrying each $h \in Epi(F_n(r), P^*_{n,k})$ into its kernel in M_k is surjective.

On the other hand, it is easy to see that if $M = Ker\ h \in M_k$, then $s^{-1}(M) = \{\alpha \circ h : \alpha \in Aut(P^*_{n,k})\}$. Consequently we have the

Lemma.

We also have

LEMMA 22. $N[Epi(F_n(r),P^*_{n,k})] = N[Epi(P_n(r),P^k_n)]$.

Proof. Let $h \in Epi(F_n(r),P^*_{n,k})$ and h' the restriction of h to $P_n(r)$. Then it follows from Lemma 17 that $h'(P_n(r)) = SP(h'(G)) = SP(h(G)) = S(h(G)) = h(F_n(r)) = P^*_{n,k}$ and $P^*_{n,k} = P^k_n$ as Post algebra, then h' belongs to $Epi(P_n(r),P^k_n)$.

Conversely, if $h' \in Epi(P_n(r),P^k_n)$, let f be the restriction of h' to G. Then f is a mapping from G into $P^*_{n,k}$ and therefore, f can be extended to a homomorphism h_f from $F_n(r)$ into $P^*_{n,k}$. It is easy to see that h_f belongs to $Epi(F_n(r),P^*_{n,k})$, and that its restriction to $P_n(r)$ coincides with h'.

If $Epi(B(P_n(r)),B(P^k_n))$ is the set of all Boolean epimorphisms from the center $B(P_n(r))$ of $P_n(r)$ into the center $B(P^k_n)$ of P^k_n, we have:

LEMMA 23. $N[Epi(P_n(r),P^k_n)] = N[Epi(B(P_n(r)),B(P^k_n))]$.

Proof. Suppose h_1 belongs to $Epi(B(P_n(r)),B(P^k_n))$. If $x \in P_n(r)$ then $x = (d_1 \wedge e_1) \vee (d_2 \wedge e_2) \vee \ldots \vee d_{n-1}$, with $d_i \in B(P_n(r))$, $1 \leqslant i \leqslant n-1$, $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_{n-1}$ and this representation is unique. If we put $h(e_i) = e_i \in P^k_n$ for $i = 0,1,\ldots,n-1$, where the elements $e_o,e_1,\ldots,e_{n-1}$ have been denoted by the same letters in both algebras for convenience reasons only, and $h(x) = (h_1(d_1) \wedge e_1) \vee (h_1(d_2) \wedge e_2) \vee \ldots \vee h_1(d_{n-1})$, h is a homomorphism from $P_n(r)$ onto P^k_n extending h_1 (See [22]).

Conversely, if $h \in Epi(P_n(r),P^k_n)$ then it is clear that $h|_{B(P_n(r))} \in Epi(B(P_n(r)),B(P^k_n))$. This is a one-to-one correspondence.

Let us observe that the free n-valued Post algebra on r generators is isomorphic to the direct product of n^r copies of P_n, and then it is clear that $B(P_n(r))$ is isomorphic to the Boolean algebra with n^r atoms B_{n^r}, the elements $a_i = (0,\ldots,\underset{i}{1},\ldots,0)$, $1 \leqslant i \leqslant n^r$, being the atoms of $B(P_n(r))$.

In a similar way $B(P_n^k)$ is isomorphic to the Boolean algebra with k atoms B_k.

Therefore [20]

$$N[Epi(B(P_n(r)),B(P_n^k))] = N[Epi(B_{n^r},B_k)] = V_{n^r,k}.$$

Finally, it is clear that $N[Aut(P^*_{n,k})] = N[Aut(P_n^k)] =$
$= N[Aut(B(P_n^k))] = N[Aut(B_k)] = k!$

By this sequence of Lemmas and remarks it follows that

$$\alpha_k = \frac{V_{n^r,k}}{k!} = \binom{n^r}{k} \qquad 1 \leqslant k \leqslant n^r$$

and consequently

$$F_n(r) = \prod_{k=1}^{n^r} (P^*_{n,k})^{\binom{n^r}{k}}$$

Finally,

$$N[F_n(r)] = \prod_{k=1}^{n^r} (n^k)^{\binom{n^r}{k}} = n^{\sum_{k=1}^{n^r} k\binom{n^r}{k}}$$

But $$\sum_{k=1}^{n^r} k\binom{n^r}{k} = \sum_{k=1}^{n^r} \frac{k.n^r!}{k!(n^r-k)!} = n^r + \sum_{k=1}^{n^r-1} \frac{n^r!}{(k-1)!\,(n^r-k)!} =$$

$$= n^r + n^r \sum_{k=1}^{n^r-1} \frac{(n^r-1)!}{(k-1)!\,(n^r-k)!} = n^r[1 + \sum_{k=1}^{n^r-1} \frac{(n^r-1)!}{(k-1)!\,(n^r-k)!}]$$

and making $t = k-1$ we have

$$n^r[1 + \sum_{t=0}^{n^r-2} \frac{(n^r-1)!}{t!\,(n^r-1-t)!}] = n^r[1 + \sum_{t=0}^{n^r-2} \binom{n^r-1}{t}] =$$

$$= n^r[\sum_{t=0}^{n^r-1} \binom{n^r-1}{t}] = n^r.2^{n^r-1} ,$$

and then $$N[F_n(r)] = n^{[n^r.2^{n^r-1}]}$$

In the case n = 3, the formula $F_3(r)$ coincides with that obtained by L.Monteiro in [14].

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NOTAS DE LOGICA MATEMATICA (*)

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ON FREE L-ALGEBRAS

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ON FREE L-ALGEBRAS*

by

Manuel Abad and Luiz Monteiro

ABSTRACT. In [4] A.Horn proved that the free L-algebra $FL(n)$ with a finite generating set of cardinal n, is finite. He determined its number of elements by logical methods.

In this paper we prove these results by an algebraic method, following a path analogous to that of L.Monteiro in [12]. We also give a formula to compute the number of elements of the set $\Pi(n)$ of all prime elements of $FL(n)$, and obtain the ordered structure of $\Pi(n)$.

We prove that $FL(n)$ is a direct product of two distributive lattices with 0 and 1, A_1 and A_2, where $A_2 - \{0\}$ is isomorphic to A_1.

1. DISTRIBUTIVE LATTICES.

We devote this section to give some definitions and known results on ordered sets and distributive lattices.

1.1. DEFINITION. Given an ordered set $(X,\leqslant)$, we say that an element $b \in X$ covers an element $a \in X$ if $a < b$, but $a < x < b$ for no $x \in X$.

If $x \in X$, the set $S(x) = \{y \in X: x \leqslant y\}$ is called the upper section determined by x. Similarly, $I(x) = \{z \in X: z \leqslant x\}$ is the lower section determined by x.

1.2. LEMMA. In any ordered set X, the following conditions are equivalent:

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T1) $I(x)$ is a chain, for every $x \in X$.

T2) For every pair of incomparable elements $x_o, x_1 \in X$, $S(x_o) \cap S(x_1) = \emptyset$.

1.3. LEMMA. Let X be a finite ordered set in which the condition T1 is verified. If $m_1, m_2, \ldots, m_t$ are the minimal elements of X, then X is the cardinal sum [2, p.55] of the sets $S(m_i)$, $1 \leqslant i \leqslant t$. We note $X = \sum_{i=1}^{t} S(m_i)$.

Proof. Clearly $S(m_i) \neq \emptyset$ for $1 \leqslant i \leqslant t$, and $\bigcup_{i=1}^{t} S(m_i) \subseteq X$.
If $x \in X$, there exists a minimal element m_i such that $m_i \leqslant x$. Then $x \in S(m_i)$. We have then $X = \bigcup_{i=1}^{t} S(m_i)$.
Since the elements m_i, $1 \leqslant i \leqslant t$, are incomparable, from 1.2 $S(m_i) \cap S(m_j) = \emptyset$ for $i \neq j$, $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant t$.
Finally, if $p \in S(m_i)$, $q \in S(m_j)$, $i \neq j$, then p and q are incomparable. Endeed, if $p \leqslant q$ then $q \in S(m_i)$ and therefore $q \in S(m_i) \cap S(m_j)$, which is a contradiction. Similarly if $q \leqslant p$.

The sets $S(m_i)$, $1 \leqslant i \leqslant t$, are the connected components of the ordered set X. For this notion, see A.Monteiro [11, p.53].

If R is a distributuve lattice, let the ordered set of its prime elements be denoted by $\Pi(R)$, and let $\mathbf{P}(R)$ stand for the set of all prime filters of R, and $F(x) = \{y \in R: x \leqslant y\}$ for the principal filter generated by x. If R is finite, every filter of R is principal, and we have that $P \in \mathbf{P}(R)$ if and only if $P = F(p)$ with $p \in \Pi(R)$.

It is well known that given a finite ordered set X, there exists a finite distributive lattice R such that $\Pi(R) \cong X$, and if R' and R'' are distributive lattices such that $\Pi(R') \cong \Pi(R'')$, then $R' \cong R''$ [1].

In what follows the number of elements of a finite set Y will be noted by $N[Y]$, and $R' \oplus R''$ will be the ordinal sum of the distributive lattices R' and R'' [2, p.198].

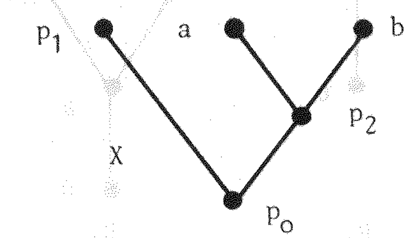
1.4. LEMMA. Let X be a finite ordered set with least element p_o and with more than one element. Let $p_1, p_2, \ldots p_t$ the elements of X which cover p_o, and suppose that $S(p_i) \cap S(p_j) = \emptyset$ for $i \neq j$, $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant t$. If R_i

is the distributive lattice such that $\Pi(R_i) \cong S(p_i)$, $1 \leqslant i \leqslant t$, then $R = \{0\} \oplus (\prod_{i=1}^{t} R_i)$ is a distributive lattice such that $\Pi(R) \cong X$. We have

$$N[R] = \prod_{i=1}^{t} N[R_i] + 1.$$

1.5. EXAMPLES.

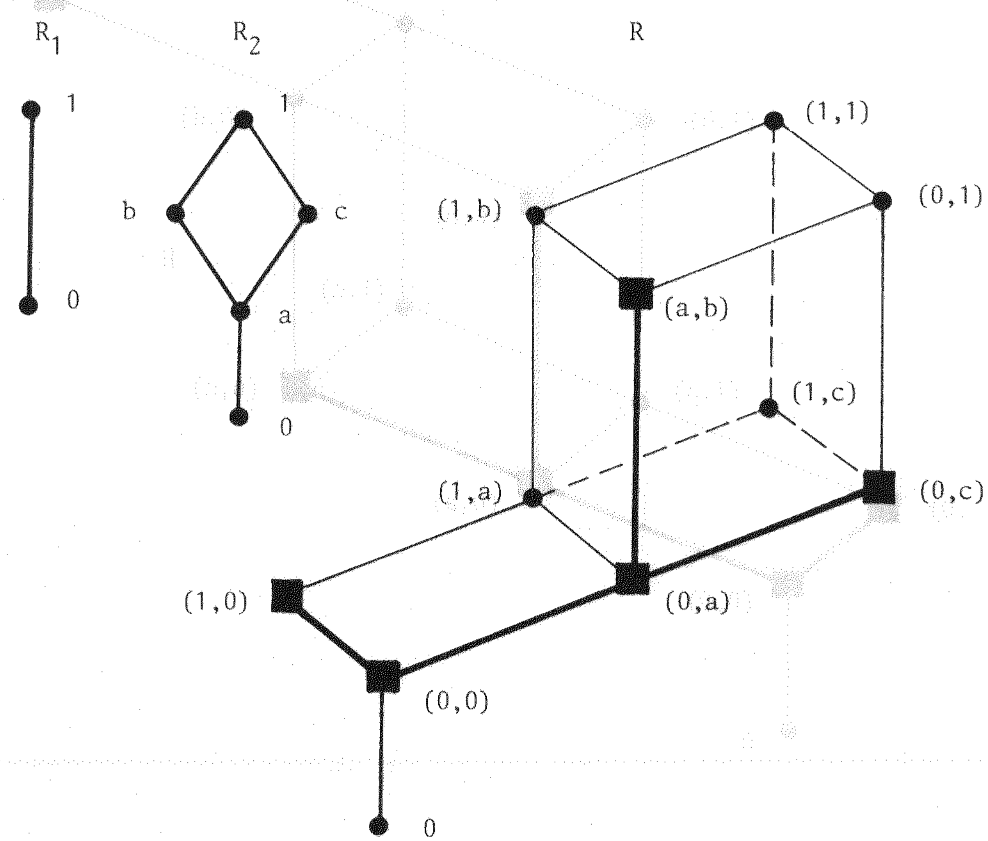
a) Let X be the ordered set of the figure:



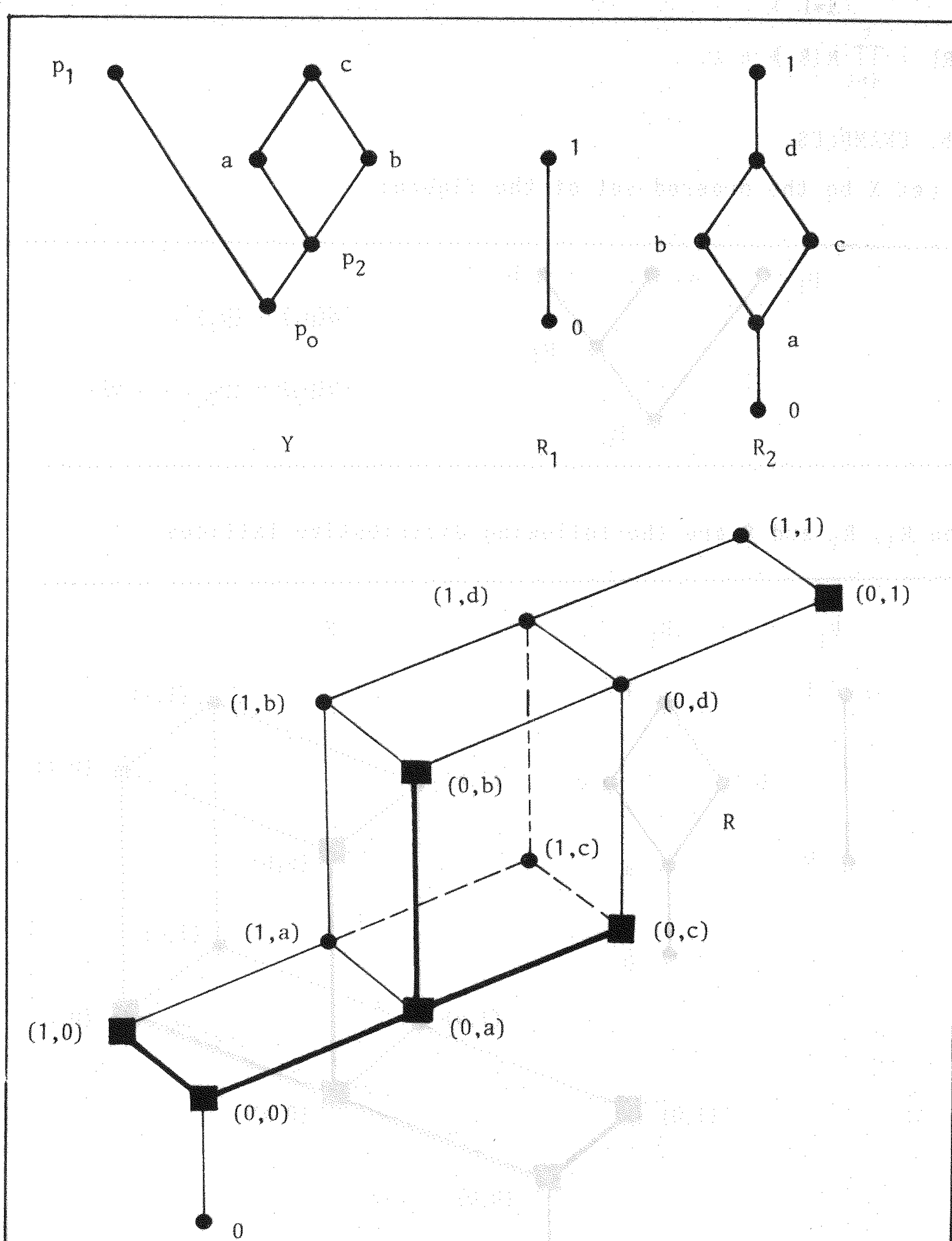
$S(p_1) = \{p_1\}$.

$S(p_2) = \{p_2 , a , b\}$.

Then R_1, R_2 and R are the following distributive lattices:



b) Consider the following set Y, then R_1, R_2 and R have the Hasse diagrams indicated in the next figure.



Observe that in the example b), the condition T1 fails to hold.

1.6. DEFINITION. Every ordered set with least element which verifies T1 is called a tree. If X is an ordered set which is a cardinal sum of trees, X is called a forest [10, p.87].

As an example, the ordered set X of 1.5 a) is a tree.

2. L-ALGEBRAS.

2.1. DEFINITION. A Heyting algebra [7,11,15] is a system $(A,\wedge,\vee,\Rightarrow,0,1)$ where A is a nonempty set, 0, 1 are two elements of A and $\wedge, \vee, \Rightarrow$ are binary operations defined on A such that the following conditions are verified:

H0) $a \wedge 0 = 0$.

H1) $a \Rightarrow a = 1$.

H2) $(a \Rightarrow b) \wedge b = b$.

H3) $a \wedge (a \Rightarrow b) = a \wedge b$.

H4) $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)$.

H5) $(a \vee b) \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c)$.

For short, we say that A is a Heyting algebra.

It is well known that any Heyting algebra is a distributive lattice with least element 0 and greatest element 1.

If X is a subset of a Heyting algebra A, the (Heyting) subalgebra of A generated by X will be noted $SH(X)$.

2.2. DEFINITION. An L-algebra [4, 11] is a Heyting algebra A such that $(a \Rightarrow b) \vee (b \Rightarrow a) = 1$, for $a,b \in A$.

Any chain A with least element 0 and greatest element 1 is an L-algebra if we define $x \Rightarrow y = 1$ if $x \leqslant y$ and $x \Rightarrow y = y$ if $x > y$.

Let n be a non negative integer. Let C_n be the chain with $n+2$ elements $a_i = \frac{i}{n+1}$, $i = 0,1,\ldots,n+1$.

The least element of C_n is $a_0 = 0$ and the greatest element is $a_{n+1} = 1$.

Then C_n is an L-algebra. It is clear that any subset X of C_n such that 0 and 1 belong to X, is a subalgebra of C_n.

2.3. For n fixed, let us note $C_i(n)$, $0 \leq i \leq n$, the following subalgebras of C_n: $C_i(n)=\{a_0 = 0, a_1, \ldots, a_i, 1\}$.

In particular, $C_0(n) = \{0,1\}$; $C_n(n) = C_n$.

In this work we identify isomorphic L-algebras.

2.4. LEMMA. If A is a chain with least element 0 and greatest element 1, and X is part of A, then $SH(X) = X \cup \{0,1\}$.

2.5. THEOREM. In any Heyting algebra the following conditions are equivalent:

(L) $(a \Rightarrow b) \vee (b \Rightarrow a) = 1$.

(L1) $a \vee b = ((a \Rightarrow b) \Rightarrow b) \wedge ((b \Rightarrow a) \Rightarrow a)$.

(L2) $(a \wedge b) \Rightarrow c = (a \Rightarrow c) \vee (b \Rightarrow c)$.

(L3) $a \Rightarrow (b \vee c) = (a \Rightarrow b) \vee (a \Rightarrow c)$.

See M.Ward [15] and A.Monteiro [8,9].

2.6. THEOREM. A Heyting algebra is an L-algebra if and only if one of the following conditions is verified:

C') Every proper filter which contains a prime filter is a prime filter.

C") The family of prime filters which contain a prime filter is a chain.

[5] , [6, p.159] , [10, p.82-84] , [11, p.118] , [3,4,14].

2.7. THEOREM. Let A be a Heyting algebra and P a prime filter of A. Then A/P is a chain if and only if the family of all proper filters of A containing P is a chain. [11, p.118].

2.8. THEOREM. Let A be a Heyting algebra and P a prime filter of A. Then A/P is a chain if and only if A is an L-algebra.

The sufficient condition was proved by A.Horn [3].

2.9. THEOREM. A non trivial Heyting algebra A (i.e. with more than one element) is an L-algebra if and only if A is isomorphic to a subalgebra of a direct product of chains. [8,9,3].

In the proof of the necessary condition of 2.9 it is considered the direct product $T = \prod_{P \in \mathbf{P}(A)} A/P$. A is isomorphic to a subalgebra A^* of T.

If A is a finite L-algebra, since $P \in \mathbf{P}(A)$ if and only if $P = F(p)$, with $p \in \Pi(A)$, from 2.6 we have:

2.10. LEMMA. If $p \in \Pi(A)$, $I(p) = \{q \in \Pi(A): q \leqslant p\}$ is a chain.

2.11. DEFINITION. Let X be a finite ordered set. We say that $p \in X$ is of level i, i positive integer, if the maximun "length" of chains in X having p for greatest element is i.

In the set Y of the example 1.5.b, p_0 is of level 1 and c is of level 4.

2.12. REMARK. If A is a finite L-algebra then $p \in \Pi(A)$ is of level i in $\Pi(A)$, i positive integer, if and only if $N[I(p)] = i$.
This is equivalent to say that $A/F(p) = C_{i-1}(n)$.

3. FREE L-ALGEBRAS.

Let FL(n) be the free L-algebra with a finite set of free generators of cardinal $n > 0$. For the sake of simplicity we will write $\mathbf{P}(n)$ instead of $\mathbf{P}(FL(n))$ and $\Pi(n)$ instead of $\Pi(FL(n))$.

From 2.9, FL(n) is isomorphic to a subalgebra of the direct product $\prod\{FL(n)/P: P \in \mathbf{P}(n)\}$. We want to prove that every quotient algebra FL(n)/P, with $P \in \mathbf{P}(n)$, is finite and also that $\mathbf{P}(n)$ is finite. From this we will have that FL(n) is finite. This result was obtained by A.Horn [4] by other method.

3.1. LEMMA. If A is an L-algebra, G a generating set of A of power n, and $P \in \mathbf{P}(A)$, then $N[A/P] \leqslant n+2$.

Proof. If h is the natural homomorphism from A onto A/P, h(G) is a generating set of A/P. Since A/P is a chain, we have $A/P = h(A) = SH(h(G)) = h(G) \cup \{0,1\}$. Then $N[A/P] = N[h(G) \cup \{0,1\}] \leqslant N[h(G)] + 2 \leqslant n+2$.

3.2. COROLLARY. If $P \in \mathbf{P}(n)$, then FL(n)/P is a finite L-algebra.

3.3. REMARK. In the conditions of lemma 3.1, if P is a prime filter of A, the family of filters containing P has at most n+2 elements, and the family of prime filters containing P has at most n+1 elements.

With $\mathbf{C}(P) = \{P = P_{t+1}, P_t, \ldots, P_1, P_0 = A\}$, where $1 \leqslant t \leqslant n$, and $P = P_{t+1} \subset P_t \subset \ldots \subset P_1 \subset P_0 = A$ we will denote the family of all filters containing P.

It is well known that the natural homomorphism h from A onto $A/P = C_t(n)$ is defined in the following way:

$$h(x) = \begin{cases} 1 & \text{if } x \in P_{t+1} = P \\ a_i = \frac{i}{n+1} & \text{if } x \in P_i - P_{i+1}\ ,\quad 0 \leqslant i \leqslant t \end{cases}$$

If $P \in \mathbf{P}(n)$ then from 2.8 and 3.1 we can state that $A/P = C_i(n)$ for some i, $0 \leqslant i \leqslant n$.

If $\mathbf{P}_i(n) = \{P \in \mathbf{P}(n) : FL(n)/P = C_i(n)\}$, $0 \leqslant i \leqslant n$, then it is clear that $\mathbf{P}(n) = \bigcup_{i=0}^{n} \mathbf{P}_i(n)$ and that $\mathbf{P}_j(n) \cap \mathbf{P}_k(n) = \emptyset$ for $j \neq k$, $0 \leqslant j \leqslant n$, $0 \leqslant k \leqslant n$.

3.4. LEMMA. Every $\mathbf{P}_i(n)$, $0 \leqslant i \leqslant n$, is a nonempty finite set.

Proof. Let $\mathbf{F}_i(n)$, $0 \leqslant i \leqslant n$, be the set of all functions f from the set G of free generators of FL(n) into $C_i(n)$ such that $SH(f(G)) = C_i(n)$. Since $i \leqslant n$, it is clear that every $\mathbf{F}_i(n)$ is nonempty (see 3.8). If $f \in \mathbf{F}_i(n)$, f can be extended to a unique homomorphism $\overline{f}$ from FL(n) into $C_i(n)$. Observe that $C_i(n) = SH(f(G)) = SH(\overline{f}(G)) = \overline{f}(FL(n))$, then $\overline{f}$ is an epimorphism from FL(n) onto $C_i(n)$. If $Ker(\overline{f})$ is the kernel of $\overline{f}$, it is well known that $Ker(\overline{f}) \in \mathbf{P}(n)$ and $FL(n)/Ker(\overline{f}) = C_i(n)$, therefore $Ker(\overline{f}) \in \mathbf{P}_i(n)$. Thus, for each i, $0 \leqslant i \leqslant n$, we have a function ψ_i from $\mathbf{F}_i(n)$ into $\mathbf{P}_i(n)$ defined by $\psi_i(f) = Ker(\overline{f})$, where $f \in \mathbf{F}_i(n)$.

Let us see that ψ_i is onto. For $P \in \mathbf{P}_i(n)$, consider h the natural homomorphism from FL(n) onto $FL(n)/P = C_i(n)$, and $f = h|_G$ the restriction of h to G. Then $SH(f(G)) = SH(h(G)) = h(SH(G)) = h(FL(n)) = C_i(n)$, and therefore $f \in \mathbf{F}_i(n)$. Let $\overline{f}$ be the extension of f.

Since $\overline{f}|_G = f = h|_G$, then $\overline{f} = h$ and therefore $\psi_i(f) = Ker(\overline{f}) = Ker(h) = P$.

Since $\mathbf{F}_i(n)$ is finite for every i, $0 \leqslant i \leqslant n$, then $\mathbf{P}_i(n)$ is finite, and we have

3.5. LEMMA. $\mathbf{P}(n)$ is a finite set.

3.6. THEOREM. $FL(n)$ is finite.

We have in addition that the function ψ_i, $0 \leqslant i \leqslant n$, is one-to-one. Endeed, if $f_1, f_2 \in \mathbf{F}_i(n)$ verify $Ker(\overline{f}_1) = Ker(\overline{f}_2)$ then, from results of universal algebra, there is an automorphism α of $C_i(n)$ such that $\alpha \circ \overline{f}_1 = \overline{f}_2$. But the only automorphism of $C_i(n)$ is $\alpha = Id.$, then $\overline{f}_1 = \overline{f}_2$ and then $f_1 = f_2$.

Then we have that $N[\mathbf{P}_i(n)] = N[\mathbf{F}_i(n)]$, $0 \leqslant i \leqslant n$, and therefore:

3.7. LEMMA. $N[\mathbf{P}(n)] = \sum_{i=0}^{n} N[\mathbf{P}_i(n)] = \sum_{i=0}^{n} N[\mathbf{F}_i(n)]$.

3.8. REMARK. The functions f from G into $C_i(n)$, $0 \leqslant i \leqslant n$, such that $SH(f(G)) = C_i(n)$, that is, such that $f(G) \cup \{0,1\} = C_i(n)$, are those which verify some of the following conditions:

1) $f(G) = \{a_1, a_2, \ldots, a_i\} = X_i$; 2) $f(G) = X_i \cup \{0\}$;

3) $f(G) = X_i \cup \{1\}$; 4) $f(G) = C_i(n)$.

Note that $X_i = \emptyset$ if $i=0$.

If $NS(a,b)$ is the number of functions from a set with a elements onto a set with b elements, then:

$$NS(a,b) = \begin{cases} \sum_{i=0}^{b-1} (-1)^i \binom{b}{i} (b-i)^a & \text{if } a \geqslant b \\ 0 & \text{if } a < b \end{cases}$$

Then we can state that:

$N[\mathbf{F}_i(n)] = NS(n,i) + 2NS(n,i+1) + NS(n,i+2)$, $0 \leqslant i \leqslant n$. In particular, $N[\mathbf{F}_0(n)] = NS(n,0) + 2NS(n,1) + NS(n,2) = 2^n$, and $N[\mathbf{F}_n(n)] = n!$.

It is easy to see that $N[\mathbf{P}(1)] = 3$ and that if $n \geqslant 2$,

$N[\mathbf{P}(n)] = 3 + 4 \sum_{s=2}^{n} NS(n,s)$.

As we have stated in section 1, any finite distributive lattice is determined, up to isomorphism, by the ordered set of its prime elements. Then the algebra FL(n) is known if we describe the ordered set $\Pi(n)$.

Consider the set $\mathbf{F}(n) = \bigcup_{i=0}^{n} \mathbf{F}_i(n)$. If $f \in \mathbf{F}(n)$, there is a unique i such that $f \in \mathbf{F}_i(n)$. If we put $\psi(f) = \psi_i(f)$ we have a one-to-one mapping from $\mathbf{F}(n)$ onto $\mathbf{P}(n)$.

Since $\mathrm{Ker}(\bar{f})$ is a prime filter of FL(n), then $\mathrm{Ker}(\bar{f}) = F(p_f)$ with $p_f \in \Pi(n)$. If we define $\varphi(f) = p_f$ we obtain a bijection between $\mathbf{F}(n)$ and $\Pi(n)$. Then each element of $\Pi(n)$ can be represented by an element of $\mathbf{F}(n)$, that is, a function f from G into $C_i(n)$, $0 \leq i \leq n$, such that $SH(f(G)) = C_i(n)$.

3.9. REMARK. If $p_f \in \Pi(n)$ is of level i, $1 \leq i \leq n+1$, then from 3.1 $FL(n)/F(p_f) = C_{i-1}(n)$, and this is equivalent to say that $F(p_f) \in \mathbf{P}_{i-1}(n)$. But $\psi_{i-1} : \mathbf{F}_{i-1}(n) \to \mathbf{P}_{i-1}(n)$ is a bijection, then $f = \psi_{i-1}^{-1}(F(p_f)) \in \mathbf{F}_{i-1}(n)$. Therefore $SH(f(G)) = C_{i-1}(n)$, that is, $f(G) \cup \{0,1\} = C_{i-1}(n)$.

3.10. LEMMA. For $p_f \in \Pi(n)$ to be of level 1 it is necessary and sufficient that $f(g) \in \{0,1\}$ for all $g \in G$.

Proof. It follows inmediately from 3.9.

As a consequence, the set $\Pi(n)$ has 2^n minimal elements, i.e., elements of level 1, as we had seen in 3.8.
In a similar way we have:

3.11. LEMMA. For $p_f \in \Pi(n)$ to be of level i, $2 \leq i \leq n+1$, it is necessary and sufficient that $f(G) \subseteq C_{i-1}(n)$ and that $a_1, a_2, \ldots, a_{i-1} \in f(G)$.

3.12. REMARK. If $f \in \mathcal{F}_t(n)$, the extension homomorphism $\bar{f}$ and the natural homomorphism h from FL(n) into $FL(n)/\mathrm{Ker}(\bar{f})$ verify $h = \bar{f}$. Then if $\mathbf{C}(\mathrm{Ker}(\bar{f})) = \{\mathrm{Ker}(\bar{f}) = P_{t+1}, P_t, \ldots, P_1, P_0 = FL(n)\}$ (see 3.3), we have

$$\overline{f}(x) = \begin{cases} 1 & \text{if } x \in \operatorname{Ker}(\overline{f}) \\ a_i = \frac{i}{n+1} & \text{if } x \in P_i - P_{i+1}, \quad 0 \leqslant i \leqslant t \end{cases}$$

3.13. LEMMA. If $p,q \in \Pi(n)$, q covers p if and only if the following conditions are verified:

(1) $F(q) \subset F(p)$

(2) $F(p) \in \mathbf{P}_t(n)$, $0 \leqslant t \leqslant n-1$

(3) $F(q) \in \mathbf{P}_{t+1}(n)$, $0 \leqslant t \leqslant n-1$

Proof. Necessary condition. If q covers p, then $p < q$ and then (1) $F(q) \subset F(p)$, and $p < p' < q$ for no $p' \in \Pi(n)$, that is (i) $F(q) \subset P \subset F(p)$ for no $P \in \mathbf{P}(n)$. Since $F(p) \in \mathbf{P}(n)$ then $F(t) \in \mathbf{P}_t(n)$, with $0 \leqslant t \leqslant n$. If $t=n$ then the family of prime filters containing $F(q)$ would have $n+2$ elements, which is a contradiction (see 3.3). Therefore $F(p) \in \mathbf{P}_t(n)$, $0 \leqslant t \leqslant n-1$, and from (i) it follows (3).

Sufficient condition. Let $p,q \in \Pi(n)$ be such that (1), (2) and (3) are verified. From (1) it follows that $p < q$. If we suppose that there is $p' \in \Pi(n)$ such that $p < p' < q$ then we have $F(q) \subset F(p') \subset F(p)$, and from (2) we have $F(q) \in \mathbf{P}_{t+2}(n)$ which contradicts (3).

3.14. THEOREM. Let $f,h \in \mathbf{F}(n)$ be. Then for $\varphi(h) = p_h = q$ covers $\varphi(f) = = p_f = p$ it is necessary and sufficient that $f \in \mathbf{F}_t(n)$, $h \in \mathbf{F}_{t+1}(n)$, $0 \leqslant t \leqslant n-1$, and the following conditions are verified:

I) $f(g) = a_j$ if and only if $h(g) = a_j$, $0 \leqslant j \leqslant t$.

II) $f(g) = 1$ if and only if $h(g) = 1$ or $h(g) = a_{t+1}$.

iii) There exists $g \in G$ such that $f(g) \neq h(g)$.

Proof. Necessary condition. From 3.13 q covers p if and only if $F(q) \subset \subset F(p)$. and $f \in \mathbf{F}_t(n)$, $h \in \mathbf{F}_{t+1}(n)$, $0 \leqslant t \leqslant n-1$. Since $SH(f(G)) = C_t(n)$, then $f(G) \cup \{0,1\} = C_t(n)$. If $t=0$ then $f(G) \subseteq \{0,1\}$ otherwise $a_1,a_2,\ldots,a_t \in f(G)$. In a similar way $a_1,a_2,\ldots,a_t,a_{t+1} \in h(G)$.

Since $P_{t+2} = F(p_h) \subset P_{t+1} = F(p_f) \subset P_t \subset \ldots \subset P_2 \subset P_1 \subset P_0 = FL(n)$ we have

$$\overline{f}(x) = \begin{cases} 1 & \text{if } x \in F(p_f) \\ a_j & \text{if } x \in P_j - P_{j+1}, \quad 0 \leqslant j \leqslant t \end{cases}$$

$$\overline{h}(x) = \begin{cases} 1 & \text{if } x \in F(p_h) \\ a_j & \text{if } x \in P_j - P_{j+1}, \quad 0 \leqslant j \leqslant t+1 \end{cases}$$

Then $\overline{f}(x) = \overline{h}(x) = a_j$, $0 \leqslant j \leqslant t$, if and only if $x \in P_j - P_{j+1}$

$\overline{f}(x) = \overline{h}(x) = 1$ if and only if $x \in F(p_h)$

$\overline{f}(x) = 1$ and $\overline{h}(x) = a_{t+1}$ if and only if $x \in F(p_f) - F(p_h)$.

Particularly, if $g \in G$:

I) $f(g) = a_j$ if and only if $h(g) = a_j$, $0 \leqslant j \leqslant t$;

II) $f(g) = 1$ if and only if $h(g) = 1$ or $h(g) = a_{t+1}$.

In addition, III) there is $g \in G$ such that $f(g) \neq h(g)$. Endeed, if $f(g) = h(g)$ for all $g \in G$, then $f=h$ and therefore $p_f = p_h$.

Sufficient condition. Let $f,h \in \mathbf{F}(n)$ be such that $f \in \mathbf{F}_t(n)$, $h \in \mathbf{F}_{t+1}(n)$, $0 \leqslant t \leqslant n-1$ and verifying I, II and III.

From $f \in \mathbf{F}_t(n)$ and $h \in \mathbf{F}_{t+1}(n)$ we have $FL(n)/F(p_f) = C_t(n)$ and $FL(n)/F(p_h) = C_{t+1}(n)$, $0 \leqslant t \leqslant n-1$. Then from 3.13, we must prove that $F(p_h) \subset F(p_f)$.

Consider $F(p_f) = P_{t+1} \subset P_t \subset \ldots \subset P_1 \subset P_0 = FL(n)$

and $\qquad F(p_h) = Q_{t+2} \subset Q_{t+1} \subset \ldots \subset Q_1 \subset Q_0 = FL(n)$

the chains of prime filters containing $F(p_f)$ and $F(p_h)$ respectively and consider the following sets:

$$C_{t+2} = Q_{t+2} \cap P_{t+1}$$

$$C_{t+1} = (Q_{t+1} - Q_{t+2}) \cap P_{t+1}$$

$$C_j = (Q_j - Q_{j+1}) \cap (P_j - P_{j+1}) \quad , \quad j = 0,1,\ldots,t$$

Then $z \in C_{t+2}$ if and only if $\overline{h}(z) = 1$ and $\overline{f}(z) = 1$

$z \in C_{t+1}$ if and only if $\overline{h}(z) = a_{t+1}$ and $\overline{f}(z) = 1$

$z \in C_j$ if and only if $\overline{h}(z) = a_j$ and $\overline{f}(z) = a_j$, $0 \leqslant j \leqslant t$.

Observe that $C_0 = (Q_0 - Q_1) \cap (P_0 - P_1) = (FL(n) - Q_1) \cap (FL(n) - P_1) =$

$$= CQ_1 \cap CP_1 = C(Q_1 \cup P_1).$$

We have that C_{t+2} is a filter, C_0 is an ideal and C_j, $0 \leqslant j \leqslant t$, are nonempty sets, being that $a_j \in \overline{h}(FL(n))$, $a_j \in \overline{f}(FL(n))$, $0 \leqslant j \leqslant t$, C_{t+1} is also nonempty. Endeed, from III, there exists $g \in G$ such that $f(g) \neq h(g)$, and then we have, from I and II, that $\overline{h}(g) = a_{t+1}$ and $\overline{f}(g) = 1$, that is, $g \in C_{t+1}$.

It is clear that the sets C_j, $0 \leqslant j \leqslant t+2$, are pairwise disjoint. Observe that $C_{t+2} \cup C_{t+1} = Q_{t+2} \cap P_{t+1}$, and so it is a filter. Using these remarks it is a routine matter to show that the set $S = \bigcup_{i=0}^{t+2} C_i$ is a subalgebra of $FL(n)$. This proof is long but computational, so it will be omitted.

Let us see that $G \subseteq S$. If $g \in G$, $h(g) \in \{0 = a_0, a_1, \ldots, a_t, a_{t+1}, 1\}$.

If $h(g) = 1$, $g \in Q_{t+2}$ and from II, $f(g) = 1$, that is, $g \in P_{t+1}$. Then $g \in Q_{t+2} \cap P_{t+1} = C_{t+2} \subseteq S$.

If $h(g) = a_{t+1}$, $g \in Q_{t+1} - Q_{t+2}$ and from II, $f(g) = 1$, that is $g \in P_{t+1}$. So $g \in (Q_{t+1} - Q_{t+2}) \cap P_{t+1} = C_{t+1} \subseteq S$.

If $h(g) = a_j$, $0 \leqslant j \leqslant t$, then $g \in Q_j - Q_{j+1}$ and from I, $f(g) = a_j$, that is, $g \in P_j - P_{j+1}$. Then $g \in (Q_j - Q_{j+1}) \cap (P_j - P_{j+1}) = C_j \subseteq S$.

Therefore, $G \subseteq S$. We then have $S = FL(n)$.

Then we can write $F(p_h) = Q_{t+2} = Q_{t+2} \cap FL(n) = Q_{t+2} \cap (\bigcup_{i=0}^{t+2} C_i) =$

$= \bigcup_{i=0}^{t+2} (Q_{t+2} \cap C_i) = Q_{t+2} \cap C_{t+2} = Q_{t+2} \cap P_{t+1} = F(q_h) \cap F(p_f)$.

That is, $F(p_h) = F(p_h) \cap F(p_f)$, and therefore $F(p_h) \subseteq F(p_f)$.

If $F(p_h) = F(p_f)$ then $Ker(\overline{h}) = Ker(\overline{f})$ and then $\overline{h} = \overline{f}$ and $h = f$, which contradicts III. Therefore $F(p_h) \subset F(p_f)$.

By virtue of 3.10, there exists a one-to-one correspondence between the set of minimal elements of $\Pi(n)$ and the set of functions f from G into $\{0,1\} \subseteq C_n(n)$.

Since the ordered set $\Pi(n)$ verifies the condition T1 of 1.2, then the set $\mathbf{F}(n)$, with the order relation induced by φ, also verifies the condition T1. Then, from 1.3, the connected components of $\mathbf{F}(n)$ are the sections S(f), with f minimal, that is, the connected components of $\mathbf{F}(n)$ are the sets S(f), where $f: G \to \{0,1\} \subseteq C_n(n)$.

In the figures (pages 17 and 18) we give the Hasse diagram of $\Pi(n)$, for n = 1,2,3.

Let $K_j(n)$ be, $0 \leqslant j \leqslant n$, the family of connected components S(f) such that $N[f^{-1}(1)] = j$.

It is clear that $N[K_j(n)] = \binom{n}{j}$.

From theorem 3.14, if $S(f) \in K_j(n)$ then $h \in S(f)$ covers f if and only if $h \in \mathbf{F}_1(n)$ and

I) $h(g) = 0$ if and only if $f(g) = 0$

II) $h(g) \in \{a_1, 1\}$ if and only if $f(g) = 1$

III) There exists $g \in G$ such that $h(g) = a_1$.

So there are $2^j - 1$ functions h covering f, $\binom{j}{t}$, $1 \leqslant t \leqslant j$, of which verify $N[h^{-1}(a_1)] = t$.

Let h be in the above conditions, that is, h covers f and $N[h^{-1}(a_1)] = t$. Then $N[h^{-1}(1)] = j-t$ and $N[h^{-1}(0)] = n-j$. If $f_1: G \to \{0,1\} \subseteq C_n(n)$ is the function defined by

$$f_1(g) = \begin{cases} 1 & \text{if} \quad h(g) = 1 \\ 0 & \text{if} \quad h(g) = a_1 \quad \text{or} \quad h(g) = 0 \end{cases}$$

f_1 is clearly a minimal element of $\mathbf{F}(n)$, and $S(f_1) \in K_{j-t}(n)$. Let us see that (*): S(h) and $S(f_1)$ are isomorphic.

First observe that if $u \in S(f)$, then $u \in \mathbf{F}_i(n)$, where $0 \leqslant i \leqslant j$.

We define $\alpha: S(h) \to S(f_1)$ by mean of: if $u \in S(h)$, $\alpha(u) = v$, where

$$v(g) = \begin{cases} 0 & \text{if } u(g) = 0 \\ 1 & \text{if } u(g) = 1 \\ a_{i-1} = \frac{i-1}{n+1} & \text{if } u(g) = a_i, \quad 1 \leqslant i \leqslant j \end{cases}$$

Clearly α is one-to-one and onto.

If $u,u' \in S(h)$ and u covers u', then $u' \in \mathbf{F}_{i-1}(n)$, $u \in \mathbf{F}_i(n)$, with $0 \leqslant i \leqslant j$, and verify I, II and III. Then it is clear that $\alpha(u) \in \mathbf{F}_{i-1}(n)$, $\alpha(u') \in \mathbf{F}_{i-2}(n)$ and verify I, II and III.

Therefore $\alpha(u)$ covers $\alpha(u')$, and α is an isomorphism.

It is clear that $N[K_0(n)] = 1$ and if $K \in K_0(n)$ then $N[K] = 1$.

For a given j, $1 \leqslant j \leqslant n$, all the connected components of $K_j(n)$ have the same number of elements. Then if $K \in K_j(n)$ we write $N(n,j) = N[K]$.

Then if $K_j(n) = \{K_1, K_2, \ldots, K_{\binom{n}{j}}\}$

$$N[\bigcup_{i=1}^{\binom{n}{j}} K_i] = \sum_{i=1}^{\binom{n}{j}} N[K_i] = \sum_{i=1}^{\binom{n}{j}} N(n,j) = \binom{n}{j} N(n,j).$$

But from (*), $N(n,j) = \sum_{t=1}^{j} \binom{j}{t} N(n,j-t) + 1$.

Therefore $N[\bigcup_{i=1}^{\binom{n}{j}} K_i] = \binom{n}{j}[\sum_{t=1}^{j} \binom{j}{t} N(n,j-t) + 1]$.

Then $N[\mathbf{F}(n)] = N[\Pi(n)] = 1 + [\sum_{j=1}^{n} \binom{n}{j}[\sum_{t=1}^{j} \binom{j}{t} N(n,j-t) + 1]]$.

Let us denote $R_j(n)$, $0 \leqslant j \leqslant n$, the distributive lattice such that $\Pi(R_j(n)) \cong K$, where $K \in K_j(n)$.

Then $R_0(n)$ is a chain with two elements, and if $K \in K_j(n)$, $1 \leqslant j \leqslant n$, since K verifies the conditions of lemma 1.4, and taking into account (*), we can state that

$$R_j(n) = \{0\} \oplus [\prod_{h=0}^{j-1} R_h(n)^{\binom{j}{h}}].$$

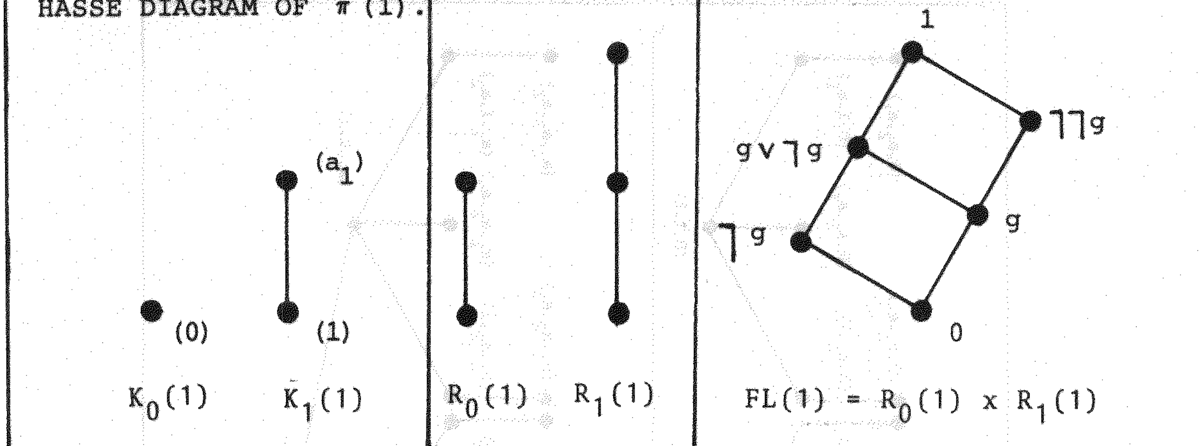
In particular $$R_n(n) = \{0\} \oplus [\prod_{j=0}^{n-1} R_j(n)^{\binom{n}{j}}].$$

Since $$FL(n) = \prod_{j=0}^{n} R_j(n)^{\binom{n}{j}}$$ then

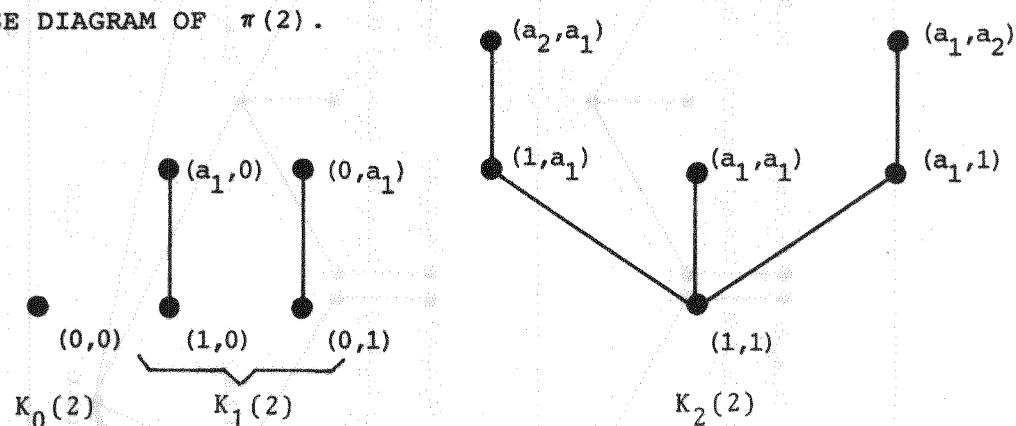
$$FL(n) = [\prod_{j=0}^{n-1} R_j(n)^{\binom{n}{j}}] \times [\{0\} \oplus \prod_{j=0}^{n-1} R_j(n)^{\binom{n}{j}}].$$

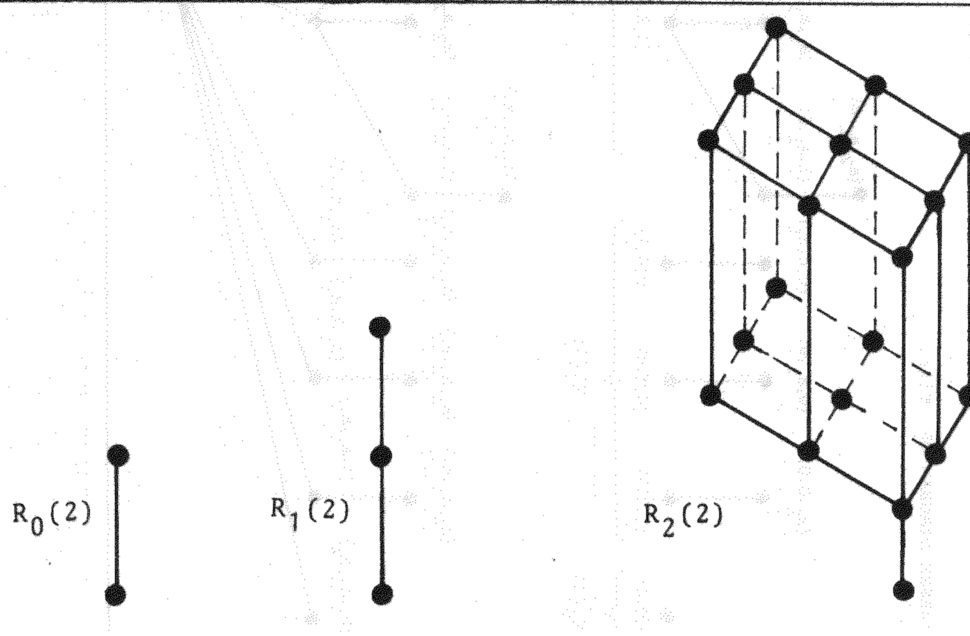
Recibido: Junio 1986

HASSE DIAGRAM OF $\pi(1)$.



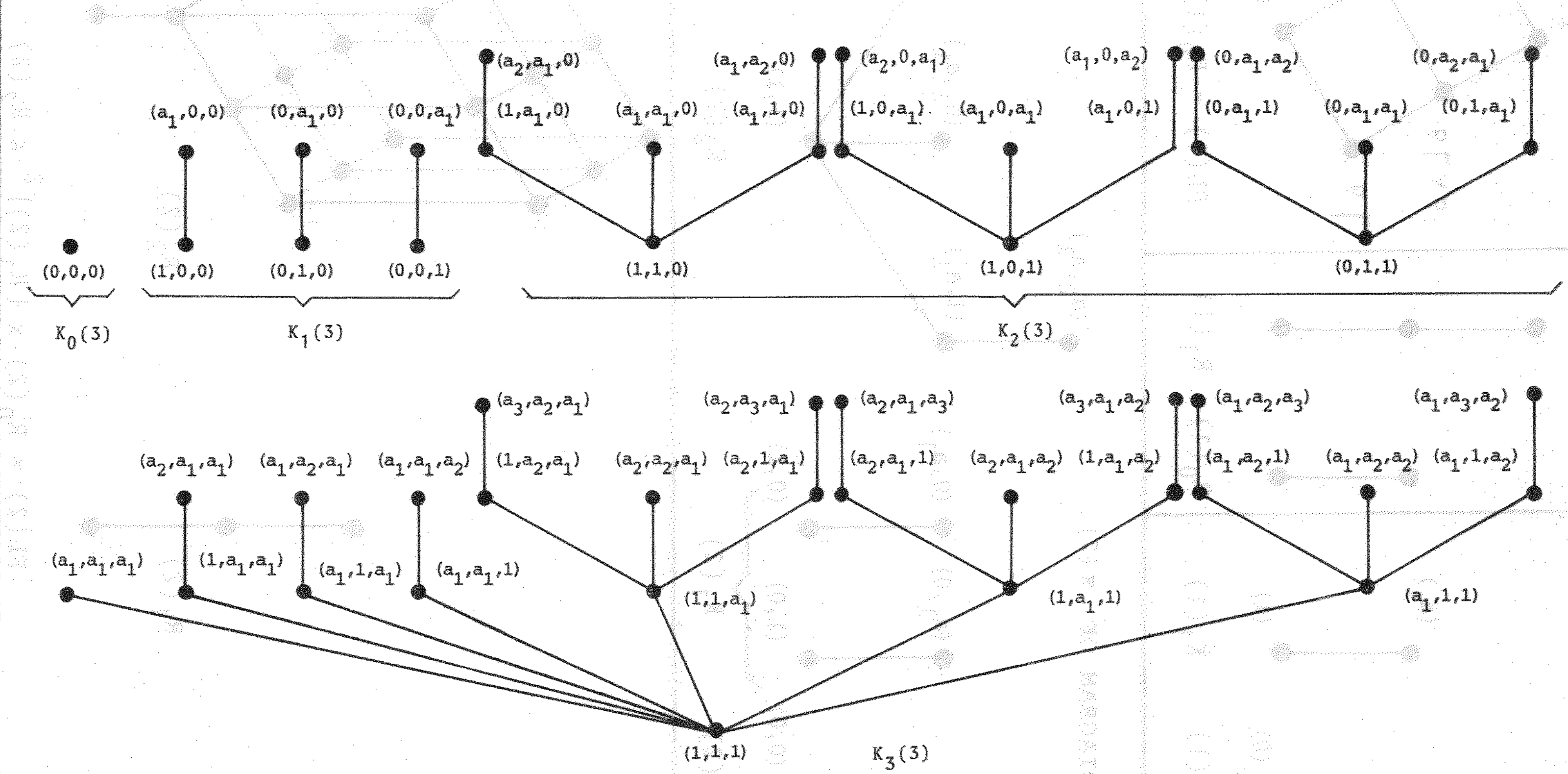
HASSE DIAGRAM OF $\pi(2)$.





$FL(2) = R_0(2) \times [R_1(2)]^2 \times R_2(2)$

HASSE DIAGRAM OF $\pi(3)$.



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