ISSN 0078 - 2017

NOTAS DE LOGICA MATEMATICA

34 - 35

MANUEL ABAD

NOTES ON n-VALUED POST ALGEBRAS

MANUEL ABAD and LUIZ MONTEIRO

ON FREE L-ALGEBRAS

1987

INMABB - CONICET
UNIVERSIDAD NACIONAL DEL SUR
BAHIA BLANCA - ARGENTINA

NOTAS DE LOGICA MATEMATICA (*)

N° 34

NOTES ON n-VALUED POST ALGEBRAS

MANUEL ABAD

INMABB - CONICET

1987

UNIVERSIDAD NACIONAL DEL SUR

BAHIA BLANCA - ARGENTINA

(*) La publicación de este volumen ha sido subsidiada por el Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.

NOTES ON n-VALUED POST ALGEBRAS

by

Manuel Abad

1. HISTORICAL INTRODUCTION AND SUMMARY OF RESULTS.

In 1920, E.Post [16] defined the most important operations of n-valued logic as a generalization of the usual 2-valued calculus, and discussed some of their properties by means of tables of values. Earlier a special concept of 3-valued logic was introduced by Lukasiewicz [9].

Rosenbloom [18] in 1942, studied Post algebras from an algebraic standpoint, giving the abstract algebraic properties of the systems independently of their interpretation as logics, and he gave the first postulate-set for Post algebras. However, Rosenbloom's system of axioms was a very difficult one and this complexity hindered the development of the theory until Epstein's paper [7], appeared in 1960. Epstein [7] and Traczyk [21] simplified Rosenbloom axiom system by making use of a greater number of operations and of the existence of an underlying Boolean algebra of a given Post algebra, and they proved the equational definability of the class of Post algebras. Since then several papers on Post algebras, as well as their generalizations and applications not only in logic but in other branches of mathematics, have been published (See Traczyk [22], Chang and Horn [5], Dwinger [6], Rouseau [19], Cignoli [4], Rasiowa [17], Monteiro [14]).

On the other hand, Halmos [8] introduced in 1955 the monadic algebras with the aim of developing an algebraic version of the logical operation of quantification, and in 1970 L.Monteiro [13] introduced the notion of three-valued monadic Lukasiewicz algebra

$1 \leqslant i \leqslant n-1$, that every element x of the Post algebra P can be written also in the form

$$x = (d_1 \wedge e_1) \vee (d_2 \wedge e_2) \vee \ldots \vee d_{n-1}$$

with $d_i \in B(P)$, $1 \leqslant i \leqslant n-1$, and $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_{n-1}$.

This representation is unique ([7] , [21]).

If we denote $d_i = D_i(x)$ and if b' is the Boolean complement of an element $b \in B(P)$, then Epstein [7] proved that the operator

$$\sim x = \bigvee_{i=1}^{n-1} (e_i \wedge (D_{n-i}(x))')$$

verifies

M1) $\sim\sim x = x$

M2) $\sim(x \vee y) = \sim x \wedge \sim y$

Also, if we write $s_i(x) = D_{n-i}(x)$ $(1 \leqslant i \leqslant n-1)$ then the following properties are satisfied:

L1) $s_i(x \vee y) = s_i x \vee s_i y$

L2) $s_i x \vee \sim s_i x = 1$

L3) $s_i s_j x = s_j x$

L4) $s_i \sim x = \sim s_{n-i} x$

L5) $s_1 x \leqslant s_2 x \leqslant \ldots \leqslant s_{n-1} x$

L6) The Moisil determination principle:

If $s_i x = s_i y$ for $i = 1,2,\ldots,n-1$, then $x=y$,

and in addition

$$s_i e_j = \begin{cases} 0 & \text{if} \quad 1+j < n \\ 1 & \text{if} \quad i+j \geqslant n \end{cases} \qquad (1)$$

Moreover, the two following properties are known to be true in any n-valued Post algebra:

$$\sim x \vee s_{n-1} x = 1 \ , \quad \text{and} \quad s_i x \wedge \sim s_i x = 0.$$

Now we re-write from Cignoli's paper the following theorem which characterizes the n-valued Post algebras.

LEMMA 4. In any n-valued monadic Post algebra the following properties hold:

1) $\exists 1 = 1$; 2) $\exists\exists x = \exists x$; 3) If $x \leqslant y$ then $\exists x \leqslant \exists y$;

4) $\sim x \vee s_{n-1} \exists x = 1$; 5) $\exists x \vee s_{n-1} \sim x = 1$.

Proof. For 1), put $x = 1$ in E1) [8]. For 2), put $x = 1$ in E2) and apply E4), $\exists(1 \wedge \exists y) = \exists 1 \wedge \exists y$, then $\exists\exists y = \exists y$.

If $x \leqslant y$ then $\exists x \leqslant \exists y$. Since $y \leqslant \exists y$, we have $x \leqslant \exists y$, then $x \wedge \exists y = x$, and by E2) $\exists(x \wedge \exists y) = \exists x \wedge \exists y = \exists x$. Then 3). From $1 = \sim x \vee s_{n-1} x \leqslant \sim x \vee s_{n-1} \exists x$ it follows 4). Finally, since $x \leqslant \exists x$, then $1 = x \vee s_{n-1} \sim x \leqslant \exists x \vee s_{n-1} \sim x$, and we have 5).

DEFINITION 5. An element k of an n-valued monadic Post algebra P is said to be a constant if $\exists k = k$.

Let K(P) be the set of constants of P. Clearly, K(P) is the range of the quantifier $\exists$, that is, $K(P) = \exists(P)$. Furthermore, 0 and 1 belong to K(P).

Throughout the following statements it is assumed that P is an n-valued monadic Post algebra and $\exists$ is a quantifier on P.

LEMMA 6. If $s_i x \in K(P)$, $1 \leqslant i \leqslant n-1$, then $x \in K(P)$.

Proof. If $s_i x \in K(P)$, $1 \leqslant i \leqslant n-1$, then $\exists s_i x = s_i x$, $1 \leqslant i \leqslant n-1$, and by E3), $s_i \exists x = s_i x$, $1 \leqslant i \leqslant n-1$, then by L6), $\exists x = x$. Therefore $x \in K$.

COROLLARY 7. The elements $e_1, e_2, \ldots, e_{n-2}$ belong to K(P).

Proof. Since $s_i e_j = 0$ or $s_i e_j = 1$ for every i and j, and since $0 \in K(P)$ and $1 \in K(P)$, it follows from the above lemma that $e_j \in K(P)$, $1 \leqslant j \leqslant n-2$.

This corollary together with next theorem shows that K(P) is a Post subalgebra of P.

Proof. If x is in $B(P)$, then $s_i x = x$ for all i, $1 \leqslant i \leqslant n-1$, and consequently $\exists s_i x = \exists x$. By E3), $s_i \exists x = \exists x$ for all i, $1 \leqslant i \leqslant n-1$, and therefore $\exists x \in B(P)$.

COROLLARY 11. The system $\langle B(P), \exists \rangle$ is a monadic Boolean algebra.

LEMMA 12. If $B(P) \subseteq K(P)$, then $\exists x = x$ for all x of P. (This quantifier is called discrete).

Proof. If x is in P, $s_i x$ is in $B(P)$ for all i, $1 \leqslant i \leqslant n-1$, and since $B(P) \subseteq K(P)$ it follows that $s_i x \in K(P)$ for all i, $1 \leqslant i \leqslant n-1$, and from lemma 6, $x \in K(P)$. Then $\exists x = x$.

As a consequence of this lemma it follows that the discrete quantifier is the only one which can be defined on the Post algebra P_n. Indeed, $B(P_n) = \{0,1\}$ and then $B(P_n) \subseteq K(P_n)$.

3. MONADIC HOMOMORPHISMS.

Let $\langle P, \exists \rangle$ and $\langle P', \exists \rangle$ be two n-valued monadic Post algebras with distinghished elements $e_o, e_1, \ldots, e_{n-1}$ and $e'_o, e'_1, \ldots, e'_{n-1}$ respectively. A mapping $h: P \longrightarrow P'$ is said to be a monadic homomorphism if h is a Post homomorphism preserving the $\exists$ operation. In other words, h is a monadic homomorphism if and only if for all $x,y \in P$

$$h(x \vee y) = h(x) \vee h(y)$$

$$h(\sim x) = \sim h(x)$$

$$h(s_i x) = s_i h(x)$$

$$h(e_i) = e'_i \ , \quad 0 \leqslant i \leqslant n-1$$

$$h(\exists x) = \exists h(x)$$

By a monadic deductive system in an n-valued monadic Post algebra $\langle P, \exists \rangle$ is understood a filter $D \subseteq P$ such that if x belongs to D then $\forall x$ and $s_1 x$ belong to D.

It can be proved in the usual way that all homomorphic images of exists a set of monadic deductive systems D_γ such that $D_\gamma \neq \{1\}$ and $\cap D_\gamma = \{1\}$. Then the corresponding filters $D_\gamma \cap K(P) \cap B(P)$ give a set of filters in the Boolean algebra $K(P) \cap B(P)$ with $D_\gamma \cap K(P) \cap B(P) \neq \{1\}$ and $\cap D_\gamma \cap K(P) \cap B(P) = \{1\}$. So $K(P) \cap B(P)$ is not subdirectly irreducible and then $K(P) \cap B(P)$ is not simple. Consequently P is not simple, a contradiction. This completes the proof of the theorem.

THEOREM 14. Every n-valued monadic Post algebra is a subdirect product of simple algebras.

Proof. This follows from the above Theorem by Birkhoff's Theorem [3].

The following theorem is an important characterization of simple n-valued monadic Post algebras.

THEOREM 15. An n-valued monadic Post algebra P is simple if and only if $K(P) = \{0,e_1,\ldots,e_{n-2},1\}$.

Proof. We always have $\{0,e_1,\ldots,e_{n-2},1\} \subseteq K(P)$.
Suppose $K(P) = \{0,e_1,\ldots,e_{n-2},1\}$. Since none of the elements $e_1,\ldots,e_{n-2}$ is complemented (see [7]) it follows that $K(P) \cap B(P) = \{0,1\}$, hence from Theorem 13, P is simple.
For the converse, suppose P is simple. Then by Theorem 13 K(P) is a simple n-valued Post algebra. But P_n is the only simple n-valued Post algebra (See [4]). Therefore K(P) and P_n are isomorphic algebras, hence $K(P) = \{0,e_1,\ldots,e_{n-2},1\}$.
Some of the above results can be improved in the case P is a finite algebra.
Let us remark that in every n-valued monadic Post algebra P, the principal filter generated by a, F(a), is a monadic deductive system if and only if a belongs to $K(P) \cap B(P)$.
If P is finite, every filter is principal and then it follows that the family of monadic deductive systems of P is the family

As a consequence, if P is a simple n-valued monadic Post algebra, $G \subseteq P$, G finite of cardinal $N[G] = r$ and $S(G) = P$, then P is finite. In fact, $n \leqslant N[P] \leqslant n^{n^r}$. This follows from the fact that P is a homomorphic image of the free n-valued Post algebra on r generators which has n^{n^r} elements (See [4]).

If P is an n-valued monadic Post algebra with a finite set G of r generators and M is a maximal monadic deductive system of P, then $P/_M$ is also r-finitely generated and from above remark, $n \leqslant N[P/_M] \leqslant n^{n^r}$. So $P/_M$ is isomorphic as n-valued Post algebra, to P_n^k, $1 \leqslant k \leqslant n^r$. Besides, $P/_M$ is simple and then $K(P/_M) = \{0, e_1, \ldots, e_{n-1}, 1\}$, therefore $P/_M$ is isomorphic, as n-valued monadic Post algebra, to $P^*_{n,k} = \langle P_n^k, \exists \rangle$, $1 \leqslant k \leqslant n^r$. Then from Theorem 14, P is isomorphic to a subalgebra of $\prod_{k=1}^{n^r} (P^*_{n,k})^{\alpha_k}$, where α_k is the number of times which the axis $P^*_{n,k}$ appears in the decomposition of P as a subdirect product of simple algebras.

Let us see that α_k is finite.

If M_k is the set of maximal monadic deductive systems M of P such that $P/_M$ is isomorphic to $P^*_{n,k}$, it is clear that $\alpha_k = N[M_k]$.

Let $\mathrm{Epi}(P, P^*_{n,k})$ be the set of all epimorphisms from P onto $P^*_{n,k}$, $F(G, P^*_{n,k})$ the set of all functions from G into $P^*_{n,k}$. The mapping $h \longrightarrow \mathrm{Ker}\, h$ carrying each $h \in \mathrm{Epi}(P, P^*_{n,k})$ into its kernel in M_k is clearly surjective, and the mapping $h \longrightarrow h|_G$ carrying each $h \in \mathrm{Epi}(P, P^*_{n,k})$ into its restriction to G is injective, being that if $h|_G = h'|_G$ then $\{x \in P : h(x) = h'(x)\}$ is a subalgebra of P which contains G.

Then $\alpha_k = N[M_k] \leqslant N[\mathrm{Epi}(P, P^*_{n,k})] \leqslant N[F(G, P^*_{n,k})] < \infty$.

Therefore, $P = \prod_{k=1}^{n^r} (P^*_{n,k})^{\alpha_k}$ and we have proved

THEOREM 18. Every finitely generated n-valued monadic Post algethen if we put $M_k = \{M \in M: {}^{F_n(r)}/_M$ is isomorphic to $P^*_{n,k}\}$, we have $M = \bigcup_{k=1}^{n^r} M_k$, $M_i \cap M_j = \emptyset$ if $i \neq j$, and by putting $\alpha_k = N[M_k]$, we can write

$$F_n(r) = \prod_{k=1}^{n^r} (P^*_{n,k})^{\alpha_k}$$

So we must to calculate α_k.

First we have the following result:

LEMMA 20. If G is a free generating set of $F_n(r)$ and if $P_n(r) = SP(G)$ is the Post subalgebra of $F_n(r)$ generated by G, then G is a free generating set of the n-valued Post algebra $P_n(r)$.

Proof. If A is any n-valued Post algebra and f is a mapping from G into A, consider the n-valued monadic Post algebra $A^* = \langle A, \exists \rangle$ where $\exists$ is the discrete quantifier on A. Since $F_n(r)$ is the free n-valued monadic Post algebra there exists a homomorphism h from $F_n(r)$ into A^* extending f. The restriction h' of h to $P_n(r)$ is a Post homomorphism from $P_n(r)$ into A extending f.

Now we are going to determine the numbers α_k. For this purpose we denote, for $1 \leqslant k \leqslant n^r$, $\mathrm{Epi}(F_n(r), P^*_{n,k})$ the set of all epimorphisms from $F_n(r)$ onto $P^*_{n,k}$, $\mathrm{Epi}(P_n(r), P_n^k)$ the set of all (Post) epimorphisms from $P_n(r)$ onto P_n^k, $\mathrm{Aut}(P^*_{n,k})$ the set of all automorphisms of $P^*_{n,k}$. Then we have

LEMMA 21. $$\alpha_k = \frac{N[\mathrm{Epi}(F_n(r), P^*_{n,k})]}{N[\mathrm{Aut}(P^*_{n,k})]}$$

Proof. We know that the mapping $h \longrightarrow s(h) = \mathrm{Ker}\, h$ carrying each $h \in \mathrm{Epi}(F_n(r), P^*_{n,k})$ into its kernel in M_k is surjective.

On the other hand, it is easy to see that if $M = \mathrm{Ker}\, h \in M_k$, then $s^{-1}(M) = \{\alpha \circ h: \alpha \in \mathrm{Aut}(P^*_{n,k})\}$. Consequently we have the dence.

Let us observe that the free n-valued Post algebra on r generators is isomorphic to the direct product of n^r copies of P_n, and then it is clear that $B(P_n(r))$ is isomorphic to the Boolean algebra with n^r atoms B_{n^r}, the elements $a_i = (0,\ldots,\underset{i}{1},\ldots,0)$, $1 \leqslant i \leqslant n^r$, being the atoms of $B(P_n(r))$.

In a similar way $B(P_n^k)$ is isomorphic to the Boolean algebra with k atoms B_k.

Therefore [20]

$$N[Epi(B(P_n(r)),B(P_n^k))] = N[Epi(B_{n^r},B_k)] = V_{n^r,k}.$$

Finally, it is clear that $N[Aut(P^*_{n,k})] = N[Aut(P_n^k)] =$
$= N[Aut(B(P_n^k))] = N[Aut(B_k)] = k!$

By this sequence of Lemmas and remarks it follows that

$$\alpha_k = \frac{V_{n^r,k}}{k!} = \binom{n^r}{k} \qquad 1 \leqslant k \leqslant n^r$$

and consequently

$$F_n(r) = \prod_{k=1}^{n^r} (P^*_{n,k})^{\binom{n^r}{k}}$$

Finally,

$$N[F_n(r)] = \prod_{k=1}^{n^r} (n^k)^{\binom{n^r}{k}} = n^{\sum_{k=1}^{n^r} k\binom{n^r}{k}}$$

But $\displaystyle\sum_{k=1}^{n^r} k\binom{n^r}{k} = \sum_{k=1}^{n^r} \frac{k.n^r!}{k!(n^r-k)!} = n^r + \sum_{k=1}^{n^r-1} \frac{n^r!}{(k-1)!\,(n^r-k)!} =$

$$= n^r + n^r \sum_{k=1}^{n^r-1} \frac{(n^r-1)!}{(k-1)!\,(n^r-k)!} = n^r\left[1 + \sum_{k=1}^{n^r-1} \frac{(n^r-1)!}{(k-1)!\,(n^r-k)!}\right]$$

and making $t = k-1$ we have

REFERENCES

[1] ABAD,M., Cyclic Post algebras of order n, An. Acad. Brasil. Ciênc., (1981) 53(2) 243-246.

[2] BASS,H., Finite monadic algebras, Proc. Amer. Math. Soc., 9 (1958) 258-268.

[3] BIRKHOFF,G., Lattice theory, Amer. Math. Soc. Coll. Publ., Vol.25, 3rd ed. Providence, (1967).

[4] CIGNOLI,R., Moisil algebras, Notas de Lógica Matemática N°27, Universidad Nacional del Sur. Bahía Blanca (1970).

[5] CHANG,C.C. and HORN,A., Prime ideals characterization of generalized Post algebras, lattices theory, Proceedings of Symp. in Pure Mathematics 2(1961) 43-48.

[6] DWINGER,P., Notes on Post algebras I,II, Indagations Math., 28(1966) 462-478.

[7] EPSTEIN,G., The lattice theory of Post algebras, Trans. Amer. Math.Soc., 95(1960) 300-317.

[8] HALMOS,P., Algebraic logic I. Monadic Boolean algebras. Compositio Mathematica, 12(1955) 217-249.

[9] LUKASIEWICZ,J., O logice trójwartościowej, Ruch Filozoficzny, 5(1922) 166-171.

[10] MOISIL,G.C., Le algebre di Lukasiewicz, Acta Logica, Bucharest, 6(1963) 97-135.

[11] MONTEIRO,A., Algebras de Lukasiewicz trivalentes, Lectures given at the Universidad Nacional del Sur, Bahía Blanca (1963).

[12] MONTEIRO,L., Sur les algèbres de Heyting trivalentes, Notas de Lógica Matemática N°19, Universidad Nacional del Sur, Bahía Blanca (1964).

[13] MONTEIRO,L., Algebras de Lukasiewicz trivalentes monádicas, Notas de Lógica Matemática N°32, Universidad Nacional del

NOTAS DE LOGICA MATEMATICA (*)

N° 35

ON FREE L-ALGEBRAS

MANUEL ABAD and LUIZ MONTEIRO

INMABB - CONICET

1987

UNIVERSIDAD NACIONAL DEL SUR

BAHIA BLANCA - ARGENTINA

(*) La publicación de este volumen ha sido subsidiada por el Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.

ON FREE L-ALGEBRAS*

by

Manuel Abad and Luiz Monteiro

ABSTRACT. In [4] A.Horn proved that the free L-algebra FL(n) with a finite generating set of cardinal n, is finite. He determined its number of elements by logical methods.

In this paper we prove these results by an algebraic method, following a path analogous to that of L.Monteiro in [12]. We also give a formula to compute the number of elements of the set $\Pi(n)$ of all prime elements of FL(n), and obtain the ordered structure of $\Pi(n)$.

We prove that FL(n) is a direct product of two distributive lattices with 0 and 1, A_1 and A_2, where $A_2 - \{0\}$ is isomorphic to A_1.

1. DISTRIBUTIVE LATTICES.

We devote this section to give some definitions and known results on ordered sets and distributive lattices.

1.1. DEFINITION. Given an ordered set $(X,\leqslant)$, we say that an element $b \in X$ covers an element $a \in X$ if $a < b$, but $a < x < b$ for no $x \in X$.

If $x \in X$, the set $S(x) = \{y \in X: x \leqslant y\}$ is called the upper section determined by x. Similarly, $I(x) = \{z \in X: z \leqslant x\}$ is the lower section determined by x.

1.2. LEMMA. In any ordered set X, the following conditions are equivalent:

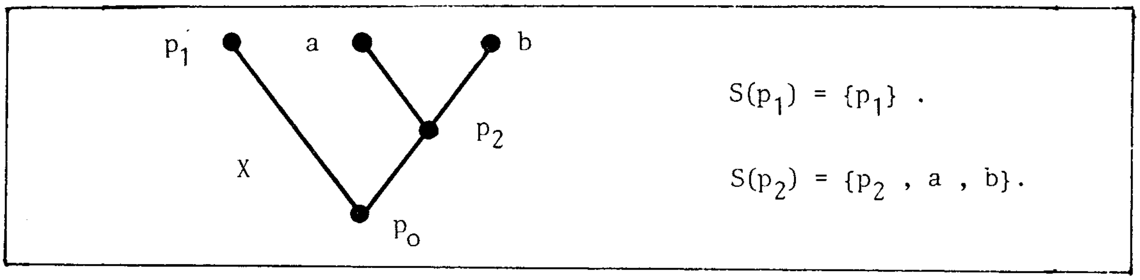
---

* The most essential results of the present paper were submitted to VII Simposio Latino Americano de Lógica Matemática (July 28 to August 3,1985) in a talk given at the University of Campinas, Brazil, by L.Monteiro. An Abstract of this work will be printed in the Journal of Symbolic Logic as part of the summary of that meeting.

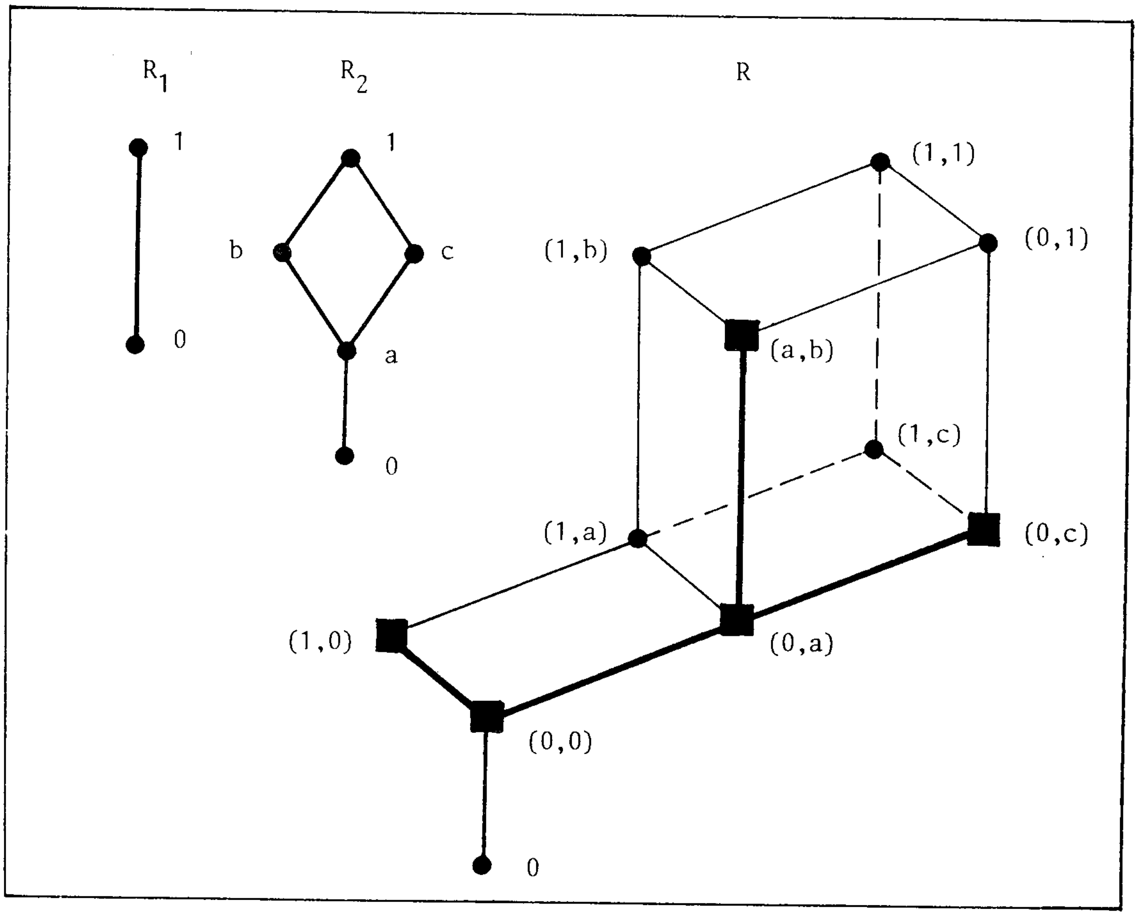
is the distributive lattice such that $\Pi(R_i) \cong S(p_i)$, $1 \leqslant i \leqslant t$, then $R = \{0\} \oplus (\prod_{i=1}^{t} R_i)$ is a distributive lattice such that $\Pi(R) \cong X$. We have $N[R] = \prod_{i=1}^{t} N[R_i] + 1$.

1.5. EXAMPLES.

a) Let X be the ordered set of the figure:



Then R_1, R_2 and R are the following distributive lattices:



Observe that in the example b), the condition T1 fails to hold.

1.6. DEFINITION. Every ordered set with least element which verifies T1 is called a tree. If X is an ordered set which is a cardinal sum of trees, X is called a forest [10, p.87].

As an example, the ordered set X of 1.5 a) is a tree.

2. L-ALGEBRAS.

2.1. DEFINITION. A Heyting algebra [7,11,15] is a system $(A,\wedge,\vee,\Rightarrow,0,1)$ where A is a nonempty set, 0, 1 are two elements of A and $\wedge, \vee, \Rightarrow$ are binary operations defined on A such that the following conditions are verified:

H0) $a \wedge 0 = 0$.

H1) $a \Rightarrow a = 1$.

H2) $(a \Rightarrow b) \wedge b = b$.

H3) $a \wedge (a \Rightarrow b) = a \wedge b$.

H4) $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)$.

H5) $(a \vee b) \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c)$.

For short, we say that A is a Heyting algebra.

It is well known that any Heyting algebra is a distributive lattice with least element 0 and greatest element 1.

If X is a subset of a Heyting algebra A, the (Heyting) subalgebra of A generated by X will be noted SH(X).

2.2. DEFINITION. An L-algebra [4. 11] is a Heyting algebra A such that $(a \Rightarrow b) \vee (b \Rightarrow a) = 1$, for $a,b \in A$.

Any chain A with least element 0 and greatest element 1 is an L-algebra if we define $x \Rightarrow y = 1$ if $x \leqslant y$ and $x \Rightarrow y = y$ if $x > y$.

Let n be a non negative integer. Let C_n be the chain with n+2 elements $a_i = \frac{i}{n+1}$, $i = 0,1,\dots,n+1$.

The least element of C_n is $a_0 = 0$ and the greatest element is $a_{n+1} = 1$. Then C_n is an L-algebra. It is clear that any subset X of C_n such that 0 and 1 belong to X, is a subalgebra of C_n.

If A is a finite L-algebra, since $P \in \mathbf{P}(A)$ if and only if $P = F(p)$, with $p \in \Pi(A)$, from 2.6 we have:

2.10. LEMMA. If $p \in \Pi(A)$, $I(p) = \{q \in \Pi(A): q \leqslant p\}$ is a chain.

2.11. DEFINITION. Let X be a finite ordered set. We say that $p \in X$ is of level i, i positive integer, if the maximun "length" of chains in X having p for greatest element is i.

In the set Y of the example 1.5.b, p_0 is of level 1 and c is of level 4.

2.12. REMARK. If A is a finite L-algebra then $p \in \Pi(A)$ is of level i in $\Pi(A)$, i positive integer, if and only if $N[I(p)] = i$.
This is equivalent to say that $A/F(p) = C_{i-1}(n)$.

3. FREE L-ALGEBRAS.

Let FL(n) be the free L-algebra with a finite set of free generators of cardinal $n > 0$. For the sake of simplicity we will write $\mathbf{P}(n)$ instead of $\mathbf{P}(FL(n))$ and $\Pi(n)$ instead of $\Pi(FL(n))$.

From 2.9, FL(n) is isomorphic to a subalgebra of the direct product $\prod\{FL(n)/P: P \in \mathbf{P}(n)\}$. We want to prove that every quotient algebra $FL(n)/P$, with $P \in \mathbf{P}(n)$, is finite and also that $\mathbf{P}(n)$ is finite. From this we will have that FL(n) is finite. This result was obtained by A.Horn [4] by other method.

3.1. LEMMA. If A is an L-algebra, G a generating set of A of power n, and $P \in \mathbf{P}(A)$, then $N[A/P] \leqslant n+2$.

Proof. If h is the natural homomorphism from A onto A/P, h(G) is a generating set of A/P. Since A/P is a chain, we have $A/P = h(A) = SH(h(G)) = h(G) \cup \{0,1\}$. Then $N[A/P] = N[h(G) \cup \{0,1\}] \leqslant N[h(G)] + 2 \leqslant n+2$.

3.2. COROLLARY. If $P \in \mathbf{P}(n)$, then $FL(n)/P$ is a finite L-algebra.

3.3. REMARK. In the conditions of lemma 3.1, if P is a prime filter of A, the family of filters containing P has at most n+2 elements, and the family of prime filters containing P has at most n+1 elements.

Since $\mathbf{F}_i(n)$ is finite for every i, $0 \leq i \leq n$, then $\mathbf{P}_i(n)$ is finite, and we have

3.5. LEMMA. $\mathbf{P}(n)$ is a finite set.

3.6. THEOREM. FL(n) is finite.

We have in addition that the function ψ_i, $0 \leq i \leq n$, is one-to-one. Endeed, if $f_1, f_2 \in \mathbf{F}_i(n)$ verify $Ker(\overline{f}_1) = Ker(\overline{f}_2)$ then, from results of universal algebra, there is an automorphism α of $C_i(n)$ such that $\alpha \circ \overline{f}_1 = \overline{f}_2$. But the only automorphism of $C_i(n)$ is $\alpha = Id.$, then $\overline{f}_1 = \overline{f}_2$ and then $f_1 = f_2$.

Then we have that $N[\mathbf{P}_i(n)] = N[\mathbf{F}_i(n)]$, $0 \leq i \leq n$, and therefore:

3.7. LEMMA. $N[\mathbf{P}(n)] = \sum_{i=0}^{n} N[\mathbf{P}_i(n)] = \sum_{i=0}^{n} N[\mathbf{F}_i(n)]$.

3.8. REMARK. The functions f from G into $C_i(n)$, $0 \leq i \leq n$, such that $SH(f(G)) = C_i(n)$, that is, such that $f(G) \cup \{0,1\} = C_i(n)$, are those which verify some of the following conditions:

1) $f(G) = \{a_1, a_2, \ldots, a_i\} = X_i$; 2) $f(G) = X_i \cup \{0\}$;

3) $f(G) = X_i \cup \{1\}$; 4) $f(G) = C_i(n)$.

Note that $X_i = \emptyset$ if $i=0$.

If NS(a,b) is the number of functions from a set with a elements onto a set with b elements, then:

$$NS(a,b) = \begin{cases} \sum_{i=0}^{b-1} (-1)^i \binom{b}{i} (b-i)^a & \text{if } a \geq b \\ 0 & \text{if } a < b \end{cases}$$

Then we can state that:

$N[\mathbf{F}_i(n)] = NS(n,i) + 2NS(n,i+1) + NS(n,i+2)$, $0 \leq i \leq n$. In particular, $N[\mathbf{F}_0(n)] = NS(n,0) + 2NS(n,1) + NS(n,2) = 2^n$, and $N[\mathbf{F}_n(n)] = n!$.

It is easy to see that $N[\mathbf{P}(1)] = 3$ and that if $n \geq 2$,

$N[\mathbf{P}(n)] = 3 + 4 \sum_{s=2}^{n} NS(n,s)$.

$$\overline{f}(x) = \begin{cases} 1 & \text{if} \quad x \in \operatorname{Ker}(\overline{f}) \\ a_i = \frac{i}{n+1} & \text{if} \quad x \in P_i - P_{i+1} \ , \quad 0 \leqslant i \leqslant t \end{cases}$$

3.13. LEMMA. If $p,q \in \Pi(n)$, q covers p if and only if the following conditions are verified:

(1) $F(q) \subset F(p)$

(2) $F(p) \in \mathbf{P}_t(n) \quad , \quad 0 \leqslant t \leqslant n-1$

(3) $F(q) \in \mathbf{P}_{t+1}(n) \quad , \quad 0 \leqslant t \leqslant n-1$

Proof. Necessary condition. If q covers p, then $p < q$ and then (1) $F(q) \subset F(p)$, and $p < p' < q$ for no $p' \in \Pi(n)$, that is (i) $F(q) \subset P \subset F(p)$ for no $P \in \mathbf{P}(n)$. Since $F(p) \in \mathbf{P}(n)$ then $F(t) \in \mathbf{P}_t(n)$, with $0 \leqslant t \leqslant n$. If $t=n$ then the family of prime filters containing $F(q)$ would have $n+2$ elements, which is a contradiction (see 3.3). Therefore $F(p) \in \mathbf{P}_t(n)$, $0 \leqslant t \leqslant n-1$, and from (i) it follows (3).

Sufficient condition. Let $p,q \in \Pi(n)$ be such that (1), (2) and (3) are verified. From (1) it follows that $p < q$. If we suppose that there is $p' \in \Pi(n)$ such that $p < p' < q$ then we have $F(q) \subset F(p') \subset F(p)$, and from (2) we have $F(q) \in \mathbf{P}_{t+2}(n)$ which contradicts (3).

3.14. THEOREM. Let $f,h \in \mathbf{F}(n)$ be. Then for $\varphi(h) = p_h = q$ covers $\varphi(f) = = p_f = p$ it is necessary and sufficient that $f \in \mathbf{F}_t(n)$, $h \in \mathbf{F}_{t+1}(n)$, $0 \leqslant t \leqslant n-1$, and the following conditions are verified:

I) $f(g) = a_j$ if and only if $h(g) = a_j$, $0 \leqslant j \leqslant t$.

II) $f(g) = 1$ if and only if $h(g) = 1$ or $h(g) = a_{t+1}$.

iii) There exists $g \in G$ such that $f(g) \neq h(g)$.

Proof. Necessary condition. From 3.13 q covers p if and only if $F(q) \subset \subset F(p)$. and $f \in \mathbf{F}_t(n)$, $h \in \mathbf{F}_{t+1}(n)$, $0 \leqslant t \leqslant n-1$. Since $SH(f(G)) = C_t(n)$, then $f(G) \cup \{0,1\} = C_t(n)$. If $t=0$ then $f(G) \subseteq \{0,1\}$ otherwise $a_1,a_2,\ldots,a_t \in f(G)$. In a similar way $a_1,a_2,\ldots,a_t,a_{t+1} \in h(G)$.

Then $z \in C_{t+2}$ if and only if $\bar{h}(z) = 1$ and $\bar{f}(z) = 1$

$z \in C_{t+1}$ if and only if $\bar{h}(z) = a_{t+1}$ and $\bar{f}(z) = 1$

$z \in C_j$ if and only if $\bar{h}(z) = a_j$ and $\bar{f}(z) = a_j$, $0 \leqslant j \leqslant t$.

Observe that $C_0 = (Q_0 - Q_1) \cap (P_0 - P_1) = (FL(n) - Q_1) \cap (FL(n) - P_1) =$

$= \complement Q_1 \cap \complement P_1 = \complement(Q_1 \cup P_1)$.

We have that C_{t+2} is a filter, C_0 is an ideal and C_j, $0 \leqslant j \leqslant t$, are nonempty sets, being that $a_j \in \bar{h}(FL(n))$, $a_j \in \bar{f}(FL(n))$, $0 \leqslant j \leqslant t$, C_{t+1} is also nonempty. Endeed, from III, there exists $g \in G$ such that $f(g) \neq h(g)$, and then we have, from I and II, that $\bar{h}(g) = a_{t+1}$ and $\bar{f}(g) = 1$, that is, $g \in C_{t+1}$.

It is clear that the sets C_j, $0 \leqslant j \leqslant t+2$, are pairwise disjoint. Observe that $C_{t+2} \cup C_{t+1} = Q_{t+2} \cap P_{t+1}$, and so it is a filter. Using these remarks it is a routine matter to show that the set $S = \bigcup_{i=0}^{t+2} C_i$ is a subalgebra of FL(n). This proof is long but computational, so it will be omitted.

Let us see that $G \subseteq S$. If $g \in G$, $h(g) \in \{0 = a_0, a_1, \ldots, a_t, a_{t+1}, 1\}$.

If $h(g) = 1$, $g \in Q_{t+2}$ and from II, $f(g) = 1$, that is, $g \in P_{t+1}$. Then $g \in Q_{t+2} \cap P_{t+1} = C_{t+2} \subseteq S$.

If $h(g) = a_{t+1}$, $g \in Q_{t+1} - Q_{t+2}$ and from II, $f(g) = 1$, that is $g \in P_{t+1}$. So $g \in (Q_{t+1} - Q_{t+2}) \cap P_{t+1} = C_{t+1} \subseteq S$.

If $h(g) = a_j$, $0 \leqslant j \leqslant t$, then $g \in Q_j - Q_{j+1}$ and from I, $f(g) = a_j$, that is, $g \in P_j - P_{j+1}$. Then $g \in (Q_j - Q_{j+1}) \cap (P_j - P_{j+1}) = C_j \subseteq S$.

Therefore, $G \subseteq S$. We then have $S = FL(n)$.

Then we can write $F(p_h) = Q_{t+2} = Q_{t+2} \cap FL(n) = Q_{t+2} \cap (\bigcup_{i=0}^{t+2} C_i) =$

$= \bigcup_{i=0}^{t+2}(Q_{t+2} \cap C_i) = Q_{t+2} \cap C_{t+2} = Q_{t+2} \cap P_{t+1} = F(q_h) \cap F(p_f)$.

That is, $F(p_h) = F(p_h) \cap F(p_f)$, and therefore $F(p_h) \subseteq F(p_f)$.

If $F(p_h) = F(p_f)$ then $Ker(\bar{h}) = Ker(\bar{f})$ and then $\bar{h} = \bar{f}$ and $h = f$, which contradicts III. Therefore $F(p_h) \subset F(p_f)$.

$$v(g) = \begin{cases} 0 & \text{if } u(g) = 0 \\ 1 & \text{if } u(g) = 1 \\ a_{i-1} = \frac{i-1}{n+1} & \text{if } u(g) = a_i, \quad 1 \leqslant i \leqslant j \end{cases}$$

Clearly α is one-to-one and onto.

If $u, u' \in S(h)$ and u covers u', then $u' \in \mathbf{F}_{i-1}(n)$, $u \in \mathbf{F}_i(n)$, with $0 \leqslant i \leqslant j$, and verify I, II and III. Then it is clear that $\alpha(u) \in \mathbf{F}_{i-1}(n)$, $\alpha(u') \in \mathbf{F}_{i-2}(n)$ and verify I, II and III. Therefore $\alpha(u)$ covers $\alpha(u')$, and α is an isomorphism.

It is clear that $N[K_0(n)] = 1$ and if $K \in K_0(n)$ then $N[K] = 1$. For a given j, $1 \leqslant j \leqslant n$, all the connected components of $K_j(n)$ have the same number of elements. Then if $K \in K_j(n)$ we write $N(n,j) = N[K]$. Then if $K_j(n) = \{K_1, K_2, \ldots, K_{\binom{n}{j}}\}$

$$N[\bigcup_{i=1}^{\binom{n}{j}} K_i] = \sum_{i=1}^{\binom{n}{j}} N[K_i] = \sum_{i=1}^{\binom{n}{j}} N(n,j) = \binom{n}{j} N(n,j).$$

But from (*), $N(n,j) = \sum_{t=1}^{j} \binom{j}{t} N(n,j-t) + 1$.

Therefore $N[\bigcup_{i=1}^{\binom{n}{j}} K_i] = \binom{n}{j}[\sum_{t=1}^{j} \binom{j}{t} N(n,j-t) + 1]$.

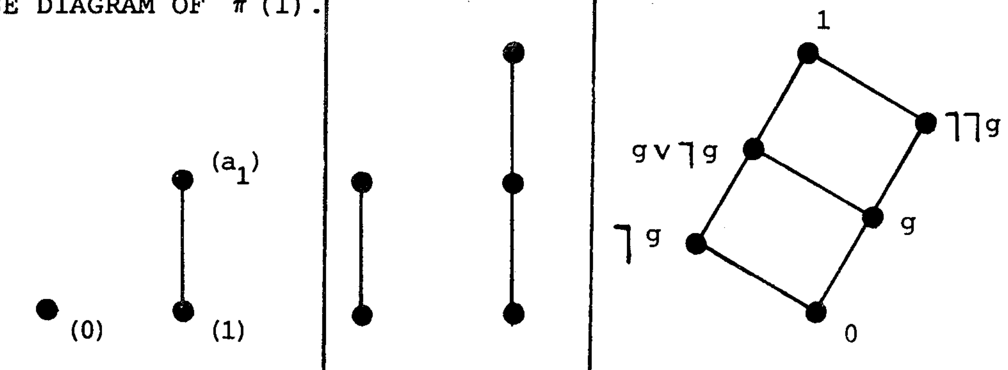
Then $N[\mathbf{F}(n)] = N[\Pi(n)] = 1 + [\sum_{j=1}^{n} \binom{n}{j}[\sum_{t=1}^{j} \binom{j}{t} N(n,j-t) + 1]]$.

Let us denote $R_j(n)$, $0 \leqslant j \leqslant n$, the distributive lattice such that $\Pi(R_j(n)) \cong K$, where $K \in K_j(n)$.

Then $R_0(n)$ is a chain with two elements, and if $K \in K_j(n)$, $1 \leqslant j \leqslant n$, since K verifies the conditions of lemma 1.4, and taking into account (*), we can state that

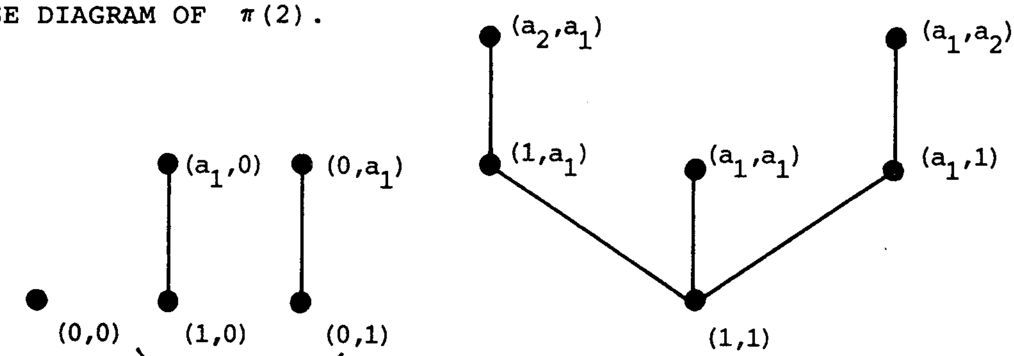
$$R_j(n) = \{0\} \oplus [\prod_{h=0}^{j-1} R_h(n)^{\binom{j}{h}}].$$

HASSE DIAGRAM OF $\pi(1)$.

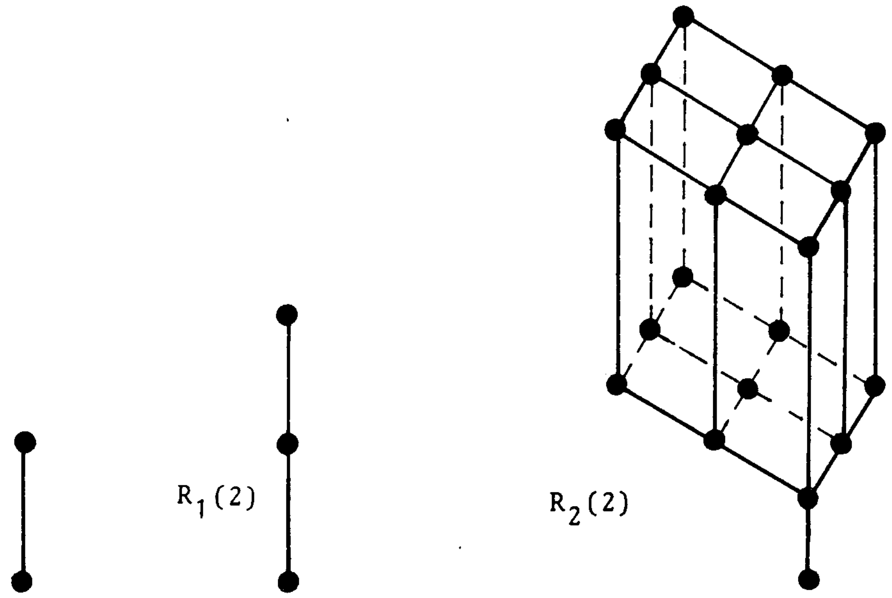


$K_0(1)$ $K_1(1)$ $R_0(1)$ $R_1(1)$ $FL(1) = R_0(1) \times R_1(1)$

HASSE DIAGRAM OF $\pi(2)$.



$K_0(2)$ $\underbrace{\qquad}_{K_1(2)}$ $K_2(2)$



$R_0(2)$ $R_1(2)$ $R_2(2)$

$FL(2) = R_0(2) \times [R_1(2)]^2 \times R_2(2)$

REFERENCES

[1] BIRKHOFF,G. On the combination of subalgebras. Proc.Camb.Phil.Soc.29 (1963), 441-464.

[2] BIRKHOFF,G. Lattice Theory, Amer.Math.Soc.Colloq.Pub.,25, 3rd.ed., Providence (1967).

[3] HORN,A. Logic with truth values in a linearly ordered Heyting algebra. Journal of Symbolic Logic 34 (1969), 395-408.

[4] HORN,A. Free L-Algebras, Journal of Symbolic Logic 34 (1969), 475-480.

[5] MONTEIRO,A. L'arithmétique des filtres et les espaces topologiques. Memoir presented in 1950 at the Soc.Math. de France in occasion of jubilee of M.Frechet. Published in 1974.

[6] MONTEIRO,A. L'arithmétique des filtres et les espaces topologiques. Segundo Symposium Latino Americano de Matemática. Centro de Cooperación Científica de la UNESCO para América Latina. Montevideo (1954), 129-162.

[7] MONTEIRO,A. Axiomes indépendants pour les algèbres de Brouwer. Revista de la U.M.A., 17 (1955), 149-160.

[8] MONTEIRO,A. Linearisation de la logique positive de Hilbert-Bernays. Revista de la U.M.A. 20(1962), 308-309.

[9] MONTEIRO,A. Lectures given at Universidad Nacional del Sur (1964).

[10] MONTEIRO,A. L'aritmétique des filtres et les Espaces Topologiques I-II. Notas de Lógica Matemática Nro.29-30. Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca, Argentina (1974).

[11] MONTEIRO,A. Sur les Algèbres de Heyting Symétriques. Portugaliae Mathematica, 39 (1980), 1-237.

[12] MONTEIRO,L. Algèbre du calcul propositionnel trivalent de Heyting. Fund. Math., 74 (1972), 99-109.

[13] RASIOWA,H. and SIKORSKI,R. The Mathematics of Metamathematics. Mo-