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38

PROCEEDINGS OF THE IX LATIN AMERICAN SYMPOSIUM ON MATHEMATICAL LOGIC

(Part 1)

1993

INMABB - CONICET

UNIVERSIDAD NACIONAL DEL SUR

BAHIA BLANCA - ARGENTINA

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The Symposium was preceded by two short courses: Non-Standard Analysis by Sergio Fajardo, and Logic and Computer Science by James Lipton.

The activities of the Symposium consisted of: i) Invited lectures delivered by C. Alchorrón (Argentina), J. Baumgartner (USA), N. Bertoglio (Chile), R. Cignoli (Argentina), M. Corrada (Chile), L. J. Corredor (Colombia), R. Chuaqui (Chile), M. Dezzani (Italy), A. Di Nola (Italy), C. Di Prisco (Venezuela), S. Fajardo (Colombia), A. Figallo (Argentina), H. Gaitan (Venezuela), J. Keisler (USA), R. Laver (USA), R. Lewin (Chile), J. Lipton (USA), J. Llopis (Venezuela), V. Marshall (Chile), D. Miller (UK), I. Mickemberg (Chile), E. Mizraji (Uruguay), L. Monteiro (Argentina), D. Mundici (Italy), J. Oikkonen (Finland), D. Pigozzi (USA), C. Pizzi (Italy), J. Roetti (Argentina), M. Sagastume (Argentina), G. Schwarze (Chile), J. Stern (France), R. Shore (USA), S. Surma (N. Zealand), C. Uzcategui (Venezuela), B. Venneri (Italy). ii) Contributed papers.

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Manuel Abad

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PATRICK SUPPES AND ROLANDO CHUAQUI

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Introduction

If someone examines, even cursorily, a treatise or advanced textbook on theoretical physics, it is apparent that the way mathematics is used is very different from what is to be found even in the most constructive formulations of mathematics. There is for example, no close connection between two such well-known books on constructive mathematics as that of Troelstra and Van Dalen, [19], and Bishop and Bridges, [3], on the one hand, and any classical textbook in quantum mechanics, or quite recent books, for example Ryder, [18], on quantum field theory.

The differences run a good deal deeper than the fact that the books on theoretical physics are not written in the definition-theorem-proof style characteristic of pure mathematics and exemplified in the books just cited. There are at least four differentiating characteristics that can immediately be discerned on closer examination. Although a good many propositions are proved in the books on physics, there are almost without exception no existential proofs, and consequently there is no really serious systematic use of quantifiers. Secondly, there are almost no systematic negative statements. Almost everything is positive and computational in character. In fact, the third characteristic is the extreme constructive character, mainly computational in terms of argument, that characterizes the theoretical development. Finally, as a fourth important characteristic there is a free use of infinitesimals without even a nod toward justification in terms of foundational work on nonstandard analysis. These characteristics of books written in the last forty some years would also be characteristic of treatises on theoretical physics written in the last hundred years. What we have said about these books is not true on the other hand of treatises of a modern sort that specifically are labeled mathematical physics. This is another story and not our focus here.

Even when theorems are mentioned, as for example in Ryder's excellent qualitative discussion of Noether's fundamental theorem on the relation between space and time symmetries and laws of conservation, the discussion is wholly informal and constructive, although the important and central features of Noether's theorem are preserved. The kind of mathematical approach exemplified by Ryder is primary support for the argument that the classical discussions of the foundations of mathematics, including the constructive ones, are very far removed from making contact in any detail with the mathematical practices of physicists. What we have to say about physicists applies just as well to theoretical chemists, engineers, and many social scientists. As a specific frame of reference, however, we shall stick with theoretical physics.

The natural foundational question that arises about the discrepancy between the way mathematics is ordinarily done in theoretical physics and the way it is built up from a foundational standpoint in any of the standard modern views, raises the question of whether it might be possible to construct quite directly a rigorous foundation that reflects very closely a large part of this standard practice in theoretical physics. As in other cases of foundational work, we do not want to suggest that what we do here is satisfactory for all that theoretical physicists do; our foundation will not hold for everything, and in other respects it is probably too inclusive. On the other hand, we think it is possible to give a foundational formulation of the differential and integral calculus and differential equations that corresponds to much of the mathematical practice in theoretical physics. But this correspondence between our foundation and mathematical practice in physics does not extend to the style of this paper, which is concerned with providing the mathematical details that justify the intuitive practice of physicists. The style of such justification is in the general mathematical spirit of foundational papers, necessarily concerned with fine points of little concern to physicists.

To reflect the features mentioned above characteristic of works in theoretical physics, the foundation approach we develop here has the following features: (i) the underlying logic is positive: there is no use of negation; we adopt Hilbert's positive propositional calculus; (ii) the formulation is a free-variable one with no use of quantifiers; (iii) we use infinitesimals in an elementary way drawn from nonstandard analysis, but the account here is axiomatically selfcontained and deliberately elementary in spirit.

We believe that the system formulated here comprises a new constructive approach to the foundations of mathematics. As Wattenberg, [20], points out there is a natural affinity between nonstandard analysis and constructivism, particularly in the extensive use of computational arguments and in the handling of real numbers that appear equal to zero by use of infinitesimals. Our system is a particularly restricted version of nonstandard analysis, however restricted in ways which are wholly in the direction of constructivism — positive logic and only free variables. It is important to note that our free-variable constructive methods yield proofs of approximate equality, rather than exact equality, but an infinitesimal difference is as good as equality for physical purposes. Indeed, finite numerical approximations are necessarily characteristic of the solutions of most complex problems in contemporary physics.

The philosophical spirit of the enterprise is actually closest to the constructive approach characteristic of geometric constructions, both ancient and modern, which are naturally free-variable in formulation and which in elementary formulation have representations over nonstandard fields, because there is no Archimedean condition that must be satisfied (for an example, see Moler and Suppes, [12]). The Greek theory of constructions arose from concrete geometric problems and was probably the part of Greek geometry closest to applications. Certainly geometric constructions were central to astronomical and architectural computations for over 2,000 years. We are motivated in the same general way by the use of mathematics in science, not a theme of any major importance in traditional philosophies of mathematics, in spite of some lip service to the contrary.

Because of the constructive restrictions we have imposed, we have had to state a very large number of axioms about the elementary properties of the real numbers, which are ordinarily proved by *reductio ad absurdum*. All of these elementary axioms however, have an immediate intuitive content. This constitutes the first group of axioms which we call the restricted elementary field axioms. The second group of axioms concern the natural numbers and open induction; again only elementary results are needed. The third group of axioms are those about infinitesimals. In addition, we have at the beginning logical rules of inference and, later as we proceed, various constructive rules of inference within the framework of the infinitesimal calculus.

As should be clear from what we have said, in principle we assume no previous theories of any kind, for we start with rules of sentential inference of a restricted kind. However, the developments move ahead at a fairly rapid pace and we intuitively assume familiarity of the reader with the topics we develop. On the other hand not only is classical logic not used, but the logic used is much weaker than intuitionistic logic, a point we discuss later in more detail.

In [6], we introduce a system for which we give a finitary proof of consistency using Herbrand's Theorem. The present system is a fragment of the system in [6], where we restrict the logic to be free-variable and with no negation. The system of [6] is finitistic in character, but not as constructive as the present system. The main difference is that system [6] uses full first-order logic. On the other hand, the present system uses only free-variable positive logic, for axioms, rules and proofs. Thus, we increase the constructivity by increasing considerably the number of axioms in order not to use negation. The systems are similar, however, and many of the proofs in one system can be obtained by obvious modifications of the corresponding proofs in the other system. In order to make this paper as self-contained as possible and assure that they can be given in a free-variable negationless form, we shall repeat some of the proofs. Also, many of the proofs in [6] were omitted or only sketched, while here we tried to give the proofs as complete as possible, without being unduly pedantic.

The organization of the paper is shown in the Table of Contents at the beginning. We note that there are two separate developments of the theory of integration, one in Section 8 and the other in Section 11. Section 14 is concerned with the finitary consistency of our system.

1. Logical Axioms and Rules

We use Church's [7, p. 141] version of Hilbert's positive propositional calculus, but with axiom schemas replacing sentential axioms. The sentential priorities are $\rightarrow$ (if... then), $\leftrightarrow$ (iff), $\wedge$ (and), and $\vee$ (or). (To make the theorems easier to read, we use in the theorems, but not in the axioms, English words rather than standard logical symbols for the connectives.) We use the Greek letters φ, ψ, θ and so forth for formulas, in particular, for sentential variables. We omit a formal definition of an inference, which is simple for formulas without quantifiers. The axiom schemas and the rule are these: For any formulas φ, ψ and θ, we assume

L 1. $\varphi \rightarrow (\psi \rightarrow \varphi)$.

L 2. $(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \theta))$.

L 3. $(\varphi \wedge \psi) \rightarrow \varphi$.

L 4. $(\varphi \wedge \psi) \rightarrow \psi$.

L 5. $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$.

L 6. $\varphi \rightarrow (\varphi \vee \psi)$.

L 7. $\psi \rightarrow (\varphi \vee \psi)$.

L 8. $(\varphi \rightarrow \theta) \rightarrow ((\psi \rightarrow \theta) \rightarrow ((\varphi \vee \psi) \rightarrow \theta))$.

L 9. $(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$.

L 10. $(\varphi \leftrightarrow \psi) \rightarrow (\psi \rightarrow \varphi)$.

L 11. $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow (\varphi \leftrightarrow \psi))$.

We end with the rule of modus ponens:

L 12. *From φ and $\varphi \rightarrow \psi$ infer ψ.*

Formulas are built up from the notation for primitive concepts, numerical variables, function variables and, as the theory develops, notation for defined concepts in familiar recursive fashion, which we shall not spell out in detail. The important restrictions are that there is no primitive notation for negation and there is no use of existential or universal quantifiers.

We assume the usual logic of *identity*:

LI 1. *$v = v$, where v is any variable.*

LI 2. *If ψ results from φ by replacing one or more occurrences of τ_1 in φ by τ_2, or by replacing τ_2 in φ by τ_1, then the following is an axiom*

$$(\varphi \wedge \tau_1 = \tau_2) \rightarrow \psi.$$

Notice that the usual restriction that φ be an open formula is not required, for all formulas are open.

Like the rules for identity, the *substitution* rule (**LS**) of terms for variables need not have restriction on quantifier capture:

LS. *From φ we may derive ψ if ψ results from φ by substituting a fixed term τ for a variable v in every occurrence in φ.*

Because of the restriction to free variables, many of the axioms we shall need in our system are in the form of rules of inference. For instance, when we would need as an axiom a statement of the form $\forall x\varphi \rightarrow \psi$, instead we shall introduce the rule: From φ infer ψ. Thus, the universal quantifier does not occur.

The deduction theorem is valid in our system, provided no rule, except for modus ponendo ponens, L 12, is used. That is, we have

Theorem 1.1 (Deduction Theorem). *If ψ can be inferred from φ and the set of formulas Σ, and no rules, except for* L 12, *are used, then $\varphi \rightarrow \psi$ can be derived from Σ.*

The usual proof works here.

We conjecture, but have not checked all details, that Axiom Schemas L 6 and L 7, which formulate the law of addition and the commutative character of inclusive *or*, could be replaced by a single axiom schema of commutativity. We believe that the classical law of addition is not needed. Moreover, we would like to eliminate if possible because it violates the intuitively desirable Aristotelian canon that no symbol should occur in the conclusion of an argument that does not appear in one of the premises.

2. Restricted axioms for a field

We state thirty open formulas as axioms, which with exceptions to be noted are standard. The nonlogical primitive concepts are addition $+$, negative operation $-$, multiplication $\cdot$, division $/$, ordering $<$ and the two constants 0 and 1. Several axioms would ordinarily have as a hypothesis $x \neq 0$. The changes made here are to avoid negation. Needless to say, the absence of negation increases the number of axioms considerably, as it is evident by comparing especially the axioms on order with standard field axioms. Axiom F 11 suggests a way of defining negation: $\neg\varphi \leftrightarrow (\varphi \to 0 = 1)$. We comment more on this definition later.

Although we use as an axiom the obvious replacement for $0 \neq 1$, i.e., the statement $0 < 1$ (Axiom F 30), in order to prove that the system is valid in a one-element model, we would have to change it to the weaker disjunction $0 < 1 \vee 0 = 1$. This one-element model is discussed in Section 14 on consistency.

F 1. $x + y = y + x$.

F 2. $(x + y) + z = x + (y + z)$.

F 3. $x + 0 = x$.

F 4. $x + (-x) = 0$.

F 5. $x \cdot y = y \cdot x$.

F 6. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

F 7. $x \cdot 1 = x$.

F 8. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

F 9. $(x < y \wedge y < z) \to x < z$.

F 10. $x = y \vee x < y \vee y < x$.

F 11. $(x < y \vee x = y) \to (y < x \to 0 = 1)$.

F 12. $(x < y \vee y < x) \to (x = y \to 0 = 1)$.

F 13. $((x < y \vee x = y) \to 0 = 1) \to y < x$.

F 14. $x < y \to x + z < y + z$.

F 15. $(x < y \wedge 0 < z) \to x \cdot z < y \cdot z$.

F 16. $(x < y \wedge z < 0) \to y \cdot z < x \cdot z$.

F 17. $(0 < x \wedge 0 < y) \to 0 < x \cdot y$.

F 18. $(0 < x \wedge y < 0) \to x \cdot y < 0$.

F 19. $(x < 0 \wedge y < 0) \to 0 < x \cdot y$.

F 20. $(0 < x \cdot y \wedge 0 < x) \to 0 < y$.

F 21. $(0 < x \cdot y \wedge x < 0) \to y < 0$.

F 22. $(x \cdot y < 0 \wedge 0 < x) \to y < 0$.

F 23. $(x \cdot y < 0 \wedge x < 0) \to 0 < y$.

F 24. $0 < x \cdot y \to ((0 < x \wedge 0 < y) \vee (x < 0 \wedge y < 0))$.

F 25. $x \cdot y < 0 \to ((0 < x \wedge y < 0) \vee (x < 0 \wedge 0 < y))$.

F 26. $(x < 0 \vee 0 < x) \to x \cdot (1/x) = 1$.

F 27. $(y < 0 \vee 0 < y) \to x/y = x \cdot (1/y)$.

F 28. $(x \cdot y < 0 \vee x \cdot y < 0) \to 1/(x \cdot y) = (1/x) \cdot (1/y)$.

F 29. $x/1 = x$.

F 30. $0 < 1$.

We also need to introduce the following function δ:

F 31.

(1) $x \geq 0 \to \delta(x) = 1$.
(2) $x < 0 \to \delta(x) = -1$.

We define

Definition 2.1.

(1) $\delta_1(x) = \dfrac{\delta(x) + 1}{2}$.
(2) $\delta_2(x) = \delta_1(-\delta(-x))$.

Then

$$\delta_1(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

and

$$\delta_2(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

We now list elementary theorems that would be expected to follow from our axioms. The first group of theorems requires one definition, that for the binary operation of subtraction.

Definition 2.2. $x - y = x + (-y)$.

Theorem 2.1. $0 + x = x$.

Theorem 2.2. $(-x) + x = 0$.

Theorem 2.3. $x - x = 0$.

Theorem 2.4. $0 - x = -x$.

Theorem 2.5. $0 = -0$.

Theorem 2.6. $x - 0 = x$.

Theorem 2.7. *If* $x + y = x + z$ *then* $y = z$.

Theorem 2.8. *If* $x + y = z$ *then* $x = z - y$.

Theorem 2.9. *If* $x = z - y$ *then* $x + y = z$.

Theorem 2.10. *If* $x + y = 0$ *then* $x = -y$.

Theorem 2.11. *If* $x = -y$ *then* $x + y = 0$.

Theorem 2.12. *If* $x + y = x$ *then* $y = 0$.

Theorem 2.13. $-(-x) = x$.

Theorem 2.14. *If* $0 < x$ *then* $-x < 0$.

Theorem 2.15. $1 \cdot x = x$.

Theorem 2.16. *If* $x < 0$ *or* $0 < x$ *then* $(1/x) \cdot x = 1$.

Theorem 2.17. *If* $x < 0$ *or* $0 < x$ *then* $x/x = 1$.

Theorem 2.18. *If* $y < 0$ *or* $0 < y$ *then* (*if* $x/y = z$ *then* $x = z \cdot y$).

Theorem 2.19. $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$.

Theorem 2.20. $x \cdot 0 = 0$.

Theorem 2.21. *If* $x < 0$ *or* $0 < x$ *then* $0/x = 0$.

Theorem 2.22. *If* $x < 0$ *or* $0 < x$, *and* $x \cdot y = x \cdot z$ *then* $y = z$.

Theorem 2.23. *If* $x < 0$ *or* $0 < x$, *and* $x \cdot y = 1$ *then* $y = 1/x$.

Theorem 2.24. *If* $x < 0$ *or* $0 < x$, *and* $x \cdot y = x$ *then* $y = 1$.

Theorem 2.25. *If* $y < 0$ *or* $0 < y$ *then* $(x/y) \cdot z = (x \cdot z)/y$.

Theorem 2.26. *If* $y < 0$ *or* $0 < y$ *then* $(x/y) \cdot z = (z/y) \cdot x$.

Theorem 2.27. *If* $y < 0$ *or* $0 < y$, *and* $u < 0$ *or* $0 < u$, *then* $(x/y) \cdot (z/u) = (z/y) \cdot (x/u)$.

Theorem 2.28. *If* $y < 0$ *or* $0 < y$, *and* $x < 0$ *or* $0 < x$, *then* $(x/y) \cdot (y/x) = 1$.

Theorem 2.29. *If* $x < 0$ *or* $0 < x$, *and* $x \cdot y = 0$ *then* $y = 0$.

Theorem 2.30. *If* $x = 0$ *or* $y = 0$ *then* $x \cdot y = 0$.

Theorem 2.31. *If* $x \cdot y = 0$ *then* $x = 0$ *or* $y = 0$.

Theorem 2.32. *If* $x \cdot y < 0$ *or* $0 < x \cdot y$ *then* $y/(x \cdot y) = 1/x$.

Theorem 2.33. *If* $x < 0$ *or* $0 < x$, *and* $y < 0$ *or* $0 < y$, *then* $(z \cdot y)/(x \cdot y) = z/x$.

Theorem 2.34. *If* $y < 0$ *or* $0 < y$, $u < 0$ *or* $0 < u$, *and* $x/y = z/u$, *then* $x \cdot u = z \cdot y$.

Theorem 2.35. *If* $y < 0$ *or* $0 < y$, *and* $x = y \cdot z$ *then* $x/y = z$.

Theorem 2.36. *If* $x < x$ *then* $0 = 1$.

Theorem 2.37. $0 < y$ *iff* $0 < 1/y$.

Theorem 2.38. $y < 0$ *iff* $1/y < 0$.

Theorem 2.39. *If* $y < 0$ *or* $0 < y$ *then* (*if* $0 < x/y$ *then* $0 < x \cdot y$).

Theorem 2.40. *If* $y < 0$ *or* $0 < y$ *then* (*if* $0 < x \cdot y$ *then* $0 < x/y$).

We now introduce three standard elementary definitions.

Definition 2.3. $x \leq y \leftrightarrow x < y \vee x = y$.

We next define the absolute value of a number and prove the standard theorems that hold in our positive logic.

Definition 2.4. $|x| = y \leftrightarrow (x \geq 0 \rightarrow x = y) \wedge (x \leq 0 \rightarrow -x = y)$.

Theorem 2.41. *If* $0 \leq x$ *then* $|x| = x$.

Theorem 2.42. *If* $x \leq 0$ *then* $|x| = -x$.

Theorem 2.43. $|x| = |-x|$.

Theorem 2.44. $|x^2| = x^2$.

Theorem 2.45. $x \leq |x|$.

Theorem 2.46. $-|x| \leq x$.

Theorem 2.47. $|x + y| = |y + x|$.

Theorem 2.48. $|x \cdot y| = |x| \cdot |y|$.

Theorem 2.49. *If* $y < 0$ *or* $0 < y$ *then* $|1/y| = 1/|y|$.

Theorem 2.50. *If* $y < 0$ *or* $0 < y$ *then* $|x/y| = |x|/|y|$.

Theorem 2.51. $|x + y| \leq |x| + |y|$.

Theorem 2.52. $|x| - |y| \leq |x - y|$.

Theorem 2.53. *If* $|x - y| < c$ *then* $|x| < c + |y|$.

Theorem 2.54. *If* $|y - x| < x$ *then* $0 < y$.

Theorem 2.55. $|x - y| \leq |x| + |y|$.

Theorem 2.56. $||x| - |y|| \leq |x - y|$.

Theorem 2.57. $|y - z| \leq |x - y| + |x - z|$.

Theorem 2.58. $-y \leq x$ *and* $x \leq y$ *iff* $|x| \leq y$.

Theorem 2.59. *If* $|x - a| < c/2$ *and* $|y - b| < c/2$ *then* $|(x + y) - (a + b)| < c$.

Theorem 2.60. *If* $|x - a| < c/2$ *and* $|y - b| < c/2$ *then* $|(x - y) - (a - b)| < c$.

3. Axioms for Natural Numbers and Open Induction

We need elementary number theory restricted as in the case of the field of real numbers to open formulas and the positive sentential rules of derivation. The axioms require the new primitive predicate $\mathcal{N}$, with $\mathcal{N}(x)$ meaning that x is a natural number.

N 1. $\mathcal{N}(1)$.

N 2. $\mathcal{N}(x) \rightarrow \mathcal{N}(x+1)$.

N 3. $\mathcal{N}(x) \rightarrow 1 \leq x$.

N 4. $(\mathcal{N}(x) \wedge \mathcal{N}(y) \wedge x \leq y \wedge \; y \leq x+1) \rightarrow (x = y \vee x+1 = y)$.

N 5. $\mathcal{N}(x) \wedge \mathcal{N}(y) \wedge y < x+1 \rightarrow y \leq x$.

The schema for open induction (N 6) is stated as a rule of derivation, where we use now the variable n for x and omit $\mathcal{N}(x)$ in standard fashion. We also use the variables i, j, k, n, m, ν, μ and η restricted to natural numbers.

We must distinguish between internal and external formulas. All the formulas introduced up to now are *internal*, but when we introduce the predicate symbol Inf, we must exclude it from internal formulas. Formulas with Inf or any predicate defined from it are *external*

N 6. *Let φ be an internal formula, where neither $\mathcal{N}$ nor* min *(to be introduced in Axiom Schema N 7) occur. Then, from*

(1) $\varphi(1)$,
(2) $\varphi(n) \rightarrow \varphi(n+1)$,

infer $\varphi(n)$.

Since we do not have negation, we need to introduce the minimum operator by an axiom. We suppose that the variables are ordered and always write k for the first variable. The minimum operator for internal formulas is introduced by the axiom schema:

N 7. *Let φ be an internal open formula, where neither $\mathcal{N}$ nor* min *occur and x_1, $\ldots$, x_n are the distinct free variables in φ, except for the first variable. We introduce an n-ary function symbol* $\min_\varphi$, *with the following axiom:*

$$\mathcal{N}(x) \wedge \varphi(x) \rightarrow \mathcal{N}(\min_\varphi(x_1, \ldots, x_n)) \wedge \min_\varphi(x_1, \ldots, x_n) \leq x \wedge \varphi(\min_\varphi(x_1, \ldots, x_n)).$$

We may omit the variables x_1, $\ldots$, x_n, when they are clear from the context.

Theorem 3.1. *If $\mathcal{N}(x)$ and $\mathcal{N}(y)$ then $\mathcal{N}(x+y)$.*

Theorem 3.2. *If $\mathcal{N}(x)$ and $\mathcal{N}(y)$ then $\mathcal{N}(x \cdot y)$.*

We also need a sort of Archimedian axiom, that is, for any x, we need to find a natural number larger than x. We introduce a new unary function symbol li (least integer), such that $\mathrm{li}(x)$ is the least natural number greater or equal to x. The symbol li may occur in internal formulas.

N 8. $\mathcal{N}(\mathrm{li}(x)) \wedge \mathrm{li}(x) \geq x$.

We also need a maximum operator, introduced by recursion:

Recursive Definition 1. Let τ be a term where min does not occur and $x_1, \ldots, x_m$ are its distinct free variables, except for the first one. We introduce an $n+1$-ary function symbol, $\max_\tau(n, x_1, \ldots, x_m)$ with the axioms

(1) $\max_\tau(1, x_1, \ldots, x_m) = 1$,
(2) $\mathcal{N}(n) \wedge \tau(n+1) \leq \tau(\max_\tau(n, x_1, \ldots, x_m)) \rightarrow \max_\tau(n+1, x_1, \ldots, x_m) = \max_\tau(n, x_1, \ldots, x_m)$,
(3) $\mathcal{N}(n) \wedge \tau(n+1) > \tau(\max_\tau(n, x_1, \ldots, x_m)) \rightarrow \max_\tau(n+1, x_1, \ldots, x_m) = n+1$.
(4) $\mathcal{N}(n) \rightarrow \mathcal{N}(\max_\tau(n, x_1, \ldots, x_m)$.

As an example, let $\tau(i)$ be $i + x$. Then, by (1), $\max_\tau(1, x) = 1 + x$. Since $\tau(n) = n + x < n + 1 + x = \tau(n+1)$, we can prove by internal induction, using (3), that $\max_\tau(n, x) = n + x$. On the other hand, if $\tau(i) = x/i$, then, by (1), $\max_\tau(1, x) = 1$, and, since $x/(n+1) < x/n$, by (2), $\max_\tau(n, x) = 1$, for every n.

We need to prove by induction a lemma about the maximum operator.

Lemma 3.3. *Let τ be a term where* min *does not occur. Then, from $\mathcal{N}(i)$ infer*

$$\max_\tau(\nu) \leq \nu \wedge (i \leq \nu \rightarrow \tau(\max_\tau(\nu)) \geq \tau(i)).$$

Proof. Let $\mathcal{N}(i)$ and let $\varphi(i, \nu)$ be the formula

$$\max_\tau(\nu) \leq \nu \wedge (i \leq \nu \rightarrow \tau(\max_\tau(\nu)) \geq \tau(i)),$$

which we shall prove by open internal induction. It is clear that it is an open internal formula, where neither $\mathcal{N}$ nor min occur. We have that $\max_\tau(1) = 1$. Hence, $i \leq 1$ implies $i = 1$, since we have $\mathcal{N}(i)$. Thus, we obtain $\varphi(i, 1)$.

Suppose $\varphi(i, n)$. Notice that if $i \leq n+1$, then, since we have $\mathcal{N}(i)$, $i \leq n$ or $i = n+1$. We have two cases:

Case 1: $\tau(n+1) \leq \max_\tau(n)$. Then $\max_\tau(n+1) = \max_\tau(n)$. Let $i \leq n+1$. If $i = n+1$, then $\tau(i) \leq \tau(\max_\tau(n)) = \tau(\max_\tau(n+1))$. On the other hand, if $i \leq n$, by the inductive hypothesis, $\tau(i) \leq \tau(\max_\tau(n)) = \tau(\max_\tau(n+1))$.

Case 2: $\tau(n+1) > \max_\tau(n)$. Then $\max_\tau(n+1) = n+1$. Thus, $\mathcal{N}(\max_\tau(n+1)$. Let $i \leq n+1$. If $i = n+1$, then $\tau(i) \leq \tau(\max_\tau(n+1))$. On the other hand, if $i \leq n$, then $\tau(i) \leq \tau(\max_\tau(n)) < \tau(n+1) = \tau(\max_\tau(n+1))$.

□

Notice that in the last lemma we need the premise $\mathcal{N}(i)$ because we do not have the Deduction Theorem, since we are using the rule of Open Internal Induction. In the system [6], we can omit this premise, since we have an open internal induction axiom instead of a rule. We now introduce the recursive definition of sum.

Recursive Definition 2. For any term τ we define

(1) $\sum_{i=1}^{1} \tau(i) = \tau(1)$,
(2) $\mathcal{N}(n) \rightarrow \sum_{i=1}^{n+1} \tau(i) = \sum_{i=1}^{n} \tau(i) + \tau(n+1)$.

We write $\sum_{i=m}^{\nu} \tau(i)$ for $\sum_{j=0}^{\nu-m} \tau(j+m)$. With the δ or δ_i functions, $i = 1, 2$, we can express different sums. For instance

Definition 3.1. Let σ and τ be terms. Then

(1) $\sum_{\substack{i=1 \\ \sigma(i)\leq x}}^{n} \tau(i) = \sum_{i=1}^{n} \delta_1(x-\sigma(i))\tau(i),$

(2) $\sum_{\substack{i=1 \\ \sigma(i)< x}}^{n} \tau(i) = \sum_{i=1}^{n} \delta_2(x-\sigma(i))\tau(i).$

We assume we have similar definitions for $\geq$ or $>$ instead of $\leq$ or combinations of these relations.

The usual properties of the sum can be proved by internal induction. In particular, we have the following propositions:

Proposition 3.4. *We have*

$$\sum_{\substack{i=1 \\ \sigma(i)\leq x}}^{1} \tau(i) = \begin{cases} \tau(1) & \text{if } \sigma(1) \leq x, \\ 0 & \text{if } \sigma(1) > x, \end{cases}$$

$$\sum_{\substack{i=1 \\ \sigma(i)\leq x}}^{n+1} \tau(i) = \begin{cases} \sum\limits_{\substack{i=1 \\ \sigma(i)\leq x}}^{n} \tau(i) + \tau(n+1) & \text{if } \sigma(n+1) \leq x, \\ \sum\limits_{\substack{i=1 \\ \sigma(i)\leq x}}^{n} \tau(i) & \text{if } \sigma(n+1) > x. \end{cases}$$

Similar statements are true for the other inequalities.

The proof is by an easy induction.

Proposition 3.5.

(1) $\sum_{i=1}^{\nu} \tau(i) = \sum_{\sigma(i)<x} \tau(i) + \sum_{\sigma(i)\geq x} \tau(i).$

(2) *Let* $x < y$. *Then*

$$\sum_{\sigma(i)\geq x} \tau(i) = \sum_{x\leq\sigma(i)<y} \tau(i) + \sum_{\sigma(i)\geq y} \tau(i).$$

Other similar equalities are also valid.

Proof. We shall prove (1) by internal induction. Let $n = 1$. We have

$$\sum_{i=1}^{1} \tau(i) = \tau(1).$$

We have, $\sigma(1) < x \vee \sigma(1) \geq x$. If $\sigma(1) < x$, then

$$\sum_{\substack{i=1 \\ \sigma(i)<x}}^{1} \tau(i) = \tau(1) \quad \text{and} \quad \sum_{\substack{i=1 \\ \sigma(i)\geq x}}^{1} \tau(i) = 0,$$

and if $\sigma(1) \geq x$, the equations are reversed, so that the proposition is proved. The proof of the inductive case is analogous.

The other cases are proved similarly. □

Proposition 3.6. (1) *From*

$$j \leq n \rightarrow \tau(j) = 0,$$

infer

$$\sum_{i=1}^{n} \tau(j) = 0.$$

(2) *From*

$$i \neq j \wedge j,\ i \leq n \rightarrow \tau(j) = 0,$$

infer

$$\sum_{j=1}^{n} \tau(j) = \tau(i).$$

Proof. The proof of (1) is by an easy induction. The proof of (2) is also by induction on n: If $n = 1$, then $i = 1$ and

$$\sum_{j=1}^{1} \tau(j) = \tau(1) = \tau(i).$$

Suppose the theorem true for n and let $i \leq n+1$ and $j \leq n+1$, $j \neq i$ implies $\tau(j) = 0$. We have that

$$\sum_{j=1}^{n+1} \tau(j) = \sum_{j=1}^{n} \tau(j) + \tau(n+1).$$

We have two cases. Consider first $i \leq n$. Then, by the inductive hypothesis

$$\sum_{j=1}^{n} \tau(j) = \tau(i) \quad \text{and} \quad \tau(n+1) = 0.$$

Thus, we obtain the conclusion. Consider, next, the case $i = n + 1$. Then, by (1)

$$\sum_{j=1}^{n} \tau(j) = 0,$$

and we also obtain the conclusion. □

In order to develop Taylor series approximations, which we shall do in Section 12, we need to define by recursion natural number powers and factorials:

Recursive Definition 3.

(1) $x^1 = x$,
(2) $\mathcal{N}(n) \rightarrow x^{n+1} = x^n x$.

Recursive Definition 4.

(1) $1! = 1$,
(2) $(n+1)! = n!(n+1)$.

While the max and $\sum$-axioms schemas are each an infinite collection of axioms, one for each term τ, the power and factorial axioms are particular formulas, where x is a variable, so that x^n has two variables, x and n and $n!$ one variable. Thus, we could have terms with min substituted for x or n. We could also add other terms defined by open internal induction.

4. Axioms for infinitesimals

We add the new predicate Inf with the intended interpretation of $\mathrm{Inf}(x)$ being 'x is an infinitesimal', that is, in this case x should be interpreted as a number which satisfies $|x| \leq r$, for any positive real number r. As we mentioned before, Inf is not internal.

I 1. $\mathrm{Inf}(x) \wedge \mathrm{Inf}(y) \rightarrow \mathrm{Inf}(x+y)$.

I 2. $\mathrm{Inf}(x) \wedge (\mathrm{Inf}(\frac{1}{y}) \rightarrow 0 = 1) \rightarrow \mathrm{Inf}(xy)$.

I 3. $(x < 0 \vee x > 0) \wedge \mathrm{Inf}(x) \rightarrow (\mathrm{Inf}(\frac{1}{x}) \rightarrow 0 = 1)$.

I 4. $\mathrm{Inf}(x) \wedge |y| \leq |x| \rightarrow \mathrm{Inf}(y)$.

I 5. $\mathrm{Inf}(\frac{1}{x}) \wedge (\mathrm{Inf}(\frac{1}{y}) \rightarrow 0 = 1) \rightarrow \mathrm{Inf}(\frac{1}{x+y})$.

I 6. $(\mathrm{Inf}(\frac{1}{x}) \rightarrow 0 = 1) \wedge (\mathrm{Inf}(\frac{1}{y}) \rightarrow 0 = 1) \rightarrow (\mathrm{Inf}(\frac{1}{x+y}) \rightarrow 0 = 1)$.

I 7. $(\mathrm{Inf}(y) \rightarrow 0 = 1) \wedge \mathrm{Inf}(x) \rightarrow |x| \leq |y|$.

I 8. $((\mathrm{Inf}(x) \rightarrow 0 = 1) \rightarrow 0 = 1) \rightarrow \mathrm{Inf}(x)$.

If we take $\varphi \rightarrow 0 = 1$ as $\neg\varphi$, the last axiom may be expressed by

$$\neg\neg\,\mathrm{Inf}(x) \rightarrow \mathrm{Inf}(x),$$

which is a special case of the logical law $\neg\neg\varphi \rightarrow \varphi$, that is classically valid, but that cannot be proved in our positive logic.

I 9. $\mathrm{Inf}(x) \vee (\mathrm{Inf}(x) \rightarrow 0 = 1)$.

This axiom may be expressed using negation by:

$$\mathrm{Inf}(x) \vee \neg\,\mathrm{Inf}(x),$$

and, thus, it is a special case of the law of excluded middle, which is also classically valid, but not provable in our system.

We introduce a constant ν_0 with the axiom:

I 10. $\mathrm{Inf}(1/\nu_0) \wedge \mathcal{N}(\nu_0)$.

We define $\varepsilon_0 = 1/\nu_0$. Then, it is clear that we can prove $\mathrm{Inf}(\varepsilon_0)$. Also, since we have that $\nu_0 > 0$, we get that $\varepsilon_0 > 0$. We introduce $\approx$, which means 'approximately equal', as a defined notion:

Definition 4.1. $x \approx y \leftrightarrow \mathrm{Inf}(|x - y|)$.

The symbol $\approx$ is not internal. We also introduce the following expressions as definitions:

Definition 4.2. $x \approx \infty \leftrightarrow \mathrm{Inf}(\frac{1}{x}) \wedge x \geq 0.$

The expression $x \approx \infty$, which is not internal, can be read 'x is positive infinite'.
We now can introduce external induction:

N 9 (External Induction). *Let φ be an open formula, not necessarily internal. Then:*
From
(1) $\varphi(1)$, *and*
(2) $((\frac{1}{n} \approx 0 \rightarrow 0 = 1) \wedge \varphi(n)) \rightarrow \varphi(n+1)$,
infer

$$(\frac{1}{n} \approx 0 \rightarrow 0 = 1) \rightarrow \varphi(n).$$

Definition 4.3. $x \lesssim y \leftrightarrow y \gtrsim x \leftrightarrow x \leq y \vee \mathrm{Inf}(x-y).$

Definition 4.4. $x \ll \infty \leftrightarrow (\mathrm{Inf}(\frac{1}{x}) \rightarrow 0 = 1) \vee x \leq 0.$

The expression $x \ll \infty$, which is not internal, can be read 'x is not positive infinite', that is, 'x is nonnegative finite or negative'.

Definition 4.5. $x \gg y \leftrightarrow (\mathrm{Inf}(x-y) \rightarrow 0 = 1) \wedge x \geq y..$

The expression $x \gg y$, which is also not internal, can be read 'x is strictly greater than y'.
Thus, 'x is infinite', is expressed in symbols $|x| \approx \infty$, 'x is finite', i.e., not infinite, in symbols $|x| \ll \infty$, and 'x is noninfinitesimal', in symbols $|x| \gg 0$.
The following proposition is a reformulation of some of the axioms, using the defined notions.

Proposition 4.1.
(1) $\mathrm{Inf}(x) \leftrightarrow x \approx 0.$
(2) $x \approx 0 \wedge |y| \ll \infty \rightarrow xy \approx 0.$
(3) $x \approx 0 \rightarrow |x| \ll \infty.$
(4) $|x| \approx \infty \wedge |y| \ll \infty \rightarrow |x+y| \approx \infty.$
(5) $|x| \ll \infty \wedge |y| \ll \infty \rightarrow |x+y| \ll \infty.$
(6) $|y| \gg 0 \wedge x \approx 0 \rightarrow |x| \leq |y|.$
(7) $(|x| \gg 0 \rightarrow 0 = 1) \rightarrow x \approx 0.$
(8) $|x| \gg 0 \vee x \approx 0.$
(9) $\mathrm{Inf}(0).$

We notice that (9) is proved from I 4 and $\varepsilon_0 > 0$. We also have the following easy proposition:

Proposition 4.2.
(1) $x = y \rightarrow x \approx y.$
(2) $x \approx y \rightarrow y \approx x.$
(3) $x \approx y \wedge y \approx z \rightarrow x \approx z.$

(4) $x_1 \approx y_1 \wedge x_2 \approx y_2 \rightarrow x_1 + x_2 \approx y_1 + y_2$.
(5) $x \approx y \wedge |z| \gg 0 \rightarrow \frac{x}{z} \approx \frac{y}{z}$.
(6) $x \approx z \wedge y \approx z \wedge x \leq u \leq y \rightarrow u \approx z$.
(7) $1 \approx 0 \rightarrow 0 = 1$ *or, simply,* $1 \gg 0$.
(8) $x \approx 0 \wedge |y| \gg 0 \rightarrow |x| \leq |y|$.
(9) $x \approx 0 \rightarrow 1 - x \geq 0$.
(10) $|x| \ll \infty \rightarrow |x+1|, |x-1| \ll \infty$
(11) $|x| \approx \infty \rightarrow |x+1|, |x-1| \approx \infty$

We introduce relative approximate equality by definition:

Definition 4.6. $x \approx z \quad (y) \leftrightarrow \frac{x}{y} \approx \frac{z}{y}$.

We note that if $|y| \ll \infty$, then $x \approx z \quad (y)$ implies $x \approx z$

The notion of relative approximate equality is especially useful when $y \approx 0$. Division of a number by an infinitesimal, makes the number much larger. Thus, $x \approx z \quad (y)$ expresses the fact that x and z are infinitely close, even when divided by y. For instance, $\varepsilon_0 \approx \varepsilon_0^2$, but we don't have $\varepsilon_0 \approx \varepsilon_0^2 \quad (\varepsilon_0)$. On the other hand, we do have $\varepsilon_0^2 \approx \varepsilon_0^3 \quad (\varepsilon_0)$.

All the parts of the previous propositions that make sense and some new ones are true with relative approximate equality. For instance:

Proposition 4.3. *Let $u < 0$ or $u > 0$. Then*

(1) $x = y \rightarrow x \approx y \quad (u)$.
(2) $x \approx y \quad (u) \rightarrow y \approx x \quad (u)$.
(3) $x \approx y \quad (u) \wedge y \approx z \quad (u) \rightarrow x \approx z \quad (u)$.
(4) $x_1 \approx y_1 \quad (u) \wedge x_2 \approx y_2 \quad (u) \rightarrow x_1 + x_2 \approx y_1 + y_2 \quad (u)$.
(5) $x \approx y \quad (u) \wedge (z \ll 0 \vee z \gg 0) \rightarrow \frac{x}{z} \approx \frac{y}{z} \quad (u)$.
(6) $x \approx z \quad (u) \wedge y \approx z \quad (u) \wedge x \leq v \leq y \rightarrow u \approx z \quad (u)$.
(7) $x \approx z \quad (u) \wedge |u| \leq |y| \rightarrow x \approx z \quad (y)$.

We also need:

Proposition 4.4.

$$x \lesssim y \wedge x \gg y \rightarrow 0 = 1.$$

Proof. Suppose $x \lesssim y$ and $x \gg y$. Then, by the definitions, $x \leq y$ or $\mathrm{Inf}(x-y)$ and $x \geq y \wedge (\mathrm{Inf}(x-y) \rightarrow 0 = 1)$. If $x \leq y$, since $x \geq y$, we obtain $x = y$, and then $\mathrm{Inf}(x-y)$. Since the other case is $\mathrm{Inf}(x-y)$, we have in any case, $\mathrm{Inf}(x-y)$. From $\mathrm{Inf}(x-y) \rightarrow 0 = 1$ we obtain $0 = 1$. □

5. Geometric subdivisions and rule for derivatives

We introduce the constant function symbol I for the identity function, and, for each constant term τ, the constant function symbol C_τ, for the constant function with value τ.

FU 1. $\mathrm{I}(x) = x$.

FU 2. $\mathrm{C}_\tau(x) = \tau$, *where τ is a constant term.*

We usually write just τ instead of C_τ.

We introduce informally, to make the paper more readable, function schemas $f(x)$, $g(x)$, $h(x)$ for terms. We abbreviate $(f+g)(x)$ for $f(x)+g(x)$, $(fg)(x)$ for $f(x)g(x)$, $\frac{f}{g}(x)$ for $\frac{f(x)}{g(x)}$, and $f \circ g(x)$ for $f(g(x))$. We also write ,

$$(df)(x,y) \quad \text{for} \quad f(x+y)-f(x).$$

We write $df(x)$ for $df(x,\varepsilon_0)$. We write dx for $d\mathrm{I}(x)$, where I is the identity function. That is, $dx = d\mathrm{I}(x) = \mathrm{I}(x+\varepsilon_0) - \mathrm{I}(x) = \varepsilon_0$. Because of the many traditional controversies about the notation dx, we remark that here d is an operator mapping a function f that has as domain and range a subset of the extended set of real numbers to a new function df with the same domain but with the range a set of infinitesimals. (We have used intuitive set theory, which lies outside our system as formulated here, to give a description in familiar language.) So in the notation dx or $d\mathrm{I}(x)$, for which dx is an abbreviation, x is a free variable, and we may replace it by any numerical term. The standard notation dx for differentials dominated the calculus during the 17th and 18th centuries. Derivatives were made basic only in the 19th century by Cauchy.

It is very easy to compute differentials. For instance, if $f(x)=x^2$, then

$$df(x,dx) = (x+dx)^2 - x^2 = 2x\,dx + dx^2.$$

(We write, as is usual, dx^2 for $(dx)^2$.) Thus

$$df(x,dx) \approx 2x\,dx \quad (dx).$$

We have the following identities for differentials, which can be proved purely algebraically.

Proposition 5.1.

(1) $$d(f \pm g)(x,y) = df(x,y) \pm dg(x,y).$$

(2) $$\begin{aligned} d(f\cdot g)(x,y) &= f(x+y)\,dg(x,y) + g(x)\,df(x,y) \\ &= f(x)\,dg(x,y) + g(x+y)\,df(x,y). \end{aligned}$$

(3) $$d(\frac{f}{g})(x,y) = \frac{g(x)\,df(x,y) - f(x)\,dg(x,y)}{g(x+y)g(x)}.$$

We need to introduce derivatives and integrals at least for all elementary functions. One of the problems is that we cannot prove that the functions defined as inverses of other functions (such as the exponential) are defined on all numbers. The most we can prove is that for any number there is an approximately equal number where the function is defined. We must, then, complicate the definition of the derivative to allow for this possibility. In order to have the transcendental functions defined on the right domains, we use, in Section 12, Taylor series. Compared to the system in [6], in the present context an additional complication arises, since we cannot use existential quantifiers or negations. Thus, we have to obtain constructively the necessary approximations. In order to avoid unnecessary complications, we shall simplify the definition of integrals and obtain the integrals of some of the transcendental functions after being defined via Taylor series. So we only need to complicate the definition of derivative, precisely for obtaining Taylor series.

The domain of inverse functions is the range of a function (for instance, in the case of the exponential, its domain is the range of the logarithm). Terms, however, are defined everywhere. So, a function in our system is determined by two terms, say τ and σ, where min does not occur: the argument is a value $\sigma(u)$, for a certain u, and the value is $\tau(\sigma(u))$. If a function f is represented by a pair of terms τ, σ in this fashion, and x is the variable for the argunent of the function, we sometimes write $f(x)$ instead of $\tau(x)$ and f_{dom} for σ. It is clear, that in case $\sigma(u)$ is $\mathrm{I}(u)$, then the domain contains all real numbers. We allow τ and σ to have other variables besides x and u, but u may not occur in τ and x may not occur in σ. The derivative of f will, then, be associated with the pair of terms which constitute f. If f corresponds to $\tau\sigma$, then the term $\tau(x)$, will be denoted just by $f(x)$ and $\sigma(y)$, by $f_{\text{dom}}(y)$.

The expression $x \in I$, where I is an interval, may be used as an abbreviation for the appropriate inequalities. For instance, if $I = [a, b]$ then $x \in I$ means $a \leq x \leq b$, and if $I = (a, b)$ then $x \in I$ stands for $a < x < b$. We shall always assume that the endpoints of I are a and b. We also use informally the subset, intersection and union notation. For instance, $[a, b] \subseteq \varphi$, where φ is a formula, is an abbreviation of the formula $x \in [a, b] \to \varphi(x)$.

The expression σ is *monotone*(x, y) *on the interval* I, is an abbreviation of the formula

$$(x, y \in I \wedge x < y \to \sigma(x) < \sigma(y)) \vee (x, y \in I \wedge x < y \to \sigma(x) > \sigma(y).$$

The expression σ *is* σ_L*-Lipschitz*(x, y) *on the interval* I, where σ_L is a term, stands for the formula

$$x, y \in I \wedge x \approx y \to |\sigma(x) - \sigma(y)| \leq \sigma_L(a, b)\,|x - y|.$$

In general, on defining an expression such as Lipschitz, we shall display the variables used in the definition. However, when using the notion, we shall omit the variables, in order to make the paper more readable. It is understood that when several of these expressions occur in a theorem, the variables are all distinct.

The definitions that follow are very technical so that we shall give informal explanations which may not be free-variable. We shall use the following abbreviation.

Definition 5.1. We say that f *is a function*(x, y) *on the interval* I, if the following conditions are satisfied:

(1) $|f_{B1}(a, b)|$, $|f_{B2}(a, b)| \ll \infty$ and $I \subseteq [f_{\text{dom}}(f_{B1}(a, b), f_{\text{dom}}(f_{B2}(a, b))]$.
(2) The term f_{dom} is monotone(x, y) and f_L-Lipschitz on

$$[f_{\text{dom}}(f_{B1}(a, b), f_{\text{dom}}(f_{B2}(a, b))].$$

(3) $x, y \in [f_{\text{dom}}(f_{B1}(a, b), f_{\text{dom}}(f_{B2}(a, b))] \wedge f_{\text{dom}}(x) \approx f_{\text{dom}}(y) \to x \approx y$

So we see that a function f involves five terms: f itself, f_{dom}, f_L, f_{B1} and f_{B2}. Also, the assertion of 'f is a function(x, y) on $[a, b]$' is the assertion of an open formula with free variables x, y, a, and b, so it can be used as a premise for a rule. In [G], we only needed two terms and a formula, since we could use existential quantifiers. We shall also use the expressions g, h for functions in the sense introduced above.

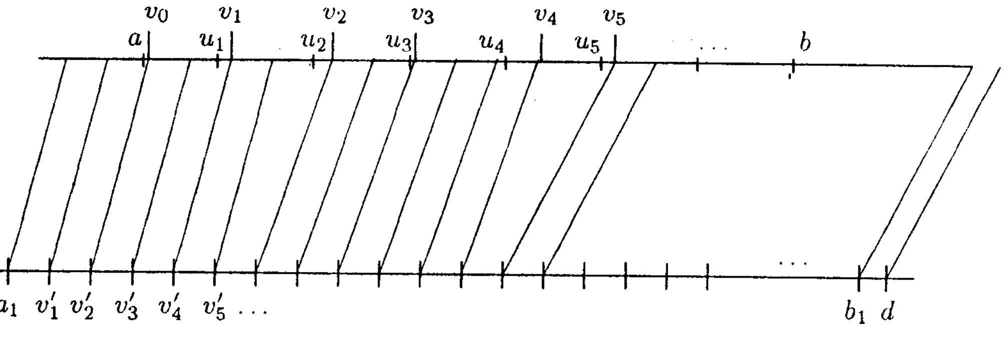

FIGURE 1. Selector

The function f is, then, defined on the range of f_{dom}. As an example, that we shall discuss in more detail in Section 12, we take the exponential. We shall define the exponential by

$$\exp(x) = y \leftrightarrow y = \log x.$$

The function $\log = \exp_{\text{dom}}$ is monotone and $\log_L$-Lipschitz, where $\log_L(a, b) = 1/a + 1$. The function exp is defined only on the range of log. The functions f_{B1} and f_{B2} select finite points $a_1 = f_{B1}(a, b)$ and $b_1 = f_{B2}(a, b)$ where f is defined, and such that $[f_{\text{dom}}(a_1), f_{\text{dom}}(b_1)] \supseteq [a, b]$. For the case of exp, we may define $\exp_{B2}$ as follows: Let $n = \text{li}(b/\log 2)$. Then $\exp_{B2}(a, b) = 2^n$. It is clear that $\log 2^n \approx n \log 2 \geq b$. We can also prove that n is finite, if b is finite. The function $\exp_{B1}$ can be defined similarly.

We require the function f_{dom} to be Lipschitz continuous so that for every $x \in [a, b]$ and every ε one can find y where f is defined (i.e., in the range of f_{dom}) such that $|x - y| \leq \varepsilon$. We shall use this property in order to define the selector, later on.

Definition 5.2. Let ν be an infinite natural number, $du = (b - a)/\nu$ and $u_i = a + i\,du$. So the u_i, for $0 \leq i \leq \nu$, form a partition of the interval $[a, b]$, what we call the *geometric subdivision of* $[a, b]$ *of order* ν. We always assume that a and b are finite and ν is infinite.

Notice that a geometric subdivision is determined by three numbers: the endpoints of the interval, a, b, and the infinite natural number ν. So when we use the variable u for a geometric subdivision, we are formally using three variables.

Our concept of geometric subdivision is closely related to ideas that began to be developed in medieval times as far back as the 14th century by Nicole Oresme and others. In the 15th century Nicholas of Cusa defined the infinitely small as that which cannot be made smaller. The early important and influential work was that of Cavalieri [4] in the first half of the 17th century on the geometry of indivisibles — or, as we would tend to say, geometry of infinitesimals. His geometric indivisibles were lines for plane surfaces and planes for solids, and building on his predecessors like Galileo, his teacher, he developed, in anticipation of the calculus, effective methods of calculation of areas and volumes. Our term *geometric subdivision* is deliberately chosen to recognize the historical point that infinitesimals were first used to solve problems in geometry.

Definition 5.3. Let f be a function on $[a, b]$ and u the geometric subdivision of $[a, b]$ of order $\nu \approx \infty$. So $u_0 = a$ and $u_\nu = b$. We define the *selector* v *for* f *and* u *on* $[a, b]$. (The selector v, which is a term with a natural number variable, is such that f is defined on v_i (i.e., $v_i = f_{\mathrm{dom}}(y)$, for a certain y) and $v_i \in [u_i, u_i + 1]$, for $0 \leq i \leq \nu - 1$.) Assume that f_{dom} is increasing. The construction for f_{dom} decreasing is similar.

Let $a_1 = f_{B1}(a, b)$, $b_1 = f_{B2}(a, b)$ and $M = f_L(a, b_1)$. (See Figure 1.) We first find a geometric subdivision v' such that $f_{\mathrm{dom}}(v'_{i+1}) - f_{\mathrm{dom}}(v'_i) \leq du$. This subdivision v' can be defined by $v'_j = a_1 + j\, du/M$. If we take $\nu_1 = \mathrm{li}(M(b_1 - a_1)/du)$ and $d = v_{\nu_1} = a_1 + \nu_1\, du/M$, then, by Axiom N 8, the natural number $\nu_1 \geq M(b_1 - a_1)/du$, and, hence

$$d \geq a_1 + \frac{M(b-a)}{du}\frac{du}{M} = b_1.$$

So that $[f_{\mathrm{dom}}(a_1), f_{\mathrm{dom}}(d)] \supseteq [a, b]$ and v' is the geometric subdivision of $[a_1, d]$ of order ν_1. So $v'_0 = a_1$ and $v'_{\nu_1} = d$.

For every i such that $a \leq u_i < b$, we shall construct $s(i)$ such that $f_{\mathrm{dom}}(v'_{s(i)}) \in [u_i, u_{i+1}]$ and take $v_i = f_{\mathrm{dom}}(v'_{s(i)})$, for $0 \leq i \leq \nu - 1$. We define $s(i) = \min_{f_{\mathrm{dom}}(v'_k) \geq u_i}$. (In Figure 1, we have $s(0) = 2$, $s(1) = 4$, $s(2) = 6$, $s(3) = 8$, $s(4) = 10$, and so on.)

Since

$$f_{\mathrm{dom}}(v'_{\nu_1}) = f_{\mathrm{dom}}(d) \geq f_{\mathrm{dom}}(b_1) \geq b \geq u_i,$$

for $0 \leq i \leq \nu - 1$, by Axiom N 7, $v_i = f_{\mathrm{dom}}(v'_{s(i)}) \geq u_i$ and $f_{\mathrm{dom}}(v'_{s(i)-1}) < u_i$. We have that

$$\begin{aligned} v_i - u_i &\leq v_i - f_{\mathrm{dom}}(v'_{s(i)-1}) \\ &= f_{\mathrm{dom}}(v'_{s(i)}) - f_{\mathrm{dom}}(v'_{s(i))-1}) \\ &\leq M|v'_{s(i)} - v'_{s(i)-1}| \\ &\leq M\frac{du}{M} \\ &= du = u_{i+1} - u_i. \end{aligned}$$

Hence, $v_i \in [u_i, u_{i+1}]$. Thus, we have achieved what was needed.

However, the min terms should be avoided in the definition of $s(i)$, in order to be able to use open induction. We redefine $s(i)$ as follows, when f_{dom} is increasing (for f_{dom} decreasing, the definition is similar):

$$s(i) = \sum_{j=0}^{\nu_1 - 1} \delta_1(f_{\mathrm{dom}}(v'_j) - u_i)\delta_2(u_i - f_{\mathrm{dom}}(v'_{j-1}))j.$$

We now prove that the two definitions of $s(i)$ yield the same number. We have that if, for a fixed i, we put h as the old $s(i)$, i.e., $h = \min_{u_i \leq f_{\mathrm{dom}}(v'_k)}$, then $u_{i+1} \geq f_{\mathrm{dom}}(v'_h) \geq u_i$ and $f_{\mathrm{dom}}(v'_{h-1}) < u_i$. Hence, $\delta_1(f_{\mathrm{dom}}(v'_h) - u_i) = 1$ and $\delta_2(u_i - f_{\mathrm{dom}}(v'_{h-1}) = 1$. On the other hand, we have the following cases:

Case 1: $j < h$. Then $v'_j \leq v_{h-1}$ and, since f_{dom} is increasing, $f_{\mathrm{dom}}(v'_j) \leq f_{\mathrm{dom}}(v'_{h-1}) < u_i$ and, hence, $\delta_1(f_{\mathrm{dom}}(v'_j) - u_i) = 0$.

Case 2: $j > h$. Then $v'_{j-1} \geq v'_h$. Since f_{dom} is increasing, $f_{\mathrm{dom}}(v'_{j-1}) \geq f_{\mathrm{dom}}(v_h) \geq u_i$ and, hence, $\delta_2(u_i - f_{\mathrm{dom}}(v'_{j-1}) = 0$.

Thus, by Proposition 3.6

$$\sum_{j=0}^{\nu_1-1} \delta_1(f_{\text{dom}}(v'_j) - u_i)\delta_2(u_i - f_{\text{dom}}(v'_{j-1}))j = h = \min_{u_i \le f_{\text{dom}}(v'_k)} = s(i).$$

Therefore, the selector v can be defined without min.

We now introduce the derivative: For a pair of terms, $\tau(x, x_1, \ldots, x_n)$ and $\sigma(y, y_1, \ldots, y_m)$, where the variables are as displayed, and a variable x, we introduce the $n+m+1$ operation $(\tau\sigma)_x$. If τ is written as the function $f(x)$, we write $f'(x)$ for the term $(\tau\sigma)_x(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_p)$.

We can now introduce the definition of the derivative f' of f as a rule:

Calculus Rule 1. *From*

(1) *f is a function on I, $|a|$, $|b| \ll \infty$, and*

(2) $x = f_{\text{dom}}(x_1) \wedge x + y = f_{\text{dom}}(y_1) \wedge \text{Inf}(y) \wedge (y < 0 \vee y > 0) \wedge x,\ x + y \in I \rightarrow$
$\dfrac{df(x,y)}{y} \approx g(x),$

infer

$$x_3 \in I \wedge x_3 = f_{\text{dom}}(x_4) \rightarrow f'(x_3) \approx g(x_3).$$

If the premises of the rule are satisfied, we shall say informally that g is a derivative of f. Notice that if $h(x) \approx g(x)$ on I and g is a derivative of f on I, then so is h, from the transitivity of $\approx$.

We have two different developments of integrals, a more algebraic development in Section 8, which does not use Recursive Definition N 2 of sums, and another, based on sums, in Section 11. We postpone the corresponding axioms for those sections.

6. Differentials and Derivatives

We introduce the assertions that a function f is 'differentiable' and 'continuous', using the notion of function of Definition 5.1. This assertions are an open formulas:

Definition 6.1. (1) We say that the function f on I is *differentiable*(y, x_1, y_1, x_2, y_2) *at* x, if and only if f is a function(x_2, y_2) on I and

$$x = f_{dom}(x_1) \wedge x + y = f_{\text{dom}}(y_1)\, \text{Inf}(y) \wedge (y < 0 \vee y > 0) \wedge x,\ x + y \in I \rightarrow$$
$$|f'(x)| \ll \infty \wedge \frac{df(x,y)}{y} \approx f'(x).$$

(2) We say that f is *differentiable*$(x, y, x_1, y_1, x_2, y_2)$ *on the interval* I, if and only if f is a function(x_2, y_2) on I and

$$\text{Inf}(y) \wedge (y < 0 \vee y > 0) \wedge x,\ x + y \in I \wedge x = f_{\text{dom}}(x_1) \wedge x + y = f_{\text{dom}}(y_1) \rightarrow$$
$$|f'(x)| \ll \infty \wedge \frac{df(x,y)}{y} \approx f'(x).$$

(3) Similarly, f is *continuous*$(x, y, x_1, y_1, x_2, y_2)$ *on the interval* I if and only if f is a function(x_2, y_2) on I and

$$x,\ y \in I \wedge x = f_{\text{dom}}(x_1) \wedge y = f_{\text{dom}}(y_1) \wedge x \approx y \rightarrow f(x) \approx f(y).$$

Thus, these statements can be used as premises or conclusions of rule-like theorems. We indicate all the variables in the statements, which in the case of the last two are the numerical variables a, b, x, y, x_1, y_1, x_3 and y_3.

The next theorem says that a differentiable function is continuous.

Theorem 6.1. *From f is differentiable on I infer f is continuous on I.*

Proof. Assume $t \approx z$, $t = f_{\text{dom}}(t_1)$, $z = f_{\text{dom}}(z_1)$. Hence, since $f'(t)$ is finite

$$f(z) - f(t) = df(t, z-t) \approx f'(t) \cdot (z-t) \approx 0.$$

□

Theorem 6.2. *From*

(1) *f is a function on the interval I,*
(2) *f is differentiable on I and*
(3) *$x \in I \to |f'(x)| \leq g(a,b)$, where $g(a,b)$ is finite,*

infer f is $(g(a,b)+1)$–Lipschitz continuous on I.

Proof. Let x, $y \in I$, $x \approx y$. Then

$$\frac{|f(x) - f(y)|}{|x-y|} \approx |f'(x)|.$$

Thus

$$\frac{|f(x) - f(y)|}{|x-y|} \leq |f'(x)| + 1 \leq g(a,b) + 1.$$

Thus, $|f(x) - f(y)| \leq g(a,b)|x-y|$. □

We can prove the following inference rules for derivatives using Propositions 4.3 and 5.1 and Theorem 6.1.

Theorem 6.3. *From f and g are differentiable on I, and $f_{\text{dom}}(x_1) = g_{\text{dom}}(x_1)$ infer*

(1) *$f + g$ is differentiable on I and $x_2 \in I \wedge x_2 = f_{\text{dom}}(x_3) \to (f+g)'(x_2) \approx f'(x_2) + g'(x_2)$.*
(2) *With the additional hypothesis that $x_1 \in I \wedge x_1 = f_{\text{dom}}(x_2) \to |f(x_1)| \ll \infty \wedge |g(x_1)| \ll \infty$, we have that $f \cdot g$ is differentiable on I and*

$$x_3 \in I \wedge x_3 = f_{\text{dom}}(x_4) \to (f \cdot g)'(x_3) = f(x_3) \cdot g'(x_3) + g(x_3) \cdot f'(x_3).$$

(3) *With the additional hypotheses $x_1 \in I \wedge x_1 = f_{\text{dom}}(x_2) \to |f(x_1)| \ll \infty \wedge |g(x_1)| \ll \infty$ and $x_3 \in I \wedge x_3 = f_{\text{dom}}(x_4) \to 0 \ll |g(x_3)| \ll \infty$, infer $\frac{f}{g}$ is differentiable on I, and*

$$x_5 \in I \wedge x_5 = f_{\text{dom}}(x_6) \to (\frac{f}{g})'(x_5) \approx \frac{g(x_5)f'(x_5) - f(x_5)g'(x_5)}{(g(x_5))^2}.$$

A similar theorem is true for differentiability at x.

We can prove the chain rule, which is also an inference rule:

Theorem 6.4 (Chain rule). *From $g(x) = f_{\text{dom}}(x_1)$, $x = g_{\text{dom}}(x_2)$, and f is differentiable at $g(x)$ and g is differentiable at x, infer $f \circ g$ is differentiable at x and $(f \circ g)'(x) \approx f'(g(x))g'(x)$.*

Proof. Assume the hypotheses of the theorem, and let $\operatorname{Inf}(y)$. Then

$$\frac{dg(x,y)}{y} \approx g'(x),$$

i.e., $dg(x,y) = g'(x) + \varepsilon_1 y$, where $\varepsilon_1 \approx 0$. Hence, since $g'(x)$ is finite, by Theorem 6.1, $\operatorname{Inf}(dg(x,y))$. Therefore, if $dg(x,y) > 0$ or $dg(x,y) < 0$

$$\frac{df(g(x), dg(x,y))}{dg(x,y)} \approx f'(g(x)).$$

Thus, in any case, $df(g(x), dg(x,y)) = f'(g(x))dg(x,y) + \varepsilon_2 dg(x,y)$ where $\varepsilon_2 \approx 0$. But, by the definition of composition

$$df \circ g(x,y) = f(g(x+y)) - f(g(x)).$$

On the other hand, $g(x+y) = g(x) + dg(x,y)$. Hence,

$$df \circ g(x,y) = f(g(x) + dg(x,y)) - f(g(x)) = df(g(x), dg(x,y)).$$

From these equations, we get the conclusion of the theorem. □

Theorem 6.5. *From f is differentiable on I, and*

$$x \in I \wedge x = f_{\text{dom}}(x_1) \rightarrow |f'(x)| \leq g(a,b)$$

with $|g(a,b)| \ll \infty$ *infer*

$$\operatorname{Inf}(y_2) \wedge x_2, x_2 + y_2 \in I \wedge x_2 = f_{\text{dom}}(x_3) \wedge x_2 + y_2 = f_{\text{dom}}(y_3) \rightarrow \\ |df(x_2, y_2)| \leq (g(a,b) + 1)|y_2|.$$

This theorem asserts: Let f be a differentiable function with a finitely bounded derivative on an interval I. Then f is Lipschitz continuous on I with $f_L = g$; that is, for a finite $M = g(a,b) + 1$ such that if $\varepsilon \approx 0$, and $x \in I$

$$|f(x+\varepsilon) - f(x)| \leq M|\varepsilon|.$$

Proof. Let $g(a,b) = N$, and let $y \approx 0$, $y < 0$ or $y > 0$. Then

$$\begin{aligned} \left|\frac{f(x+y) - f(x)}{y}\right| &\approx |f'(x)| \\ &\lesssim N \\ &\leq N + 1. \end{aligned}$$

The case $y = 0$ is obvious. □

7. Theorems on continuous functions

In order to make more readable axioms, theorems and definitions, we use, as we have been doing, t, u, and v, for terms when their main variables refer to natural numbers. We write u_n instead of $u(n)$.

We will need the notions of geometric subdivisions and selectors defined in Definitions 5.2 and 5.3.

We notice that all the theorems of this section are derived rules of inference. We state them explicitly as rules to make this clear.

We have the following proposition:

Proposition 7.1. *From f is continuous on $[a, b]$, u is the geometric subdivision of order ν of $[a, b]$, and v is the selector for u and f on $[a, b]$, infer*

$$a \leq x \leq b \wedge x = f_{\text{dom}}(x_3) \wedge \nu \approx \infty \rightarrow f(x) \approx f(\min_{v_k \geq x}).$$

Proof. Let $n = \min_{u_k \geq x}$. We have, $b = u_\nu \geq x$. Hence, by Axiom N 7, $u_n \geq x$, and $0 \leq n \leq \nu$. If $x = a$ then, since $v_0 \approx a$, $f(x) \approx f(v_0)$. So assume that $x > a$; then $u_n > a = u_0$. Also, $n = 0 \rightarrow u_n = u_0$, and, hence, $n = 0 \rightarrow 0 = 1$. Then, $n - 1 \geq 0$ is a natural number.

Suppose that $u_{n-1} \geq x$. Then, by Axiom N 7, $n \leq n - 1$, and so, $0 = 1$. Thus, $u_{n-1} \geq x \rightarrow 0 = 1$, and, hence, $u_{n-1} \leq x$. Since $u_{n-1} \approx u_n$, we have that $u_n \approx x$, and, hence, $v_n \approx u_n \approx x$. Since f is continuous, we obtain the theorem. □

We have the approximate Intermediate Value Theorem:

Theorem 7.2 (IVT). *From f is continuous on $[a, b]$, u is the geometric subdivision of order ν of $[a, b]$, and v is the selector for u and f on $[a, b]$, infer*

$$f(v_0) \leq 0 \leq f(v_\nu) \wedge \nu \approx \infty \rightarrow f(v_{\min_{f(v_k) \geq 0}}) \approx 0.$$

Proof. Let $n = \min_{f(v_k) \geq 0}$. We have $f(v_\nu) \geq 0$, and thus, by Axiom N 7, $f(v_n) \geq 0$ and $0 \leq n \leq \nu$. Suppose that $n = 0$, and, hence, $u_n = a$. Then, we have $f(v_0) \leq 0$ and $f(v_0) \geq 0$, and, thus, $f(v_0) = 0$. Suppose, now, that $n > 0$. Then $n - 1 \geq 0$. Suppose that $f(v_{n-1}) \geq 0$. Then, by Axiom N 7, $n - 1 \geq n$, and so $0 = 1$. Therefore, $f(v_{n-1}) \leq 0 \leq f(v_n)$. Since f is continuous and $v_{n-1} \approx v_n$, $f(v_{n-1}) \approx f(v_n)$. Thus, $f(v_n) \approx 0$. □

We say that $f(x)$ is a *near maximum*(y, x_1, y_1) *for f on* $[a, b]$ if and only if $x = f_{\text{dom}}(x_1)$ and

$$a \leq y \leq b \wedge y = f_{\text{dom}}(y_1) \rightarrow f(y) \lesssim f(x).$$

Theorem 7.3. *From f is continuous on $[a, b]$, u is the geometric subdivision of order ν of $[a, b]$, and v is the selector for u and f on $[a, b]$, infer*

$$a \leq x \leq b \wedge \nu \approx \infty \rightarrow f(x) \lesssim f(v_{\max_{f(v_k)}(\nu)}).$$

This theorem says that a maximum on a selector of a geometric subdivision is a near maximum of f. There is a similar statement for minima.

Proof. Let $n = \max_{f(u_k)}(\nu)$. Then, $f(u_n) \geq f(u_i)$, for all $0 \leq i \leq \nu$. By Proposition 7.1, $f(x) \approx f(u_i)$, for $i = \min_{u_k \geq x}$, so that $f(x) \approx f(u_i) \leq f(u_n)$. □

We have the following theorem on local maxima, which is, in fact, an approximate version of Rolle's theorem. If u is the geometric subdivision of order ν of $[a,b]$ and v is the selector for f and u on $[a,b]$, we write $du = u_{i+1} - u_i$ and $dv_i = v_{i+1} - v_i$

Theorem 7.4 (Rolle). *From u is the geometric subdivision of order ν of $[a,b]$, v the selector for f and u on $[a,b]$, $f_{\mathrm{dom}}(x) = g_{\mathrm{dom}}(x)$ and $n = \max_{f(v_k)}(\nu)$, infer if $0 < n < \nu \approx \infty$,*

$$df(v_n, dv_n) \approx g(v_n)\, dv_n \quad (dv_n)$$

and

$$df(v_n, -dv_{n-1}) \approx g(v_n)\,(-dv_{n-1}) \quad (-dv_{n-1})$$

then

$$g(v_n) \approx 0.$$

Proof. We have dv_n, $dv_{n-1} > 0$, and $v_n + dv_n = v_{n+1}$ and $v_n - dv_{n-1} = v_{n-1}$. Then

$$\frac{f(v_n + dv_n) - f(v_n)}{dv_n} \leq 0$$

and

$$\frac{f(v_n - dv_{n-1}) - f(v_n)}{-dv_{n-1}} \geq 0$$

But

$$\frac{f(v_n + dv_n) - f(v_n)}{dv_n} \approx g(v_n) \approx \frac{f(v_n - dv_{n-1}) - f(v_n)}{-dv_{n-1}}.$$

Thus, $g(v_n) \approx 0$. □

From Rolle's theorem we derive an approximate version of the Mean Value Theorem:

Theorem 7.5 (MVT). *From $b - a \gg 0$, u is the geometric subdivision of order ν of $[a,b]$, v the selector for f and u on $[a,b]$,, $\nu \approx \infty$, f is differentiable on (a,b), and*

$$a \leq x \leq b \wedge x = f_{\mathrm{dom}}(t) \rightarrow h(x) = (v_\nu - v_0)(f(x) - f(v_0)) - (x - v_0)(f(v_\nu) - f(v_0)),$$

infer

$$\frac{f(v_\nu) - f(v_0)}{v_\nu - v_0} \approx f'(v_{\max_{h(v_k)}(\nu)}).$$

Since, by Axiom I 1, any derivative of f is approximately equal to f', the Mean Value Theorem is true for any derivative. Although for any derivative, we only have this approximate form of the Mean Value Theorem, if f_{dom} is the identity function, given an interval $[a,b]$, one can always define a derivative g of f such that

$$(*) \qquad \frac{f(b) - f(a)}{b - a} = g(u_{\max_{f(u_k)}(\nu)}).$$

It is enough to define

$$g(x) = f'(x) + \left(\frac{f(b) - f(a)}{b - a} - f'(u_{\max_{h(v_k)}(\nu)})\right).$$

The function g is also a derivative of f, because

$$\frac{f(b)-f(a)}{b-a} - f'(u_{\max_{h(v_k)}(\nu)}) \approx 0.$$

For most applications, (*) is sufficient.

Proof. We have $h(v_0) = h(v_\nu) = 0$. Assume, first, that $b-a \gg 0$. Let $n = \max_{h(v_k)}(\nu)$. Then

$$dh(v_n, dv_n) \approx (b-a)f'(v_n) - (f(b)-f(a))dv_n \quad (dv_n),$$

and

$$dh(v_n, -dv_{n-1}) \approx (b-a)f'(v_n) - (f(b)-f(a))(-dv_{n-1}) \quad (-dv_{n-1}).$$

We can apply Rolle's theorem to h and obtain

$$((v_\nu - v_0)f'(v_n) - (f(v_\nu) - f(v_0)) \approx 0.$$

Since $b-a \gg 0$ and $v_\nu - v_0 \approx b-a$, we obtain the result. □

Definition 7.1. We say that the function f is *nearly increasing*$(x, y, x_1, y_1, x_2, y_2)$ (*decreasing*$(x, y, x_1, y_1, x_2, y_2)$) *on the interval* I if and only if f is a function(x_2, y_2) on I and

$$x, y \in I \wedge x = f_{\mathrm{dom}}(x_1) \wedge y = f_{\mathrm{dom}}(y_1) \wedge x \lesssim y \rightarrow f(x) \lesssim (\gtrsim) f(y).$$

As corollaries of MVT, we obtain:

Corollary 7.6. *From f is continuous on $[a,b]$, differentiable on (a,b), and*

$$a < x < b \rightarrow f'(x) \gtrsim 0,$$

($f'(x) \lesssim 0$), *infer f is nearly increasing (decreasing) on $[a,b]$.*

Proof. Let $x, y \in [a,b]$, $x < y$, and $y - x \gg 0$. Let u be the geometric subdivision of order ν_0 of $[x,y]$, v the selector for f and u on $[x,y]$ and

$$h(t) = (v_{\nu_0} - v_0)(f(t) - f(v_0)) - (t - v_0)(f(v_{\nu_0}) - f(v_0)),$$

for $t = f_{\mathrm{dom}}(t_1)$. By MVT, Theorem 7.5

$$f(y) - f(x) \approx f'(v_{\max_{h(v_k)}(\nu_0)})(y-x) \gtrsim 0.$$

Thus, we have proved the conclusion for the case $y \gg x$.

On the other hand, if $x \approx y$, since f is continuous, $f(x) \approx f(y)$, and, hence, $f(x) \lesssim f(y)$. Thus, the corollary follows, also in the case $x \approx y$. □

Corollary 7.7. *From f is differentiable on the interval I, and $y \in I \wedge y = f_{\mathrm{dom}}(y_1) \rightarrow f'(y) \approx 0$, infer*

$$x, z \in I \wedge x = f_{\mathrm{dom}}(x_1) \wedge z = f_{\mathrm{dom}}(z_1) \rightarrow f(x) \approx f(z).$$

Proof. Consider the constant function C_0, such that $C_0(x) = 0$ (Axiom FU 2). Then, we have that C_0 is a derivative of f on I. Let u be the geometric subdivision of $[a, b]$ of order ν_0, v the selector for f and u on $[a, b]$ and

$$h(x) = (v_\nu - v_0)(f(x) - f(v_0)) - (x - v_0)(f(v_\nu) - f(v_0)).$$

By MVT, Theorem 7.5

$$f(x) - f(z) \approx C_0(v_{\max_{h(v_k)}}(\nu_0))(x - z) = 0.$$

□

We shall need the next theorem for defining inverse functions.

Definition 7.2. We say that a function f is *strictly increasing*$(x, y, x_1, y_1, x_2, y_2)$ (*decreasing*$(x, y, x_1, y_1, x_2, y_2)$) *on* $[a, b]$ if and only if

(1) f is a function(x_1, y_1) on $[a, b]$,
(2) $x,\ y \in [a, b]$, $x < y$, $x = f_{\mathrm{dom}}(x_2)$, $y = f_{\mathrm{dom}}(y_2)$ implies $f(x) < f(y)$ $(f(x) > f(y))$, and
(3) $x,\ y \in [a, b]$, $x \ll y$, $x = f_{\mathrm{dom}}(x_2)$, $y = f_{\mathrm{dom}}(y_2)$ implies $f(x) \ll f(y)$ $(f(x) \gg f(y))$.

Theorem 7.8. *From*

(1) *f is a function on $[a, b]$ with f_{dom} the identity function,*
(2) $-\infty \ll a < b \ll \infty$,
(3) *f is differentiable on (a, b),*
(4) $x \in [a, b] \to |f(x)| \ll \infty$,
(5) *f is continuous on $[a, b]$, and*
(6) $x \in (a, b) \to f'(x) \gg 0$ ($x \in (a, b) \to f'(x) \ll 0$),

infer

f is strictly increasing (strictly decreasing) on $[a, b]$.

With the additional hypothesis

$$y_1 \in [a, b] \to (y_1 = g(x_1) \leftrightarrow x_1 = f(y_1))$$

infer

$$x_3 \approx x_4 \wedge x_3 = f(y_3) \wedge x_4 = f(y_4) \wedge y_3, y_4 \in [a, b] \to g(x_3) \approx g(x_4),$$

and

$$y_5 = f(x_5) \wedge x_5 \in [a, b] \wedge z \approx 0 \wedge y_5 + z = f(z_1) \wedge z_1 \in [a, b] \to \frac{dg(y_5, z)}{z} \approx \frac{1}{f'(x_5)}.$$

The last two conclusions of the theorem say that the inverse of f restricted to $[a, b]$ in its domain is continuous and differentiable. We cannot prove, however, that the domain of the inverse of a function whose domain is an interval is also an interval. By the Intermediate Value Theorem, 7.2, we can only prove that for every c in the interval between $f(a)$ and $f(b)$ we can construct an $x \approx c$ such that the inverse is defined at x.

Proof. We have to consider two cases. First, let $x, y \in [a, b]$ with $x \ll y$. It is clear that $y - x \ll \infty$. By the Mean Value Theorem, 7.5, one can construct z between x and y such that

$$\frac{f(y) - f(x)}{y - x} \approx f'(z) \gg 0.$$

Since $0 \ll y - x \ll \infty$, we obtain

$$f(y) - f(x) \approx f'(z)(y - x) \gg 0.$$

Second, assume that $x < y$ with $x \approx y$. Then

$$\frac{f(y) - f(x)}{y - x} \approx f'(x).$$

if $f(y) - f(x) \leq 0$, then

$$\frac{f(y) - f(x)}{y - x} \leq 0,$$

and, then, $f'(x) \lesssim 0$. Since we have $f'(x) \gg 0$, by Proposition 4.4, we obtain $0 = 1$. Thus, we have $f(y) \leq f(x) \rightarrow 0 = 1$. Hence, by Axiom F 3 and other field axioms, $f(y) > f(x)$.

We now obtain the derivative of the inverse function g. Let $y + z = f(u)$ and $z < 0 \vee z > 0$. Then, $g(y + z) = u = x + v \approx g(y) = x$ and $v < 0 \vee v > 0$, by the first part of the theorem. Thus, $v \approx 0$ and $y + z = f(x + v)$. Therefore

$$\frac{z}{v} = \frac{y + z - y}{v} = \frac{f(x + v) - f(x)}{v} \approx f'(x),$$

i.e., since $f'(x) \gg 0$ (or $f'(x) \ll 0$)

$$\frac{v}{z} \approx \frac{1}{f'(x)}.$$

This is equivalent to

$$\frac{dg(y, z)}{z} \approx \frac{1}{f'(x)}.$$

□

We define by recursion the nth derivative, using the definition of derivative, Calculus Rule 1, and write it $f^{(n)}$. Thus, we take $f^{(0)}(x) = f(x)$, and assuming $f^{(n)}$ to be defined and differentiable, we define $f^{(n+1)} = f^{(n)\prime}$. Recall that differentiable means that f' exists and is finite.

We say that $f^{(n)}$ *is an nth order derivative*(m, x, y, x_1, y_1) *on* I if and only if

$$1 \leq m \leq n \rightarrow (\mathrm{Inf}(y) \wedge (y < 0 \vee y > 0) \wedge x, x + y \in I \wedge x = f_{\mathsf{dom}}(x_1) \wedge$$
$$x + y = f_{\mathsf{dom}}(y_1) \rightarrow |f^{(m)}(x)| \ll \infty \wedge df^{(m-1)}(x, y) \approx f^{(m)}(x) \cdot y \quad (y)),$$

Theorem 7.9 (Taylor). *From*

(1) f *is a function on* $[a, b]$,
(2) $n \ll \infty$,
(3) $a \leq x_1 \leq b \wedge x_1 = f_{\mathsf{dom}}(x_2) \rightarrow |f(x_1)| \ll \infty$,
(4) $f^{(n+1)}$ *is an* $n+1$*th order derivative of* f *on* $[a, b]$, *and* $f^{(n+1)}$ *is continuous on* $[a, b]$,

(5) $a \leq c \leq d \leq b$ (*or* $a \leq d \leq c \leq b$) *and* $c = f_{\mathrm{dom}}(c_1)$, $d = f_{\mathrm{dom}}(d_1)$.
(6) $a \leq x \leq b \to p(x) = f(c) + f'(c)(x-c) + f''(c)(x-c)^2 + \cdots + f^{(n)}(c)(x-c)^n$,
(7) $a \leq x_3 \leq b \wedge x_3 = f_{\mathrm{dom}}(x_4) \to f(x_3) = p(x_3) + R_{n,c}(x_3)$, *and*
(8) u *is the geometric subdivision of order* ν *of* $[c,d]$ (*or of* $[d,c]$), $\nu \approx \infty$, *and* v *is the selector for* u *and* f *on* $[c,d]$ (*or* $[d,c]$)

infer

$$|R_{n,c}(d)| \lesssim \frac{1}{(n+1)!}|f^{(n+1)}(v_{\max_{|f(v_k)|}(\nu)})|\,|d-c|^{n+1}.$$

The proof is similar to the one appearing in [14].

Proof. Let $I = [a,b]$ and $e = |f^{(n+1)}(v_{\max_{|f(v_k)|}(\nu)})|$. Then e is a quasi-maximum or quasi-minimum of $f^{(n+1)}$. Suppose that it is a quasi-maximum; the proof for a quasi-minimum is similar. Assume, first, that $d \geq c$, and define

$$g(x) = \frac{1}{(n+1)!}e\,(x-c)^{n+1} + p(x) - f(x) = \frac{1}{(n+1)!}e\,(x-c)^{n+1} - R_{n,c}(x).$$

We have that e is a near maximum of $f^{(n+1)}$, because $f^{(n+1)}$ is continuous. We have $0 = g(c) \approx g'(c) \approx \cdots \approx g^{(n)}(c)$, and

$$g^{(n+1)}(x) = e - f^{(n+1)}(x) \gtrsim 0.$$

Then, by Corollary 7.6, $g^{(n)}$ is nearly increasing, and, hence, $g^{(n)}(x) \gtrsim 0$ when $x \geq c$. But $g^{(n-1)}(c) \approx 0$, and, hence, $g^{(n-1)}(x) \gtrsim 0$. By external induction we prove $g^{(k)}(x) \gtrsim 0$ if $k \leq n$. Hence, $g(d) \gtrsim 0$, and thus

$$|R_{n,c}(d)| \lesssim \frac{1}{(n+1)!}e\,|x-c|^{n+1}.$$

Suppose now that $d \leq c$. Take

$$h(x) = f(x) - p(x) - \frac{1}{(n+1)!}e\,(x-c)^{n+1} = R_{n,c}(x) - \frac{1}{(n+1)!}e\,(x-c)^{n+1}.$$

Then, repeating the same argument with h instead of g, we obtain the result for $d \leq c$. □

8. ALGEBRAIC DEVELOPMENT OF THE INTEGRAL

We have two alternative developments of the integral. The first, which we call algebraic, is contained in this section, and the second, in Section 11. In Sections 8–11 we assume that our functions f have f_{dom} the identity function I, so that we do not need to mention f_{dom}. We could make the exposition more general, but it is not necessary for our purposes.

We need the following two rules for the indefinite integral. For each term $g(x, x_1, \ldots, x_n)$, we introduce the term $(\int g(x)\,dx)(x, x_1, \ldots, x_n)$, where the variables are as displayed.

INT 1. *For each constant term C, we have the following rule: Let I be an interval. Then from*

$$\mathrm{Inf}(y) \wedge z, z + y \in I \rightarrow |g(z)| \ll \infty \wedge df(z, y) \approx g(z)\, y \quad (y)$$

and g is continuous on I infer

$$x \in I \rightarrow \left(\int g(x)\, dx\right)(x) \approx f(x) + C.$$

We write $(\int g(x)\, dx)(x)$ instead of the more usual notation $\int g(x)\, dx$ to stress the fact that $\int g(x)\, dx$ is a functional term.

We could let the term C not be constant but also possibly contain variables different from x. However, in this case we would have to introduce rather complicated rules of substitution.

INT 2. *From f is continuous on the interval I, infer $\int f(x)\, dx$ is differentiable on the interval I and*

$$x \in I \rightarrow (\int f(x)\, dx)'(x) \approx f(x).$$

We add the following definition of the definite integral:

INT 3. $\int_c^d g \approx (\int g(x)\, dx)(d) - (\int g(x)\, dx)(c)$

With these axioms, we can develop the transcendental functions as in Section 12.

We shall prove in Section 14 that the addition of these axioms is a conservative extension of the system in [6], so that the finitary consistency proof is still valid.

Building on the theory of integration of elementary functions, which began at least with Liouville and has been developed by Ritt, [17], Risch, [16], Davenport, [8], and others we will develop the free-variable theory of integration begun here in a separate paper.

9. OVERFLOW AND UNDERTOW

In this section we prove the nonstandard principles of overflow and undertow.

Theorem 9.1 (Undertow). *Let φ be an internal formula where neither* min *nor* $\mathcal{N}$ *occur. Then*

$$(\mu \approx \infty \wedge \mu \leq \nu) \rightarrow \varphi(\mu)) \rightarrow \min_{\varphi(k)} \ll \infty.$$

The conclusion says that $\min_{\varphi(k)}$ is finite.

Proof. Let $m = \min_{\varphi(k)}$, and assume that $\frac{1}{m} \approx 0$, i.e., that m is infinite. Then, $m - 1$ is also infinite. We have, $\varphi(\nu)$, and, thus, by Axiom N 7, $\varphi(m)$ and $m \leq \nu$. Hence, $m - 1 \leq \nu$ and, since $m - 1 \approx \infty$, $\varphi(m-1)$. Thus, by Axiom N 7, $m \leq m - 1$, and, thus, $0 = 1$. That is, we have proved that

$$\frac{1}{m} \approx 0 \rightarrow 0 = 1,$$

which is the definition of $m \ll \infty$. □

In order to make more understandable the following and other statements, we introduce the maximum of a hyperfinite set:

Definition 9.1. $\mathrm{Max}_{\varphi(k)}(\nu) = \nu - \min_{\varphi(\nu-k)\wedge k\leq\nu}(\nu)$.

We have:

Proposition 9.2. $\varphi(m) \wedge m \leq \nu \rightarrow (\varphi(\mathrm{Max}_{\varphi(k)}(\nu)) \wedge m \leq \mathrm{Max}_{\varphi(k)}(\nu) \leq \nu)$.

Proof. Suppose $\varphi(m)$ and $m \leq \nu$. Then, we have $\varphi(\nu-(\nu-m))$ and $\nu - m \leq \nu$. By Axiom N 7, $\varphi(\nu-\min_{\varphi(\nu-k)\wedge k\leq\nu})$ and $\min_{\varphi(\nu-k)\wedge k\leq\nu} \leq \nu-m$. Thus, $\varphi(\mathrm{Max}_{\varphi(k)}(\nu)$ and $\mathrm{Max}_{\varphi(k)} \geq m$. □

Theorem 9.3 (Overflow). *Let φ be an internal formula where neither* min *nor* $\mathcal{N}$ *occur. Then*

$$((m \leq n \ll \infty \rightarrow \varphi(n)) \wedge \nu \approx \infty) \rightarrow \mathrm{Max}_{\varphi(k)}(\nu) \approx \infty.$$

The conclusion of the rule says that the maximum η such that $\varphi(\mu)$, for all $m \leq \mu \leq \eta$, is infinite.

Proof. Suppose that $(m \leq n \ll \infty \rightarrow \varphi(n))$. Let $\mu = \mathrm{Max}_{\varphi(k)}(\nu)$, and assume that μ is finite. Then $\mu+1$ is also finite and $\mu + 1 \geq \mu \geq m$. Thus, we have $\varphi(\mu+1)$ and $\mu+1 \leq \nu$. Hence, by Proposition 9.2, $\mu \geq \mu+1$, and so $0 = 1$. Therefore, by Axiom I 8, μ is infinite. □

We prove by overflow:

Proposition 9.4. *From*

$$n \ll \infty \rightarrow |x| \leq \frac{1}{n},$$

infer $x \approx 0$.

Proof. By overflow, we have that, $\mu = \mathrm{Max}_{1/k\geq|x|}(\nu_0)$ is infinite. Hence, $|x| \leq \dfrac{1}{\mu} \approx 0$. □

10. Hyperfinite sums

Most of the theorems of this section, in a somewhat different form, are proved in [6].

We begin with the theorems on approximately equal infinite sums. The contents of the rest of this section is influenced by [15]. As earlier, we use u, v, t, for terms and write u_n for $u(n)$, where n is meant to be a natural number.

In order to simplify the notation, we write a sum of the form

$$\sum_{i=1}^{\nu} u_i + v_1 + v_2 + \cdots + v_n,$$

just as $\sum_{i=1}^{\mu} t_i$, including the finitely many terms with ordinary addition in the $\sum$ term. Strictly speaking, this may not be possible, since the terms v_j may contain min. We shall be careful, however, that the operator min occurs only in finitely many terms of the sum. The theorems of this and the next section, should be understood with Σ-terms interpreted in this way.

We need the following lemma:

Lemma 10.1. *From*

(1) $1 \leq i \leq \nu \rightarrow u_i \approx 0$,
(2) $1 \leq i \leq \nu \rightarrow t_i > 0$ *and*
(3) $1 \leq i \leq \nu \rightarrow v_i \approx u_i t_i \quad (t_i)$,

infer

$$\sum_{i=1}^{\nu} v_i \approx 0 \quad (\sum_{i=1}^{\nu} t_i).$$

In particular, we have

$$\sum_{i=1}^{\nu} u_i t_i \approx 0 \quad (\sum_{i=1}^{\nu} t_i).$$

Therefore, from the same hypotheses we can infer

$$\left|\sum_{i=1}^{\nu} t_i\right| \ll \infty \rightarrow \sum_{i=1}^{\nu} v_i \approx 0 \qquad \text{and} \qquad \sum_{i=1}^{\nu} u_i t_i \approx 0.$$

Proof. Let n be finite. Then, since

$$\frac{v_i}{t_i} \approx u_i \approx 0,$$

we have

$$|v_i| \leq \frac{1}{n} t_i,$$

for $i \leq \nu$. Thus

$$\left|\sum_{i=1}^{\nu} v_i\right| \leq \sum_{i=1}^{\nu} |v_i|$$
$$\leq \frac{1}{n} \sum_{i=1}^{\nu} t_i.$$

Since $\sum_{i=1}^{\nu} t_i$ is positive and n is finite and arbitrary, by Proposition 9.4

$$\sum_{i=1}^{\nu} v_i \approx 0 \quad (\sum_{i=1}^{\nu} t_i).$$

The last conclusion is clear from the definitions of approximate equality and relative approximate equality. □

The theorem behind the theorems for integrals is:

Theorem 10.2. *From*

(1) $1 \leq i \leq \nu \rightarrow |u_i| \leq M$, *where M is finite,*
(2) $1 \leq i \leq \nu \rightarrow t_i > 0 \wedge t_i \approx 0$,
(3) $1 \leq i \leq \nu \rightarrow u_i t_i \approx v_i \quad (t_i)$,
(4) $\sum_{i=1}^{\nu} t_i \ll \infty$,

infer

$$\sum_{i=1}^{\nu} u_i t_i \approx \sum_{i=1}^{\nu} v_i.$$

Proof. We have

$$\left|\sum_{i=1}^{\nu} u_i t_i\right| \le \sum_{i=1}^{\nu} |u_i| t_i \le M \sum_{i=1}^{\nu} t_i$$

so that $\sum_{i=1}^{\nu} u_i t_i$ or any partial sum is finite.

We shall prove the approximate equation (note that the meaning of the condition $u_i \geq 0$ in the sum is given in Definition 3.1)

$$(*) \qquad \sum_{\substack{i=1\\u_i\geq 0}}^{\nu} v_i \approx \sum_{\substack{i=1\\u_i\geq 0}}^{\nu} u_i t_i.$$

Similarly, it can be shown that

$$\sum_{\substack{i=1\\u_i<0}}^{\nu} v_i \approx \sum_{\substack{i=1\\u_i<0}}^{\nu} u_i t_i$$

and, since, by Proposition 3.5

$$\sum_{i=1}^{\nu} v_i = \sum_{\substack{i=1\\u_i\geq 0}}^{\nu} v_i + \sum_{\substack{i=1\\u_i<0}}^{\nu} v_i$$

and

$$\sum_{i=1}^{\nu} u_i t_i = \sum_{\substack{i=1\\u_i\geq 0}}^{\nu} u_i t_i + \sum_{\substack{i=1\\u_i<0}}^{\nu} u_i t_i,$$

we obtain the conclusion of the theorem.

We now prove the approximate equation (*).

Consider, for m a natural number

$$\sum_{\substack{i=1\\u_i\geq 1/m}}^{\nu} u_i.$$

Assume that m is a finite natural number. For $1 \leq i \leq \nu$ and $u_i \geq 1/m$, $\frac{v_i}{t_i} \approx u_i$. Since $u_i \geq 1/m$, $u_i \gg 0$. Then, $\frac{v_i}{u_i t_i} \approx 1$, and so, since $u_i t_i$ is finite

$$u_i t_i(1 - \frac{1}{n}) \leq v_i \leq u_i t_i(1 + \frac{1}{n}),$$

if n is finite, $1 \leq i \leq \nu$, and $u_i \geq 1/m$. Thus

$$(1-\frac{1}{n}) \sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} u_i t_i \leq \sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} v_i \leq (1+\frac{1}{n}) \sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} u_i t_i.$$

Thus

$$1-\frac{1}{n} \leq \frac{\sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} v_i}{\sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} u_i t_i} \leq 1+\frac{1}{n},$$

and, hence, by Proposition 9.4

$$\frac{\sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} v_i}{\sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} u_i t_i} \approx 1.$$

But $\sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} u_i t_i$ is finite. Therefore

$$\sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} v_i \approx \sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} u_i t_i,$$

and, thus

$$z_m = \left| \sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} v_i - \sum_{\substack{i=1 \\ u_i \geq 1/m}}^{\nu} u_i t_i \right| \approx 0,$$

is true if m is finite. We have, then

$$z_m \leq \frac{1}{m},$$

if m is finite. Hence, by overflow, Theorem 9.3, we can define an $\eta \approx \infty$ such that

$$\sum_{\substack{i=1 \\ u_i \geq 1/\eta}}^{\nu} v_i \approx \sum_{\substack{i=1 \\ u_i \geq 1/\eta}}^{\nu} u_i t_i$$

We have $0 \leq u_i < 1/\eta$ implies that $u_i \approx 0$. Hence, by Lemma 10.1

$$\sum_{\substack{i=1 \\ 0 \leq u_i < 1/\eta}}^{\nu} u_i t_i \approx 0 \qquad \text{and} \qquad \sum_{\substack{i=1 \\ 0 \leq u_i < 1/\eta}}^{\nu} v_i \approx 0.$$

Thus, we have, by Proposition 3.5

$$\begin{aligned}
\sum_{\substack{i=1\\u_i\geq 0}}^{\nu} v_i &= \sum_{\substack{i=1\\u_i\geq 1/\eta}}^{\nu} v_i + \sum_{\substack{i=1\\0\leq u_i<1/\eta}}^{\nu} v_i\\
&\approx \sum_{\substack{i=1\\u_i\geq 1/\eta}}^{\nu} v_i\\
&\approx \sum_{\substack{i=1\\u_i\geq 1/\eta}}^{\nu} u_i t_i\\
&\approx \sum_{\substack{i=1\\u_i\geq 1/\eta}}^{\nu} u_i t_i + \sum_{\substack{i=1\\0\leq u_i<1/\eta}}^{\nu} u_i t_i\\
&= \sum_{\substack{i=1\\u_i\geq 0}}^{\nu} u_i t_i
\end{aligned}$$

□

We need an abbreviation that will be useful for the introduction of integrals in the next section. Let $d \leq a < c \leq b$, ν a natural number, $du = (b-d)/\nu$ and $u_i = d + i\,du$, i.e., u is the geometric subdivision of order ν of $[d, b]$. We abbreviate

$$\sum\nolimits_a^c f[d, b, \nu] = \sum_{\substack{i=1\\a\leq u_i, u_{i+1}<c}}^{\nu} f(u_i)\,du.$$

We also abbreviate, for any x, $\mu_x = \min_{u_k > x}$, $\bar{\mu}_x = \min_{u_k \geq x} - 1$. Then

$$\sum\nolimits_a^c f[d, b, \nu] = \sum_{i=\mu_a}^{\bar{\mu}_c} f(u_i)du.$$

The next theorem gives an approximate form of a version for sums of one of the fundamental theorems of calculus.

Corollary 10.3. *From*

(1) f *and* F *are functions defined on the finite interval* $[a, b]$,

(2) u *is a geometric subdivision of* $[a, c]$ *of order* ν, *with* $\nu \approx \infty$, *where* $c \geq b$ *and* $c - a$ *is finite,*

(3) f *is finite on* $[a, b]$, *and*

(4) $0 \leq i < \nu \rightarrow \dfrac{dF(u_i, du_i)}{du_i} \approx f(u_i)$,

infer

$$\sum_a^b f[a, c, \nu] \approx F(b) - F(a).$$

Proof. The proof is obtained from Theorem 10.2 and the fact that by internal induction one can prove, if $\mu_a < m \leq \mu_b$, that

$$\sum_{i=\mu_a}^{m-1} dF(u_i, du_i) = F(u_m) - F(u_{\mu_a}).$$

Thus

$$\begin{aligned} F(b) - F(a) &\approx F(u_{\mu_b - 1}) - F(\mu_a) \\ &\approx \sum_{i=\mu_a}^{\mu_b} f(u_i)\, du \\ &= \sum\nolimits_a^b f[a, c, \nu] \end{aligned}$$

□

We also have the following useful lemma, which implies an approximate version for sums of the second form of the Fundamental Theorem, which we give below.

Lemma 10.4. *From*

(1) $1 \leq i \leq \nu \rightarrow u_i \approx x$ *and*
(2) $1 \leq i \leq \nu \rightarrow t_i > 0$,

infer

$$\frac{\sum_{i=1}^{\nu} u_i t_i}{\sum_{i=1}^{\nu} t_i} \approx x.$$

Proof. We have, $u_i = x + v_i$ with $v_i \approx 0$ for $i = 1, \ldots, \nu$. Then

$$\sum_{i=1}^{\nu} u_i t_i = \sum_{i=1}^{\nu} (x + v_i) t_i = x \sum_{i=1}^{\nu} t_i + \sum_{i=1}^{\nu} v_i t_i.$$

By Lemma 10.1

$$\frac{\sum_{i=1}^{\nu} v_i t_i}{\sum_{i=1}^{\nu} t_i} \approx 0.$$

□

Corollary 10.5. *From f is a finite continuous function on the interval $[x, x+y]$ with $y \approx 0$, and u is a geometric subdivision of order ν of $[a, b]$ with $a < x$, $x + y \leq b$ and $b - a$ finite such that $\frac{du}{y} \approx 0$, infer*

$$\frac{\sum_x^{x+y} f[a, b, \nu]}{y} \approx f(x),$$

and

$$\frac{\sum_a^{x+y} f[a, b, \nu] - \sum_a^x f[a, b, \nu]}{y} \approx f(x).$$

Proof. The second conclusion is obtained as follows. Let $j = \bar{\mu}_x$. We have that $x \in [u_j, u_{j+1}]$. By the assumption $du/y \approx 0$, it is clear that $a < u_j \leq x \leq u_{j+1} < x + y$. We then have

$$\sum\nolimits_a^{x+y} f[u, v] - \sum\nolimits_a^x f[u, v] = \sum\nolimits_x^{x+y} f[u, v] + f(u_j)\, du.$$

Let $p = \bar{\mu}_{x+y}$. We have

$$\sum_{\substack{i=0 \\ x < u_i, u_{i+1} < x+y}} du_i + (u_j + 1) - x) + (x + y - u_p) = y.$$

Hence, by Lemma 10.4

$$\frac{\sum\limits_{\substack{i=0 \\ x < u_i, u_{i+1} < x+y}}^{\nu} f(u_i)\, du_i + f(u_j)(u_{j+1} - x) + f(x)(x + y - u_p)}{y} \approx f(x).$$

Now

$$\frac{\sum_x^{x+y} f[u, v] + f(u_j) du}{y} = \frac{\sum\limits_{\substack{i=0 \\ x < u_i, u_{i+1} < x+y}}^{\nu} f(u_i)\, du + f(u_j)(u_{j+1} - x) + f(x)(x + y - u_p)}{y} + \frac{f(u_j)(x - u_j)}{y} - \frac{f(x)(x + y - u_p)}{y}.$$

Since $du/y \approx 0$ and f is finite

$$\frac{f(u_j)(x - u_j)}{y} \approx 0 \approx \frac{f(x)(x + y - u_p)}{y}.$$

The first conclusion is obtained similarly. □

11. Definite integrals

We now introduce the definite integral. The development in this section is completely independent of Section 8.

Since we are restricting functions f to those with f_{dom} the identity, the axiom introduced here is simpler than the one discussed in [6]. We could use the same axiom as in [6], but, since we do not need it, we use only the simpler form.

For each term $\tau(x, x_1, \ldots, x_n)$, where the variables are displayed, we introduce the $n+2$–ary operation symbol $\int \tau_x$, which is called *the integral of τ with respect to x*. If the term $\tau(x, x_1, \ldots, x_n)$ is written $f(x)$, we write $\int_y^z f$ for $\int \tau_x(x_1, \ldots, x_n, y, z)$.

Calculus Rule 2. *From $a \leq b < c$, $|a| \ll \infty$, $c \leq d$, $|d| \ll \infty$, ν, μ infinite, $\frac{d-a}{\nu(c-b)} \approx 0 \approx \frac{d-a}{\mu(c-b)}$,*

(1) $\dfrac{\sum_b^c f[a, d, \nu]}{c - b} \approx \dfrac{\sum_b^c f[a, d, \mu]}{c - b}$, *and*

(2) $\dfrac{\sum_a^b f[a, d, \nu] + \sum_b^c f[a, d, \nu]}{c - b} \approx \dfrac{\sum_a^c f[a, d, \nu]}{c - b}$,

infer

(a) $c \leq e \wedge |e| \ll \infty \wedge \eta \approx \infty \wedge \dfrac{e-a}{\eta(c-b)} \approx 0 \rightarrow \dfrac{\int_b^c f}{c-b} \approx \dfrac{\sum_b^c f[a,e,\eta]}{c-b}$, *and*

(b) $\dfrac{\int_a^b f + \int_b^c f}{c-b} \approx \dfrac{\int_a^c f}{c-b}$.

As usual, we assume that all terms, unless explicitly excepted, do not contain min. We now prove that if f is continuous and $b-a$ is finite, then the definition of the integral gives what we need:

Theorem 11.1. *From u is the geometric subdivision of $[a,e]$ of order ν such that $\frac{du}{c-b} \approx 0$, and f is continuous and finite on $[a,b]$, $a \leq b < c \leq e$ $|a|$, $|e| \ll \infty$ and $\nu \approx \infty$, infer*

(1) $\dfrac{\int_b^c f}{c-b} \approx \dfrac{\sum_b^c f[a,e,\nu]}{c-b}$, *and*

(2) $\dfrac{\int_a^b f + \int_b^c f}{c-b} \approx \dfrac{\int_a^c f}{c-b}$.

Proof. We must prove that if u is the geometric subdivision of $[a,e]$ of order ν, v, of order μ, $\nu \approx \infty \approx \mu$, $e \geq c > b \geq a$, and a, b, c and e are finite, then

(1) $\dfrac{\sum_a^b f[a,e,\nu] + \sum_b^c f[a,e,\nu]}{c-b} \approx \dfrac{\sum_a^c f[a,e,\nu]}{c-b}$, and

(2) $\dfrac{\sum_b^c f[a,e,\nu]}{c-b} \approx \dfrac{\sum_b^c f[a,e,\mu]}{c-b}$.

If $c-b \approx 0$, then the result is immediately obtained from Corollary 10.5. So assume that $c \gg b$. We must prove

(1) $\sum_a^b f[a,e,\nu] + \sum_b^c f[a,e,\nu] \approx \sum_a^c f[a,e,\nu]$, and

(2) $\sum_b^c f[a,e,\nu] \approx \sum_b^c f[a,e,\mu]$.

Since $c-b \not\approx 0$, we from these results we obtain the premises (1) and (2) of the integrals Axiom 2. The result (1) is easy to see, so we prove (2): Let t be the geometric subdivision of $[a,e]$ of order $\nu\mu$. Then $dt/du \approx 0$. We also have, $t_{\mu_{u_i}-1} = u_i$ and $t_{\bar{\mu}_{u_{i+1}}+1} = u_{i+1}$. Hence, by Corollary 10.5

$$f(u_i) \approx \frac{\sum_{j=\mu_{u_i}-1}^{\bar{\mu}_{u_{i+1}}+1} f(t_j)dt}{du}.$$

Then, by Theorem 10.2

$$\sum_b^c f[a,e,\nu\mu] = \sum_b^c \left(\sum_{j=\mu_{u_i}-1}^{\bar{\mu}_{u_{i+1}}+1} f(t_j)dt \right) [a,e,\nu] \approx \sum_b^c f[a,e,\nu].$$

Similarly, we prove

$$\sum_b^c f[a,e,\nu\mu] \approx \sum_b^c f[a,e,\mu],$$

and, hence

$$\sum_b^c f[a, e, \nu] \approx \sum_b^c f[a, e, \mu],$$

and thus by Calculus Rule 2 we obtain (1) and (2) of the theorem. □

We can prove the usual theorems on integrals in a approximate form.

Corollary 11.2. *From f is continuous on the finite interval I and a, b, and $c \in I$, $a < c < b$, infer*

$$\int_a^b f \approx \int_a^c f + \int_c^b f.$$

This is an immediate corollary of Theorem 11.1 (2).

We now prove in approximate form the two Fundamental Theorems of Calculus.

Theorem 11.3 (Fundamental Theorem 1). *From*

$$\mathrm{Inf}(y) \wedge (y > 0 \vee y < 0) \wedge x,\ x + y \in I \rightarrow \frac{dF(x, y)}{y} \approx f(x),$$

where a and b are finite and $b - a \gg 0$, and f is continuous on $[a, b]$, infer

$$\int_a^b f \approx F(b) - F(a).$$

Proof. Suppose that a and b are finite and let u be a geometric subdivision of $[a, c]$ of order $\nu_0 \approx \infty$ with $c \geq b$ and $c - a$ finite. Then, by Theorem 11.1 and Theorem 10.3, since $b - a$ is noninfinitesimal

$$\int_a^b f \approx \sum_a^b f(u_i)\, du \approx F(b) - F(a).$$

□

Theorem 11.4 (Fundamental Theorem 2). *From f is a continuous function on the finite interval I and $a \in I$ infer*

$$\mathrm{Inf}(y) \wedge (y > 0 \vee y < 0) \wedge x,\ x + y \in I \rightarrow \frac{\int_a^{x+y} f - \int_a^x}{y} \approx f(x).$$

Proof. Assume that $y \approx 0$, x, $x + y \in I$ and $y > 0$. The case $y < 0$ is done similarly. Let u be the geometric subdivision of $[a, b] \supset [x, x + y]$ of order ν with $\nu \approx \infty$. Then, by Theorem 11.2

$$\frac{\int_a^{x+y} f - \int_a^x f}{y} \approx \frac{\sum_x^{x+y} f[a, b, \nu]}{y}.$$

and, by Corollary 10.5

$$\frac{\sum_x^{x+y} f[a, b, \nu]}{y} \approx f(x).$$

□

We now prove a theorem due to Keisler, [10], which we shall use in one of the applications.

Theorem 11.5 (Infinite sum theorem). *Let W and h be terms. From*

(1) *If $a \leq x \leq y \leq z \leq b$, where a, b are finite, then $W(x,z) = W(x,y) + W(y,z)$.* (This can be paraphrased as 'W is additive'.)

(2) *If x, $x+y \in I$, $\mathrm{Inf}(y)$, $y > 0$, then*

$$W(x, x+y) \approx h(x)\, y \quad (y),$$

(3) *h is a continuous function on $[a,b]$,*

infer

$$W(a,b) \approx \int_a^b h.$$

Proof. We prove by internal induction on ν

$$W(a,b) = \sum_{i=0}^{\nu-1} W(a + i\frac{b-a}{\nu}, a + (i+1)\frac{b-a}{\nu}).$$

Take $\nu = \nu_0 \approx \infty$ and let $a_i = a + i(b-a)/\nu_0$ and $dx = (b-a)/\nu_0 \approx 0$. We have

$$W(a_i, a_i + dx) \approx h(a_i)\, dx \quad (dx).$$

Then, by Theorem 10.2

$$W(a,b) \approx \sum_{i=0}^{\nu-1} h(a_i)\, dx.$$

But, by Theorem 11.1

$$\sum_{i=0}^{\nu-1} h(a_i)\, dx \approx \int_a^b h.$$

□

12. Series and transcendental functions

This section is also contained in a modified form in [6]. We return to the notion of functions f with arbitrary domain functions f_{dom}.

We first introduce convergence of sequences: u *ν-converges(μ) to x* if and only if

$$\mu \leq \nu \wedge \mu \approx \infty \rightarrow u_\mu \approx x,$$

and u is *ν-convergent(μ)* if and only if

$$\mu \leq \nu \wedge \mu \approx \infty \rightarrow u_\mu \approx u_\nu.$$

Similarly, u is convergent(ν, μ) if and only if $\nu \approx \infty$ implies that u is ν–convergent(μ).

A series is just a sequence v such that

$$v_n = \sum_{i=1}^{n} u_i.$$

Then the series ν-converges or converges when it ν-converges as a sequence. We also define, the series $\sum_{i=1}^n u_i$ *ν converges absolutely* if and only if the series $\sum_{i=1}^n |u_i|$

ν-converges. We have the following easy proposition, which in both directions is a rule:

Proposition 12.1. *From the fact that the series $\sum_{i=1}^{n} u_i$ ν-converges, infer that the tails $\sum_{i=\mu}^{\nu} x_i$ are infinitesimal for all infinite $\mu \leq \nu$, and vice-versa.*

From this proposition, we derive the comparison test:

Theorem 12.2 (Comparison test). *From $\nu \approx \infty$, $(n \leq \nu \wedge n \approx \infty) \rightarrow |u_n| \leq |v_n|$, and $\sum_{i=1}^{n} |v_i|$ ν-converges, infer that $\sum_{i=1}^{\nu} |u_i|$ also ν-converges.*
Therefore, from the fact that the series $\sum_{i=1}^{n} u_i$ ν-converges absolutely, infer that it ν-converges.

Proof. We have that if $\mu \leq \nu$ and $\mu \approx \infty$, $\sum_{i=\mu}^{\nu} |v_i| \approx 0$. Then

$$0 \leq \sum_{i=\mu}^{\nu} |u_i| \leq \sum_{i=\mu}^{\nu} |v_i| \approx 0,$$

for $\mu \leq \nu$, $\mu \approx \infty$. □

We also have:

Theorem 12.3 (Nelson, [15]). *From $\nu \approx \infty$, $i \ll \infty \rightarrow |u_i| \ll \infty$, and $\sum_{i=1}^{n} |u_i|$ ν-converges, infer $\sum_{i=1}^{\nu} |u_i| \ll \infty$.*

Proof. $\sum_{i=n}^{\nu} |u_i| \leq 1$ is true if n is an infinite number. Then by undertow, Theorem 9.1, the minimum m must be finite. But

$$\sum_{i=1}^{m-1} |u_i| \leq (m-1) \max_{|u_k|}(m).$$

Hence

$$\sum_{i=1}^{\nu} |u_i| = \sum_{i=1}^{m-1} |u_i| + \sum_{i=m}^{\nu} |u_i|$$

is finite. □

In order to prove the ratio test for convergence, we need some theorems about natural numbers.

From we prove by external induction:

Proposition 12.4. *If $\nu \approx \infty$ and $n \ll \infty$, then $\frac{\nu}{n} \approx \infty$.*

Proof. We must prove that $\frac{n}{\nu} \approx 0$, if n is finite. This is easily done by external induction. The proof by internal induction requires Euclid's algorithm, which can be proved by internal induction. □

We assume that we have proved by internal induction (which is easily done):

$$x > 0 \rightarrow (y \geq x \rightarrow y^n \geq x^n).$$

We also can give an easy internal inductive proof of the inequality

$$(1 + \frac{1}{n})^\nu \geq 1 + \frac{\nu}{n},$$

for n and ν natural numbers.

Proposition 12.5. *If $r \gg 1$ and $\nu \approx \infty$, then $r^\nu \approx \infty$. Therefore:*
If $r \ll 1$ and $\nu \approx \infty$, then $r^\nu \approx 0$.

Proof. Let $n = \mathrm{li}(1/(r-1))$. Then, by undertow, we show that $n \ll \infty$. We also have $r \geq 1 + \frac{1}{n}$. Thus

$$r^\nu \geq \left(1 + \frac{1}{n}\right)^\nu \geq 1 + \frac{\nu}{n} \approx \infty.$$

□

We prove by internal induction, as usual, for $r > 0$ and $a > 0$

$$\sum_{i=0}^{n} ar^i = a\frac{1 - r^{n+1}}{1 - r}.$$

Then, if $\nu \approx \infty$, a is finite, and $0 < r \ll 1$, the series $\sum_{i=0}^{\nu} ar^i$ ν–converges to $\dfrac{a}{1-r}$. Then, we can prove the ratio test:

Theorem 12.6 (Ratio test). *From*

(1) *if i is finite, $u_i \geq 0$ is finite, and*
(2) *if μ is infinite, $\mu \leq \nu$, then $\dfrac{u_{\mu+1}}{u_\mu} \approx r \ll 1$,*

infer $\sum_{i=1}^{\nu} u_i$ ν-converges.

Proof. Let $\mu \approx \infty$, $\mu \leq \nu$. Then

$$\frac{u_{\mu+1}}{u_\mu} \approx r$$
$$\leq r + \frac{1}{n} \ll 1,$$

where n is a finite number. (We may define $n = \mathrm{li}(1/(1-r))$.)

Let $s = 1 + \frac{1}{n}$. Then, we have

$$u_{\mu+1} \leq su_\mu,$$

if μ is infinite, $\mu \leq \nu$. Hence, by undertow, Theorem 9.1, the minimum number m such that if $m \leq p \leq \nu$, then

$$u_{p+1} \leq su_p,$$

is finite. By internal induction we prove

$$u_{m+p} \leq u_m s^p,$$

for $p \leq \nu - m$. Since u_m is finite, the geometric series $\sum_{p=0}^{\nu-m} u_m s^p$ converges. Then, by the comparison test, $\sum_{p=0}^{\nu-m} u_{m+p}$ converges, and so $\sum_{i=1}^{\nu} u_i$ converges. □

From the ratio test we can prove:

Proposition 12.7. *If $s \ll 1$ and $\nu \approx \infty$, then $\nu s^\nu \approx 0$.*

Proof. By the ratio test, the series $\sum_{i=1}^{n} n s^n$ ν-converges. The result is then obtained from Proposition 12.1. □

We need a few theorems about series of functions, in our case of terms with a variable, say x.

Theorem 12.8. *From*

(1) *$x \in I$, where I is a finite interval, implies that the series $\sum_{i=1}^{n} u_i(x)$ ν-converges, $\nu \approx \infty$, and*

(2) *$1 \leq i \leq \nu$ implies $u_i(x)$ is continuous on I,*

infer that $\mu \leq \nu$ implies that $\sum_{i=1}^{\mu} u_i(x)$ is continuous on I.

Proof. Let x, $y \in I$, $x \approx y$. We prove by external induction, if n is finite

$$\left|\sum_{i=0}^{n} u_i(x) - \sum_{i=0}^{n} u_i(y)\right| \leq \sum_{i=0}^{n} |u_i(x) - u_i(y)| \leq \frac{1}{n}.$$

By overflow, Theorem 9.3, an infinite $\eta \leq \nu$ can be constructed such that if μ is infinite and $\mu \leq \eta$

$$\left|\sum_{i=0}^{\mu} u_i(x) - \sum_{i=0}^{\mu} u_i(y)\right| \leq \frac{1}{\mu} \approx 0.$$

If $\eta = \nu \geq \mu$, we are done. Assume that $\eta < \mu \leq \nu$. Then

$$\begin{aligned}\left|\sum_{i=0}^{\mu} u_i(x) - \sum_{i=0}^{\mu} u_i(y)\right| &\leq \left|\sum_{i=0}^{\eta} u_i(x) - \sum_{i=0}^{\eta} u_i(y)\right| + \left|\sum_{i=\eta+1}^{\mu} u_i(x) - \sum_{i=\eta+1}^{\mu} u_i(y)\right| \\ &\leq \left|\sum_{i=0}^{\eta} u_i(x) - \sum_{i=0}^{\eta} u_i(y)\right| + \left|\sum_{i=\eta+1}^{\mu} u_i(x)\right| + \left|\sum_{i=\eta+1}^{\mu} u_i(y)\right| \\ &\approx 0.\end{aligned}$$

□

We assume that the definite integral has been extended to lower limit a, and upper limit b, both finite, with $a \geq b$, as it is usually done:

Definition 12.1.

(1) $\displaystyle\int_a^a f = 0$,

(2) $\displaystyle a < b \wedge |a|, |b| \ll \infty \rightarrow \int_b^a f = -\int_a^b f.$

We need a theorem about power series:

Theorem 12.9. *Let $I = (-r, r)$, where r is finite. From the series $\sum_{i=1}^{n} |a_i|x^i$ ν-converges absolutely on I, $\nu \approx \infty$, infer*

(1) *The series $\sum_{i=1}^{n} a_i i x^{i-1}$ and $\sum_{i=1}^{n}(a_i/(i+1))x^{i+1}$ ν-converge absolutely on any $|x| \ll r$.*

(2) *The series $\sum_{i=1}^{n} a_i x^i$ is differentiable and*

$$\left(\sum_{i=1}^{n} a_i x^i\right)'(x) \approx \sum_{i=1}^{n} a_i i x^{i-1},$$

if $n \leq \nu$ and $|x| \ll r$.

(3) $\int_0^a \sum_{i=1}^{n} a_i x^i \approx \sum_{i=1}^{n} \frac{a_i}{i+1} x^{i+1}$, *if $n \leq \nu$ and $|a| \ll r$.*

Proof. We first prove (1). Let $\mu \approx \infty$ and let $|x| \ll c \ll r$. (For $|x| \ll r$, we can always construct such a c as $c = |x| + \frac{1}{n}$, where $n = \text{li}(1/(r - |x|))$; by undertow, Theorem 9.1, we show that n is finite.) Then, by Proposition 12.7, $\mu|x/c|^\mu \approx 0$. Thus, since $1/x$ is finite

$$\mu|a_\mu||x|^{\mu-1} = \mu\left|\frac{x}{c}\right|^\mu \cdot \left|\frac{1}{x} \cdot a_\mu c^\mu\right| \leq |a_\mu c^\mu|.$$

Since the series $\sum_{i=1}^{n} a_i c^i$ ν-converges, by the comparison test we obtain (1). For the other series in (1), the proof is similar.

Next, we prove (2). Let $f_\mu(x) = \sum_{i=1}^{\mu} a_i x^i$. We claim that the series

$$df_n(x,y) = \sum_{i=1}^{n} a_i \frac{d(x^i y)}{y} = \sum_{i=1}^{n} a_i \frac{(x+y)^i - x^i}{y},$$

ν–converges if $|x| \ll r$ and y is infinitesimal. Let $\mu \approx \infty$. We have

$$\begin{aligned}\left|a_\mu \frac{(x+y)^\mu - x^\mu}{y}\right| &= \left|a_\mu \left(\mu x^{\mu-1} + \sum_{j=1}^{\mu-1} \binom{\mu}{j+1} x^{\mu-(j+1)} y^j\right)\right| \\ &= \left|a_\mu \left(\mu x^{\mu-1} + \mu \sum_{j=1}^{\mu-1} \binom{\mu-1}{j} \frac{1}{j+1} x^{\mu-1-j} y^j\right)\right| \\ &\leq |a_\mu||\mu||x|^{\mu-1} + |a_\mu|\mu(|x|+|y|)^{\mu-1},\end{aligned}$$

The series $\sum_{i=1}^{n} |a_i||i||x|^{i-1}$ and $\sum_{i=1}^{n} |a_i|i(|x|+|y|)^{i-1}$ converge, since $|x|$, $|x|+|y| \ll r$, so we obtain the claim.

We then prove by external induction, if n is finite

$$\left|\sum_{i=1}^{n} \frac{a_i((x+y)^i - x^i)}{y} - \sum_{i=1}^{n} a_i i x^{i-1}\right| \leq \frac{1}{n},$$

for y infinitesimal. By overflow, Theorem 9.3, if $n \leq \eta \approx \infty$

$$\sum_{i=1}^{n} \frac{a_i(x+y)^i - x^i}{y} \approx \sum_{i=1}^{n} a_i i x^{i-1}.$$

Let $\mu \leq \nu$. If $\eta \geq \mu$, we are done. Suppose that $\mu > \eta$. Then

$$\left|\sum_{i=1}^{\mu} \frac{a_i(x+y)^i - x^i}{y} - \sum_{i=1}^{\mu} a_i i x^{i-1}\right| \leq \left|\sum_{i=1}^{\eta} \frac{a_i(x+y)^i - x^i}{y} - \sum_{i=1}^{\eta} a_i i x^{i-1}\right| + \left|\sum_{i=\eta+1}^{\mu} \frac{a_i(x+y)^i - x^i}{y} + \sum_{i=\eta+1}^{\mu} a_i i x^{i-1}\right| \approx 0.$$

We finally prove (3). By the previous theorem, we know that the series $\sum_{i=1}^{n} a_i x^i$ is continuous, so that the integral is well defined. Let $r \gg c \gg |a|$ be obtained as above and let $\mu \leq \nu$. Then if $-c \leq t \leq c$

$$\left|\sum_{i=m+1}^{\mu} a_i t^i\right| \leq \sum_{i=m+1}^{\mu} |a_i t^i| \leq \sum_{i=m+1}^{\mu} |a_i| c^i = E_m.$$

Let $f(x) = \sum_{i=1}^{\mu} a_i x^i$. Thus

$$-E_m \leq f(x) - \sum_{i=1}^{m} a_i x^i \leq E_m.$$

If m is finite, we prove by external induction, integrating

$$-E_m a \lesssim \int_0^a f - \sum_{i=1}^{m} \frac{a_i}{i+1} x^{i+1} \lesssim E_m a.$$

Then, if m is finite

$$\left|\int_0^a f - \sum_{i=1}^{m} \frac{a_i}{i+1} x^{i+1}\right| \leq |E_m| + \frac{1}{m}.$$

By overflow, Theorem 9.3, for an $\eta \approx \infty$, $\eta \leq \mu$, which can be constructed

$$\left|\int_0^a f - \sum_{i=1}^{\eta} \frac{a_i}{i+1} x^{i+1}\right| \leq |E_\eta| + \frac{1}{\eta}.$$

Hence

$$\int_0^a f \approx \sum_{i=1}^{\eta} \frac{a_i}{i+1} x^{i+1}.$$

But

$$\sum_{\eta+1}^{\mu} \frac{a_i}{i+1} x^{i+1} \approx 0,$$

because, by (1), the series is ν–convergent. □

We can now define the main transcendental functions. We may use either the algebraic (Section 8) or the sum (Section 11) treatment of integrals. for this purpose. We first introduce the natural logarithm by the following axiom:

$$\log x = \int_1^x \frac{1}{t}\, dt.$$

We consider log as a function, take $\log_{\mathrm{dom}}(x) = \mathrm{I}(x)$, the identity function. In the usual way, we can obtain the main properties of the logarithm for finite x in approximate form, in particular, $\log x^n \approx n \log x$, for x and n finite.

By Theorem 11.4, if $\infty \gg x \gg 0$, then $\log'(x) \approx 1/x$. Then, if we take any interval $[a, b]$, with $0 \ll a < b \ll \infty$, $\log'$ is bounded on the interval by $1/a$, which is finite. Thus, by Theorem 6.2, log is $\log_L$-Lipschitz continuous on $[a, b]$, where $\log_L(a, b) = 1/a + 1$. Also, $1/x \gg 0$, if $x \gg 0$. Thus, by Theorem 7.8, log is strictly increasing on noninfinitesimal finite numbers. Therefore, log can be f_{dom} for a function f. We now calculate $\log_{B1}$ and $\log_{B2}$. Let $a < b$, finite, be given. We have, from the definition of log, that $\log 2 > 0$. Let

$$\log_{B2}(a, b) = 2^{\mathrm{li}(b/\log 2)+1}.$$

Then

$$\log \log_{B2}(a, b) = \mathrm{li}(\frac{b}{\log 2}) + 1 \log 2 \geq \frac{b}{\log 2} \log 2 = b.$$

On the other hand, let

$$\log_{B1}(a, b) = 2^{-(\mathrm{li}(-a/\log 2)+1)}.$$

We have

$$(\mathrm{li}(\frac{-a}{\log 2}) + 1) \log 2 \geq \frac{-a}{\log 2} \log 2 = -a,$$

so that

$$\log \log_{B1}(a, b) = -(\mathrm{li}(\frac{-a}{\log 2}) + 1) \log 2 \leq a.$$

We then define an "almost" exponential function:

$$\mathrm{aexp}(x) = y \leftrightarrow x = \log y,$$

with $\mathrm{aexp}_{\mathrm{dom}} = \log$. Using Theorem 7.8, we get that aexp is differentiable on the finite numbers and we can calculate its derivative, $\mathrm{aexp}'(x) \approx \mathrm{aexp}(x)$, for finite x. Thus aexp is increasing.

In a similar way, we define the arctan:

$$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt.$$

As above, we consider arctan as a function, with $\arctan_{\mathrm{dom}}(x) = \mathrm{I}(x)$. We take $\pi = 4 \arctan 1$. As for the logarithm, by Theorem 11.4, if $|x| \ll \infty$, then $\arctan'(x) \approx 1/(1+x^2)$. Thus, in any finite interval $[a, b]$, arctan is $\arctan_L$-Lipschitz on $[a, b]$, with

$$\arctan_L(a, b) = \frac{1}{1 + \delta_1(|b| - |a|)a^2 + \delta_2(|a| - |b|)b^2}.$$

i.e.

$$\operatorname{arctan}_L(a,b) = \begin{cases} \dfrac{1}{1+a^2}, & \text{if } |a| \le |b|, \\ \dfrac{1}{1+b^2}, & \text{otherwise.} \end{cases}$$

For the definition of $\operatorname{arctan}_{B1}$ and $\operatorname{arctan}_{B2}$ we may use, for instance, the formula

$$\arctan x + \arctan y \approx \arctan \frac{x+y}{1-xy},$$

so that, for any finite n, we obtain a formula of the form

$$n \arctan x = \arctan f(n,x),$$

where $f(n,x)$ is a rational function of x. The function f is defined recursively by

$$f(1,x) = x$$

$$f(n+1,x) = \frac{x + f(n,x)}{1 - xf(n,x)}.$$

We proceed similarly as for the log and define $\operatorname{arctan}_{B2}(a,b) = f(\operatorname{li}(b/\arctan 2), 2)$ and $f_{B1}(a,b) = -f(\operatorname{li}(-a/\arctan 2), 2)$.

Thus, arctan can be f_{dom}. Thus, we define the inverse, the almost tangent:

$$y = \operatorname{atan} x \longrightarrow x = \arctan y.$$

We extend this function periodically by taking $\operatorname{atan}(x + n\pi) = \operatorname{atan} x$. The domain function is $\operatorname{atan}_{\text{dom}} = \arctan$.

The definitions of the inverse functions (almost exponential and tangent) are justified by Theorem 7.8. We must use the same theorem for obtaining the derivatives. With the definitions introduced here, the proofs of the approximate form of the algebraic properties of these functions are the usual ones.

We cannot prove, however, that the inverse functions, i.e., the almost exponential and tangent, have the right domains, i.e., all finite numbers for the almost exponential and the finite numbers different from $(2n+1)\pi/2$, for the almost tangent. The most one can do, for the almost exponential for instance, is to prove that for any finite number x, there is a $y \approx x$ in its domain (see remark after Theorem 7.8), which is probably sufficient for most theoretical physics. In order to obtain functions defined everywhere, we use Taylor series approximations.

As an example, we take the series for aexp. We observe that, by the ratio test, if x is finite, the series

$$\sum_{i=0}^{n} \frac{x^i}{i!}$$

converges. Since $\log 1 \approx 0$, we have that if $x \approx 0$, $x \in \operatorname{dom} \operatorname{aexp}$, $\operatorname{aexp}(x) \approx 1$. We then prove, by external induction, that if $x \approx n+1$ and $y \approx n$, where n is a finite natural number and $x, y \in \operatorname{dom} \operatorname{aexp}$, then $\operatorname{aexp}(x) \approx \operatorname{aexp}(y) \operatorname{aexp}(1)$, and $\operatorname{aexp}(x)$ is finite. By Taylor's Theorem 7.9, if n is finite

$$\left| \operatorname{aexp}(x) - \sum_{i=0}^{n} \frac{x^i}{i!} \right| \le \frac{\operatorname{aexp}(|x|)|x|^{n+1}}{(n+1)!} + \frac{1}{n}.$$

By overflow, Theorem 9.3, the same is true if $\mu \le \nu$, for a certain infinite ν.

Let $\mu \approx \infty$ and let $m - 1 = \mathrm{li}(k > 2|x|)$. Then m is finite and, by Axiom N 8, $m > 2|x|$. Hence $|x|/n < 1/2$, for $n \geq m$. We have

$$\frac{|x|^\mu}{\mu!} = \frac{|x|^m}{m!}\frac{|x|}{m+1}\frac{|x|}{m+2}\cdots\frac{|x|}{\mu} \leq \frac{x^m}{m!}\frac{1}{2^{\mu-m}} \leq \frac{|2x|^m}{m!}\frac{1}{2^\mu} \approx 0.$$

Therefore, we have proved that if μ is infinite, $\mu \leq \nu$

$$\mathrm{aexp}(x) \approx \sum_{i=0}^{\mu} \frac{x^i}{i!}.$$

Since the series $\sum_{i=0}^{\mu} \frac{x^i}{i!}$ is convergent, we have that this is true if μ is infinite.

We now define the exponential function:

$$\exp x = e^x = \sum_{i=0}^{\nu_0} \frac{x^i}{i!}.$$

Let x be finite. Let $[a, b]$ be a finite interval such that $x \in (a, b)$ and let u be the geometric subdivision of $[a, b]$ of order ν_0 and v the selector for u and log on $[a, b]$. Let $n = \min_{u_k \geq x}$. Then $y = v_n = \log v'_m$. Thus, $y \approx x$ and y is in the domain of aexp. Hence, by Theorem 12.8

$$e^x = \sum_{i=0}^{\nu_0} \frac{x^i}{i!} \approx \sum_{i=0}^{\nu_0} \frac{y^i}{i!} \approx \mathrm{aexp}(y).$$

By the definition of aexp

$$\mathrm{aexp}(y) = z \leftrightarrow y = \log z.$$

Hence, if $e^x \approx z_1$, then $y \approx x$ and $\mathrm{aexp}(y) \approx z_1$. Let $\mathrm{aexp}(y) = v'_m = z$. Then $z \approx z_1$ and $y = \log z \approx \log z_1$. Thus, $x \approx \log z_1$. On the other hand, if $x \approx \log z_1$ and $y = \log z_1$, then by the Taylor approximation of aexp, $z_1 = \mathrm{aexp}(y) \approx e^x$. Hence, we have proved, if x is finite

$$e^x \approx z_1 \leftrightarrow x \approx \log z_1.$$

In the case of the trigonometric functions, it seems simpler to define first an almost sine an almost cosine by the formulas:

$$\mathrm{asin}\, x = \frac{2\,\mathrm{atan}\,\frac{x}{2}}{1 + \mathrm{atan}^2\,\frac{x}{2}}.$$

$$\mathrm{acos}\, x = \frac{1 - \mathrm{atan}^2\,\frac{x}{2}}{1 + \mathrm{atan}^2\,\frac{x}{2}},$$

with their domains the same as the domain of the atan. We then use Taylor series to give a definition of sin and cos on all finite numbers. The procedure is similar to that described above for the exponential function.

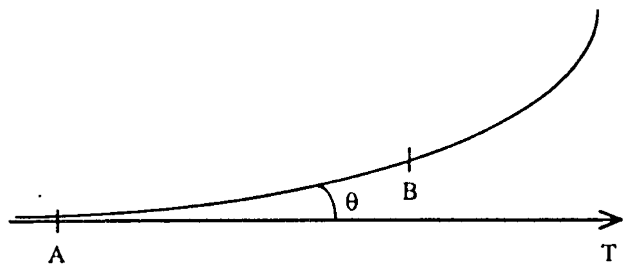

FIGURE 2. Catenary

13. EXAMPLES OF DERIVATIONS OF DIFFERENTIAL EQUATIONS AND INTEGRALS

To substantiate the claim that the foundations we have developed are adequate for much of the mathematics used in theoretical physics, we consider briefly in this section two aspects of ordinary differential equations and integrals: their derivation and their solution. This application to differential equations and integrals is possible and just as easy in system [6]. We could have include this section almost verbatim in [6].

Derivations. Physicists and engineers ordinarily use differentials without apology or explanation in a direct fashion in deriving a differential equation from physical assumptions. Almost without exception no limiting arguments are used, as would be required to make the derivations rigorous without use of infinitesimals.

On the other hand, by using infinitesimals the derivations can be both rigorous and conform to longstanding patterns of inference used in science. Moreover, unabashed use of differentials which will often eliminate any need for derivatives, is usually algebraically simpler. Derivatives are ratios and to carry out complicated manipulations with ratios can be awkward, as is well exemplified in ancient Greek geometry.

We do not offer here a general theory of derivations, but illustrate the points just made by redoing in modified form the derivation of a few differential equations and integrals.

13.1. Equation of the catenary. The catenary is the curve of a uniform cable on rope suspended at two ends, with ends possibly at different elevations.

Let

$$\begin{aligned} A &= \text{point } (x, y) \\ B &= \text{point } (x + dx, y + dy) \\ T &= \text{tension of the cable at point } A \\ ds &= \text{infinitesimal length of curve } AB \\ w\,ds &= \text{weight of } AB \\ \theta &= \text{angle of the tangent at } A. \end{aligned}$$

We shall use the following convention: when saying that a variable y is a function of another variable x, we shall assume that $y = f(x)$, and $dy = df(x, dx)$, for $dx \approx 0$, $dx > 0$ or $dx < 0$. We also assume, for any g, $dg(y) = dg(y, dy) =$

$dg(f(x), df(x, dx)) = g(f(x + dx)) - g(f(x))$. In the case of this example, we assume that variables (in particular, y, T, s, and θ) are functions of x.

We also assume the relation

(1) $$dy \approx \tan\theta\, dx \quad (dx).$$

The cable is in horizontal equilibrium, so that

(2) $$T\cos\theta = (T + dT)\cos(\theta + d\theta)$$

so

$$d(T\cos\theta) = 0,$$

i.e.

(3) $$H = T\cos\theta = \text{constant} = \text{horizontal tension}.$$

The cable is in vertical equilibrium, so the vertical forces must balance

(4) $$T\sin\theta + w\, ds = (T + dT)\sin(\theta + d\theta).$$

Dividing (4) by (2) and rearranging, we get

$$\tan(\theta + d\theta) - \tan\theta = \frac{w\, ds}{T\cos\theta} = \frac{w\, ds}{H}$$

by (3). But the left-hand side is just $d\tan\theta$, so we have

(5) $$d\tan\theta = \frac{w\, ds}{H}$$

By (1)

(6) $$dy^2 \approx \tan^2\theta\, dx^2 \quad (dx^2).$$

Also $ds \approx \sqrt{dx^2 + dy^2} \quad (dx)$, so

$$ds \approx \sqrt{dx^2 + \tan^2\theta\, dx^2} \quad (dx)$$
$$\approx \sqrt{1 + \tan^2\theta}\, dx \quad (dx)$$

Substituting in (5)

$$d\tan\theta \approx \frac{w}{H}\sqrt{1 + \tan^2\theta}\, dx \quad (dx),$$

i.e.

$$\frac{1}{\sqrt{1 + \tan^2\theta}} d\tan\theta \approx \frac{w}{H} dx \quad (dx).$$

The last approximate equation is true if x, $x + dx$ are positive. By the usual rules of integration, we obtain that

$$\sinh^{-1}\tan\theta \approx \frac{w}{H}x + C_1,$$

i.e.

$$\tan\theta\, dx \approx \sinh(\frac{w}{H}x + C_1)\, dx \quad (dx),$$

and so, by (6)

$$dy \approx \sinh(\frac{w}{H}x + C_1)\,dx \quad (dx)$$

Applying the rules for integration again, we obtain

$$y \approx \frac{H}{w}\cosh(\frac{w}{H}x + C_1) + C_2.$$

13.2. Poisson process. As a second example of derivation of a differential equation, we take the Poisson process. Let X_t, $t \in (0, \infty)$ be a nonnegative integer-valued random variable. Intuitively we think of X_t as a counting random variable, so that the value of X_t is the number of counting events that have occurred in the interval $(t_0, t]$. Examples would be the number of telephone calls or number of decaying atoms (as measured by a Geiger counter or similar device).

We assume that $X_{t_0} = 0$, and that the process is one of independent increments and satisfies stationarity, but in deriving the basic differential equation of the process we shall not use these assumptions explicitly. The basic assumptions we do use are these. If $dt > 0$, $dt \approx 0$, then

$$\Pr(X_{t+dt} = n | X_t = n - 1) = \lambda dt \tag{1}$$

and

$$P([X_{t+dt} > n + 1] \,|\, [X_t = n]) = \lambda_n\, dt \leq \gamma\,(dt)^2, \tag{2}$$

where λ_n is an unknown finite positive coefficient and γ is a finite positive number. The explicit use of infinitesimals avoids the limiting arguments and permits the derivation to have an algebraic form, of the sort much used in physics and engineering, but now without any apology for lack of rigor.

Most of the rest of the derivation is elementary probability but in an infinitesimal setting. Let $P_n(t) = \Pr(X_t = n)$. We first deal with the case $n = 0$.

$$\begin{aligned}
\Pr[X_{t+dt} = 0] &= \Pr[X_t = 0] - \Pr([X_{t+dt} > 0] \,|\, [X_t = 0]) \Pr[X_t = 0] \\
&= \Pr[X_t = 0] - \Pr([X_{t+dt} = 1] \,|\, [X_t = 0]) \Pr[X_t = 0] \\
&\quad - \Pr([X_{t+dt} > 1] \,|\, [X_t = 0]) \Pr[X_t = 0] \\
&= \Pr[X_t = 0](1 - \lambda\, dt - \lambda_0\, dt)
\end{aligned}$$

Thus

$$P_0(t + dt) = P_0(t)(1 - \lambda\, dt - \lambda_0\, dt),$$

and so, by (2)

$$|P_0(t + dt) - P_0(t)(1 - \lambda\, dt)| \leq \lambda_0\, dt \leq \gamma\, dt^2.$$

Dividing by dt

$$\frac{|P_0(t + dt) - P_0(t)(1 - \lambda\, dt)|}{dt} \leq \gamma\, dt \approx 0.$$

This is equivalent to

$$P_0(t + dt) \approx P_0(t)(1 - \lambda\, dt) \quad (dt), \tag{3}$$

Suppose, now, that $dt > 0$ and $t > t_0$ and $n > 0$. By the theorem on total probability

$$\begin{aligned}\Pr(X_{t+dt} = n) &= \sum_{i=0}^{n} \Pr(X_{t+dt} = n | X_t = i) \Pr(X_t = i) \\ &= \Pr(X_{t+dt} = n | X_t = n) \Pr(X_t = n) + \lambda dt \Pr(X_t = n-1) \\ &\quad + \sum_{i=0}^{n-2} \lambda_i dt \Pr(X_t = i) \\ &= (1 - \lambda dt - \sum_{i=0}^{n-2} \lambda_i dt) \Pr(X_t = n) + \lambda dt \Pr(X_t = n-1) \\ &\quad + \sum_{i=0}^{n-2} \lambda_i dt \Pr(X_t = i).\end{aligned}$$

Thus, we have

(4) $$P_n(t + dt) = (1 - \lambda dt - \sum_{i=0}^{n-2} \lambda_i dt) P_n(t) + \lambda dt P_{n-1}(t) + \sum_{i=0}^{n-2} \lambda_i dt P_i(t).$$

Substituting t by $t - dt$ in the above derivation, we get

(5) $$P_n(t) = (1 - \lambda dt - \sum_{i=0}^{n-2} \lambda_i dt) P_n(t - dt) + \lambda dt P_{n-1}(t - dt) + \sum_{i=0}^{n-2} \lambda_i dt P_i(t - dt)$$

The following are immediate consequence of the assumptions.

(6) $$\lambda_{n-i} dt \leq \gamma (dt)^2$$

for $2 \leq i \leq n$.

(7) $$0 \leq P_n(\tau) \leq 1$$

where τ is an arbitrary term.

(8) $$P_n(t_0) = 0, \quad \text{if } n \geq 1 \text{ and } P_0(t_0) = 1.$$

We have, by (4), if $dt > 0$

$$P_n(t + dt) = P_n(t) + \lambda dt (P_{n-1}(t) - P_n(t)) + \sum_{i=0}^{n-2} \lambda_i dt (P_i(t) - P_n(t)).$$

By (7) and (8)

$$\left| \sum_{i=0}^{n-2} \lambda_i (P_i(t) - P_n(t)) \right| \leq (n-2) \gamma dt$$

and the right side is an infinitesimal. Hence, we have proved for $dt > 0$

(9). $$P_n(t + dt) \approx P_n(t) + \lambda (P_{n-1}(t) - P_n(t)) dt \quad (dt)$$

We also have, from (5), if $dt > 0$

$$P_n(t) = P_n(t-dt) + \lambda dt(P_{n-1}(t-dt) - P_n(t-dt)) + \sum_{i=0}^{n-2} \lambda_i dt(P_i(t-dt) - P_n(t-dt)),$$

that is

$$P_n(t-dt) = P_n(t) - (\lambda(P_{n-1}(t-dt) - P_n(t-dt))\,dt + \sum_{i=0}^{n-2} \lambda_i(P_i(t-dt) - P_n(t-dt)\,dt).$$

As above, by (7) and (8)

$$\left|\sum_{i=0}^{n-2} \lambda_i(P_i(t-dt) - P_n(t-dt))\right| \leq (n-2)\gamma\,dt,$$

where the right side is an infinitesimal. So

$$P_n(t-dt) \approx P_n(t) - \lambda(P_{n-1}(t-dt) - P_n(t-dt))\,dt \quad (dt).$$

Let $dt < 0$. Changing dt by $-dt$ in the formula above, we get

(10). $$P_n(t+dt) \approx P_n(t) + \lambda(P_{n-1}(t+dt) - P_n(t+dt))\,dt \quad (dt)$$

We shall use, from now on, (9) and (10) instead of (4) and (5). Conditions (9) and (10) give immediately the continuity, because, by (7),

$$\lambda(P_{n-1}(t+dt) - P_n(t+dt))\,dt \leq \lambda dt$$

and

$$\lambda(P_{n-1}(t) - P_n(t))\,dt \leq \lambda dt.$$

So for any $dt > 0$ or $dt < 0$

$$P_n(t+dt) \approx P_n(t).$$

Thus, from (10) for $dt < 0$

(11). $$P_n(t+dt) \approx P_n(t) + \lambda(P_{n-1}(t) - P_n(t))\,dt \quad (dt)$$

Therefore, we just need (11) if $dt < 0$ or $dt > 0$.

By the definition of differentials, we have

$$dP_n(t) = P_n(t+dt) - P_n(t).$$

Thus, from (10), if $dt > 0$ or $dt < 0$

$$dP_n(t) \approx \lambda(P_{n-1}(t) - P_n(t))\,dt \quad (dt).$$

This is the differential equation in our form. For the given initial conditions may be shown to have the solution

$$\Pr(X_t = n) = P_n(t) \approx e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

13.3. Flux in a blood vessel. This is an example of a derivation of a definite integral. Poselle's law for a laminar fluid is

$$v(r) = \frac{P}{4\eta L}(R^2 - r^2),$$

where R is the radius of the blood vessel, L, its length, P, the difference in pressure, η the viscosity of the blood, and $v(r)$, the velocity of the blood at distance r of the center of the vessel. We set $K = P/(4\eta L)$ and the formula becomes $v(r) = K(R^2 - r^2)$. The problem is to calculate the flux of blood (volume per unit time) in the blood vessel. Let $Q(r, s)$ be the flux of blood in the vessel from a distance r of the center to a distance s of the center. Then, we want to calculate $Q(0, R)$. It is clear that we may assume that Q is additive, that is

$$Q(r, s) = Q(r, t) + Q(t, s),$$

for $0 \leq r \leq t \leq s \leq R$. Thus, Condition (1) of the Infinite Sum Theorem 11.5 is satisfied.

Let $A(r, s)$ be the area of the surface of the blood from distance r to s of the center. If we take an infinitesimal increment dr, then $A(r, r + dr)$ is approximately the area of the annulus between r and $r + dr$, i.e.

$$\begin{aligned} A(r, r + dr) &\approx \pi(r + dr)^2 - \pi r^2 \quad (dr) \\ &\approx 2\pi r\, dr + \pi dr^2 \quad (dr). \end{aligned}$$

Since $\pi\, dr^2 \approx 0 \quad (dr)$, we get

(1) $$A(r, r + dr) \approx 2\pi r\, dr \quad (dr).$$

We have that v is decreasing on r. Thus, on the interval $[r, r + dr]$ the maximum velocity is $v(r)$ and the minimum is $v(r + dr)$. The infinitesimal flux $Q(r, r + dr)$ satisfies then

(2) $$v(r + dr)\, A(r, r + dr) \leq Q(r, r + dr) \leq v(r) A(r, r + dr).$$

From the formula for v, we get

$$v(r + dr) = v(r) - K(2r\, dr + dr^2,$$

and so

$$v(r + dr)A(r, r + dr) \approx v(r)A(r, r + dr) \quad (dr).$$

Thus, by the infinitesimal axioms, (2) and (3)

$$Q(r, r + dr) \approx v(r)A(r, r + dr) \quad (dr).$$

Hence, by (1)

$$Q(r, r + dr) \approx 2\pi r v(r)\, dr \quad (dr).$$

Since $2\pi r v(r)$ is continuous on $[0, R]$, by the Infinite Sum Theorem 11.5

$$Q(0, R) \approx \int_0^R 2\pi r v(r).$$

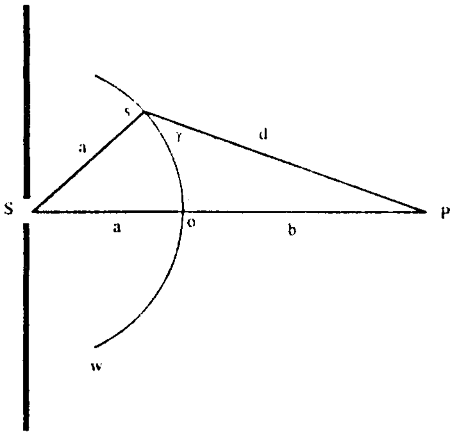

FIGURE 3. Diffraction phenomena

Integrating, we get

$$Q(0, R) \approx \frac{\pi P R^4}{8\eta L}.$$

13.4. Fresnel integrals for diffraction phenomena. Diffraction effects obtained when the source of light or the observing screen are at a finite distance from the diffracting aperture such as a slit are called Fresnel diffraction after the great 19th century French physicist. Now the wave front from a point source at a finite distance is spherical, but in the case of a slit S whose length is very large compared to its width, the envelope of the secondary waves is cylindric, and the slit is the axis of the cylinder. Figure 3 shows the relative geometry of a cross-section perpendicular to the long dimension of the slit. We derive the Fresnel integrals in this setup.

Let the point P be as shown in the figure, and let $f(P)$ be the amplitude of the diffraction wave at P for time t. Consider first the line SP intersecting the wave front at $s = 0$. At a point s on the wave front $w_1 w_2$, the amplitude is $r \sin(2\pi t/T)$, where r is the unknown constant amplitude and T is the period of the wave.

The contribution at P from an infinitesimal element ds of the wave front at $s = 0$ is then

(1) $$df_P(0, t) \approx r\, ds \sin 2\pi \left(\frac{t}{T} - \frac{b}{\lambda}\right) \quad (ds)$$

where λ is the wave length of the monochromatic wave. Now for an infinitesimal element ds at a distance s above $s = 0$ measured along the wave front we have

(2) $$df_P(s, t) \approx r ds \sin 2\pi \left(\frac{t}{T} - \frac{d}{\lambda}\right) = r ds \sin 2\pi \left(\frac{t}{T} - \frac{b + \gamma}{\lambda}\right) \qquad (ds).$$

Expanding the sine term, we get

(3)

$$df_P(s, t) \approx r \sin 2\pi \left(\frac{t}{T} - \frac{b}{\lambda}\right) \cos 2\pi \frac{\gamma}{\lambda} ds - r \cos 2\pi \left(\frac{t}{T} - \frac{b}{\lambda}\right) \sin 2\pi \frac{\gamma}{\lambda} ds \qquad (ds).$$

To obtain the contribution of the wave front from 0 to s on the wavefront we integrate:

$$
\begin{aligned}
f_P(t) \approx\ & r \sin 2\pi \left(\frac{t}{T} - \frac{b}{\lambda}\right) \int_0^s \cos \frac{2\pi}{\lambda}\gamma ds \\
& - r \cos 2\pi \left(\frac{t}{T} - \frac{b}{\lambda}\right) \int_0^s \sin \frac{2\pi}{\lambda}\gamma ds
\end{aligned} \tag{1}
$$

To first approximation, for small angles reflecting small γ,

$$\gamma = d - b = \frac{a+b}{2ab} s^2 \tag{5}$$

Now let

$$s = \sqrt{\frac{ab\lambda}{2(a+b)}}\, v, \tag{6}$$

and so

$$ds \approx \sqrt{\frac{ab\lambda}{2(a+b)}}\, dv. \tag{7}$$

From (5) and (6), we have:

$$\frac{2\pi}{\lambda}\gamma = \frac{2\pi}{\lambda}\,\frac{a+b}{2ab}\,\frac{ab\lambda}{2(a+b)}v^2 = \frac{\pi v^2}{2} \tag{8}$$

Substituting (7) and (8) into (4) we get

$$f_P(t) = r\sqrt{\frac{ab\lambda}{2(a+b)}}\left[\sin 2\pi \left(\frac{t}{T} - \frac{b}{\lambda}\right) \int_0^v \cos \frac{\pi v^2}{2} dv - \cos 2\pi \left(\frac{t}{T} - \frac{b}{\lambda}\right) \int_0^v \sin \frac{\pi v^2}{2} dv\right]$$

The integrals

$$\int_0^v \cos \frac{\pi v^2}{2} dv \quad \text{and} \quad \int_0^v \sin \frac{\pi v^2}{2} dv$$

are known as the Fresnel integrals, and play an important role in a range of diffraction problems.

Solutions. We cannot formulate or prove in our framework quite general existence theorems for ordinary differential equations, but most of the general solutions of the well-known equations of classical physics can be handled in the style familiar in physics and engineering. Most of the solutions to such equations given in handbooks like those of [1] or [21] can be reached by our methods because of the highly constructive nature of the solutions.

14. Consistency

The system of axioms of this paper are all true in the system of [6], and, thus, the present system is also finitarily consistent. Thus, we have

Theorem 14.1. *The system of axioms, rules of inference* (L 1–12, LI 1–2, LS, F 1–31, N 1–8, FU 1–2, I 1–10, INT 1–2, Recursive Definitions 1–4, Calculus Rules 1, 2), *and all definitions is finitarily consistent.*

Proof. Most of the axioms are easy to verify as axioms or theorems of the system in [6]. The only two that need a little more work are INT 1 and INT 2, for the indefinite integral. For those rules, we pick a point c in the interval I and put

$$\int f(x)\,dx = \int_c^x f.$$

Then INT 1 and 2 become theorems.

The finitary proof of consistency in [6] uses finite fragments of the field of rational numbers, which are similar to fragments of a nonarchimedian field. □

If we change Axiom F 30 to $0 \leq 1$ and omit Axiom F 13, one can obtain an absolute proof of consistency, i.e., the proof that there is a formula of the systems that cannot be derived from the axioms. Most of the theorems of this paper are true in the reduced system.

For the proof we use the one-element model with $0 = 1$. In this model the primitive concepts are given the following interpretation:

(1) the relation $<$ is the empty relation,
(2) $0 + 0 = 0 \cdot 0 = 0/0 = \sqrt{0} = 0$,
(3) $\mathcal{N}(0)$ holds,
(4) any function variable f or constant or term is interpreted by the function $\{\langle 0, 0\rangle\}$,
(5) $\min_\varphi = 0$, for any formula φ,
(6) $\mathrm{li}(\tau) = 0$, for any term τ,
(7) $\max_\tau(\nu) = 0$, for any term τ and any ν,
(8) $\mathrm{Inf}(0)$ holds,
(9) $\epsilon_0 = 0$, $\nu_0 = 0$,
(10) $f'(x)$ is the same as $f(x)$,
(11) $\int f(x)\,dx$ is the same as $f(x)$
(12) $\int_a^b f = 0$, for any function f and any a, b.

The various definitions and conditional definitions are also easily shown to be satisfied in this one-element model following the given interpretations for the primitive concepts. Given this model as specified, it is seen at once that the formula $0 < 1$ is not satisfied in the model, and therefore cannot be derived from the axioms, which proves the absolute consistency of the system.

As we mentioned in Section 2 negation can be defined in our system by the formula

$$\neg\varphi \leftrightarrow (\varphi \rightarrow 0 = 1).$$

With negation so defined, we can derive the law of *reductio ad absurdum*:

$$(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi,$$

which, when added to Hilbert's positive propositional calculus constitutes a formulation of the minimal propositional calculus of Kolmogorov [11] and Johansson [9]. Also, as Church points out [7, p. 142], the decision problem for this minimal calculus has a positive solution, which can be easily shown as an extension of classical results of Gentzen and Wajsberg.

The derivation in our system of the law of *reduction ad absurdum* is as follows:

(1)	$\varphi \to \psi$	Premise
(2)	$\varphi \to \neg\psi$	Premise
(3)	$\varphi \to (\psi \to 0 = 1)$	Def. of negation
(4)	φ	Premise
(5)	ψ	Premise
(6)	$\psi \to 0 = 1$	1, 4 M.P.
(7)	$0 = 1$	3, 5 M.P.
(8)	$\varphi \to 0 = 1$	4, 7 C.P.
(9)	$\neg\varphi$	Def. of negation
(10)	$(\varphi \to \neg\psi) \to \neg\varphi$	2, 9 C.P.
(11)	$(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$	1, 10 C.P.

It is obvious that the form of negation embodied in the minimal propositional calculus is much weaker than intuitionistic negation.

All axioms are also true in nonstandard models of analysis, as it is easy to see. Thus, we have relative consistency with analysis (see, for instance, [10], [2, Chapter 1], or [5, Appendix A]).

References

1. M. Abramowitz and I. A. Stegun (editors), *Handbook of mathematical functions*, Dover Publications Inc., New York, 1972. Originally published in 1965.
2. S. Albeverio, J. E. Fenstad, R. Hoegh-Krohn, and T. Lindstrøm, *Nonstandard methods in stochastic analysis and mathematical physics*, Academic Press, New York, 1986.
3. E. Bishop and D. Bridges, *Constructive analysis*, Springer-Verlag, Berlin, 1985.
4. B. Cavalieri, *Geometria indivisibilibus continuorum nova quadam ratione promota*, Bologna, 1635 (2nd edition, 1653).
5. R. Chuaqui, *Truth, possibility and probability. New logical foundations of probability and statistical inference*, North-Holland Mathematics Studies 166, Amsterdam 1991.
6. Foundations of infinitesimal analysis for scientists, a fragment with finitary consistency proof, to appear.
7. A. Church, *Introduction to mathematical logic*, Princeton University Press, Princeton, 1956.
8. J. H. Davenport, *On the integration of algebraic functions*, Lecture Notes in Computer Science, 102, Spinger-Verlag, 1981.

9. I. Johansson, *Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus*, Compositio Mathematica 4 (1936), 119–136.

10. H. J. Keisler, *Foundations of infinitesimal analysis*, Prindle, Weber, and Schmidt, Boston, 1976.

11. A. N. Kolmogorov, *Sur le principe de tertium non datur*, (in Russian, French summary) Recueil Mathématique de la Société Mathématique de Moscou, 32 (1924–1925), 646–667.

12. N. Moler and P. Suppes, *Quantifier-free axioms for constructive plane geometry*, Compositio Mathematica, 20 (1968), 143–152.

13. J. Mycielski, *Analysis without actual infinity*, J. of Symbolic Logic, **46** (1981), 625–633.

14. E. Nelson, *Internal set theory: A new approach to nonstandard analysis*, Bull. of the American Mathematical Society, **83** (1977), 1165–1198.

15. E. Nelson, *Radically elementary probability theory*, Annals of Mathematics Studies, Princeton U. Press, Princeton, NJ, 1987.

16. R. H. Risch, *Implicitly elementary integrals*, Proceedings of the American Mathematical Society, **57** (1976), 1–7.

17. J. F. Ritt, *On the integrals of elementary functions*, Transactions of the American Mathematical Society, **25** (1923), 211–222.

18. L. H. Ryder, *Quantum field theory*, Cambridge University Press, Cambridge, 1986.

19. A. S. Troelstra and D. van Dalen, *Constructivism in mathematics, an introduction*, vols. I & II, North-Holland, Amsterdam, 1968.

20. F. Wattenberg, *Nonstandard analysis and constructivism?*, Studia Logica **47** (1980), 303–309.

21. D. Zwillinger *Handbook of differential equations*, Academic Press, New York, 1989.

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The Theories of the T, tt and wtt R. E. Degrees: Undecidability and Beyond

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Abstract

We discuss the structure of the recursively enumerable sets under three reducibilities: Turing, truth-table and weak truth-table. Weak truth-table reducibility requires that the questions asked of the oracle be effectively bounded. Truth-table reducibility also demands such a bound on the the length of the computations. We survey what is known about the algebraic structure and the complexity of the decision procedure for each of the associated degree structures. Each of these structures is an upper semilattice with least and greatest element. Typical algebraic questions include the existence of infima, distributivity, embeddings of partial orderings or lattices and extension of embedding problems such as density. We explain how the algebraic information is used to decide fragments of the theories and then to prove their undecidability (and more). Finally, we discuss some results and open problems concerning automorphisms, definability and the complexity of the decision problems for these degree structures.

In this paper we will discuss the structure of the recursively enumerable sets, those that can be effectively enumerated, under various reducibilities. The primary reducibility is that of Turing:

$B \leq_T A \equiv$ There is Turing machine Φ which, when equipped with an oracle for A, can compute (the characteristic function of) B, $\Phi^A = B$.

(We refer the reader to Rogers [1967] or Odifreddi [1989] for basic information on recursion theory and any unexplained notation.)

This reducibility is the most general effective one and allows for computations potentially unbounded in both the amount of information they require

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of the oracle and the number of steps they take to converge. Indeed, for various machines, oracles and inputs, $\Phi^A(x)$ may not converge at all. We wish to consider two other reducibilities that restrict access to the oracle by imposing recursive *a priori* bounds on the questions that can be asked, i.e. we specify a recursive function f and require that the computation of $B(x)$ from A via Turing machine Φ use only information about the initial segment of A of length $f(x)$. If this is the only restriction put on the reduction of A to B, the resulting reducibility is called weak truth table, wtt, reducibility: $B \leq_{wtt} A$.

Here too, there is no *a priori* bound on the length of the computation and some computations $\Phi^A(x)$ may still diverge. If, in addition, we recursively bound the length of the computations (and give some default output such as 0 if the computation has not converged within the specified time bound), or equivalently require that $\Phi^A(x)$ converge for every A and x, then we get the more familiar notion of truth table reducibility:

$B \leq_{tt} A \equiv$ There is a recursive function h which for, every x, specifies a truth table $h(x)$ based on elements of size at most $f(x)$ such that $x \in B$ iff A satisfies the truth table given by $h(x)$.

(Bounding the length of the computations in this way obviously implies that they always converge. For the other direction of the equivalence (due to Nerode) look at the tree of all possible computations for any set oracle. If the computations with input x halt along every path (i.e. for every oracle) then, by König's Lemma, the whole tree is finite. Thus we can recursively find a bound on the length of all possible computations on input x for any oracle.)

Each of these reducibilities, r, induces, in the usual way, an equivalence relation on sets with equivalence classes given by $\deg_r(A) = \{B | A \leq_r B \;\&\; B \leq_r A\}$, the class of sets equicomputable (with respect to r-reducibility) with A. The induced partial ordering $\leq_r$ on the equivalence classes defines the associated degree structure $\mathcal{R}_r$. These three partial orderings, $\mathcal{R}_T$, $\mathcal{R}_{tt}$ and $\mathcal{R}_{wtt}$, share several basic algebraic properties:

1. All of the structures are upper semi-lattices with least element **0** (the degree of the recursive sets) and greatest element **1**, the degree of the halting problem, i.e. of the complete r.e. set

$$K = \{\langle x, y\rangle |\; \textit{the } x^{th} \textit{ Turing machine}, \Phi_x, \textit{ halts on input } y\}.$$

2. Every countable partial ordering can be embedded in each structure and so the three structures have the same decidable $\exists$-Theory. (That is, there is an effective procedure for determining which sentences in the language with just ordering which consist of an initial string of existential qualifiers followed by a quantifier free matrix are true in the structure: Any sentence in the language of partial orderings of the form $\exists x_1 \exists x_2 ... \exists x_n \Psi(x_1, x_2, ..., x_n)$ with Ψ quantifier free is true in $\mathcal{R}_r$ iff it is consistent with the theory of partial orderings, i.e. there is a partial ordering of size at most n in which it is true. The truth of this last assertion can clearly be determined effectively.)

The three degree orderings, however, are very different once one goes up even slightly in the complexity of the questions one is considering to either the $\forall\exists$-Theories of the structures (sentences with one alternation of quantifiers, i.e. of the form $\forall y_1 ... \forall y_m \exists x_1 ... \exists x_n \Psi(x_1, ..., x_n, y_1, ..., y_m)$) or even to the extension of embedding problem (when, given two partial orderings $X \subseteq Y$, is it always possible to find, for every embedding f of X into $\mathcal{R}_r$, an extension g of f which maps all of Y into $\mathcal{R}_r$). The archetypic example of such questions is whether the structures are dense. In the first format, this is the question of the truth of the sentence $\forall x \forall y \exists z (x < y \rightarrow x < z < y)$. In the second format, the question is if, for every embedding of the partial order X with two elements $x < y$, there is an extension to the partial order Y with three elements $x < z < y$. The investigation of these sorts of problems has been a source a much of our knowledge about the structures $\mathcal{R}_r$.

Sacks [1964] answered this archetypic problem for $\mathcal{R}_T$ by proving that it is dense. This prompted Shoenfield [1965] to conjecture that $\mathcal{R}_T$ might be dense even as an usl with 0 and 1 (or as one might prefer to say now, saturated with respect to finite sets of quantifier free formulas consistent with the theory of an usl with least and greatest elements 0 and 1). As with Cantor's theorem for dense linear orderings, this conjecture would have implied, by the usual back and forth argument, that $\mathcal{R}_T$ is a model of a theory of usl's which has, up to isomorphism, only one countable model. On general model theoretic grounds, its theory, like that of dense linear orderings, would then be decidable.

The first counterexample to Shoenfield's conjecture was the existence of minimal pairs proven by Lachlan [1966] and Yates [1966]: There exist nonzero r.e. T-degrees **a** and **b** such that there the only r.e. degree below both of them is 0. (Thus the formula $\Psi(x) = (x < \mathbf{a} \;\&\; x < \mathbf{b} \;\&\; x \neq \mathbf{0})$, although consistent with the theory has no realization in $\mathcal{R}_T$.) The constructions of Lachlan and Yates began, in terms of both structural analysis and technology of proofs, the long road to the proof of the undecidability of $\mathcal{R}_T$:

Theorem (Harrington & Shelah [1982]): *$\mathcal{R}_T$ is undecidable, i.e. there is no recursive procedure for determining the truth of sentences (in the language with $\leq_T$) in $\mathcal{R}_T$.*

Proof Plan: Given a Δ^0_2 (or, equivalently, recursive in K) partial ordering $\mathcal{P} = \langle\{p_i | i \in \omega\}, \leq\rangle$, one constructs r.e. degrees $\mathbf{a}$, $\mathbf{a}_i$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ such that

1. $\{\mathbf{a}_i\}$ is the set of maximal degrees $\mathbf{x} \leq_T \mathbf{a}$ such that $\mathbf{c} \not\leq_T \mathbf{b} \vee \mathbf{x}$.
2. $p_i \leq p_j \Leftrightarrow \mathbf{a}_i \leq_T \mathbf{a}_j \vee \mathbf{d}$.

Note that any sentence true in some partial ordering is true in a Δ^0_2 one by an analysis of the standard Henkin completeness proof. Thus this construction provides an interpretation of the theory of partial orderings in that of $\mathcal{R}_T$. The undecidability of the theory of partial orderings then gives the undecidability of $\mathcal{R}_T$. In fact, it suffices to code all finite partial orderings into a structure to show that its theory is undecidable. The proof of this fact relies on the hereditary undecidability of the theory of partial orderings. (A general exposition of these procedures for proving undecidability can be found in Ambos-Spies, Nies and Shore [1992]).

The situation for $\mathcal{R}_{tt}$ is quit different. Indeed, Degtev [1973] and Marchenkov [1975] proved that there is a minimal r.e. tt-degree. The proof they provide, however, is quite indirect and does not lend itself to the construction of other initial segments of $\mathcal{R}_{tt}$. A direct construction of such a degree was found by Fejer and Shore [1989]. This construction was then extended to prove the undecidability of $\mathcal{R}_{tt}$:

Theorem (Haught & Shore [1990]): *For every $n \in \omega$, there are r. e. tt degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that, in $\mathcal{R}_{tt}$,* $[\mathbf{a},\mathbf{b}]$ $(= \{\mathbf{x} | \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\})$ *is isomorphic to the lattice of all equivalence relations on a set of n elements. Indeed,* $\mathbf{a}$ *and* $\mathbf{b}$ *can be constructed so that* $\{0\} \cup [\mathbf{a},\mathbf{b}]$ *is an initial segment of $\mathcal{R}_{tt}$ (or even of all the tt-degrees).*

Corollary (Haught & Shore [1990]): *$\mathcal{R}_{tt}$ is undecidable.*

At first glance, this approach to the undecidability of $\mathcal{R}_{tt}$ seems somewhat ad hoc or forced. Why not prove that every finite lattice (or at least some reasonable collection of them) is isomorphic to an initial segment of $\mathcal{R}_{tt}$ and so get the undecidability of its theory. This after all was the route to the undecidability of the theory of the r-degrees of all sets for many reducibilities including Turing, tt and wtt. (See Lachlan [1968] for the distributive lattices and Lerman [1971] for all finite ones and Nerode and Shore [1980] for transferring the results to reducibilities other than Turing.) As it turns out, this is not possible. Various restrictions on the initial segments of $\mathcal{R}_{tt}$ have been found by Harrington and Haught [1993] including the following: Every finite initial segment of $\mathcal{R}_{tt}$ has a least nonzero element.

Now $\mathcal{R}_{wtt}$ lies between $\mathcal{R}_T$ and $\mathcal{R}_{tt}$ in many ways that defeat both types of attempts at proving undecidability. Like $\mathcal{R}_T$, $\mathcal{R}_{wtt}$ is dense and has minimal pairs (the same proofs work), but $\mathcal{R}_{wtt}$ is much more homogeneous than $\mathcal{R}_T$:

1. In $\mathcal{R}_T$ some degrees are branching (i.e. they are the infimum of two other degrees) but not all; in $\mathcal{R}_{wtt}$ all degrees are branching. (See Lachlan [1966], Yates [1966] and Ladner and Sasso [1975].)

2. In $\mathcal{R}_T$ some degrees b can be split over all lower degrees (i.e. for every $\mathbf{c} < \mathbf{b}$ there are $\mathbf{b}_0$ and $\mathbf{b}_1$ such that $\mathbf{c} < \mathbf{b}_0, \mathbf{b}_1 < \mathbf{b}$ and $\mathbf{b}_0 \vee \mathbf{b}_1 = \mathbf{b}$) but not all; in $\mathcal{R}_{wtt}$ every degree $\mathbf{b}$ splits over every $\mathbf{c} < \mathbf{b}$. (See Sacks [1963], Lachlan [1975] and Ladner and Sasso [1975].)

The most striking algebraic difference between the structures is that $\mathcal{R}_{wtt}$ is distributive (as an usl):

$$\textit{If } \mathbf{a}_0 \vee \mathbf{a}_1 \geq \mathbf{b}, \textit{ then } (\exists \mathbf{b}_i \leq \mathbf{a}_i)(\mathbf{b}_0 \vee \mathbf{b}_1 = \mathbf{b}).$$

(See Lachlan [1972] and Stob [1983].) (To see that this corresponds to the notion of distributivity in a lattice, suppose we actually had a lattice structure and consider $c = (a_0 \vee a_1) \wedge b$. Distributivity would say that $c = (a_0 \wedge b) \vee (a_1 \wedge b)$. Thus the required degrees would be $b_0 = (a_0 \wedge b)$ and $b_1 = (a_1 \wedge b)$.) On the other hand, both basic nondistributive lattices are embeddable (as lattices) in $\mathcal{R}_T$ (Lachlan [1972]).

As Shoenfield pointed out to me in 1984, the proofs of undecidability of $\mathcal{R}_T$ can not work for $\mathcal{R}_{wtt}$. The type of constructions used in the proofs for $\mathcal{R}_T$ inherently produce unbounded Turing reductions. The problem is that they are tree arguments at the level of $\mathbf{0}'''$ which directly build the required reductions in complicated ways. Moreover, as Stob [1983] also remarks, the codings themselves are inherently nondistributive. (In a dense, distributive usl there is no maximal $x < a$ such that $c \not\leq x \vee b$, i.e. for every $x < a$, and every b and c with $c \leq a \vee b$ but $c \not\leq x \vee b$, there is a y such that $x < y < a$ and $c \not\leq y \vee b$: By hypothesis, $x \vee b < a \vee b = a \vee b \vee c$; by density, $\exists d(x \vee b < d < a \vee b)$; by distributivity, $(\exists y < a)(y \vee b = d)$; finally, as $x < d$ and $c \not\leq d$, we may take $y \vee x$ to be the required counterexample to maximality.)

There are really two problems:

1. Find an "easier" proof that $\mathcal{R}_T$ is undecidable.
2. Prove that $\mathcal{R}_{wtt}$ is undecidable.

Two years ago two answers to the first problem were found. Both had the same basic plan as the Harrington & Shelah proof: Find a definition $\Phi(x, \vec{a})$ from parameters $\vec{a}$ such that, for enough partial orderings $\mathcal{P}$, there are degrees $\vec{\mathbf{a}}$ and $\mathbf{d}$ such that $\langle \{\mathbf{x} \vee \mathbf{d} | \Phi(\mathbf{x}, \vec{\mathbf{a}})\}, \leq_T \rangle \cong \mathcal{P}$:

Slaman & Woodin [1994]: $\Phi(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c}) \equiv x$ is minimal $\leq \mathbf{a}$ such that $\mathbf{c} \leq_T \mathbf{x} \vee \mathbf{b}$; "enough" = all Δ^0_2.

Ambos-Spies & Shore [1993]: $\Phi(\mathbf{x}, \mathbf{a}, \mathbf{b}) \equiv x$ is maximal such that there is a $\mathbf{b}$ with $\mathbf{b} \wedge \mathbf{x} = \mathbf{a}$; "enough" = all finite.

The second construction supplies a particularly simple proof that uses only the branching and nonbranching degree constructions (as in Soare [1987, IX]) in a standard $\mathbf{0}''$ priority argument. However, in the setting of the wtt-degrees, the nontriviality of either of these definable sets also violates distributivity.

More recently, a quite different approach to the undecidability of $\mathcal{R}_{wtt}$ has been found:

Theorem (Ambos-Spies, Nies & Shore [1992]): *$\mathcal{R}_{wtt}$ is undecidable.*

Proof Plan: 1. (Ambos-Spies & Soare [1989]): There exists a uniformly r.e. sequence of sets A_i such that, in both $\mathcal{R}_T$ and $\mathcal{R}_{wtt}$, their degrees $\mathbf{a}_i$ are pairwise minimal pairs but no one of them bounds a minimal pair.

2. The ideals $\mathcal{I}$ of r.e. wtt-degrees with a uniformly r.e. (or equivalently a Σ^0_3) sequence of representatives are precisely those with exact pairs $\mathbf{x}$ and $\mathbf{y}$, i.e. $\mathcal{I} = \{\mathbf{z} | \mathbf{z} \leq_{wtt} \mathbf{x} \ \& \ \mathbf{z} \leq_{wtt} \mathbf{y}\}$.

3. By an algebraic argument, the distributivity of $\mathcal{R}_{wtt}$ now guarantees that the set of degrees $\mathcal{A} = \{\mathbf{a}_i\}$ given in (1) is independent (no element is below the join of any finite number of other elements of the set) and definable from the exact pair for the ideal it generates. It then follows that the class of subsets

of $\mathcal{A}$ which generate ideals determined by exact pairs is isomorphic to $\mathcal{E}^3$ the lattice of all Σ_3^0 subsets of the natural numbers $\mathcal{N}$:

$$\langle\{\mathcal{C} \subseteq \mathcal{A} | \exists \mathbf{x}, \mathbf{y} \forall \mathbf{z}(\mathbf{z} \in \mathbf{C} \Leftrightarrow \mathbf{z} < \mathbf{x} \ \& \ \mathbf{z} > \mathbf{y}\}, \subseteq\rangle \cong \langle \mathcal{E}^3, \subseteq \rangle.$$

4. (Herrmann [1983] and [1984]): $\mathcal{E}$, the lattice of all r.e. (Σ_1^0) subsets of $\mathcal{N}$ and indeed, for each n, $\mathcal{E}^n$, the lattice of all Σ_n^0 subsets of $\mathcal{N}$, is hereditarily undecidable. Thus any structure in which we can interpret $\mathcal{E}^n$ with parameters is undecidable. Of course, the above steps show that we can interpret $\mathcal{E}^3$ in $\mathcal{R}_{wtt}$ using as parameters the exact pair defining the ideal generated by $\mathcal{A}$.

We must admit, however, that the proofs of the results used here from Ambos-Spies & Soare [1989] and Herrmann [1983] and [1984] are quite difficult and so we can hardly claim to have an elementary proof of the undecidability of $\mathcal{R}_{wtt}$. In addition, this proof does not work for $\mathcal{R}_T$ and so we still have no uniform proof for the two structures.

We must now explain the word "beyond" of our title. We have in mind several aspects of the theories of the degree structures that we are discussing. First, what more can we about the complexity of the theories than that they are undecidable. As recursion theorists we are not satisfied simply with the assertion that they are not recursive. We want to know the precise degree of the theory of each structure; to characterize their "true theories". For both the r.e. Turing and truth-table degrees the theories of their structures are as complicated as possible. Of course, both are definable in first order arithmetic and so are reducible to the true theory of $\mathcal{N}$, the natural numbers with addition and multiplication. The degree of this theory is that of $\emptyset^{(\omega)} = \{\langle x, n\rangle | n \in \emptyset^{(n)}\}$, the recursive join of all finite iterations of the halting problem. This is also the degree of each of these theories. Indeed they are 1-1 equivalent. (This is equivalent to the existence of a recursive permutation of $\mathcal{N}$ that takes one set of sentences to the other.)

Theorem (Harrington & Slaman; Slaman & Woodin [1994]):

$$Th(\mathcal{R}_T) \equiv_{1-1} Th(\mathcal{N}, +, x, 0, 1) \equiv_{1-1} 0^{(\omega)}.$$

Theorem (Nies & Shore [1993]):

$$Th(\mathcal{R}_{tt}) \equiv_{1-1} Th(\mathcal{N}, +, x, 0, 1) \equiv_{1-1} 0^{(\omega)}.$$

Proof Plan: In addition to coding models of arithmetic, we must definably pick out some (codes for) standard models. The proofs for $\mathcal{R}_T$ use the previous codings and pick out some standard models as ones whose natural numbers are embeddable in all other models. It uses among other ideas the definability of prompt simplicity (a property of enumerations of r.e. sets) in degree theoretic terms (Ambos-Spies et al. [1984]). The proof for $\mathcal{R}_{tt}$ extends the previous embedding results to include certain recursive lattices of equivalence relations that are used to code nicely generated models of arithmetic. The standard ones are then picked out as the ones all of whose proper initial segments which are defined by exact pairs have greatest elements. It need a new exact pair theorem for $\mathcal{R}_{tt}$: If I is a Σ_3^0 ideal in $\mathcal{R}_{tt}$ and every member of I is strictly below K in Turing degree, then $\mathcal{I}$ has an exact pair in $\mathcal{R}_{tt}$.

Along these lines, we mention two, perhaps related, open questions:

Question: $Th(\mathcal{R}_{wtt}) \equiv_{1-1} Th(\mathcal{N}, +, x, 0, 1) \equiv_{1-1} 0^{(\omega)}$?

Question: $Th(\mathcal{E}) \equiv_{1-1} Th(\mathcal{N}, +, x, 0, 1) \equiv_{1-1} 0^{(\omega)}$?

Another measure of the complexity of a theory is the number of 1-types consistent with it. The results of Ambos-Spies and Soare [1989] show that $\aleph_0$ many 1-types are realized in $\mathcal{R}_T$ and $\mathcal{R}_{wtt}$ while those of Haught and Shore [1990] give the same result for $\mathcal{R}_{tt}$. The proof of undecidability of $\mathcal{R}_T$ in Ambos-Spies and Shore [1993] also shows that its theory has as many 1-types as possible, 2^ω.

Question: Are there continuum many 1-types over the theories of $\mathcal{R}_{tt}$ and $\mathcal{R}_{wtt}$?

Finally, we come to our last topic beyond undecidability, the related issues of definability and automorphisms. There are one or two examples of classes of r.e. degrees with natural nonorder theoretic definitions which are definable from the ordering on degrees:

Theorem (Ambos-Spies, Jockusch, Shore & Soare [1984]): *The promptly simple r.e. Turing degrees are the noncappable ones (i.e. those degrees* $\mathbf{a}$ *such that there is no* $\mathbf{b}$ *with* $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$*).*

Theorem (Downey & Shore [1993]): *The* low_2 *r.e. tt-degrees (i.e. those* $\mathbf{a}$ *such that* $\mathbf{a}'' = \mathbf{0}''$*) are precisely those with minimal covers (i.e. those* $\mathbf{a}$ *such that there is a* $\mathbf{b} < \mathbf{a}$ *with no* $\mathbf{c}$ *between* $\mathbf{a}$ *and* $\mathbf{b}$*).*

There are some hopes for defining the low_2 r.e. T-degrees as well as Slaman and Shore [1990], [1993] have definably separated the low_2 from the high degrees in $\mathcal{R}_T$.

Question: Is any degree other than 0 and $0'$ definable in any of these structures? Are any of the jump classes definable in $\mathcal{R}_T$?

The last issue we want to address is the problem of the existence of automorphisms. A purely algebraic argument based on distributivity supplies us with isomorphic intervals in R_{wtt}. (Suppose $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$. The map taking $\mathbf{x} \in [\mathbf{c}, \mathbf{a}]$ to $\mathbf{x} \vee \mathbf{b} \in [\mathbf{b}, \mathbf{a} \vee \mathbf{b}]$ is an isomorphism. It is onto by a direct application of distributivity. To see that it is one-one, suppose that $\mathbf{x} < \mathbf{y}$ but $\mathbf{x} \vee \mathbf{b} = \mathbf{y} \vee \mathbf{b}$. As $\mathbf{y} < \mathbf{x} \vee \mathbf{b}$, there is a $\mathbf{d} < \mathbf{b}$ such that $\mathbf{d} \vee \mathbf{x} = \mathbf{y}$. Note, however, that as $\mathbf{d} < \mathbf{y}$, $\mathbf{d} < \mathbf{a}$. Thus $\mathbf{d} < \mathbf{a}, \mathbf{b}$ and so $\mathbf{d} \leq \mathbf{c}$. As this would imply that $\mathbf{d} \vee \mathbf{x} = \mathbf{x}$, we have the desired contradiction.) Similarly, the initial segment results for $\mathcal{R}_{tt}$ supply isomorphic intervals. Otherwise, almost nothing is known about the possible existence of automorphisms for any of the structures. This leaves us with the obvious questions:

Question: Are there any nontrivial automorphisms of $\mathcal{R}_T$, $\mathcal{R}_{tt}$ or $\mathcal{R}_{wtt}$? Indeed, are there any nontrivial isomorphic initial segments of $\mathcal{R}_T$ or $\mathcal{R}_{wtt}$?

There is, however, a quite remarkable result connecting this problem with that for the Turing degrees of all sets:

Theorem (Slaman & Woodin [1994] see also Slaman [1991]): *If $\mathcal{R}_T$ is rigid, i.e. has no nontrivial automorphisms, then so is $\mathcal{D}_T$ the structure of all the Turing degrees.*

The most intriguing suggestion is Harrington's far reaching proposal that $\mathcal{R}_T$ might be interdefinable (or biinterpretable) with (the standard model of) arithmetic, that is not only can we define the standard model of arithmetic in $\mathcal{R}_T$ but we can define a map taking each r.e. degree **a** to (a code for) an index e for a representative $W_e \in \mathbf{a}$ in the model. Now Simpson [1977] and Shore [1982] provide such outright (parameterless) interpretations of second order arithmetic in $\mathcal{D}'_T$ and $\mathcal{D}_T$ respectively that are correct on a cone (i.e. on the set of degrees above a fixed degree $\mathbf{z}$). Slaman & Woodin prove the above result on rigidity by constructing such an interpretation which is correct for all of $\mathcal{D}_T$ from r.e. parameters.(See Slaman [1991], where Slaman and Woodin conjecture that this proposal is true, for a discussion of this notion in various degree structures and many applications.) In particular, a proof of the interdefinability of first order arithmetic and $\mathcal{R}_T$ would show that every r.e. Turing degree is definable in $\mathcal{R}_T$, that $\mathcal{R}_T$, $\mathcal{D}_T$ and many other degree structures are rigid and indeed that $\mathcal{D}_T$ is interdefinable with second order arithmetic. We are thus lead to our final open problem, or perhaps better, program:

OPEN PROBLEM (PROGRAM): Work towards proving the interdefinability of $\mathcal{R}_T$ ($\mathcal{R}_{tt}$ and $\mathcal{R}_{wtt}$) and first order arithmetic!

References

Ambos-Spies, K., Jockusch, C. G., Shore, R. A. and Soare, R. I. [1984], An algebraic decomposition of the recursively enumerable degrees and the coincidence of several degree classes with the promptly simple degrees, *Trans. Am. Math. Soc.* **281**, 109-128.

Ambos-Spies, K., Nies, A. and Shore, R. A. [1992], The theory of the recursively enumerable weak truth-table degrees in undecidable, *J. Symb. Logic* **57**, 864-874.

Ambos-Spies, K. and Shore, R. A, Undecidability and 1-types in the recursively enumerable degrees [1993], *Ann. Pure and Applied Logic*, to appear.

Ambos-Spies, K. and Soare. R. I. [1989], The recursively enumerable degrees have infinitely many one-types, *Ann. Pure and Applied Logic* **44**, 1-23.

Downey, R. and Shore, R. A. [1993], Degree theoretic definitions of the low_2 recursively enumerable sets, to appear.

Harrington, L. and Haught, C. [1993], Limitations on initial segment embeddings in the r.e. tt-degrees, to appear.

Harrington, L. and Shelah, S. [1982], The undecidability of the recursively enumerable degrees, *Bull. Am. Math. Soc.* (N. S) **6**, 79-80.

Haught, C. and Shore, R. A. [1990], Undecidability and initial segments of the (r.e.) tt-degrees, *J. Symb. Logic* **55**, 987-1006.

Herrmann, E. [1983], Definable Boolean pairs and the lattice of recursively enumerable sets, *Proc. Conf. Model Th.*, Seminarberichte **49**, Sektion Mathematik, Humboldt-Universität, Berlin, 42-67.

Herrmann, E. [1984], The undecidability of the elementary theory of the lattice of recursively enumerable sets (abstract), in *Proc. 2^{nd} Frege Conference at Schwerin, GDR, 1984*, G. Weschung, ed., Akademie-Verlag, Berlin, Math. Forschung **20**, 66-72.

Lachlan, A. [1966], Lower bounds for pairs of recursively enumerable degrees, *Proc. London Math. Soc.* 16, 537-569.

Lachlan, A. [1968], Distributive initial segments of the degrees of unsolvability, *Z. Math. Logik Grundlag. Math.* **14**, 457-472.

Lachlan, A. [1972], Embedding nondistributive lattices in the recursively enumerable degrees, in *Conf. in Math. Logic, London (1970)*, W. Hodges, ed., LNMS **255**, Springer-Verlag, Berlin, 149-177.

Lachlan, A. [1975], A recursively enumerable degree which will not split over all lesser ones, *Ann. Math. Logic* **9**, 307-365.

Ladner, R. E. and Sasso, L. P. [1975], The weak truth table degrees of recursively enumerable sets, *Ann. Math. Logic* **8**, 429-448.

Lerman, M. [1971], Initial segments of the degrees of unsolvability, *Ann. Math.* (2) **93**, 365-389.

Nerode, A. and Shore, R. A. [1980], Second order logic and first order theories of reducibility orderings, in *The Kleene Symposium*, J. Barwise, H. J. Keisler and K. Kunen, eds., North-Holland, Amsterdam, 181-200.

Nies, A. and Shore, R. A. [1993], Interpreting true arithmetic in the theory of the r.e. truth-table degrees, to appear.

Odifreddi, P. [1989], *Classical Recursion Theory*, North-Holland, Amsterdam.

Rogers H. Jr. [1967], *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York (reprinted MIT Press, Cambridge, Mass., 1988).

Sacks, G. E. [1963], On the degrees less than $\mathbf{0}'$, *Ann. Math.* (2) **77**, 211-231.

Sacks, G. E. [1964], The recursively enumerable degrees are dense, *Ann. Math.* (2) **80**, 300-312.

Shoenfield, J. R. [1965], Applications of model theory to degrees of unsolvability, in *Symposium on the Theory of Models*, J. W. Addison, L. Henkin and A. Tarski eds., North-Holland, Amsterdam, 359-363.

Shore, R. A. [1982], On homogeneity and definability in the first order theory of the Turing degrees, *J. Symb. Logic* **47**, 8-16.

Shore, R. A. and Slaman, T. [1990], Working below a low_2 recursively enumerable degree, Archive for Math. Logic 29, 201-211.

Shore, R. A. and Slaman, T. [1993], Working below a high recursively enumerable degree, *J. Symb. Logic*, to appear.

Simpson, S. [1977], First-order theory of the degrees of recursive unsolvability, *Ann. Math.* (2) **105**, 121-139.

Slaman, T. [1991], Degree structures, in *Proc. Int. Congress of Math., Kyoto, 1990*, Springer-Verlag, Tokyo, 303-316.

Slaman, T. and Woodin, H. [1994], *Definability in Degree Structures*, in preparation.

Soare, R. I. [1987], *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin.

Stob, M. [1983], wtt-degrees and T-degrees of r.e. sets, *J. Symb. Logic* **48**, 921-930.

Yates, C. E. M. [1966], A minimal pair of recursively enumerable degrees, *J. Symb. Logic* **31**, 159-168.

Ehrenfeucht-Fraïssé-games and nonstructure theorems

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Abstract

We describe here some aspects of the research in model theory, which has been carried out by the Helsinki Logic Group during the last few years. Our exposition combines uncountable models, Ehrenfeucht-Fraïssé-games and infinitary logic. It turns out that in the theory of uncountable structures and uncountable infinitary logic, there is an analogue to some aspects of the interplay between $L_{\infty\omega}$ and countable structures. The most interesting results in this theory are however perhaps those related to stability theory and algebra, in which a drastic difference to the countable situation can be seen. There is an excellent overview of the work of the Helsinki Group by Väänänen, see [17]. The work discussed here is due to (in alphabetical order) Heikki Heikkilä, Taneli Huuskonen, Tapani Hyttinen, Alan Mekler, Juha Oikkonen, Saharon Shelah, Heikki Tuuri and Jouko Väänänen.

1 Background

Infinitary logic gives an elegant and well-understood way of approximating the relation of being isomorphic among countable structures. Let $\mathcal{A}$ and $\mathcal{B}$

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be two countable structures. They are isomorphic, if and only if they are *partially isomorphic*, $\mathcal{A} \simeq_{\surd} \mathcal{B}$, i.e., there is a set I of partial isomorphisms between substructures of $\mathcal{A}$ and $\mathcal{B}$ which can be extended within I "back and forth". The latter relation is by *Karp's theorem* equivalent to that the two structures are elementarily equivalent in the infinitary logic $L_{\infty\omega}$,

$$\mathcal{A} \equiv_{\infty\omega} \mathcal{B}.$$

(See [2] and [9].) The set of sentences of $L_{\infty\omega}$ can be filtered according to a notion of quantifier rank which assigns an ordinal to every formula. Thus in connection to every ordinal α, we have a relation

$$\mathcal{A} \equiv^{\alpha}_{\infty\omega} \mathcal{B},$$

which holds exactly when the structures $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of quantifier rank $\leq \alpha$. This means that the countable structures $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, if and only if $\mathcal{A} \equiv^{\alpha}_{\infty\omega} \mathcal{B}$ holds for all α. There is a refined version of Karp's theorem, according to which $\mathcal{A} \equiv^{\alpha}_{\infty\omega} \mathcal{B}$ holds, if and only if $\mathcal{A}$ and $\mathcal{B}$ are α-partially isomorphic in a certain sense, $\mathcal{A} \simeq^{\alpha}_{\surd} \mathcal{B}$. (See [2].)

Infinitary logic gives also *invariants* for countable structures. If $\mathcal{A}$ is a countable structure (over a countable language) then by *Scott's theorem*, there is a countable ordinal α which satisfies for every structure $\mathcal{B}$ that $\mathcal{A} \equiv^{\alpha}_{\infty\omega} \mathcal{B}$ implies that the two structures are isomorphic. The smallest such α is called the *Scott height* of $\mathcal{A}$. Moreover, the structure $\mathcal{A}$ has a *Scott sentence* $\sigma(\mathcal{A})$ in $L_{\omega_1\omega}$ which is such that for every countable $\mathcal{B}$, $\mathcal{B} \models \sigma(\mathcal{A})$ implies $\mathcal{A} \simeq \mathcal{B}$.

After recalling these well-known properties of countable structures we shall discuss the question whether this picture can be carried over for uncountable structures.

2 Ehrenfeucht-Fraïssé-games

We consider here only structures of cardinality $\leq \omega_1$ for the sake of concreteness. Let $\mathcal{A}$ and $\mathcal{B}$ be two structures. The Ehrenfeucht-Fraïssé-game $EF(\mathcal{A}, \mathcal{B})$ of length ω_1 is defined as follows. In it there are two players $\forall$ and $\exists$ which make the following kind of moves in ω_1 rounds. On round 0, player $\forall$ first picks an element x_0 from one of the structures ($\mathcal{A}$ or $\mathcal{B}$). Then $\exists$ replies with another element y_0 from the other structure ($\mathcal{B}$ or $\mathcal{A}$). Similarly, on round ν player player $\forall$ first picks an element x_ν from one of the structures and then $\exists$ replies with an element y_ν from the other.

When all the ω_1 rounds have been played, the players have produced an ω_1-sequence $(a_\nu)_{\nu<\omega_1}$ of elements of $\mathcal{A}$ and an ω_1-sequence $(b_\nu)_{\nu<\omega_1}$ of elements of $\mathcal{B}$; here a_ν is that one of x_ν and y_ν which is an element of $\mathcal{A}$ and b_ν that one which is and element of $\mathcal{B}$. Player $\exists$ *wins* if $a_\nu \mapsto b_\nu$ induces a partial isomorphism between the two structures. Otherwise $\forall$ *wins*. The notion of a *winning strategy* is defined in the obvious way.

The game $EF(\mathcal{A}, \mathcal{B})$ determines whether $\mathcal{A}$ and $\mathcal{B}$ are isomorphic or not and it is *determined*, i.e., one of the two players has a winning strategy. Indeed, if $\mathcal{A} \simeq \mathcal{B}$, then $\exists$ can win by playing according to any fixed isomorphism. If $\mathcal{A} \not\simeq \mathcal{B}$, then $\forall$ can enumerate the union $A \cup B$ of the domains of the structures and player $\exists$ cannot win, since otherwise the moves of $\exists$ would induce an isomorphism.

Notice that we relied here very heavily on the assumption that the cardinalities of $\mathcal{A}$ and $\mathcal{B}$ are at most ω_1. If we consider models of higher cardinality, then the situation becomes more complicated, as the following result due to Mekler, Shelah and Väänänen from [12] shows.

Theorem 1 *1. There are models $\mathcal{A}$ and $\mathcal{B}$ of cardinality ω_3 for which the Ehrenfeucht-Fraïssé-game $EF(\mathcal{A}, \mathcal{B})$ of length ω_1 is not determined.*

2. It is consistent relative to the consistency of a measurable cardinal that the game $EF(\mathcal{A}, \mathcal{B})$ of length ω_1 is determined for all models $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\leq \omega_2$.

3. It is consistent relative to the consistency of ZFC that the game $EF(\mathcal{A}, \mathcal{B})$ of length ω_1 is not determined for some models of cardinality $\leq \omega_2$.

Since the full game $EF(\mathcal{A}, \mathcal{B})$ of length ω_1 is not very interesting in studying structures of cardinality $\leq \omega_1$, we shall consider suitable shorter games. They lead us to some very interesting questions.

Given a countable ordinal α, we denote by $EF_\alpha(\mathcal{A}, \mathcal{B})$ the truncated version of $EF(\mathcal{A}, \mathcal{B})$ in which the players go through only α rounds and produce α-sequences $(a_\nu)_{\nu<\alpha}$ and $(b_\nu)_{\nu<\alpha}$ of the two structures.

If for example, $\mathcal{A} = (\mathcal{BQ}, \leq)$ and $\mathcal{B} = (\mathcal{BR}, \leq)$, then player $\exists$ has a winning strategy in $EF_\omega(\mathcal{A}, \mathcal{B})$, but $\forall$ has a winning strategy in $EF_{\omega+1}(\mathcal{A}, \mathcal{B})$.

This suggests the idea that we might measure the similarity of two structures $\mathcal{A}$ and $\mathcal{B}$ by the length of those versions of the Ehrenfeucht-Fraïssé-game in which player $\exists$ has a winning strategy.

In case $\alpha = \omega$ we obtain the usual Ehrenfeucht-Fraïssé-game much discussed in the literature. Especially, player $\exists$ has a winning strategy in $EF_\omega(\mathcal{A}, \mathcal{B})$, if and only if $\mathcal{A} \underset{\surd}{\simeq} \mathcal{B}$. The truncated versions $EF_n(\mathcal{A}, \mathcal{B})$ of $EF_\omega(\mathcal{A}, \mathcal{B})$, when put together characterize elementary equivalence in $L_{\omega\omega}$ (or more generally in the portion of $L_{\infty\omega}$ containing only sentences of finite quantifier rank). To capture also infinite quantifier ranks, we need richer approximations of $EF_\omega(\mathcal{A}, \mathcal{B})$. We give here a general definition which is interesting also for the full game $EF(\mathcal{A}, \mathcal{B})$ of length ω_1.

Let T be a tree with only countable branches. Such trees will be called *bounded* below. The *approximation* $EF_T(\mathcal{A}, \mathcal{B})$ is like $EF(\mathcal{A}, \mathcal{B})$ with the addition that at the beginning of each round α, player $\forall$ has to play an element $t_\alpha \in T$ in such a way that $t_0 < t_1 < \ldots < t_\alpha < \ldots$. If this is not possible, then the game ends and the winner is decided by inspecting the sequences $(a_\nu)_{\nu<\alpha}$ and $(b_\nu)_{\nu<\alpha}$. Notice that every play of $EF_T(\mathcal{A}, \mathcal{B})$ is countable.

We say that $\mathcal{A}$ and $\mathcal{B}$ are *T-equivalent* and write $\mathcal{A} \simeq_T \mathcal{B}$, if player $\exists$ has a winning strategy in $EF_T(\mathcal{A}, \mathcal{B})$.

In case $T = \alpha$ consists of a single α-branch, the game $EF_T(\mathcal{A}, \mathcal{B})$ is essentially the truncated game $EF_\alpha(\mathcal{A}, \mathcal{B})$ defined above. If $T = B_\alpha$ is the tree of descending sequences of the ordinal α (ordered according to the initial segment relation), then $EF_T(\mathcal{A}, \mathcal{B})$ is a game theoretic version of Karp's characterization of elementary equivalence in $L_{\infty\omega}$ up to quantifier rank α, in the sense that $\mathcal{A} \underset{\surd}{\simeq^\alpha} \mathcal{B}$ holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are T-equivalent. Especially, two countable structures $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, if and only if they are B_α-equivalent for all ordinals α.

A similar result due to Hyttinen [5] holds also for $EF(\mathcal{A}, \mathcal{B})$ and all bounded trees of cardinality $\leq \omega_1$.

Theorem 2 *Assume the CH. Two structures $\mathcal{A}$ and $\mathcal{B}$ (of cardinality $\leq \omega_1$) are isomorphic, if and only if they are T-equivalent for all bounded trees T of cardinality $\leq \omega_1$.*

This means that we have here a theory of approximations of isomorphism among structures of cardinality $\leq \omega_1$. We shall see below that every one of the relations $\simeq_T$ is strictly weaker than $\simeq$.

3 Infinitary logic

The approximations of the Ehrenfeucht-Fraïssé-game are closely connected to infinitary logic. Indeed, the logic $L_{\infty\omega_1}$ has an extension $M_{\infty\omega_1}$ in which the sentences are labelled trees (T_ϕ, l_ϕ) where T_ϕ is the *syntax tree* of ϕ and l_ϕ assigns to each node $t \in T_\phi$ a label $l_\phi(t)$ where

1. $l_\phi(t)$ is $\bigwedge$ or $\bigvee$ if t has more than one immediate successor;
2. $l_\phi(t)$ is or the form $\forall x$ or $\exists x$ if t has exactly one immediate successor;
3. $l_\phi(t)$ is an atomic formula or the negation of an atomic formula, if t is maximal.

This means that ϕ looks locally like a sentence of $L_{\infty\omega}$. We require moreover that every bounded chain (linearly ordered subset) in T_ϕ has a unique supremum and that every branch (maximal chain) of T_ϕ is countable. If moreover, every node $t \in T_\phi$ has $< \kappa$ immediate successors, then we say that ϕ belongs to the fragment $M_{\kappa\omega_1}$.

The satisfaction relation $\mathcal{A} \models \phi$ is defined in terms of an obvious semantical game with players $\forall$ and $\exists$. This game is rather similar to the Ehrenfeucht-Fraïssé-game and it is not hard to show that the Ehrenfeucht-Fraïssé-game can be interpreted as a semantical game for a suitable sentence. Indeed, given a structure $\mathcal{A}$ and a bounded tree T, we can construct a sentence $\sigma_T(\mathcal{A})$ of $M_{\infty\omega_1}$ which satisfies

$$\mathcal{B} \models \sigma_T(\mathcal{A}), \text{ if and only if } \mathcal{A} \simeq_T \mathcal{B}$$

for all structures $\mathcal{B}$. The logics $M_{\infty\omega_1}$ and $M_{\omega_2\omega_1}$ have several other interesting properties, too.

1. The inclusions $L_{\infty\omega_1} \subsetneq M_{\infty\omega_1}$ and $L_{\omega_2\omega_1} \subsetneq M_{\omega_2\omega_1}$ hold.
2. The logics $M_{\infty\omega_1}$ and $M_{\omega_2\omega_1}$ are closed under $\bigwedge$, $\bigvee$, $\forall x$ and $\exists x$, but by a recent result of Taneli Huuskonen, there is a sentence of $M_{\omega_2\omega_1}$ which has no negation in $M_{\infty\omega_1}$.

3. The relation $\mathcal{A} \simeq_T \mathcal{B}$ holds, if and only if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences $\phi \in M_{\infty\omega_1}$ with $qr(\phi) \leq T$. (Here $qr(\phi)$ is the subset of T_ϕ consisting of those elements whose label is of the form $\forall x$ or $\exists x$. The sign $\leq$ refers to the existence of an order preserving function $qr(\phi) \longrightarrow T$.)

4. If we assume CH, then a form of the Craig interpolation theorem holds for $M_{\omega_2\omega_1}$. (See [5], [13] and also [16] and [15].)

After this short excursion in infinitary logic we shall go back to our main theme.

4 From Scott heights to nonstructure

Recall that every countable structure $\mathcal{A}$ has a Scott height α and that for all countable $\mathcal{B}$,

$$\mathcal{A} \simeq_{\mathcal{B}_\alpha} \mathcal{B} \text{ implies } \mathcal{A} \simeq \mathcal{B}.$$

For uncountable structures the situation is much more complicated.

We call a bounded tree T a *universal equivalence tree* for a structure $\mathcal{A}$, if for all $\mathcal{B}$ (of cardinality $\leq \omega_1$),

$$\mathcal{A} \simeq_T \mathcal{B} \text{ implies } \mathcal{A} \simeq \mathcal{B}.$$

Our aim is to discuss below some *nonstructure theorems* which tell us that a class K of structures contains a structure $\mathcal{A}$ which has no universal equivalence tree. The reader can find other aspects of the problem of generalizing Scott heights to uncountable structures in [8] and [17].

What do we really know, if there is a structure $\mathcal{A}$ with no universal equivalence tree in a class K? How is this connected to the nonstructure part of Shelah's *Classification Theory*? The following remarks show that such a result tells us in several ways that in such a case, there cannot be a nice system of complete invariants, which seems to be the main aim of the nonstructure theorems of Shelah, too.

1. In this case $EF_T(\mathcal{A},.)$ cannot, for any bounded tree T, carry enough information to fix $\mathcal{A}$ up to isomorphism. Recall that when put together, these games characterize the notion of being isomorphic among structures of cardinality $\leq \omega_1$.

2. No sentence ϕ of $M_{\omega_2\omega_1}$ can serve as a complete invariant for $\mathcal{A}$ (in the sense that $\mathcal{B} \models \phi$ is equivalent to $\mathcal{A} \simeq \mathcal{B}$).

3. There is no notion of a complete invariant for K which satisfies a certain reasonable set theoretic definability condition. Indeed, Heikki Heikkilä shows in his forthcoming dissertation [3] that K cannot for instance have a complete notion of an invariant which is Δ_1 relative to the predicate "x is the set of all countable sequences of elements of y". (Notice that being isomorphic is Σ_1. So Δ_1 relative to some predicates is the widest level of definability which is of interest here.)

The starting point to the nonstructure theorems described below is the following result due to Hyttinen and Tuuri [7].

Theorem 3 *Assume the CH. There is a linear ordering η of cardinality ω_1 and for every bounded tree T of cardinality $\leq \omega_1$ there is a linear ordering η_T of cardinality $\leq \omega_1$, where*

1. $\eta \simeq_T \eta_T$

2. η contains an uncountable descending sequence; and

3. η_T contains no uncountable descending sequences (especially, $\eta \not\simeq \eta_T$).

Notice that if T has a branch of length $\omega + \omega$, then T-equivalence implies $\equiv_{\infty\omega_1}$, elementary equivalence in $L_{\infty\omega_1}$. So the above theorem gives extremely similar nonisomorphic linear orderings. An earlier but weaker result about similar nonisomorphic linear orderings is proved in [14].

When this result is used to construct Ehrenfeucht-Mostowski models as in Shelah's work on nonstructure theory, we have the following more general and very strong result which should be compared with those of Shelah. Indeed, Shelah shows that there are too many nonisomorphic models whereas the following theorem tells that there is one single model which is too complicated. This result due to [7] refers to the terminology of Stability Theory which the reader can find either in [7] or in a standard text on Stability like [1].

Theorem 4 *Assume the CH. Let Σ be a first order theory and assume that Σ is unstable or has the OTOP. Then Σ has a model of cardinality ω_1 which has no universal equivalence tree of cardinality ω_1.*

A similar result is proved in [7] also for superstable theories with DOP, but in this case one has to consider larger structures.

Next we shall discuss the case of theories which have neither of the properties $OTOP$ and DOP. Such theories are said to have the properties $NOTOP$ ad $NDOP$. A bound for nonstructure behavior is given by the following result of Shelah.

Theorem 5 *If Σ is superstable and has $NOTOP$ and $NDOP$, then*

$$\mathcal{A} \equiv_{\infty\omega_\infty} \mathcal{B} \text{ implies } \mathcal{A} \simeq \mathcal{B}$$

holds for all models of Σ of cardinality $\leq \omega_1$. Especially, when CH holds, every model of Σ has universal equivalence tree of cardinality $\leq \omega_1$.

But if the theory Σ is only assumed to be stable, then a strong nonstructure property is possible, as the following result from [10] shows.

Theorem 6 *Assume the CH. There is a theory Σ which is stable and has $NOTOP$ and $NDOP$, but which has a model of cardinality ω_1 with no universal equivalence tree of cardinality $\leq \omega_1$.*

This theorem is proved by considering suitable abelian p-groups. Let first H be the direct sum

$$\bigoplus_{\nu<\omega_1} H_\nu$$

of a sequence of countable p-groups of ascending Ulm-lengths. We consider the following kind of *Hahn powers*. Let η be a linear ordering and let H^η be the group consisting of all those functions $x : \eta \longrightarrow H$ where the carrier $\{t \in \eta : x(t) \neq 0\}$ has for some $\alpha < \omega_1$ the form $\{t_\nu : \nu < \alpha\}$ where $t_0 > t_1 > \ldots$. The groups used to prove Theorem 6 are the torsion subgroups $G(H, \eta)$ of the groups H^η. They are p-groups and they have the following properties.

Lemma 7

1. *For every bounded tree T of cardinality ω_1 there is another bounded tree T' also of cardinality ω_1 where $\eta \simeq_{T'} \theta$ implies $G(H, \eta) \simeq_T G(H, \theta)$.*

2. *If exactly one of the orderings η and θ contains an uncountable descending sequence, then $G(H, \eta) \not\simeq G(H, \theta)$.*

3. *If $\eta \simeq \eta + \eta$, then the theory of $G(H, \eta)$ is stable and has the properties NPTOP and NDOP. Here $\eta + \eta$ consists of two adjacent copies of η.*

The crucial part is here 2.: 1. and 3. hold also for H^{η} and H^{θ} and all abelian groups H. Theorem 6 now follows from Theorem 3 and Lemma 7, since we can replace in the assertion of Theorem 3 the orderings η and η_T by $\eta \otimes BQ$ and $\eta_T \otimes BQ$. Here $\otimes BQ$ means BQ copies put one after the other according to the ordering of BQ.

The nonstructure theorems described above depend very heavily on the CH as the following result of Shelah from [6] shows.

Theorem 8 *It is consistent with ZFC that every linear ordering of cardinality ω_1 has a universal equivalence tree of the form $T + 1$ where T is (bounded and) of cardinality $\leq \omega_1$.*

Here $T + 1$ means the tree which is obtained from T when a new element is added at the end of every branch of T.

There are also some slightly weaker nonstructure theorems which can be expressed in terms of the Ehrenfeucht-Fraïssé-game. We say that a tree T is a *universal nonequivalence tree* for $\mathcal{A}$, if for all $\mathcal{B}$, the condition

$\forall$ has no winning strategy in $EF_T(\mathcal{A}, \mathcal{B})$

implies $\mathcal{A} \simeq \mathcal{B}$. So every universal nonequivalence tree for $\mathcal{A}$ is also a universal nonequivalence tree for $\mathcal{A}$. Hence a structure which has no universal equivalence trees has no universal nonequivlance trees. An interesting example of a nonstructure theorem stated in terms of universal nonequivalence trees is the following one from Mekler and Shelah [11]. There a *canary tree* is a bounded tree of cardinality 2^{ω_1} which gets uncountable branches whenever some stationary sets are killed in forcing which adds no new reals.

Theorem 9 *The free abelin group of cardinality ω_1 has a universal nonequivalence tree if and only if there is a canary tree.*

Also [7] contains some results about universal nonequivalence trees.

References

[1] J. Baldwin, Fundamentals of Stability Theory, Perspectives in Mathematical Logic, Springer-Verlag, (1988).

[2] J. Barwise, Admissible Sets and Structures, Perspectives in Mathematical Logic, Springer-Verlag, (1975).

[3] H. Heikkilä, Dissertation, (199?).

[4] T. Huuskonen, Comparing notions of similarity for uncountable models, dissertation, Helsinki (1991).

[5] T. Hyttinen, Model theory for infinite quantifier languages, Fund. Math.vol. 134.2, pp. 125–142, (1990).

[6] T. Hyttinen, S. Shelah and H. Tuuri, On strong nonstructure theorems, (19??).

[7] T. Hyttinen and H. Tuuri, Constructing strongly equivalent nonisomorphic models for unstable theories, Ann. Pure and Applied Logic, vol. 52, 203–248, (1991).

[8] T. Hyttinen and J. Väänänen, On Scott and Karp trees of uncountable models, Journal of Symbolic Logic 55, 897–908, (1990).

[9] C. Karp, Finite quantifier equivalence, The Theory of Models, J. Addison, L Henkin and A. Tarski (eds.), North-Holland, p. 407 - 412, (1965).

[10] Abelian p-groups with no invariants, Journal of Pure and Applied Algebra, (199?).

[11] The canary tree, (199?).

[12] A. Mekler, S. Shelah and J. Väänänen, The Ehrenfeucht-Fraïssé-game of length ω_1, (19??).

[13] J. Oikkonen, How to obtain interpolation for $L_{\kappa^+\kappa}$, in: F. Drake, J. Truss (eds.) Logic Colloquium '86 North–Holland, p. 175–209, (1988).

[14] J. Oikkonen. Ehrenfeucht-Fraïssé equivalence of linear orderings. Journal of Symbolic Logic, Vol. 55, No. 1, p. 65 - 73, (1990).

[15] J. Oikkonen, Chain models and infinitely deep logics, Reports from the Department of Mathematics, University of Helsinki, Preprint 13, (1992)

[16] H. Tuuri, Relative separation theorems for $L_{\kappa^+\kappa}$, Notre Dame Journal of Formal Logic. Vol. 33. 3. (1992).

[17] J. Väänänen. Games and trees in infinitary logic: a survey, Reports from the Department of Mathematics. University of Helsinki, Preprint 7. (1992).

K_0, RELATIVE DIMENSION, AND THE C^*-ALGEBRAS OF POST LOGIC

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Summary

As shown in an earlier paper, Grothendieck's functor K_0 yields an interpretation of AF C^*-algebras as Lindenbaum algebras of the many-valued sentential calculus of Lukasiewicz. In particular, in this paper we show that subhomogeneous AF C^*-algebras with Hausdorff spectrum correspond to the finite-valued calculus, and that homogeneous AF C^*-algebras correspond to Post calculus. Fell's notion of one-dimensional projection field is shown to be equivalent to the notion of greatest singular element in lattice-group theory. The K_0 group of every liminary AF C^*-algebra A with Hausdorff spectrum is conveniently visualized as the l-group of rational-valued continuous functions over Prim(A) generated by the *relative* dimension functions of projections.

Preliminaries

We assume the reader to be familiar with Elliott's classification of AF C^*-algebras [8], [7], and with the categorical equivalence Γ between lattice-ordered abelian groups (for short, l-groups) with strong unit, and MV algebras [14]. We refer to [4] for background on MV algebras, to [1] for l-groups, and to [6] for C^*-algebras. For each $n = 2, 3, \ldots$, we let I_n denote the algebra $\{0, 1/(n-1), 2/(n-1), \ldots, (n-2)/(n-1), 1\}$ with distinguished constants 0 and 1, and operations $x^* = 1-x$, $x \oplus y = \min(1, x+y)$, $x \cdot y = (x^* \oplus y^*)^* = \max(0, x+y-1)$. Algebras in the variety generated by I_n are called MV$_n$ algebras [12]. Boolean algebras are the same as MV$_2$ algebras. An MV algebra B is said to be *of finite order* iff for some n, B happens to be an MV$_n$ algebra. MV$_n$ algebras stand to the n-valued sentential calculus of Lukasiewicz as boolean algebras stand to the classical sentential calculus.

Let X be a *boolean* space, i.e., a totally disconnected compact Hausdorff space. Then by $C(X, I_n)$ we denote the MV$_n$ algebra of all continuous functions from X into I_n, the latter being equipped with the discrete topology. In the light of Epstein's representation theorem [10, Theorem 16], we say that an MV algebra B is a *Post* MV *algebra of order* n iff B is isomorphic to some MV algebra $C(X, I_n)$.

Following [2], by an AF C^*-algebra we mean the norm closure of the union of an ascending sequence of finite dimensional C^*-algebras, all with the same unit. The results of [14, § 3] yield a one-one correspondence between (all isomorphism classes of) AF C^*-algebras whose Grothendieck group is lattice-ordered, and (all isomorphism classes of) countable MV algebras. For any such AF C^*-algebra A, the corresponding MV algebra B is given by $B = \Gamma(K_0(A), [A])$, where $[A]$ is the image of the A-module A in $K_0(A)$. Some instances of the correspondence are given in the following table:

COUNTABLE MV ALGEBRA	*ITS AF C* CORRESPONDENT*
Lukasiewicz chain I_{n+1}	$M_n(\mathbf{C})$, the $n \times n$ complex matrices
finite	finite dimensional
boolean	commutative
totally ordered	with Murray, von Neumann comparability
MV_3	3-subhomogeneous with Hausdorff spectrum
subalgebra of $\mathbf{Q} \cap [0,1]$	Glimm's UHF algebra
dyadic rationals in the unit interval	CAR algebra of the Fermi gas
all rationals in $[0,1]$	Glimm's universal UHF algebra
generated by $\rho \in [0,1] \setminus \mathbf{Q}$	Effros-Shen algebra F_ρ
real algebraic numbers in $[0,1]$	Blackadar algebra B
Chang algebra C	Behncke-Leptin algebra $A_{0,1}$
of finite order	subhomogeneous with Hausdorff spectrum
Post algebra of order n+1	homogeneous of order n
finite product of Post algebras	continuous trace

Our aim is to establish the correspondences given in the last three lines of the above table. Recall that a C^*-algebra A is *subhomogeneous* iff there is an integer n such that $\dim \pi < n$ for all irreducible representations π of A.

1. THEOREM. *The map* $A \to \Gamma(K_0(A), [A])$ *is a one-one correspondence between subhomogeneous AF C^*-algebras with Hausdorff primitive spectrum, and countable MV algebras of finite order.*

PROOF. ($\to$) Let n be an integer such that for each $J \in \mathrm{Prim}(A)$ the quotient A/J is isomorphic to $M_m(\mathbf{C})$, for some $m < n$. Since K_0 preserves exact sequences, and primitive ideals in A correspond to prime ideals in the dimension group $G = K_0(A)$, it follows that each prime quotient of G is isomorphic to the additive group of integers with the natural order. This, together with the assumed Hausdorff property of Prim(A) suffices to show that G is an l-group, by [9, Theorem 1]. An application of the functor Γ in the light of [14, §3] yields a countable MV algebra $B = \Gamma(K_0(A), [A])$. Since A is liminary, the preservation properties of K_0 and Γ ensure that in B prime ideals coincide with maximal ideals. It follows that the intersection of all maximal ideals of B is $\{0\}$, whence by [16, 3.1], B can be represented as an MV algebra of continuous real-valued functions over the compact Hausdorff space $X \cong$ Prim(A) $\cong$ Prim(B) $\cong$ Prim($K_0(A)$). Accordingly, each prime ideal J of B is in canonical one-one correspondence with a point $x_J \in X$, and the quotient map $B \to B/J$ amounts to evaluating each $f \in B$ at x_J. From the assumed subhomogeneity of A it follows that each prime quotient B/J is isomorphic to $I_{m+1} = \{0, 1/m, ..., (m-1)/m, 1\}$ for some $m < n$. Since any such I_{m+1} is a subalgebra of $I_{n!+1}$, we obtain that B is a subalgebra of a product of copies of $I_{n!+1}$, one copy for each point $x \in X$. By definition, B is an $MV_{n!+1}$ algebra.

($\leftarrow$) Suppose B is a countable MV_n algebra. By [12, 2.4, 2.17, p. 89], in B all prime ideals are maximal, and for each prime ideal J the quotient B/J is isomorphic to I_m, for some m such that $m-1$ divides $n-1$. By [14, 3.12], a unique AF C^*-algebra A with lattice-ordered K_0 corresponds to B via K_0 and Γ. The preservation properties of these two functors ensure that each primitive quotient A/J is isomorphic to some $M_r(\mathbf{C})$, where r divides $n-1$. Therefore, A is subhomogeneous. We also have canonical homeomorphisms Prim(A) $\cong$ Prim(B) $\cong$ Prim($K_0(A)$). This latter space is Hausdorff by [1, 10.2.2], since also in $K_0(A)$ prime l-ideals coincide with maximal l-ideals. QED.

Recall [6], [11] that a C^*-algebra A is *homogeneous of order* n iff every irreducible representation of A is of the same dimension n.

2. THEOREM. *For each* $n = 1, 2, ...,$ *the map* $A \to \Gamma(K_0(A), [A])$ *is a one-one correspondence between homogeneous AF* C^**-algebras of order* n, *and countable Post MV algebras of order* $n+1$.

PROOF. ($\to$) Suppose A is a homogeneous AF C^*-algebra of order n. Since by assumption A is unital, by [13, 4.2], Prim(A) is a compact Hausdorff space, whence by [3, p.76], it is a boolean space. By Theorem 1 above, A corresponds via Γ and $K_0^{\cdot}$ to a unique countable MV algebra B of finite order. Following [14, 4.13-16], let L denote the MV algebra of all [0,1]-valued McNaughton functions over the Hilbert cube $[0,1]^{\omega}$. L is the free MV algebra over denumerably many free generators. Thus B is a quotient of L, and since in B the intersection of all maximal ideals is $\{0\}$, by [16, 3.1] there exists a boolean subspace X of the Hilbert cube such that B is the MV algebra of restrictions to X of the McNaughton functions of L. Each prime ideal J of B canonically corresponds to a point $x_J \in X$, and the quotient map $B \to B/J$ is the evaluation map at x_J. From the assumed homogeneity of A we have that for each prime ideal J of B, B/J is isomorphic to I_{n+1}. Therefore, B is an MV algebra of continuous I_{n+1}-valued functions over the boolean space $X \subseteq [0,1]^{\omega}$. To complete the proof we must settle the following

Claim. Each continuous function $h: X \to I_{n+1}$ actually belongs to B.

For each point $x_J = x = (x_0, x_1,...) \in X$, every coordinate x_i must be a rational number—for otherwise the quotient B/J would be infinite. So let us write $x_i = a_i/b_i$, $a_i, b_i \in \mathbf{Z}$, $a_i \geq 0$, $b_i > 0$, $\gcd(a_i, b_i) = 1$. Let d be least common multiple of the denominators $\{b_i \mid i \in \omega\}$. Then d must be equal to n. Indeed, the quotient B/J is isomorphic to the finite MV algebra $\{k/d \mid k = 0,1,...,d\}$. It follows that for each $k = 0, 1,...,n$ and $x \in X$ there is a McNaughton function $f \in L$ such that $f(x) = k/n$. By continuity, f equals k/n in an open neighbourhood $N \subseteq X$ of x, and we can safely assume N to be a clopen subspace of X. Turning to our function h, for each $x \in X$ there is a McNaughton function $f_x \in L$ such that $f_x = h$ in a clopen neighbourhood $N_x \subseteq X$ of x. By compactness, there are points $x_1, ..., x_t$ in X such that $N_{x_1} \cup ... \cup N_{x_t}$ covers X. We can safely assume the N_{x_i}'s to be pairwise disjoint. By a routine compactness argument, together with [15, 3.3], for each $i = 1,...,t$ there is a McNaughton function $g_i \in L$ which is equal to 1 over N_{x_i}, and equal to 0 over $X \setminus N_{x_i}$. Thus, h coincides with the function $(f_{x_1} \wedge g_1) \vee ... \vee (f_{x_t} \wedge g_t)$ over all of X, and $h \in B$.

($\leftarrow$) If $B = C(X, I_{n+1})$ is a countable Post MV algebra, then B is an MV_{n+1} algebra, and by [12, 2.4] each prime ideal J of B is maximal, and uniquely corresponds to a point $x_J \in X$. The quotient map $B \to B/J$ is the evaluation map at x_J, and B/J is isomorphic to I_{n+1}. In the unique AF C^*-algebra A corresponding to B, as given by [14, 3.12], all primitive quotients will be isomorphic to $\mathrm{M}_n(\mathbf{C})$. QED.

Remark. From the above proof one has that an MV algebra B is a Post MV algebra of order n iff all prime quotients of B have the same cardinality n.

3. COROLLARY. *The map $A \to \Gamma(K_0(A), [A])$ is a one-one correspondence between continuous trace* AF *C^*-algebras, and finite products of countable Post MV algebras of finite order.*

PROOF. From Theorem 2, using standard arguments [6, 10.8.8, 10.5.4] and the fact that Prim(A) is boolean [3, p.76], [6, 4.5.3]. QED.

(Possibly nonunital) continuous trace AF C^*-algebras play an interesting role among all separable continuous trace C^*-algebras [17]. Turning to our (unital) continuous trace AF C^*-algebras, a routine argument shows that any such C^*-algebra has—in Fell's terminology, [11, p.259]—a *one-dimensional projection field*, i.e., an element p such that $\pi(p)$ is a one-dimensional projection for every irreducible representation π of A. We are interested in the image of p under K_0. Recall [1, 11.2.7] that an element s in an l-group G is called *singular* iff $s > 0$ and whenever $s = u+w$ with $u,w \geq 0$, then $u \wedge w = 0$. An element q in an MV algebra B is called *supersingular* iff for each prime ideal J of B, the image q/J of q under the quotient map is an atom of the totally ordered MV algebra B/J. It is easy to see that B has at most one supersingular element.

4. LEMMA. *For every MV algebra B the following are equivalent:*
(i) *For some k, $B \cong P_1 \times ... \times P_k$, where each P_i is a Post MV algebra of finite order;*
(ii) *B has a supersingular element q.*

PROOF. (i)→(ii) Write each P_i as $C(X_i, I_{n_i})$, with pairwise disjoint boolean spaces X_i. Let $q: \cup_i X_i \to \mathbf{Q}$ be the function constantly equal to $1/(n_i-1)$ on each X_i. Since B is an MV algebra of finite order, for each prime ideal J, letting x_J be its corresponding point in $\cup_i X_i$, we have $B/J \cong I_{n_i}$, where i is the only integer such that $x_J \in X_i$. Then q/J is an atom of B/J.

(ii)→(i) Observe that in B prime ideals coincide with maximal ideals. It follows that the intersection of all maximal ideals of B is $\{0\}$. By [16, 3.1] there is a cardinal κ, and a compact Hausdorff subspace X of the cube $[0,1]^{\kappa}$, such that B is the MV algebra of restrictions to X of the McNaughton functions in the free MV algebra L_{κ} over κ-many free generators. Each prime ideal J of B now corresponds to a point $x_J \in X$, and the quotient map B/J is evaluation at x_J. Since q/J is an atom of the MV algebra $B/J \subseteq [0,1]$, it follows from [4, 3.12] that there is an integer $n_J > 0$ such that $q/J = 1/n_J$, and $B/J = I_{n_J+1}$. Thus each function $g \in B$ is continuous and rational-valued over X. Let G be the abelian l-group with strong unit corresponding to B via Γ, as in [14, §3]. Then G is the l-group of rational-valued continuous functions over X generated by B, with the constant function 1 as the strong unit. The preservation properties of Γ ensure that in G prime l-ideals (i.e., those l-ideals J of G such that G/J is totally ordered, [1, 2.4.1]) are maximal. By [1, 14.1.2], G is hyperarchimedean.

Claim 1. For each $f \in G$, range(f) is finite.

It is sufficient to show that each rational $a/b \in$ range(f) is an isolated point in range(f). For otherwise, suppose without loss of generality that for every $\varepsilon > 0$ the interval $[a/b, a/b + \varepsilon]$ contains infinitely many points of range(f). Let $h \in G$ be the function $h = (bf-a) \vee 0$. Then for each $\eta > 0$ infinitely many points of range(h) are in $[0,\eta]$, contradicting the hyperarchimedean property of G, [1, 14.1.2].

Applying our claim to the supersingular element $q \in B \subseteq G$, we have that q partitions X into finitely many clopen subspaces $X = X_1 \cup \ldots \cup X_k$, where $X_i = q^{-1}(1/n_i)$.

Claim 2. $B \cong \prod_i C(X_i, I_{n_i+1})$.

We have only to show that $\prod_i C(X_i, I_{n_i+1})$ is contained in B. Each function $g \in \prod_i C(X_i, I_{n_i+1})$ partitions X into finitely many clopen sets $Y_1,\ldots,Y_t$, in such a way that g is constant on each Y_j. We can safely assume that for each Y_j there is an index i such that $Y_j \subseteq X_i$. It follows that over each Y_j, g is a multiple of q, say, $g = n_j q$. Arguing as in the proof of Theorem 2 above, and again using [16, 3.3], for each Y_j there exists a [0,1]-valued McNaughton function $g_j \in L_{\kappa}$ such that $g_j = 1$ on Y_j, and $g_j = 0$ on $X \setminus Y_j$. Therefore, over all of X we have $g = \sum_j (n_j q \wedge g_j)$, whence $g \in \{r \in G \mid 0 \leq r \leq 1\} = B$. QED.

5. LEMMA. *Let B be an MV algebra with a (necessarily unique) supersingular elèment q. Let (G,u) be the abelian l-group with strong unit u corresponding to B via Γ, say without loss of generality $B = \{g \in G \mid 0 \leq g \leq u\}$. Then q is the greatest singular element of G.*

PROOF. By Lemma 4, there is a compact Hausdorff space $X \subseteq [0,1]^\kappa$ such that B is the MV algebra of restrictions to X of the McNaughton functions of the free MV algebra L_κ over κ free generators. Each function $f \in B$ is rational-valued and continuous. G is the l-group of continuous rational-valued functions on X generated by B, with the constant function 1 as the strong unit u. Each $g \in G$ has a finite range. The supersingular element $q \in B \subseteq G$ partitions X into finitely many clopen subspaces $X = X_1 \cup ... \cup X_k$, and there are positive integers $n_1,...,n_k$ such that $X_i = q^{-1}(1/n_i)$. For each $g \in G$ and $x \in X$ there is an integer n such that $g(x) = nq(x)$. It is now easy to see that q is singular in G. To see that q is the greatest singular element in G, suppose $s \in G$ to be singular, and $s(x) > q(x)$ for some $x \in X$. Since s is continuous and range(s) is finite, there is clopen neighbourhood $N \subseteq X$ of x such that both s and q are constant on N. Without loss of generality, $N \subseteq X_i$ for some i. As in the proof of Lemma 4, there is a McNaughton function $f \in L_\kappa$ such that $f = 1$ on N, and $f = 0$ on $X \setminus N$. The function $t = s \wedge 2f$ is still singular in G, by [1, 11.2.9], and $t > q$ in N. Therefore for some integer $n \geq 2$ we can write $t = nq \wedge 2f = (q \wedge 2f) + ((n-1)q \wedge 2f)$, contradicting the singularity of t. QED.

6. LEMMA (Elliott). *Let A be a C^*-algebra with boolean primitive spectrum. Then two projections p and q are equivalent if they are equivalent in each primitive quotient.*

PROOF. By continuity of the norm, a partial isometry connecting p and q at some $J \in$ Prim(A) can be lifted to a neighbourhood, which we may safely suppose to be clopen. By compactness of the set $S \subseteq$ Prim(A) where the projections are nonzero, a finite number of such neighbourhoods $N_1,...,N_k$ covers S, and we may assume $N_i \cap N_j$ to be empty whenever $i \neq j$. Adding up the partial isometries, we obtain that p and q are equivalent. QED.

7. LEMMA. *Let A be a liminary AF C^*-algebra with Hausdorff spectrum. For each projection $p \in A$ let the* relative dimension *function* d_p: Prim$(A) \to \mathbf{Q}$ *be defined by* $d_p(J)$ = (dim range π (p)) / (dim π), *where* π *is an irreducible representation of* A *such that* $\ker(\pi) = J$. *We then have*:

(i) *Two projections p and q of A have the same image in $K_0(A)$ iff $d_p = d_q$.*

(ii) *$(K_0(A), [A])$ is the l-group of rational-valued functions over* Prim(A) *generated by the relative dimension functions of projections of A, with the constant function* 1 *as the strong unit.*

(iii) *For each projection $p \in A$, d_p is a continuous function having a finite range.*

PROOF. (i) By [3, p. 76], Prim(A) is a boolean space. By Lemma 6, two projections p and q are equivalent if they are equivalent in each primitive quotient, i.e., if $d_p(J) = d_q(J)$ for each $J \in$ Prim(A). The converse is trivial.

(ii) By [9, Theorem 1], $K_0(A)$ is an l-group, and by [14, §3] the MV algebra $B = \Gamma(K_0(A), [A])$ is well defined. By [8] together with (i) above, B is the MV algebra of relative dimension functions of projections in A, with natural pointwise operations. Again by [14, § 3], we have that $(K_0(A), [A])$ is the l-group of rational-valued functions on Prim(A) generated by B, with the constant function 1 as the strong unit.

(iii) Since in B prime ideals are maximal, identifying Prim(A) with a boolean subspace X of the Hilbert cube, by (ii) and [16, 3.1] we can represent B as an MV algebra of continuous rational-valued (McNaughton) functions over X. Since $K_0(A)$ is hyperarchimedean [1,14.1.2], and each function f of $K_0(A)$ is rational-valued, the same argument of Claim 1 in Lemma 4 above shows that the range of f is finite. QED.

Remark. Liminary AF C^*-algebras with Hausdorff spectrum are further investigated in [5]. Here the authors consider the analogue of Kaplansky's problem for these algebras, and prove that the Murray von Neumann order of projections alone is sufficient to uniquely recover the C^*-algebraic structure.

8. THEOREM. *Let A be a continuous trace AF C^*-algebra, and suppose p is a projection in A. Then p is mapped by K_0 into the greatest singular element of $K_0(A)$ iff p is a one-dimensional projection field.*

PROOF. Suppose p is a one-dimensional projection field. Using Corollary 3 we can write $\Gamma(K_0(A), [A]) \cong B \cong P_1 \times ... \times P_k$, where each P_i is a Post MV algebra of finite order. For each primitive ideal J in A, and each representation π such that $\ker(\pi) = J$, we have that $K_0(A/J) = K_0(\pi(A)) \cong \mathbf{Z}$, and $\pi(p) = p/J$ is mapped by K_0 into the only atom of $K_0(A/J)$. Thus p corresponds to *the* supersingular element q of $B = \{g \in K_0(A) \mid 0 \le g \le [A]\}$. By Lemma 5, q is the greatest singular element in $K_0(A)$. By Lemma 7 (i), a projection $r \in A$ has the same image as p in $K_0(A)$ iff $d_p = d_r$, iff r is a one-dimensional projection field. QED.

References

[1] A.BIGARD, K.KEIMEL, S.WOLFENSTEIN, "Groupes et anneaux réticulés", Lecture Notes in Mathematics, vol. 608, Springer-Verlag, Berlin, 1977.

[2] O. BRATTELI, Inductive limits of finite-dimensional C^*-algebras, *Trans. Amer. Math. Soc.*, **171** (1972) 195-234.

[3] O. BRATTELI, G.A.ELLIOTT, Structure spaces of approximately finite dimensional C^*-algebras II, *J. Functional Analysis*, **30** (1978) 74-82.

[4] C.C.CHANG, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.*, **88** (1958) 467-490, *ibid.*, **93** (1959) 74-80.

[5] R. CIGNOLI, G.A.ELLIOTT, D.MUNDICI, Reconstructing C^*-algebras from their Murray von Neumann order, *Advances in Mathematics*, to appear.

[6] J.DIXMIER, "C^*-algebras", North-Holland, Amsterdam, 1977.

[7] E.G.EFFROS, "Dimensions and C^*-algebras", CBMS Regional Conf. Series in Math., vol. 46, Amer. Math. Soc., Providence, R.I., 1981.

[8] G.A.ELLIOTT, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, *J. Algebra*, **38** (1976) 29-44.

[9] G.A.ELLIOTT, D. MUNDICI, A characterization of lattice-ordered abelian groups, *Math. Zeit.*, to appear.

[10] G. EPSTEIN, The lattice theory of Post algebras, *Trans. Amer. Math. Soc.*, **95** (1960) 300-317.

[11] J.M.G.FELL, The structure of algebras of operator fields, *Acta Math.*, **106** (1961) 233-280.

[12] R.GRIGOLIA, Algebraic analysis of Lukasiewicz Tarski's n-valued logical systems, In: "Selected Papers on Lukasiewicz Sentential Calculi" (R.Wòjcicki and G. Malinowski, Eds.) pp. 81-92, Polish Acad. of Sciences, Ossolineum, Wroclaw, 1977.

[13] I.KAPLANSKY, The structure of certain operator algebras, *Trans. Amer. Math. Soc.*, **70** (1951) 219-255.

[14] D. MUNDICI, Interpretation of AF C^*-algebras in Lukasiewicz sentential calculus, *J. Functional Analysis,* **65** (1986) 15-63.

[15] D.MUNDICI, Free products in the category of abelian l-groups with strong unit, *J.Algebra,* **113** (1988) 89-109.

[16] D.MUNDICI, The derivative of truth in Lukasiewicz sentential calculus, Proc. VII Latin Amer. Symp. on Math. Logic, Contemporary Math., vol. 69, pp. 209-227, Amer. Math. Soc., Providence, R.I., 1988.

[17] J.S. SPIELBERG, Embedding C^*-algebra extensions into AF algebras, *J.Functional Analysis,* **81** (1988) 325-344.

AN INTRODUCTION TO LAMBDA ABSTRACTION ALGEBRAS

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ABSTRACT. The concept of a *lambda abstraction algebra* (LAA) has recently been defined. It is designed to algebraize the untyped lambda calculus in the same way cylindric and polyadic algebras algebraize the first-order predicate logic. Like cylindric and polyadic algebras, LAA's can be defined by true identities and thus form a variety in the sense of universal algebra. They provide a distinctly algebraic alternative to the highly combinatorial lambda calculus. The paper is largely expository. We attempt to motivate the theory in terms of the relevant notions of lambda calculus and cylindric algebras. As an example of the technical part of the theory we give a complete proof of one of the main representation results: that every locally finite lambda abstraction algebra is isomorphic to a functional lambda abstraction algebra.

Introduction. Lambda abstraction algebras are intended to provide a full algebraization of the untyped lambda calculus in much the same way cylindric and polyadic algebras algebraize the first-order predicate logic. Like cylindric and polyadic algebras they form a variety in the sense of universal algebra and thus provide a distinctly algebraic alternative to the highly combinatorial methods of the lambda calculus.

The untyped lambda calculus is a formalization of an intensional as opposed to an extensional theory of functions, that is, a theory of functions viewed as "rules" rather than "sets of ordered pairs". Its basic feature is the lack of distinction that is made between functions and the elements of the domains on which the functions act.

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Thus a function can, in theory, take other functions, including itself, as legitimate arguments. There are two primitive notions: *application*, the operation of applying a function to an argument, and *lambda* (*functional*) *abstraction*, the process of forming a function from the "rule" that defines it.

As one would expect there are no simple models of the untyped lambda calculus, but one can imagine idealized models that are "constructed" in the following way: Start with any set S (possibly empty), and successively form the sets $T_0 = S$, $T_1 = S^S$, $T_2 = (S \cup S^S)^{S \cup S^S}$, $T_3 = (T_1 \cup T_2)^{T_1 \cup T_2}, \ldots$. Iterate the construction until a "fixpoint" is reached, giving a set V satisfying the "domain equation" $V^V = V$. We know of course that if V^V is interpreted as the set of all functions from V to itself in the usual set-theoretical sense, then the above iterative process can never reach a fixpoint since no set can satisfy the domain equation. By restricting the functions we consider to certain admissible ones, and interpreting V^V accordingly, domains satisfying the domain equation, or a somewhat weaker form of it the guarantees that there is in some sense enough admissible functions, have have been found.[1] Such domains are the "natural" models of the untyped lambda calculus. They are called *environment models* in the literature of the lambda calculus ([10]). They are closely related to the *syntactical lambda models* of [1]. They can be characterized by means of an injective partial mapping $\lambda \colon V^V \xrightarrow{\mathrm{P}} V$ whose domain is the set of admissible functions. λ may be thought of as the process of encoding admissible functions as elements of V. With functions encoded this way, application can be viewed as a binary operation on V. Let $\mathbf{V}$ be the domain V enriched by the application operation and the encoding mapping. We will denote the application operation by $\cdot^{\mathbf{V}}$ and the encoding mapping by $\lambda^{\mathbf{V}}$.

Intuitively, each admissible function in V^V has two forms, an *intensional* one and an *extensional* one. In its intensional form it is represented by a term $t(x)$ of the lambda calculus with a free variable x. (The exact nature of terms will be spelled out later.) For each $v \in V$, let $t^{\mathbf{V}} [\![v]\!]$ be the value $t(x)$ takes in $\mathbf{V}$ when x is interpreted as v. Then its extensional form is the function $\langle t^{\mathbf{V}} [\![v]\!] : v \in V\rangle \in V^V$, which is encoded as the element $\lambda^{\mathbf{V}} \left(\langle t^{\mathbf{V}} [\![v]\!] : v \in V\rangle\right)$ of V. It is represented by the term $\lambda x.t(x)$. Note that $t(x)$ and $\lambda x.t(x)$ both represent the same function, but in environment models only the extensional form corresponds to an actual element of the universe of the model; this is an essential difference between the models of lambda calculus and lambda abstraction algebras, as we shall see.

The two forms of the function are connected by the operation of application. Intuitively, the value $t^{\mathbf{V}} [\![v]\!]$ of the function at a particular argument v is obtained by applying its extensional form to v; symbolically, $\langle t^{\mathbf{V}} [\![v]\!] : v \in V\rangle(v) = t^{\mathbf{V}} [\![v]\!]$. Expressed in the environment model this becomes

$$\lambda^{\mathbf{V}} \left(\langle t^{\mathbf{V}} [\![v]\!] : v \in V\rangle\right) \cdot^{\mathbf{V}} v = t^{\mathbf{V}} [\![v]\!].$$

[1]The first such model was constructed by Scott [15].

In the lambda calculus itself this relationship is represented by the fundamental axiom of *β-conversion:*

(β) $(\lambda x.t)s = t[s/x]$, for all terms t, s and variable x such that s is free for x in t.

Terms of the lambda calculus are constructed as follows: There is an infinite set of variables called *λ-variables.* Every λ-variable is a term; if t and s are terms, so are $t \cdot s$ and $\lambda x(t)$ for each variable x. By convention we write ts for $t \cdot s$ and $\lambda x.t$ for $\lambda x(t)$. An occurrence of a variable x in a term is *bound* if it lies within the scope of a lambda abstraction λx; otherwise it is *free.* s is *free for* x in t if no free occurrence of x in t lies within the scope of a lambda abstraction with respect to a variable that occurs free in s. $t[s/x]$ is the result of substituting s for all free occurrences of x in t.

The other axioms of the lambda calculus are as follows: t, s, r, u are terms and x, y variables.

(α) $\lambda x.t = \lambda y.t[y/x]$, for every y that does not occur free in t;

- $t = t$;
- $t = s \implies s = t$;
- $t = s, s = r \implies t = r$;
- $t = r, s = u \implies ts = ru$;
- $t = s \implies \lambda x.t = \lambda x.s$.

(α)-conversion says that bound variables can be replaced in a term under the appropriate condition. A *lambda theory* is any set of equations that is closed under α and β conversion and the five congruence rules.

The following *completeness theory* is a basic result of lambda calculus; see [10].

Every lambda theory consists of precisely the equations valid in some environment model.

The axioms are all in the form of equations, but the lambda calculus is not a true equational theory since in β-conversion, for instance, the term-variables t and s cannot be substituted for freely, without restriction. The situation is similar to that in the standard formalism of first-order logic. In both cases the source of the problem is the way substitution is handled. By dealing with substitution at the level of the object language rather than the metalanguage, i.e., by abstracting it, two different but closely related pure equational formalizations of first-order predicate calculus have been developed giving rise to the theories of polyadic Boolean algebras and cylindric algebras. This abstraction of substitution is a characteristic feature of first-order algebraic logic. In polyadic Boolean algebras abstract substitution is primitive, but in cylindric algebras it is a defined notion. For any first-order formula $\varphi(x)$ and any individual variables x and y such that y is free for x in φ,

$$\exists x\big(x = y \ \& \ \varphi(x)\big) \leftrightarrow \varphi[y/x] \tag{1}$$

is a logically valid formula. The key idea of cylindric algebras is the use of this equivalence as the basis of a definition of abstract substitution. Another important feature of cylindric algebras is the suppression of individual variables. More accurately, the variables $x_0, x_1, \dots, x_\kappa, \dots$, $\kappa < \omega$, of the standard formulation of first-order logic are transformed into the indices $0, 1, \dots, \kappa, \dots$ of a Cartesian-like coordinate system. In their place are the ***diagonal elements***, an infinite system of constant symbols (i.e., nullary operations) $\mathsf{d}_{\kappa\lambda}$, for all $\kappa, \lambda < \omega$. $\mathsf{d}_{\kappa\lambda}$ corresponds to the atomic equality formula $x_\kappa = x_\lambda$. The actual variables of the object language of cylindric algebras correspond to the variables of the metalanguage of first-order logic that range over formulas. The other primitive notions are the unary operations of *cylindrification* c_κ, $\kappa < \omega$, corresponding existential quantifications $\exists x_\kappa$, and the standard Boolean operations $+$, $\cdot$, and $-$ that correspond respectively to disjunction, conjunction, and logical negation. When transformed into the language of cylindric algebras the logical equivalence (1) becomes the definition of the abstract substitution operation S^κ_λ ([8, Part 1, Def. 1.5.1]):

$$S^\kappa_\lambda x := \mathsf{c}_\kappa(\mathsf{d}_{\kappa\lambda} \cdot x), \qquad \text{if } \kappa \neq \lambda.$$

The axioms of cylindric algebras are similar in form, but of course involve only primitive operations. They express the basic properties of the propositional connectives, quantification, and equality predicate in algebraic form. Since substitution plays an important role in the metatheory of first-order logic, a key function of the axioms is to express the fundamental properties of the substitution operator S^κ_λ. The most significant feature of the axioms is that they are true identities in the sense that they continue to hold when arbitrary terms are substituted for the variables. Thus the theory of cylindric algebras gives a pure equational theory of first-order logic, and cylindric algebras form a variety in the universal-algebraic sense.

The way in which lambda abstraction theory arises from the lambda calculus closely parallels the way cylindric algebras are obtained from first-order logic. Again the key is the abstraction of substitution. But β-conversion plays the role of (1). λ-variables are transformed into members of an abstract index set I rather than of an ordinal. In this regard we follow the lead of polyadic Boolean algebras instead of cylindric algebras. I can be of arbitrary, possibly finite, cardinality. Lambda abstraction algebras contain an individual constant for each λ-variable. For simplicity, we identify these constants with elements of the index set I. This has the added advantage of making the language of lambda abstraction algebras conform more closely to the language of lambda calculus.

Lambda abstraction algebras. Let I be a nonempty set. The similarity type of lambda abstraction theory over I is $\langle \cdot, \langle \lambda x : x \in I \rangle, \langle x : x \in I \rangle \rangle$, where $\cdot$ is a binary operation symbol, λx is a unary operation symbol for every $x \in I$, and x is a constant (i.e., nullary operation) symbol for every $x \in I$. Note that λx is to be viewed as an indivisible complex symbol; alternatively, $\langle \lambda x : x \in I \rangle$

can be viewed as a system of unary operations index by elements of I. Similarly, $\langle x : x \in I \rangle$ can be viewed a system of constants indexed by I. The elements of I are to be thought of as the variables of lambda calculus and we will refer to them as *λ-variables*. However in their algebraic transformation they no longer play the role of variables in the usual sense. The actual variables of lambda abstraction theory will be referred to as *context variables* and denoted by the greek letters ξ, ν, and μ, possibly with subscripts. Context variables correspond to variables of the metalanguage of lambda calculus that range over terms. The terms of the language of lambda abstraction theory are called *λ-terms*. Every λ-variable x and context variable ξ is a λ-term; if t and s are λ-terms, then so are $t \cdot s$ and $\lambda x(t)$. Thus, formally, terms of lambda abstraction theory differ from those of lambda calculus only to the extent that they may contain context variables. An occurrence of a λ-variable x in a λ-term is *bound* if it falls within the scope of the operation symbol λx; otherwise it is *free*. (The "x" in the operation symbol "λx" does not count as an occurrence of x.) The *free variables* of a λ-term are the λ-variables that have at least one free occurrence. A λ-term without any context variables is said to be *pure*. A λ-term without free variables is said to be *closed*.

Because of their similarity to the terms of the lambda calculus we use the standard notational conventions of the latter. The application operation symbol "·" is normally omitted, and the application of $t \cdot s$ of two terms is written as juxtaposition ts. When parentheses are omitted, association to the left is assumed. For example, $(((ts)(ru))v)$ will be written $ts(ru)v$. The left parenthesis delimiting the scope of a λ-abstraction is replaced with a period and the right parenthesis is omitted. For example, $\lambda x\big(ts(ru)v\big)$ is written $\lambda x.ts(ru)v$. Successive λ-abstractions $\lambda x \lambda y \lambda z \ldots$ are written $\lambda xyz \ldots$. Here is an example of a term that makes use of all of these conventions: $\lambda x_1 \cdots x_n.t_1 \cdots t_m$ is shorthand for $\lambda x_1[\lambda x_2[\ldots[\lambda x_n[(t_1 \cdot t_2) \cdots t_m]\ldots]]]$ We occasionally revert to universal-algebraic notational conventions when the context seems to warranted it.

We now give the formal definition of lambda abstraction algebras. Readers unfamiliar with the notation of the lambda calculus may want to go directly to the reformulation of the axioms in terms the substitution operator that is given later.

Definition. By a *lambda abstraction algebra of dimension I* we mean an algebraic structure of the form

$$\boldsymbol{A} := \big\langle A, \cdot^{\mathsf{A}}, \langle \lambda x^{\mathsf{A}} : x \in I \rangle, \langle x^{\mathsf{A}} : x \in I \rangle \big\rangle$$

satisfying the following quasi-identities for all $x, y, z \in I$ (subject to the indicated conditions) and all $\xi, \mu, \nu \in A$.

(β_1) $(\lambda x.x)\xi = \xi$;
(β_2) $(\lambda x.y)\xi = y$, if $x \neq y$;
(β_3) $(\lambda x.\xi)x = \xi$;
(β_4) $(\lambda xx.\xi)\mu = \lambda x.\xi$;

(β_5) $(\lambda x.\xi\mu)\nu = (\lambda x.\xi)\nu\big((\lambda x.\mu)\nu\big)$;
(β_6) $(\lambda y.\mu)z = \mu \rightarrow (\lambda xy.\xi)\mu = \lambda y.(\lambda x.\xi)\mu$, if $x \neq y, z \neq y$;
(α) $(\lambda y.\xi)z = \xi \rightarrow \lambda x.\xi = \lambda y.(\lambda x.\xi)y$, if $z \neq y$.

I is called the *dimension set* of $\boldsymbol{A}$. $\cdot^{\mathsf{A}}$ is called *application* and λx^{A} *λ-abstraction* with respect to x.

Axioms (β_1)–(β_6) constitute a definition of the abstract substitution that corresponds roughly to the axiomatic definition of metalinguistic substitution that can be found in [2]. This will become much more apparent below. (α) is a direct algebraic translation of (α)-conversion. The five congruence axioms of lambda calculus are not represented since by tradition they are implicit in every algebraic theory.

The class of lambda abstraction algebras of dimension I is denoted by LAA_I and the class of all lambda abstraction algebras of any dimension by LAA. We also use LAA_I as shorthand for the phrase "lambda abstraction algebra of dimension I", and similarly for LAA. A LAA_I is *infinite dimensional* if I is infinite.

In the sequel $\boldsymbol{A}$ will be an arbitrary LAA_I, unless otherwise noted. The dimension set I is also arbitrary; in particular it can be finite. We assume however that it contains at least three variables, since many of the results we obtain in this section require this.

We often omit the superscript $^{\mathsf{A}}$ on $\cdot^{\mathsf{A}}$, λx^{A}, and x^{A} when we are sure we can do so without confusion. This will also apply to defined notions introduced below, such as $\boldsymbol{\Delta}^{\mathsf{A}}$.

We note here a useful immediate consequence of the axioms: in any LAA $\boldsymbol{A}$ the functions λx^{A} are always one-one, i.e.,

$$\lambda x.a = \lambda x.b \quad \text{iff} \quad a = b, \quad \text{for all } a, b \in A.$$

For if $\lambda x.a = \lambda x.b$, then by ($\beta_3$), $a = (\lambda x.a)x = (\lambda x.b)x = b$.

A LAA with only one element is said to be *trivial.* λx^{A} is onto (i.e., its range is all of A) only if $\boldsymbol{A}$ is trivial. In particular, if $\boldsymbol{A}$ is nontrivial, then x^{A} cannot be in the range of λx^{A}. For if $\lambda x.b = x$ for some $b \in A$, then for every $a \in A$ we have by (β_1) and (β_4),

$$a = (\lambda x.x)a = (\lambda xx.b)a = \lambda x.b = x.$$

It follows that every nontrivial LAA is infinite.

Substitution and dimension. When transformed into the equational language of lambda abstraction theory, (β)-conversion becomes the definition of abstract substitution. It takes the following form: For any set S, let S^* be the set of all finite sequences of elements of S.

Definition. Let $\boldsymbol{A}$ be a LAA_I.

(i) $S_b^x(a) = (\lambda x.a)b$, for all $x \in I$ and $a, b \in A$.

(ii) $S_{\mathbf{b}}^{\mathbf{x}}(a) = S_{b_1}^{x_1}(\ldots(S_{b_n}^{x_n}(a))\ldots)$, for all $a \in A$, $\boldsymbol{x} = x_1 \cdots x_n \in I^*$, and $\boldsymbol{b} = b_1 \cdots b_n \in A^*$.

S is called the *(abstract) substitution operator.*

Definition. Let $\boldsymbol{A}$ be an LAA_I. Let $a \in A$ and $x \in I$. a is said to be *algebraically dependent on* x *(over* $\boldsymbol{A}$*)* if $(\lambda x.a)z \neq a$ for some $z \in I$; otherwise a is *algebraically independent of* x *(over* $\boldsymbol{A}$*)*. The set of all $x \in I$ such that a is algebraically dependent on x over $\boldsymbol{A}$ is called the *dimension set* of a and is denoted by $\Delta^{\mathsf{A}} a$; thus

$$\Delta^{\mathsf{A}} a = \{ x \in I : (\lambda x.a)z \neq a \text{ for some } z \in I \}.$$

a is *finite (infinite) dimensional* if Δa is finite (infinite).

It is convenient to treat algebraic dependency as a symmetric relation and speak of "x being algebraically dependent on (independent of) a". We shall see in Lem. 3 below that $x \notin \Delta a$ iff $(\lambda x.a)z = a$ for some $z \in I \setminus \{x\}$. Thus the axioms for lambda abstraction algebras can be reformulated in the following way:

(β_1) $S_\xi^x(x) = \xi$;
(β_2) $S_\xi^x(y) = y$, if $x \neq y$;
(β_3) $S_x^x(\xi) = \xi$;
(β_4) $S_\mu^x(\lambda x.\xi) = \lambda x.\xi$;
(β_5) $S_\nu^x(\xi\mu) = S_\nu^x(\xi) S_\nu^x(\mu)$;
(β_6) $y \notin \Delta\mu \implies S_\mu^x(\lambda y.\xi) = \lambda y.S_\mu^x(\xi)$, if $x \neq y$;
(α) $y \notin \Delta\xi \implies \lambda x.\xi = \lambda y.S_y^x(\xi)$.

Note that the two occurrences of x in (β_3) have different meanings, something that is hidden by our streamlined notation. This becomes apparent when we interpret (β_3) in an actual LAA_I and explicitly relativize all the operations:

$$(S^{\mathsf{A}})_{x^{\mathsf{A}}}^x(a) = ((\lambda x)^{\mathsf{A}}.a)x^{\mathsf{A}} = a, \quad \text{for all } a \in A \text{ and } x \in I.$$

We will avoid notation like "$(S^{\mathsf{A}})_{x^{\mathsf{A}}}^x$" because it is so cumbersome. We leave it to context to determine the particular algebra in which S is being applied.

If x and y are distinct λ-variables, then $x^{\mathsf{A}} \neq y^{\mathsf{A}}$ in any nontrivial LAA $\boldsymbol{A}$. To see this choose any $a \in A$ such that $a \neq y^{\mathsf{A}}$. Then by (β_2) and (β_3), $S_a^x(x^{\mathsf{A}}) = a \neq y^{\mathsf{A}} = S_a^x(y^{\mathsf{A}})$. So $x^{\mathsf{A}} \neq y^{\mathsf{A}}$.

The admissible functions of an environment model of the lambda calculus that have an intensional form, i.e., that can be represented by terms, must all be of finite rank since a term can have only a finite number of variables. There may be admissible functions that cannot be represented in this way and that in fact are of infinite rank. In terms of lambda abstraction theory this means that functional LAA's exist with elements of infinite dimension. The fact that each term contains only a finite number of variables from an infinite reservoir of variables is a critical property of the lambda calculus; the situation in first-order logic is similar. One would expect then that the algebraic analogue of this property would have a significant effect on the structure of a LAA, and this is indeed the case.

Definition. Let $\boldsymbol{A}$ be a LAA_I. $\boldsymbol{A}$ is *locally finite* if it is of infinite dimension (i.e., I is infinite) and every $a \in A$ is of finite dimension (i.e., $|\boldsymbol{\Delta} a| < \omega$).

LAA's, even locally finite ones, are no easier to construct than models of the lambda calculus. Indeed, we shall see below that the most natural LAA's all arise directly from environment models.

We will show in the next proposition that in the presence of the other axioms, (β_5) and (β_6) are equivalent to identities.

Lemma 1. *Axioms (β_4) and (β_5) imply $S^y_c S^y_b(a) = S^y_{S^y_c(b)}(a)$ for all $x \in I$ and $a, b, c \in A$.*

Proof.

$$\begin{aligned} S^y_c S^y_b(a) &= S^y_c\big((\lambda y.a)b\big) \\ &= S^y_c(\lambda y.a)S^y_c(b), && \text{by } (\beta_5) \\ &= (\lambda y.a)S^y_c(b), && \text{by } (\beta_4) \\ &= S^y_{S^y_c(b)}(a). \quad \square \end{aligned}$$

Proposition 2. *In the presence of (β_2), (β_4), and (β_5), the quasi-identities (β_6) and (α) are logically equivalent to the following identities, respectively,*

(β_6') $S^x_{S^y_z(\mu)}(\lambda y.\xi) = \lambda y.S^x_{S^y_z(\mu)}(\xi)$, *if $x \neq y, z \neq y$.*

(α') $\lambda x.S^y_z(\xi) = \lambda y.S^x_y S^y_z(\xi)$, *if $z \neq y$.*

Thus LAA_I is a variety for every dimension set I.

Proof. Clearly (β_6') implies (β_6). For the opposite implication, substitute $S^y_z(\mu)$ for μ in (β_6) and observe that the antecedent of the resulting quasi-identity, $S^y_z S^y_z(\mu) = S^y_z(\mu)$, is a consequence of the Lem. 1 and (β_2). The equivalence of (α) and (α') is established in the same way. $\square$

The following lemma shows that an element a depends on x if $S^x_b(a) \neq b$ for some b of $\boldsymbol{A}$, and, if a depends on x, then $S^x_z(a) \neq z$ for all $z \in I \setminus \{x\}$.

Lemma 3. *Let $A \in \mathsf{LAA}_I$, and let $a \in A$ and $x \in I$. The following are equivalent.*

(i) $S^x_z(a) = a$ *for some* $z \in I \setminus \{x\}$*;*

(ii) $S^x_z(a) = a$ *for all* $z \in I$*, i.e.,* $x \notin \mathbf{\Delta}a$*;*

(iii) $S^x_b(a) = a$ *for all* $b \in A$*.*

Proof. It clearly suffices to prove that (i) implies (iii). Assume $S^x_z(a) = a$ for some $z \in I \setminus \{x\}$. Then

$$\begin{aligned} S^x_b(a) &= S^x_b S^x_z(a) \\ &= S^x_{S^x_b(z)}(a), && \text{by Lem. 1} \\ &= S^x_z(a), && \text{by } (\beta_2) \\ &= a. && \text{by assumption.} \quad \square \end{aligned}$$

Proposition 4. *Let $A \in \mathsf{LAA}_I$, $a, b \in A$, and $x \in I$.*

(i) $\mathbf{\Delta}(ab) \subseteq \mathbf{\Delta}a \cup \mathbf{\Delta}b$.

(ii) $\mathbf{\Delta}(\lambda x.a) = \mathbf{\Delta}a \setminus \{x\}$.

(iii) $\mathbf{\Delta}\big(S^x_b(a)\big) \subseteq (\mathbf{\Delta}a \setminus \{x\}) \cup \mathbf{\Delta}b$.

(iv) $\mathbf{\Delta}x \subseteq \{x\}$, *with equality holding if A is nontrivial.*

Proof. (i) follows immediately from (β_5).

For the inclusion $\mathbf{\Delta}(\lambda x.a) \subseteq \mathbf{\Delta}a \setminus \{x\}$ of (ii) use (β_4) and (β_6). To get the opposite inclusion, suppose $y \notin \mathbf{\Delta}(\lambda x.a)$ and $y \neq x$. Then for any λ-variable $z \neq x, y$, we have $S^y_z(\lambda x.a) = \lambda x.a$. But by (β_2) and (β_6), $S^y_z(\lambda x.a) = \lambda x.S^y_z(a)$. Hence $\lambda x.S^y_z(a) = \lambda x.a$, which implies $S^y_z(a) = a$ since λx is one-one. Thus $y \notin \mathbf{\Delta}a$.

(iii) is a direct consequence of (i) and (ii). Finally, the inclusion of (iv) follows from (β_2) and the equality from (β_1). $\square$

In the following lemma we give some basic properties of substitution that will be used repeatedly in the sequel.

Lemma 5. *For all $x, y, z \in I$ and $a, b, c \in A$ we have:*

(i) $x \notin \mathbf{\Delta}c \implies S^y_c S^x_b(a) = S^x_{S^y_c(b)} S^y_c(a)$;

(ii) $y \notin \mathbf{\Delta}b \implies S^y_b S^x_y(a) = S^y_b S^x_b(a)$;

(iii) $x \notin \mathbf{\Delta}c, y \notin \mathbf{\Delta}b \implies S^y_c S^x_b(a) = S^x_b S^y_c(a)$, *if* $x \neq y$;

(iv) $z \notin \mathbf{\Delta}a \cup \mathbf{\Delta}b \implies S^x_b(a) = S^z_b S^x_z(a)$.

Proof. (i) If $x = y$ the equation reduces to Lem. 1. Assume $x \neq y$.

$$\begin{aligned} S^y_c S^x_b(a) &= S^y_c\big((\lambda x.a)b\big) \\ &= S^y_c(\lambda x.a)S^y_c(b), && \text{by } (\beta_4) \\ &= \big(\lambda x.S^y_c(a)\big)S^y_c(b), && \text{by } (\beta_6) \text{ since } x \notin \mathbf{\Delta}c \text{ and } x \neq y \\ &= S^x_{S^y_c(b)}\big(S^y_c(a)\big). \end{aligned}$$

(ii) By part (i) and (β_1) we have $S_b^y S_y^x(a) = S^x_{S_b^y(y)} S_b^y(a) = S_b^y S_b^x(a)$.

(iii)
$$\begin{aligned} S_c^y S_b^x(a) &= S^x_{S_c^y(b)} S_c^y(a), && \text{by (i) since } x \notin \Delta c \\ &= S_b^y S_c^y(a), && \text{since } y \notin \Delta b. \end{aligned}$$

(iv)
$$\begin{aligned} S_b^z S_z^x(a) &= S_b^z S_b^x(a), && \text{by (ii) since } z \notin \Delta b \\ &= S_b^x(a), && \text{since } z \notin \Delta a \cup \Delta b \supseteq \Delta S_b^x(a). \quad \square \end{aligned}$$

Simultaneous substitution. We abstract the process of simultaneously substituting a finite sequence $t_1, \ldots, t_n$ of terms for the variables $x_1, \ldots, x_n$ in a term s. Such a substitution can be simulated by a sequence of single substitutions provided the free occurrences of the x_i in s are first replaced by new variables that do not conflict with the free variables of $t_1, \ldots, t_n$. This is the basis of our abstraction. Implicit is the assumption that a reservoir of new variables is always available. Consequently simultaneous substitution can only be abstracted under some kind of dimension-restricting assumption. Here we assume the strongest such assumption, local finiteness. Thus for the remainder of the paper, unless specifically indicated otherwise, all LAA's will be locally finite.

We introduce some useful notation. $I^\star$ denotes the set of all finite sequences of λ-variables without repetitions. Let A be a (locally finite) LAA_I. Let $a \in A$, $\boldsymbol{x} = x_1 \cdots x_n \in I^\star$, and $p \in A^I$. Recall that a variable z is independent of a if $z \notin \Delta a$. Let $p \circ \boldsymbol{x} = p_{x_1} \cdots p_{x_n}$. We say that z is *independent* of $p \circ \boldsymbol{x}$ if $z \notin \Delta p_{x_1} \cup \cdots \cup \Delta p_{x_n}$.

Definition. Assume A is a locally finite LAA and $p \in A^I$. For each $a \in A$, let $\boldsymbol{x} = x_1 \cdots x_n \in I^\star$ such that $\Delta a \subseteq \{x_1, \ldots, x_n\}$, and choose $\boldsymbol{z} = z_1 \cdots z_n \in I^\star$ such that the z_i are distinct from the x_j and independent of both a and $p \circ \boldsymbol{x}$. We define

$$\tilde{S}_p(a) := S^{\mathbf{z}}_{p \circ \mathbf{x}} S^{\mathbf{x}}_{\mathbf{z}}(a) \quad \text{for all } a \in A.$$

Lemma 6. *The definition $\tilde{S}_p(a)$ is independent both of the choice of $\boldsymbol{x} = x_1 \cdots x_n$ such that $\{x_1, \ldots, x_n\} \subseteq \Delta a$ and the choice of $\boldsymbol{z} = z_1 \cdots z_n$ independent of a and $p \circ \boldsymbol{x}$.*

Proof. The proof is in three parts. We first prove that, with $\boldsymbol{x}$ fixed, the definition is independent of the choice of $\boldsymbol{z}$. Let $\boldsymbol{w} = w_1 \cdots w_n \in I^\star$ be any other sequence of variables that are distinct from the x_j and independent of both a and $p \circ \boldsymbol{x}$. We assume without loss of generality that they are also distinct from the z_i; otherwise we could consider a third sequence of variables disjoint from both $\boldsymbol{z}$ and $\boldsymbol{w}$. Let $\boldsymbol{x}' = x_2 \cdots x_n$; $\boldsymbol{z}'$ and $\boldsymbol{w}'$ are similarly defined.

$$\begin{aligned}
S^{\mathsf{z}}_{p\circ\mathsf{x}} S^{\mathsf{x}}_{\mathsf{z}}(a) &= S^{\mathsf{z}'}_{p\circ\mathsf{x}'} S^{z_1}_{p_{x_1}} S^{x_1}_{z_1} S^{\mathsf{x}'}_{\mathsf{z}'}(a), && \text{by Lem. 5(iii)}\\
&= S^{\mathsf{z}'}_{p\circ\mathsf{x}'} S^{w_1}_{p_{x_1}} S^{x_1}_{w_1} S^{\mathsf{x}'}_{\mathsf{z}'}(a), && \text{by Lem. 5(iv)}\\
&= S^{w_1}_{p_{x_1}} S^{\mathsf{z}'}_{p\circ\mathsf{x}'} S^{\mathsf{x}'}_{\mathsf{z}'} S^{x_1}_{w_1}(a), && \text{by Lem. 5(iii)}\\
&= S^{w_1}_{p_{x_1}} S^{\mathsf{w}'}_{p\circ\mathsf{x}'} S^{\mathsf{x}'}_{\mathsf{w}'} S^{x_1}_{w_1}(a), && \text{by induction hypothesis}\\
&= S^{\mathsf{w}}_{p\circ\mathsf{x}} S^{\mathsf{x}}_{\mathsf{w}}(a). && \text{by Lem. 5(iii)}
\end{aligned}$$

In the second part of the proof we show that the definition of $\tilde{S}_p(a)$ does not depend on the order of variables in the sequence $\boldsymbol{x} = x_1 \cdots x_n$. Let $\boldsymbol{x}'$ be any permutation of $\boldsymbol{x}$ and let $\boldsymbol{z}'$ be the corresponding permutation of $\boldsymbol{z}$. Then by repeated application of Lem. 5(iii) we have $S^{\mathsf{z}}_{p\circ\mathsf{x}} S^{\mathsf{x}}_{\mathsf{z}}(a) = S^{\mathsf{z}'}_{p\circ\mathsf{x}'} S^{\mathsf{x}'}_{\mathsf{z}'}(a)$.

Finally, assume $\boldsymbol{y} = y_1 \cdots y_m \in I^\star$ is another sequence of variables such that $\{y_1, \ldots, y_m\} \subseteq \boldsymbol{\Delta} a$, and let $\boldsymbol{w} = w_1 \cdots w_m \in I^\star$ be distinct from the y_j and independent of both a and $p \circ \boldsymbol{y}$. We must show that

$$S^{\mathsf{z}}_{p\circ\mathsf{x}} S^{\mathsf{x}}_{\mathsf{z}}(a) = S^{\mathsf{w}}_{p\circ\mathsf{y}} S^{\mathsf{y}}_{\mathsf{w}}(a).$$

Clearly we can assume without loss of generality that $\{y_1, \ldots, y_m\} = \boldsymbol{\Delta} a$. In view of the second part of the proof, we also assume without lost of generality that $\boldsymbol{x} = \boldsymbol{yu} = y_1 \cdots y_m u_1 \cdots u_{n-m}$ where $\{u_1, \ldots, u_{n-m}\} = \{x_1, \ldots, x_n\} \setminus \boldsymbol{\Delta} a$. Finally, in view of the first part of the proof, we assume without loss of generality that $\boldsymbol{w} = z_1 \cdots z_m$. Let $\boldsymbol{v} = z_{m+1} \cdots z_n$ so that $\boldsymbol{z}$ $\boldsymbol{wv}$.

$$\begin{aligned}
S^{\mathsf{z}}_{p\circ\mathsf{x}} S^{\mathsf{x}}_{\mathsf{z}}(a) &= S^{\mathsf{w}\,\mathsf{v}}_{p\circ(\mathsf{y}\,\mathsf{u})} S^{\mathsf{y}\,\mathsf{u}}_{\mathsf{w}\,\mathsf{v}}(a)\\
&= S^{\mathsf{w}}_{p\circ\mathsf{y}} S^{\mathsf{v}}_{p\circ\mathsf{u}} S^{\mathsf{y}}_{\mathsf{w}} S^{\mathsf{u}}_{\mathsf{v}}(a)\\
&= S^{\mathsf{w}}_{p\circ\mathsf{y}} S^{\mathsf{v}}_{p\circ\mathsf{u}} S^{\mathsf{y}}_{\mathsf{w}}(a), && \text{by Lem. 3 since } z_i \notin \boldsymbol{\Delta} a \text{ for all } i\\
&= S^{\mathsf{w}}_{p\circ\mathsf{y}} S^{\mathsf{y}}_{\mathsf{w}}(a).
\end{aligned}$$

The last equality holds by Lem. 3 since $z_i \notin \boldsymbol{\Delta} a \cup \{z_1, \ldots, z_m\} \supseteq S^{\mathsf{y}}_{\mathsf{w}}$ for $i = z_{n-m}, \ldots, z_n$. □

Thus $\tilde{S}_p$ is a well defined mapping from A into itself for all every $p \in A^I$. $\tilde{S}$ is called the *simultaneous substitution operator* and $\tilde{S}_p(a)$ is the result of *simultaneously substituting* p_x for x in a for each $x \in \boldsymbol{\Delta} a$.

Lemma 7. *Assume $\boldsymbol{A}$ is a locally finite* LAA_I. *Let $\varepsilon \in A^I$ be defined by $\varepsilon_x = x^{\mathsf{A}}$ for each $x \in I$. Then $\tilde{S}_\varepsilon(a) = a$ for all $a \in A$.*

Proof. Let $\boldsymbol{x} = x_1 \cdots x_n$ be an enumeration without repetitions of $\boldsymbol{\Delta} a$ and let $\boldsymbol{z} = z_1 \cdots z_n$ be disjoint from $\boldsymbol{x}$ and independent of a. Note that $\varepsilon \circ \boldsymbol{x} = \boldsymbol{x}$. Let $\boldsymbol{x}' = x_2 \cdots x_n$ and $\boldsymbol{z}' = z_2 \cdots z_n$. Then

$$
\begin{aligned}
\tilde{S}_\varepsilon(a) &= S^{\mathsf{z}}_{\varepsilon\circ\mathsf{x}} S^{\mathsf{x}}_{\mathsf{z}}(a) \\
&= S^{\mathsf{z}}_{\mathsf{x}} S^{\mathsf{x}}_{\mathsf{z}}(a) \\
&= S^{\mathsf{z}'}_{\mathsf{x}'} S^{z_1}_{x_1} S^{x_1}_{z_1} S^{\mathsf{x}'}_{\mathsf{z}'}(a), && \text{by Lem. 5(iii)} \\
&= S^{\mathsf{z}'}_{\mathsf{x}'} S^{\mathsf{x}'}_{\mathsf{z}'}(a), && \text{by Lem. 5(iv) and } (\beta_3) \\
&= a, && \text{by the induction hypothesis.} \quad \square
\end{aligned}
$$

Let $p \in A^I$, $a \in A$, and $x \in I$. Then $p(a/x) \in A^I$ is the mapping such that for all $y \in I$

$$
p(a/x)_y := \begin{cases} a, & \text{if } y = x \\ p_y, & \text{otherwise.} \end{cases}
$$

Lemma 8. *Assume $\boldsymbol{A}$ is a locally finite* LAA_I. *Let $p \in I$.*

(i) $\tilde{S}_p(ab) = \tilde{S}_p(a)\tilde{S}_p(b)$ *for all $a, b \in A$.*

Let $a \in A$ and $x \in I$. Let $z \in I$ be distinct from x and independent of a and of p_y for every $y \in \boldsymbol{\Delta} a$.

(ii) $\tilde{S}_p(\lambda x.a) = \lambda z.\tilde{S}_{p(z/x)}(a)$.

(ii) $S^z_b \tilde{S}_{p(z/x)}(a) = \tilde{S}_{p(b/x)}(a)$ *for every $b \in A$.*

Proof. (i) is an immediate consequence of (β_5) and the definition of $\tilde{S}$.

(ii) Let $\boldsymbol{y} = y_1 \cdots y_n \in I^\star$ such that $\{y_1, \ldots, y_n\} = \boldsymbol{\Delta} a \setminus \{x\}$. Let $\boldsymbol{w} = w_1 \cdots w_n \in I^\star$ be disjoint from $\boldsymbol{y}$ and $\{x, z\}$, and independent of both a and $p \circ \boldsymbol{y}$. Then

$$
\begin{aligned}
\lambda z.\tilde{S}_{p(z/x)}(a) &= \lambda z.S^{\mathsf{w}}_{p\circ\mathsf{y}} S^{z}_{p(z/x)_x} S^{\mathsf{y}}_{\mathsf{w}} S^{x}_{z}(a) \\
&= \lambda z.S^{\mathsf{w}}_{p\circ\mathsf{y}} S^{z}_{z} S^{\mathsf{y}}_{\mathsf{w}} S^{x}_{z}(a) \\
&= \lambda z.S^{\mathsf{w}}_{p\circ\mathsf{y}} S^{\mathsf{y}}_{\mathsf{w}} S^{x}_{z}(a), && \text{by } (\beta_3) \\
&= S^{\mathsf{w}}_{p\circ\mathsf{y}} S^{\mathsf{y}}_{\mathsf{w}} \left(\lambda z.S^{x}_{z}(a)\right), && \text{by } (\beta_6) \\
&= \tilde{S}_p(\lambda x.a), && \text{by } (\alpha).
\end{aligned}
$$

(iii) If $x \notin \mathbf{\Delta}a$ the result is obvious. So we assume $x \in \mathbf{\Delta}a$. Let $\boldsymbol{y}$ and $\boldsymbol{w}$ be as in part (ii), and assume in addition that $\boldsymbol{w}$ is independent of b. Then

$$\begin{aligned} S^z_b \tilde{S}_{p(z/x)}(a) &= S^z_b S^{\mathrm{w}\,z}_{p(z/x)\circ(\mathrm{y}\,x)} S^{\mathrm{y}\,x}_{\mathrm{w}\,z}(a) \\ &= S^z_b S^{\mathrm{w}}_{p\circ\mathrm{y}} S^z_{p(z/x)_x} S^{\mathrm{y}}_{\mathrm{w}} S^x_z(a) \\ &= S^{\mathrm{w}}_{p\circ\mathrm{y}} S^z_b S^{\mathrm{y}}_{\mathrm{w}} S^x_z(a), \qquad \text{by } (\beta_3) \text{ and Lem. 5(iii)} \\ &= S^{\mathrm{w}\,z}_{p(b/x)\circ(\mathrm{y}\,x)} S^{\mathrm{y}\,x}_{\mathrm{w}\,z}(a) \\ &= \tilde{S}_{p(b/x)}(a). \quad \square \end{aligned}$$

Functional LAA's and representation. The main difference between lambda abstraction theory and the lambda calculus is that one is a pure equational theory while the other is not. This accounts for the distinctly algebraic nature of the lambda abstraction theory in contrast to the highly combinatorial lambda calculus. Another important distinction between them is that it is apparent what the models of lambda abstraction theory are: they are the algebras that satisfy the laws (β_1)–(β_6), (α), i.e., lambda abstraction algebras. In this sense there is a close analogy with the theories of groups and rings. On the other hand, there are no simple LAA's (in the nontechnical sense); in particular, there are no finite LAA's. In fact, as we have already mentioned, nontrivial lambda algebras are no easier to construct than nontrivial models of the lambda calculus.

There is a notion of a "natural" lambda abstraction algebra–the algebras that the axioms of lambda abstraction theory are intended to characterize. They correspond to functional polyadic algebras and, more loosely, to representable cylindric algebras; we call them *functional lambda abstraction algebras.* Not surprisingly, they are closely related to the environment models of lambda calculus. Functional LAA's are obtained by *coordinatizing* environment models by the λ-variables in a natural way. We will try to explain this intuitively before we give the formal definition.

Let $\boldsymbol{V}$ be an environment model and let $\lambda^{\mathrm{V}} : V^V \xrightarrow{\mathrm{p}} V$ be the encoding of the admissible functions of $\boldsymbol{V}$ into V. Let I be the set of λ-variables. Elements of V^I (i.e., assignments of elements of V to λ-variables) are called *environments.* Let $f \in V^{V^I}$. Each λ-variable x and environment p determines a function $f_{x,p} = \langle f(p(v/x)) : v \in V\rangle$ in V^V; recall that $p(v/x)$ is the environment obtained from p by reassigning v to x. f is *admissible* if each of the functions $f_{x,p}$ is admissible in $\boldsymbol{V}$, i.e., is in the domain of λ^{V}. Every functional lambda abstraction algebra $\boldsymbol{A}$ consists of a set of admissible functions in V^{V^I} for some environment model $\boldsymbol{V}$. $\boldsymbol{V}$ is called the *value domain* of $\boldsymbol{A}$. The λ-abstraction of a member f of $\boldsymbol{A}$ is defined as follows: $\lambda x^{\mathrm{A}}(f)(p) = \lambda^{\mathrm{V}}(f_{x,p})$ for every environment p. f and $\lambda x^{\mathrm{A}}(f)$ can be viewed as two different forms of the same function in V^{V^I} with the former being the intensional and the latter the extensional form. The important point here is that, in contrast to environment models, the intensional form of the function corresponds to an actual element of the functional LAA. This is similar to cylindric algebras where the propositional function associated with a first-order formula (with free variables) corresponds to an actual element of the algebra. In the lambda calculus the only way of accessing the intensional form of a function is by means of a λ-term that defines it.

Definition. Let $\boldsymbol{V} = \langle V, \cdot^{\mathsf{V}}, \lambda^{\mathsf{V}} \rangle$ be a structure where V is a nonempty set, $\cdot^{\mathsf{V}}$ is a binary operation on V, and $\lambda^{\mathsf{V}} : V^V \xrightarrow{\mathrm{P}} V$ is a partial function assigning elements of V to certain functions from V into itself. $\boldsymbol{V}$ is called a *functional domain* if for each f in the domain of λ^{V},

$$f(v) = \big(\lambda^{\mathsf{V}}(f)\big) \cdot^{\mathsf{V}} v, \quad \text{for all } v \in V. \tag{2}$$

Note that λ^{V} must be injective.

Functional domains have an alternative characterization that is taken to be their definition in Meyer [10].

Let $\boldsymbol{V}$ be a functional domain. Define $\Phi\colon V \to V^V$ by setting

$$\Phi(u) := \langle u \cdot^{\mathsf{V}} v : v \in V \rangle, \quad \text{for all } u \in V.$$

(2) can be reformulated in terms of Φ as $f(v) = \Phi\big(\lambda^{\mathsf{V}}(f)\big)(v)$, for all $v \in V$, i.e.,

$$\Phi\big(\lambda^{\mathsf{V}}(f)\big) = f, \quad \text{for all } f \text{ in the domain of } \lambda^{\mathsf{V}}. \tag{3}$$

Conversely, let V is a nonempty set, $\Phi\colon V \to V^V$, and $\lambda\colon V^V \xrightarrow{\mathrm{P}} V$ such that (3) holds. Define $u \cdot v = \big(\Phi(u)\big)(v)$, for all $u, v \in V$. Then (2) holds and hence $\langle V, \cdot, \lambda \rangle$ is a functional domain. Thus a functional domain can be alternatively defined as a structure $\langle V, \lambda^{\mathsf{V}}, \Phi^{\mathsf{V}} \rangle$ satisfying (3).

Definition. Let $\boldsymbol{V} = \langle V, \cdot^{\mathsf{V}}, \lambda^{\mathsf{V}} \rangle$ be a functional domain and let I be a nonempty set. Let $V_I = \{ f : f\colon V^I \xrightarrow{\mathrm{P}} V \}$, i.e., the set of all partial functions from V^I to V. By the *I-coordinatization* of $\boldsymbol{V}$ we mean the algebra

$$\boldsymbol{V}_I = \big\langle V_I, \cdot^{\mathsf{V}_I}, \langle \lambda x^{\mathsf{V}_I} : x \in I \rangle, \langle x^{\mathsf{V}_I} : x \in I \rangle \big\rangle,$$

where, for all $a, b\colon V^I \xrightarrow{\mathrm{P}} V$, $x \in I$, and $p \in V^I$:

- $(a \cdot^{\mathsf{V}_I} b)(p) := a(p) \cdot^{\mathsf{V}} b(p)$, provided $a(p)$ and $b(p)$ are both defined; otherwise $(a \cdot^{\mathsf{V}_I} b)(p)$ is undefined.
- $(\lambda x^{\mathsf{V}_I}.a)(p) := \lambda^{\mathsf{V}}\big(\langle a(p(v/x)) : v \in V \rangle\big)$, provided $\langle a(p(v/x)) : v \in V \rangle$ is in the domain of λ^{V} (note this implies $a(p(v/x))$ is defined for all $v \in V$); otherwise $(\lambda x^{\mathsf{V}_I}.a)(p)$ is undefined.
- $x^{\mathsf{V}_I}(p) := p_x$.

Definition. Let $\boldsymbol{V}$ and I be as in the preceding definition. A subalgebra $\boldsymbol{A}$ of total functions of $\boldsymbol{V}_I$, i.e., a subalgebra such that $(\lambda x^{\mathsf{V}_I}.a)(p)$ is defined for all $a \in A$ and $p \in V^I$, is called a *functional lambda abstraction algebra.* I is the *dimension set* of $\boldsymbol{A}$ and $\boldsymbol{V}$ is its *value domain.*

In the sequel a subalgebra of $\boldsymbol{V}_I$ of total algebras will be called a *total subalgebra* of $\boldsymbol{V}_I$.

Lemma 9. *Let $\boldsymbol{A}$ be a functional lambda abstraction algebra with dimension set I and value domain $\boldsymbol{V}$. Let $a \in A$ and $x \in I$.*

(i) *For all $b \in A$ and $p \in V^I$,*

$$\big(S_b^x(a)\big)(p) = a\big(p(b(p)/x)\big).$$

(ii) *x is algebraically independent of a (i.e., $x \notin \boldsymbol{\Delta} a$) iff*

$$a\big(p(v/x)\big) = a(p), \quad \textit{for all } v \in V.$$

Proof.

(i)

$$\begin{aligned}\big(S_b^x(a)\big)(p) &= \big((\lambda x^{\mathsf{A}}.a) \cdot^{\mathsf{A}} b\big)(p)\\ &= \big((\lambda x^{\mathsf{V}_I}.a) \cdot^{\mathsf{V}_I} b\big)(p)\\ &= (\lambda x^{\mathsf{V}_I}.a)(p) \cdot^{\mathsf{V}} b(p)\\ &= \lambda^{\mathsf{V}}\big(\langle a(p(v/x)) : v \in V\rangle\big) \cdot^{\mathsf{V}} b(p)\\ &= a\big(p(b(p)/x)\big), \quad \text{by (2).} \quad \Box\end{aligned}$$

(ii) Assume $x \notin \boldsymbol{\Delta} b$, i.e., $S_z^x(b) = b$ for some $z \neq x$. Then, for all $q \in V^I$, $b\big(q(q_z/x)\big) = b\big(q(z^{\mathsf{A}}(q)/x)\big) = b\big(q\big)$ by part (i). First taking $q = p$ we get

$$(5) \qquad b\big(p(p_z/x)\big) = b\big(p\big).$$

Now take $q = p(v/x)$ for arbitrary $v \in V$, and note that $q_z = p_z$ and hence $q(q_z/x) = p(v/x)(p_z/x) = p(p_z/x)$. We thus get

$$(5) \qquad b\big(p(p_z/x)\big) = b\big(p(v/x)\big).$$

The combination of (4) and (5) gives the conclusion of the implication from left to right. The proof of the implication in the opposite direction is straightforward and is omitted. $\Box$

Proposition 10. *Every functional lambda abstraction algebra is a lambda abstraction algebra.*

Proof. Let $\boldsymbol{A}$ be a functional LAA_I. We verify each of the axioms (β_1)–(β_6), (α). We will use Lem. 9 repeatedly, often without comment. Let $a, b, c \in A$, $x, y \in I$ with $x \neq y$, and $p \in V^I$.

(β_1) $\big(S^x_a(x^{\mathsf{A}})\big)(p) = x^{\mathsf{A}}\big(p(a(p)/x)\big) = p(a(p)/x)_x = a(p)$.

(β_2) $\big(S^x_a(y^{\mathsf{A}})\big)(p) = y^{\mathsf{A}}\big(p(a(p)/x)\big) = p(a(p)/x)_y = p_y = y^{\mathsf{A}}(p)$.

(β_3) $\big(S^x_{x^{\mathsf{A}}}(a)\big)(p) = a\big(p(x^{\mathsf{A}}(p)/x)\big) = a\big(p(p_x/x)\big) = a(p)$.

(β_4)

$$\begin{aligned}\big(S^x_b(\lambda x^{\mathsf{A}}.a)\big)(p) &= (\lambda x^{\mathsf{A}}.a)\big(p(b(p)/x)\big)\\ &= \lambda^{\mathsf{V}}\big(\langle a\big(p(b(p)/x)(v/x)\big) : v \in V\rangle\big)\\ &= \lambda^{\mathsf{V}}\big(\langle a\big(p(v/x)\big) : v \in V\rangle\big), \quad \text{since } p(b(p)/x)(v/x) = p(v/x)\\ &= (\lambda x^{\mathsf{A}}.a)(p).\end{aligned}$$

(β_5)

$$\begin{aligned}\big(S^x_c(a \cdot^{\mathsf{A}} b)\big)(p) &= (a \cdot^{\mathsf{A}} b)\big(p(c(p)/x)\big)\\ &= a\big(p(c(p)/x)\big) \cdot^{\mathsf{V}} b\big(p(c(p)/x)\big)\\ &= \big(S^x_c(a)\big)(p) \cdot^{\mathsf{V}} \big(S^x_c(b)\big)(p)\\ &= \big(S^x_c(a) \cdot^{\mathsf{A}} S^x_c(b)\big)(p).\end{aligned}$$

(β_6) Assume $y \notin \Delta b$. For every $v \in V$ set $p^v = p(v/y)$. Then by Lem. 9(ii), $b(p) = b\big(p(v/y)\big) = b(p^v)$, and hence,

(6) $\quad p\big(b(p)/x\big)(v/y) = p(v/y)\big(b(p)/x\big) = p(v/y)\big(b(p(v/y))/x\big) = p^v\big(b(p^v)/x\big)$.

$$\begin{aligned}\big(S^x_b(\lambda y^{\mathsf{A}}.a)\big)(p) &= (\lambda y^{\mathsf{A}}.a)\big(p(b(p)/x)\big)\\ &= \lambda^{\mathsf{V}}\big(\langle a\big(p\big(b(p)/x\big)(v/y)\big) : v \in V\rangle\big)\\ &= \lambda^{\mathsf{V}}\big(\langle a\big(p^v\big(b(p^v)/x\big)\big) : v \in V\rangle\big), \qquad \text{by (6)}\\ &= \lambda^{\mathsf{V}}\big(\langle S^x_b(a)\big(p^v\big) : v \in V\rangle\big)\\ &= \lambda^{\mathsf{V}}\big(\langle S^x_b(a)\big(p(v/y)\big) : v \in V\rangle\big)\\ &= \big(\lambda y^{\mathsf{A}}.S^x_b(a)\big)(p).\end{aligned}$$

The verification of (α) is similar to that of (β_6) and is omitted. □

Let $\boldsymbol{A} = \langle A, \cdot^{\mathsf{A}}, \lambda x^{\mathsf{A}}, x^{\mathsf{A}} \rangle_{x \in I}$ be an arbitrary LAA_I. We define the functional domain $\boldsymbol{V} = \langle V, \cdot^{\mathsf{V}}, \lambda^{\mathsf{V}} \rangle$ *associated with* $\boldsymbol{A}$ as follows: $V = A$ and $\cdot^{\mathsf{V}} = \cdot^{\mathsf{A}}$. The domain of $\lambda^{\mathsf{V}} : V^V \xrightarrow{\mathrm{p}} V$ is

$$D_{\mathsf{A}} = \left\{ \left\langle S^x_v(a) : v \in V \right\rangle : a \in A \text{ and } x \in I \right\},$$

and for each function in this set we define

$$\lambda^{\mathsf{V}} \left(\left\langle S^x_v(a) : v \in V \right\rangle \right) := \lambda x^{\mathsf{A}} . a.$$

The following lemma shows that λ^{V} is well-defined.

Lemma 11. $\left\langle S^x_v(a) : v \in V \right\rangle = \left\langle S^y_v(b) : v \in V \right\rangle$ *implies* $\lambda x^{\mathsf{A}} .a = \lambda y^{\mathsf{A}} .b$.

Proof. Assume $S^x_v(a) = S^y_v(b)$, for all $v \in V$. If $x = y$, then taking $v = x^{\mathsf{A}} = y^{\mathsf{A}}$, we get $a = S^x_v(a) = S^y_v(b) = b$ by (β_3), and hence $\lambda x^{\mathsf{A}} .a = \lambda y^{\mathsf{A}} .b$. Suppose now that $x \neq y$. Taking $v = x^{\mathsf{A}}$, we get $a = S^x_v(a) = S^y_v(b) = S^y_{x^{\mathsf{A}}}(b)$. Thus a is independent of y by Prop. 4(iii). Now taking $v = y^{\mathsf{A}}$, we get $b = S^y_v(b) = S^x_v(a) = S^x_{y^{\mathsf{A}}}(a)$. Hence, by (α), $\lambda x^{\mathsf{A}} .a = \lambda y^{\mathsf{A}} .S^x_{y^{\mathsf{A}}}(a) = \lambda y^{\mathsf{A}} .b$. □

So λ^{V} and hence the structure $\boldsymbol{V}$ are well defined. Recall that $V = A$.

Lemma 12. $\boldsymbol{V}$ *is a functional domain.*

Proof. We must show that, for each f in the domain of λ^{V}, condition (2) is satisfied. f is of the form $\langle S^x_v(a) : v \in V \rangle$, for some $a \in A$ and $x \in I$. Thus

$$\left(\lambda^{\mathsf{V}}(f)\right) \cdot^{\mathsf{V}} v = (\lambda x^{\mathsf{A}} .a) \cdot^{\mathsf{A}} v = S^x_v(a) = f(v). \quad \square$$

The following theorem is the main result of the paper. It is the algebraic analogue of the completeness theorem for lambda calculus.

Theorem 13 (Functional Representation of Locally Finite LAA's). *Every locally finite lambda abstraction algebra* $\boldsymbol{A}$ *is isomorphic to a functional lambda abstraction algebra. More precisely,* $\boldsymbol{A}$ *is isomorphic to a total subalgebra of the* I*-coordinatization of its associated functional domain.*

Proof. Let $\boldsymbol{V}$ be the functional domain associated with $\boldsymbol{A}$ and let $\boldsymbol{V}_I$ be its I-coordinatization. Define $\Psi : A \to V_I$ as follows (recall that V_I is the set of all partial functions from V^I to V):

$$\Psi(a)(p) = \tilde{S}_p(a), \quad \text{for every } a \in A \text{ and } p \in V^I.$$

Note that $\Psi(a)$ is a total function. Recall that $\varepsilon = \langle x^{\mathsf{A}} : x \in I \rangle \in V^I$. $\Psi(a)(\varepsilon) = \tilde{S}_\varepsilon(a) = a$ by Lem. 7. So Ψ is one-to-one. We complete the proof by verifying that

Ψ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{V}_I$ and hence an isomorphism between $\boldsymbol{A}$ and a total subalgebra of $\boldsymbol{V}_I$.

Let $a, b \in A$, $x \in I$, and $p \in V^I$.

$$\begin{aligned}
\Psi(a \cdot^{\mathsf{A}} b)(p) &= \tilde{S}_p(a \cdot^{\mathsf{A}} b) \\
&= \tilde{S}_p(a) \cdot^{\mathsf{V}} \tilde{S}_p(b), && \text{by Lem. 8(i)} \\
&= \Psi(a)(p) \cdot^{\mathsf{V}} \Psi(b)(p) \\
&= \big(\Psi(a) \cdot^{\mathsf{V}_I} \Psi(b)\big)(p).
\end{aligned}$$

Choose $z \in I$ distinct from x and independent of both a and p_y for every $y \in \boldsymbol{\Delta} a$.

$$\begin{aligned}
\Psi(\lambda x^{\mathsf{A}}.a)(p) &= \tilde{S}_p(\lambda x^{\mathsf{A}}.a) \\
&= \lambda z^{\mathsf{A}}.\tilde{S}_{p(z/x)}(a), && \text{by Lem. 8(ii)} \\
&= \lambda^{\mathsf{V}}\big(\langle S^z_v \tilde{S}_{p(z/x)}(a) : v \in V\rangle\big) \\
&= \lambda^{\mathsf{V}}\big(\langle \tilde{S}_{p(v/x)}(a) : v \in V\rangle\big), && \text{by Lem. 8(iii)} \\
&= \lambda^{\mathsf{V}}\big(\langle \Psi(a)(p(v/x)) : v \in V\rangle\big) \\
&= \big(\lambda x^{\mathsf{V}_I}.\Psi(a)\big)(p).
\end{aligned}$$

Finally, the interpretations of the λ-variables are preserved (recall that the λ-variables are constant symbols in the language of LAA's).

$$\Psi(x^{\mathsf{A}})(p) = \tilde{S}_p(x^{\mathsf{A}}) = S^x_{p_x}(x^{\mathsf{A}}) = p_x = x^{\mathsf{V}_I}(p). \quad \square$$

Further results. Lambda abstraction theory is much more extensively developed than we have been able to indicate here. As in the case of the theory of cylindric algebras, which it parallels to a large extent, the emphasis is on representation results. There exist LAA's, even of infinite dimension, that are not isomorphic to any functional LAA. But there are much weaker dimension-restricting conditions than local finiteness that guarantee functional representability.

A LAA is said to be *dimension-complemented* if it is of infinite dimension and $\boldsymbol{\Delta} a \neq I$ for all $a \in A$. Every dimension-complemented LAA is isomomorphic to a certain kind of generalized functional LAA called a *point-relativized functional* LAA. The point-relativized functional LAA's turn out to be (up to isomorphism) exactly the LAA's that can be *neatly embedded* in an LAA of infinitely higher dimension. (This notion is the exact analogue of that of the same name in the theory of cylindric algebras; see [8, Part I].) Using this result we can show that the point-relativized functional LAA_I's form a variety and are thus axiomatized by pure identities. Every functional LAA_I is isomorphic to a point-relativized functional LAA_I, but we do not know at this time if the converse is true. However they do generate the same variety.

A large part of the theory is devoted to exploring the connection with models of lambda calculus. The two kinds of models of lambda calculus of most interest are the ***lambda algebras*** and the ***lambda models***;[2] see Barendregt [1] and Meyer [10]. Let $\boldsymbol{A}$ be a LAA. The *zero-dimensional part* of $\boldsymbol{A}$ is the set of elements with empty dimension set, together with the appropriate restriction of the application operation. If $\boldsymbol{A}$ is of infinite dimension, then its zero-dimensional part is a lambda algebra, and every lambda algebra can be obtained this way. Using this characterization we give a new proof, using Birkhoff's theorem (see Grätzer [5, p.152]), that lambda algebras form a variety. Lambda models are special kinds of lambda algebras and can be identified with those functional domains $\boldsymbol{V}$ for which there exists a functional LAA with value domain $\boldsymbol{V}$. A functional LAA_I is *full* if it is the largest possible LAA_I over its value domain. A full functional LAA_I exists over every lambda model and is obviously unique. A natural notion of a *full* point-relativized functional LAA can also be defined and there is a corresponding existence result. Lambda models can be characterized (up to isomorphism) as the zero-dimensional parts of full point-relativized functional LAA's.

Connections with other work. As we have tried to emphasize in this paper, lambda abstraction algebras can be viewed as a contribution to the theory abstract substitution. Cylindric and polyadic algebras are two early contributions to this theory that have greatly influenced our work. The main reference for cylindric algebras is [8]; for polyadic algebras it is [7], see especially [6]. We also mention here Németi [11]. It contains an extensive survey of the various algebraic versions of quantifier logics; it also includes a comprehensive bibliography.

None of these systems presents a theory of pure substitution. In lambda abstraction and cylindric algebras, abstract substitution is a defined operation. In polyadic algebras it is a primitive notion, but there are other primitive notions present (viz., abstract quantification and the Boolean operations). In the ***transformation algebras*** and ***substitution algebras*** of LeBlanc [9] and Pinter [13] substitution is primitive and abstract quantification is defined in terms of it. A pure theory of abstract substitution has been developed by Feldman [3,4] (see the additional references given in his first paper). This work parallels ours in many respects and we acknowledge our indebtedness to it.

Finally, we mention that some work that has been done on a theory of substitution in combination with abstract variable-binding operators. See [12], [14].

[2] Lambda models are essentially the same as environment models; see [10].

References

1. H.P. Barendregt, *The lambda Calculus. Its syntax and semantics*, Revised edition, Studies in Logic and the Foundations of Mathematics, Vol. 103, North-Holland Publishing Co., Amsterdam, 1985.
2. H.B. Curry and R. Feys, *Combinatory Logic, Vol. I*, North-Holland Publishing Co., Amsterdam, 1958.
3. N. Feldman, *Axiomatization of polynomial substitution algebras*, J. Symbolic Logic **47** (1982), 481–492.
4. ______, *Representation of substitution algebras*, Abstract, J. Symbolic Logic **49** (1984), 677–678.
5. G. Grätzer, *Universal Algebra*, Second Edition, Springer-Verlag, New York, 1979.
6. P. Halmos, *Homogeneous locally finite polyadic Boolean algebras of infinite degree*, Fund. Math. **43** (1956), 255-325, See also [7], pp. 97–166.
7. ______, *Algebraic logic*, Chelsea Publishing Co., New York, 1962.
8. L. Henkin, J.D. Monk and A. Tarski, *Cylindric algebras , Parts I and II*, North-Holland Publishing Co., Amsterdam, 1971, 1985.
9. L. LeBlanc, *Transformation algebras*, Canad. J. Math. **13** (1961), 602–613.
10. A.R. Meyer, *What is a model of the lambda calculus?*, Inform. Control **52** (1982), 87–122.
11. I. Németi, *Algebraizations of quantifier logics. An introductory overview*, Studia Logica **50** (1991), 485–569.
12. D. Pigozzi and A. Salibra, *The abstract variable-binding calculus*, Manuscript.
13. C. Pinter, *Cylindric algebras and algebras of substitution*, Trans. Amer. Math. Soc. **175** (1973), 167–179.
14. A. Salibra, *A general theory of algebras with quantifiers*, Algebraic Logic (Proc. Conf. Budapest 1988) (H. Andréka, J.D. Monk, I. Németi, eds.), Colloq. Math. Soc. J. Bolyai, Vol. 54, North-Holland Publishing Co., Amsterdam, 1991, pp. 573–620.
15. D.S. Scott, *Data types as lattices*, SIAM J. Comput. **5** (1976), 522–587.

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Free Lattice-Ordered Abelian Groups and Varieties of MV-Algebras

Roberto Cignoli

MV-algebras were introduced by Chang [4], [5] as the algebraic counterparts of the Lukasiewicz infinite valued propositional logic.

They can be defined as algebras $\langle A, \oplus, \neg, 0\rangle$ of type $\langle 2,1,0\rangle$ satisfying the following equations:

MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2) $x \oplus y = y \oplus x$

MV3) $x \oplus 0 = x$

MV4) $\neg\neg x = x$

MV5) $x \oplus \neg 0 = \neg 0$

MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$

Indeed, by taking $y = \neg 0$ in MV6 we obtain:

MV7) $x \oplus \neg x = \neg 0$.

Therefore, if we set $1 = \neg 0$ and $x \odot y = \neg(\neg x \oplus \neg y)$, then the system $\langle A, \oplus, \odot, \neg, 0, 1\rangle$ satisfies all the axioms given in [13], Lema 2.6, and hence the above definition of an MV-algebra is equivalent to Chang's original definition (cf.[6]).

Moreover, if we define $x \Rightarrow y = \neg x \oplus y$, then the system $\langle A, \Rightarrow, \neg, 1\rangle$ is a *CN-algebra* [9], [11] or a *Wajsberg algebra* [8]. Conversely, if in a Wajsberg algebra $\langle A, \Rightarrow, \neg, 1\rangle$ we put $x \oplus y = \neg x \Rightarrow y$ and $0 = \neg 1$, then the system $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra (see [8]).

MV-algebras also coincide with *bounded commutative BCK-algebras* (see [8] and [14]), and hence they also coincide with Bosbach's *bricks* [3].

As usual, we are going to denote an MV-algebra $\langle A, \oplus, \neg, 0\rangle$ by its underlying set A.

Let G be a lattice-ordered abelian group (abelian l-group for short) and $u \in G$, $u > 0$. Then the segment $[0,u] = \{x \in G \mid 0 \leq x \leq u\}$ equipped with the operations: $x \oplus y = (x+y) \wedge u$, and $\neg x = u - x$, is an MV-algebra, which we will denote by $\Gamma(G,u)$. This is the most general example of an MV-algebra, because Lacava [12], generalizing a previous result of Chang [5], showed that for any MV-algebra A there is an abelian l-group G and $0 < u \in G$ such that A is isomorphic to $\Gamma(A,u)$. If H is another abelian l-group and $0 < v \in H$, and $h : G \to H$ is an l-group homomorphism such that $h(u) = v$, then the restriction of h to the segment $[0,u]$ is a homomorphism $\Gamma(h)$ from $\Gamma(G,u)$ into $\Gamma(H,v)$, and it is easy to check that Γ is in fact a functor from the category $\mathcal{A}$ of abelian l-groups to the category $\mathcal{M}$ of MV-algebras.

The relations between MV-algebras and abelian l-groups were further developed by Mundici [13]. An element u of an l-group G is said to be a *unit* if for each x in G there is a natural number n such that $x < nu$. Let $\mathcal{G}$ denote the category whose objects are the pairs (G,u) such that G is an abelian l-group and u is a unit of G, and whose morphisms are the l-group homomorphisms which preserve the units. Mundici [13] proved that Γ defines a natural equivalence from $\mathcal{G}$ to $\mathcal{M}$. Mundici observed that our present notation is consistent with the notation used in [13], where Γ is applied only to abelian l-groups with unit: as a matter of fact, letting G' be the l-subgroup of G given by those elements of G whose absolute value is dominated by some multiple of u, it follows that u is a unit of G', and $\Gamma(G',u) = \Gamma(G,u)$.

The equivalence between the categories $\mathcal{G}$ and $\mathcal{M}$ makes MV-algebras useful to classify approximately finite-dimensional C^* algebras ([13], [15], [17], [18]), and to investigate the Murray von Neumann order of projections in operators algebras on Hilbert spaces ([6], [19]). Mundici also discovered applications of MV-algebras to the study of the complexity of adaptive error-correcting codes [16].

A fundamental result of Chang ([5], Lemma 8) asserts that the algebra $\Gamma(Q,1)$ generates the variety of MV-algebras, where Q denotes the additive group of the rational numbers with natural order. To prove this result Chang made explicit use of the completeness of the (first-order) theory of the totally ordered divisible abelian groups. On the other hand, Komori [11] characterized all subvarieties of MV-algebras by making explicit use of the completeness of the theory of a special class of totally ordered abelian groups, previously introduced by him in [10].

The aim of this note is to show that these results of Chang and Komori can be derived, via the properties of the functor Γ, from the fact that the variety of abelian l-groups is generated by Z, the additive group of the integers with natural order. This result is in turn a consequence of Weinberg's characterization of free abelian l-groups (see the Appendice of [1], and the references given there). A similar approach was already used by Di Nola and Lettieri ([7], Lemma 3.8) to characterize the variety generated by perfect MV-algebras.

We hope that the proofs we present here may render the theory of MV-algebras more accessible to people interested in these algebras as a tool to be applied outside the field of mathematical logic, for instance in Functional Analysis or in Coding Theory.

We are going to use the following notations: R will denote the additive group of the real numbers with natural order. For each integer $n \geq 2$, L_n will denote the finite chain $0, 1/(n-1), \cdots, (n-2)/(n-1), 1$, considered as a subalgebra of the MV-algebra $\Gamma(R,1)$. Note that for each $n \geq 2$, the algebra L_n is isomorphic to $\Gamma(Z, n-1)$.

Lemma 1 *Let G, H be abelian l-groups, $0 < u \in G$ and $h : H \longrightarrow G$ be a surjective l-group homomorphism. Then there is $0 < v \in H$ such that $h(v) = u$ and $\Gamma(h) : \Gamma(H,v) \longrightarrow \Gamma(G,u)$ is a surjective homomorphism.*

Proof. Since h is surjective, there is $x \in H$ such that $h(x) = u$. Then $v = x \vee 0$ satisfies $h(v) = u$ and $v > 0$. Therefore $\Gamma(h) : \Gamma(H,v) \longrightarrow \Gamma(G,u)$ is a homomorphism.

To check that it is onto, take $y \in [0,u]$. There is $x \in H$ such that $h(x) = y$, and hence $t = v \wedge (x \vee 0) \in [0,v]$ and $h(t) = y$. □

Lemma 2 *Let H be an abelian l-group and $0 < v \in H$. If $\{K_i\}_{i\in I}$, $I \neq \emptyset$, is a family of l-groups and*

$$h : H \longrightarrow \prod_{i\in I} K_i$$

is an l-group embedding, then $J = \{j \in I | h(v)_j \neq 0\} \neq \emptyset$ and the correspondence $[o,v] \ni x \mapsto \{h(x)_j\}_{j\in J}$ defines an embedding

$$\Phi : \Gamma(H,v) \longrightarrow \prod_{j\in J} \Gamma(K_j, h(v)_j).$$

Proof. Since h is an l-group embedding and $v > 0$, $h(v) > 0$ and hence $J \neq \emptyset$. The remainder of the proof is obvious. □

Theorem 3 *The variety of MV-algebras is generated by the MV-algebras L_n, $n \geq 1$.*

Proof. Let A be a non trivial MV-algebra. Then there is an abelian l-group G and $0 < u \in G$ such that A is isomorphic to $\Gamma(G,u)$. By ([1], Corollarie A.1.7), G is a homomorphic image of a subdirect product, say H, of l-groups isomorphic to Z. Hence by Lemma 1, A is a homomorphic image of $\Gamma(H,v)$ for some $0 < v \in H$, and by Lemma 2 $\Gamma(H,v)$ is embeddable in a product of algebras $\Gamma(Z,n_i)$ for some $n_i \in Z$, $n_i \geq 1$, and the algebras $\Gamma(Z,n_i)$ and L_{n_i} are isomorphic for each $n_i \geq 1$. □

Remark: Since Chang ([5], Lemma 3) proved that each MV-algebra is a subdirect product of totally ordered MV-algebras, in the above proof we can take A to be a totally ordered MV-algebra. Hence the existence of a (totally ordered) abelian l-group G and a unit $u \in G$ such that A is isomorphic to $\Gamma(G,u)$ is guaranteed by Lemmas 5 and 6 in [5].

Corollary 4 ([5]) *The variety of MV-algebras is generated by the algebra $\Gamma(Q,1)$.*

Proof. Whenever an equation $\sigma(x_1, \cdots x_k) = \tau(x_1, \cdots, x_k)$ is falsified in $\Gamma(Q,1)$, and, say, $r_1, \cdots, r_k$ are rationals falsifying the equation, then letting d be their least common denominator, the equation is also falsified in the finite chain $0, 1/d, 2/d, \cdots, (d-1)/d$, 1 (because the MV operations of negation and addition do not change common denominators). Thus, if an equation holds in all MV-algebras L_n, then it holds in the MV-algebra $\Gamma(Q,1)$. The converse is trivial. □

For each abelian l-group G, let $\Lambda(G)$ be the lexicographic product $Z \otimes G$. It is well known that $\Lambda(G)$ is an abelian l-group (see, for instance, [2], Chapter XIII, Section 2, Lemma 3). It is easy to check that for each l-group homomorphism $h : G \longrightarrow H$, the function $\Lambda(h) : \Lambda(G) \longrightarrow \Lambda(H)$ defined by the prescription $\Lambda(f)((m,a)) = (m, h(a))$ for each $(m,a) \in Z \otimes G$, is an l-group homomorphism. It is easy to check that Λ is a functor from the category $\mathcal{A}$ of abelian l-groups to $\mathcal{A}$. Note also that h is injective (surjective) if and only if $\Lambda(h)$ is injective (surjective).

For each $1 \leq n \in Z$ and each element x of an l-group G, (n, x) is a unit of $\Lambda(G)$, and hence $\Gamma(\Lambda(G), (n,x))$ is an MV-algebra. The MV-algebras $\Gamma(\Lambda(Z),(n,0))$ and $\Gamma(\Lambda(Z),(n,1))$ will be denoted respectively by K_n and H_n. The algebra K_n coincides with the algebra S_n^ω considered in [11].

The next lemma was proved by Komori ([11], Lemma 4.9) by means of some direct computations.

Lemma 5 (Komori) *For each $k \in Z$, $Var(\Gamma(\Lambda(Z),(n,k)) = Var(K_n)$.*

Theorem 6 *For each abelian l-group G and each $b \in G$, $b > 0$, $Var(\Gamma(\Lambda(G),(n,b))) = Var(K_n)$.*

Proof. Let G be an abelian l-group and $G \ni b > 0$. By ([1]), Corollaire A.1.7, G is a homomorphic image of a subdirect product, say H, of l-groups isomorphic to Z. Suppose $h : H \longrightarrow G$ is a surjective l-group homomorphism. By Lemma 1 there is $0 < c \in H$ such that $h(c) = b$. Therefore, $\Lambda(h) : \Lambda(H) \longrightarrow \Lambda(G)$ is a surjective l-group homomorphism such that $\Lambda(h)(n,c) = (n,b)$, and hence $\Gamma(\Lambda(h)) : \Gamma(\Lambda(H),(n,v)) \longrightarrow \Gamma(\Lambda(G),(n,u))$ is surjective. Suppose now that I is a nonempty set and $h : H \longrightarrow Z^I$ is an l-group embedding. Then the function $f : \Lambda(H) \longrightarrow \Lambda(Z)^I$ defined for each $(m,x) \in \Lambda(H)$ by the prescription $f(m,x) = \{(m, h(x)_i)\}_{i\in I}$ is also an l-group embedding, and by Lemma 2 we have that $\Gamma(\Lambda(H),(n,c))$ is embeddable in a product of algebras $\Gamma(\Lambda(Z),(n,k_i))$. Hence, by Lemma 5, $\Gamma(\Lambda(G),(n,b)) \in Var(K_n)$. On the other hand, since the correspondence $(m,k) \longmapsto (m,kb)$ defines an l-group embedding $h : \Lambda(Z) \longrightarrow \Lambda(G)$ such that $h(n,1) = (n,b)$, it follows that $\Gamma(h) : H_n \longrightarrow \Gamma(\Lambda(G),(n,b))$ is an embedding. Hence, $H_n \in Var(\Gamma(\Lambda(G),(n,b)))$, and since by Lemma 5, $Var(H_n) = Var(K_n)$, we finally have $Var(\Gamma(\Lambda(G),(n,b))) = Var(K_n)$. □

From the above theorem we can derive the key result used by Komori to determine the varieties of MV-algebras ([11], Theorem 4.10):

Theorem 7 (Komori) *Let A be a totally ordered nonsimple MV-algebra and let M be its unique maximal ideal. If A/M is isomorphic to L_{n+1}, then $Var(A) = Var(K_n)$.*

Proof. By ([11], Lemma 4.4) there is a totally ordered abelian l-group G and $u \in G$, $u > 0$, such that A is isomorphic to $\Gamma(\Lambda(G),(n.u))$. Then by Theorem [?], $Var(A) = Var(K_n)$. □

References

[1] A.Bigard, K.Keimel and S.Wolfenstein, *Groupes et anneaux réticulés*, Lecture Notes in Mathematics, Vol. 608, Springer-Verlag, 1977.

[2] G.Birkhoff, **Lattice theory**, 3rd.edition, American Mathematical Society, Providence, R.I., 1967.

[3] B.Bosbach, *Concerning bricks*, Acta Mathematica Academiae Scientiarum Hungaricae, 38 (1981), 89-104.

[4] C.C.Chang, *Algebraic analysis of many-valued logics*, Transactions of the American Mathematical Society, 88 (1958), 467-490.

[5] C.C.Chang, *A new proof of the completeness of the Lukasiewicz axioms*, Transactions of the American Mathematical Society, 93 (1959), 74-80.

[6] R.Cignoli, G.A.Elliott and D.Mundici, *Reconstructing C^*-algebras from their Murray-von Neumann orders*, Advances in Mathematics, to appear.

[7] A.Di Nola and A.Lettieri, *Perfect MV-algebras are categorically equivalent to abelian l-groups*, Preprint.

[8] J.M.Font, A.J.Rodríguez and A.Torrens, *Wajsberg algebras*, Stochastica, 8 (1984), 5-31.

[9] Y.Komori, *The separation theorem of the $\aleph_o$-valued Lukasiewicz propositional logic*, Reports of the Faculty of Sciences, Shizuoka University, 12 (1978), 1-5.

[10] Y.Komori, *Completeness of two theories on ordered abelian groups and embedding relations*, Nagoya Mathematical Journal, 77 (1980), 33-39.

[11] Y.Komori, *Super Lukasiewicz propositional logics*, Nagoya Mathematical Journal, 84 (1981), 119-133.

[12] F.Lacava, *Alcune proprietá delle l-algebre e delle l-algebre esistenzialmente chiuse*, Bollettino Unione Matematica Italiana, A(5), 16 (1979), 360-366.

[13] D.Mundici, *Interpretation of AFC^*-algebras in Lukasiewicz sentential calculus*, Journal of Functional Analysis, 65 (1986), 15-63.

[14] D.Mundici, *MV-algebras are categorically equivalent to bounded commutative BCK-algebras*, Mathematica Japonica, 31 (1986), 889-894.

[15] D.Mundici, *The C^*-algebras of three-valued logics*, In: Logic Colloquium 88, Ferro, Bonotto, Valentini and Zanardo (Editors). Elsevier Science Publishers, B.V. (North-Holland), 1989, pp.61-77.

[16] D.Mundici, *The complexity of adaptative error-correcting codes*, In: E.Börger, H.Kleine Büning, M.M.Richter and W.Schönfeld (Eds). Computer science logic, Lectures Notes in Computer Science, Vol. 533, Springer-Verlag 1991, pp. 300-307.

[17] D.Mundici, *Turing complexity of Behncke-Leptin C^*-algebras with a two-point dual*, Annals of Mathematics and Artificial Intelligence, 6 (1992), 287-294.

[18] D.Mundici, *K_o, relative dimension, and the C^*-algebras of Post logic*, These Proceedings.

[19] D.Mundici and G.Panti, *Extending addition in Elliott's local semigroup*, **Journal of** Functional Analysis, to appear.

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PARTITION PROPERTIES AND PERFECT SETS

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We present an example of the use of partition relations in consistency proofs. We will use a partition relation of pairs of real numbers to obtain a consistency result about perfect sets of real numbers. Section 1 contains some definitions and the statement of the main result. Section 2 is devoted to a proof of the fact that Ramsey ultrafilters are "preserved" by Sacks forcing. More general cases appear in the literature ([HP] treates the case of finite product of Sacks forcing, and [L] infinite products, preservation of Ramsey ultrafilters by iterations of Sacks forcing is proved in [BL]) but for the sake of completeness we include this simpler case. In section 3. contains the proof of the consistency of a partition property of $[\omega]^\omega \times \omega^\omega$ using an argument from [Mi], and in section 4. the proof of the main theorem is completed.

§1. PERFECT SET PROPERTIES.

A set of real numbers is perfect if it is closed and contains no isolated points. It is easy to show that a perfect set has cardinality 2^ω, the cardinality of the continuum.

The Axiom of Choice implies that there are totally imperfect sets, this is, sets which neither contain nor are disjoint from a perfect set. This result is due to Bernstein [B], who also noticed that a totally imperfect set cannot be Lebesgue measurable nor can it have the property of Baire. We will say that a set of reals A has the Bernstein property if either A or its complement contains a perfect set.

It has been shown by Solovay [So] that if the theory ZFC+"There is a inaccessible cardinal" is consistent then it is also consistent ZFC+DC+ "every set of reals is Lebesgue measurable, every set of reals has the property of Baire and every uncountable set of reals contains a perfect subset". The assumption regarding an inaccessible cardinal is necessary to obtain the consistency of both the property about Lebesgue measure and the perfect subset property. The case of the Baire property is different: the consistency of ZFC is enough to show that "every set of reals has the property of Baire" is consistent with ZFC+DC [Sh].

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DC is a form of the Axiom of Choice which is strictly weaker than the full axiom.

Clearly if every uncountable set contains a perfect subset, then every set has the Bernstein propery, but these are not equivalent propositions (not even in consistency strength).

Instead of working directly with the set of real numbers, we will work with the Baire Space, ω^ω, the set of sequences of natural numbers, with the product topology obtained when giving the discreet topology to the set of natural numbers. A base for this tolopogy is given by the sets $U_s = \{\alpha \in \omega^\omega : (\exists k \in \omega)\alpha \upharpoonright k = s\}$ where s is a finite sequence of natural numbers.

The Baire Space is homomeorphic to the irrational numbers, and therefore the elements of ω^ω will be called reals.

We will address the question of whether the existence of an ultrafilter on the set of natural numbers is consistent with Bernstein's property for all sets of reals. (The existence of an ultrafilter on the set of natural numbers is also strictly weaker than AC and it does not follow from DC).

We will prove the following theorem after reviewing some results about Ramsey ultrafilters (Section 2) and after establishing the consistency of a partition relation of pairs (Section 3).

Theorem 1. *Con(ZFC+"There is an inaccessible cardinal") implies*

Con(ZF+DC+ "There is an ultrafilter on ω" + "Every set of reals contains or is disjoint from a perfect set").

The results presented here are not the best possible (see remarks at the end of the paper), but the proofs given are illustrative of the way partition properties can be used to establish consistency results in Set Theory. The consistency of the partition property discussed in section 3 is proved by an argument given by Miller in [Mi], where he shows that the partition relation holds for Borel partitions.

This consistency result also follows from a result of James Henle [H] which makes use of the dual Ramsey property of Carlson and Simpson [CS].

§2. THE HALPERN-LÄUCHLI THEOREM AND RAMSEY ULTRAFILTERS

In next section we will establish the consistency of a partition relation. One of the main ingredients we will use in the proof is the fact that Ramsey ultrafilters are preserved by Sacks forcing. We will proceed to prove this fact, using a variant of a weak version of a combinatorial result due to Halpern and Läuchli [HL].

Definition. An ultrafilter U on ω is Ramsey (or selective) if for every decreasing sequence $A_0 \supseteq A_1 \supseteq \ldots$ of sets in U, there is an increasing function $f : \omega \to \omega$ such that $range(f) \in U$, and for every $k \in \omega$, $f(k+1) \in A_{f(k)}$.

The existence of Ramsey ultrafilters on ω is independent from ZFC, but follows from the Continuum Hypothesis. (See [J1] for equivalent formulations of the property of being Ramsey for ultrafilters on ω).

A tree on ω is a subset T of $\omega^{<\omega}$ ordered by extension with the property that if $s \in T$ then $s \restriction i \in T$ for all $i < lenght(s)$. We say that a tree is finite branching if each node has finitely many inmediate successors. The set of (infinite) branches of a tree T is a closed subset of ω^{ω}, it is denoted by $[T]$. This set is perfect if each node has at least two incomparable successors. The height of a node of a tree is defined as the order type of the set of predecessors (in the tree) of that node. The height of the tree is the supremum of the heights of its nodes. The n^{th} level of a tree is the set of nodes of height n.

A tree is said to be a finite branching tree is each node has finitely many inmediate successors. A tree on ω is perfect if each node has incompatible successors, in other words, each path eventually branches).

Let T be a finite branching tree of height ω with the property that each node has at least two inmediate successors. Denote by $T(n)$ be the n^{th} level of T. $S \subseteq T$ is a level subtree if it is a subtree (i.e. the order relation of T restricted to S is the order relation of S) and $\forall n \exists m[S(n) \subseteq T(m)]$ We say that S is dense if $(\forall s \in S)(\forall t \in Tt$ an immediate successor of $s)(\exists s' \in S)[t \leq s']$.

If U is an ultrafilter on ω, we say that $S \subseteq T$ is a $U-full$ subtree of T if $\{m : \exists n S(n) \subseteq T(m)\} \in U$.

The following version of the Halpern-Läuchli Theorem is due to Halpern and Pincus [HP].

Theorem. *If U is a Ramsey ultrafilter on ω and T is a finite branching perfect tree any node of which has at least two inmediate successors, for any $F : T \to 2$, there is a level dense U-full subtree $S \subseteq T$ on which F is constant.*

Proof (Pincus [P]). Consider the folowing statement.

$$(*) \forall k \in \omega \forall a \in T(k)\{n \in \omega : \exists b \in T(n)[F(b) = 0 \text{ and } a \leq b]\} \in U.$$

Suppose (*) is true. For each $k \in \omega$ and each $a \in T(k)$, let $A_{k,a}$ satisfy

$$\forall n \in A_{k,a} \exists b \in T(n)[F(b) = 0 \text{ and } a \leq b].$$

Let $A_k = \cap_{a \in T(k)} A_{k,a}$. Notice that $A_k \in U$. Also, $\forall n \in A_k \forall a \in T(k) \exists b \in T(n)[F(b) = 0 \text{ and } a \leq b]$.

Without loss of generality we can assume that the A_k's are decreasing, and if we set $B_k = A_{k+1}$, since U is Ramsey, there is a function $f : \omega \to \omega$ with $range(f) \in U$ and

$$\forall k \in \omega(f(k+1) \in B_{f(k)} = A_{f(k)+1}).$$

The desired subtree $S \subseteq T$ is now built with levels in $range(f)$. Let $S(0)$ be any $a \in T(f(0))$ with $F(a) = 0$. Such an a exists since any member of $T(1)$ has such a successor in $T(f(0))$. If $S(k)$ has been defined, consider $t \in T(f(k)+1)$. Since $f(k+1) \in A_{f(k)+1}$, there is $b_t \in T(f(k+1))$ satisfying $F(b_t) = 0$ and $b_t \geq t$. Set $S(k+1) = \{b_t : \exists s \in S(k) s < t \leq b_t]\}$.

The subtree S is dense, U-full, and all its nodes have F image 0.

Suppoose now that (*) does not hold. Then we have

$$(\exists k \in \omega)(\exists a \in T(k))\{n \in \omega : \forall b \in T(n)[a \leq b \Rightarrow F(b) = 1]\} \in U.$$

Fix such a k and a, and $A \in U$ such that $\forall n \in A \forall b \in T(n)[b \geq a \Rightarrow F(b) = 1]$.

Set $S = \{b \in T : \exists n \in A[b \in T(n) \text{ and } a \leq b]\}$, S is level dense, U-full and F is constantly 1 on its nodes.

■

In [HP] Halpern and Pincus prove a more general version of the theorem we state below, namely, if a finite number of Sacks reals is added by product forcing to a model in which there is a Ramsey ultrafilter, this ultrafilter generates a Ramsey ultrafilter in the generic extension. Laver [L] generalized this to contably many Sacks reals. The difficulty in obtaining these more general versions lies in the corresponding Halpern-Läuchli results needed to prove them.

Let $\mathbb{S}$ is the partial order of Sacks perfect set forcing (i.e. the set of perfect trees on ω ordered by inclusion, $p \leq q$ if and only if q is included in p). This forcing adds a real to the ground model; if G is $\mathbb{S}$-generic over M, then $x_G = \cap G$ is the real added. G is recovered from x_G as the set of all perfect trees of which x_G is a branch.

For each $n \in \omega$ one can define an order relation $\leq_n$ between perfect trees saying that $p \leq_n q$ if $p \leq q$ and p and q coincide up to their nth splitting level. A well known fact about Sacks forcing is that for every sequence $p_0 \geq_1 p_1 \geq_2 p_2 \geq_3 \dots$ there is a condition q called the fusion fo the sequence such that for every $n \in \omega$, $q \leq_n p_n$.(See, for example, [J2]).

The next theorem was proved by Solovay (see [HP]) and Baumgartner and Laver [BL].

Theorem. *Let M be a model of ZFC+" U is a Ramsey ultrafilter on ω" then for any G $\mathbb{S}$-generic over M $M[G] \models U$ generates a Ramsey ultrafilter.*

Proof. We will first show that every subset of ω in M[G] contains or is disjoint from a set in U. From this follows that U generates an ultrafilter U^* in M[G]. Then, we will show that this ultrafilter is Ramsey. Both things are accomplished via fusion arguments.

a) Let τ be a term for a subset of ω, and let $p \in \mathbb{S}$ be such that $p \Vdash \tau \subseteq \omega$. We will find $r \leq p$ and $X \in U$ such that $r \Vdash X \subseteq \tau$ or $r \Vdash X \cap \tau = \emptyset$.

Extend p below each node in its first splitting level to decide $0 \in \tau$, obtaining this way a condition p_1 with the same first splitting level as p. Once p_n has been obtained, extend it below each node in the $(n+1)-th$ splitting level to decide $n \in \tau$. This way we construct a sequence of conditions $p \geq_1 p_1 \geq_2 p_2 \geq_3 \dots$. Let q be the fusion of the sequence (i.e. for all n, $q \leq_n p_n$). The splitting nodes of q form a perfect tree, and its nodes are divided in two classes: $F(s) = 0$ if s is in the $(n+1)-th$ splitting level and $q_s \Vdash n \in \tau$, and $F(s) = 1$ if $q_s \Vdash n \notin \tau$. Notice that by extending q if necessary, we may assume q is finite branching, and by the previous result, there is a level dense U-full subtree T of S. This subtree of splitting nodes determines a condition $r \leq q$, and X is the set $\{m : \exists n T(n) \subseteq S(m)\} \in U$.

b) To show that this ultrafilter is Ramsey, first notice that for any function $f : \omega \to M$ in $M[G]$, there is a function $g \in M$ such that $g : \omega \to [M]^{<\omega}$ such that for every $n \in \omega$, $f(n) \in g(n)$. This is an easy fact about Sacks forcing (see [Sa]) proved also by a fusion argument. If $p \Vdash f : \omega \to M$, we define a fusion sequence by deciding the value of $f(n)$ below each node in the $(n+1)-th$ spliting level of the previous forcing condition. To obtain a finite number of possible values for each $f(n)$, it is enough to assume that p is a finite splitting tree.

Now, if $A_0 \supseteq A_1 \supseteq \dots$ is, in $M[G]$, a decreasing sequence of elements of the ultrafilter U^*, let $h : \omega \to M$ be a function in $M[G]$, such that if $h(n) = Y_n$, $Y_n \in U$ and $Y_n \subseteq A_n$ (we are assuming AC in M, and therefore we have it in M[G]). By the previous remark, there is a function $g \in M$, $g : \omega \to M$, such that for each n, $g(n)$ is finite and $h(n) \in g(n)$. Put $h'(n) = \cap g(n)$, the function h' is in M and for each n, $g'(n) \in U$ and $h'(n) \subseteq A_n$. If we put now $B_n = \cap_{i \leq n} h'(i)$, we obtain a decreasing sequence $B_0 \supseteq B_1 \supseteq \dots$ in M, and using the fact that U is a Ramsey ultrafilter in M, we know there is an increasing function $f : \omega \to \omega$ such that $f \in m$, and for each n $f(n+1) \in B_{f(n)} \subseteq A_{f(n)}$. The function f works for the sequence $A_0, A_1, \dots$.

■

§3. A PARTITION PROPERTY FOR PAIRS OF REALS

As customary, $[\omega]^\omega$ denotes the collection infinite sets of natural numbers. Notice that $[\omega]^\omega$ can be viewed as a subspace of ω^ω by identifying every infinite set of natural numbers with its natural enumeration.

Consider the following partition relation for pairs of reals.

RP:

For every $F : [\omega]^\omega \times \omega^\omega \to 2$ there is an infinite $x \in [\omega]^\omega$ and a perfect set $P \subseteq \omega^\omega$ such that F is constant on the product $[x]^\omega \times P$.

This can be viewed as a parametrized version of the Bernstein property.

Theorem. *Con(ZFC+There is an inaccessible cardinal) implies Con(ZF+DC+ RP).*

R. Laver has shown that an n-dimensional version of this parametrized partition property (finite powers of ω^ω are considered) holds for Borel partitions. A proof is contained in [LSV], where Laver's result is obtained as a consecuence of a partition theorem for Borel partitions of trees.

The theorem will follow from the next Lemma by the methods developed by Solovay in [So] (See [Je]). The lemma shows that the property RP holds for partitions of certain specific form. Following Solovay, to obtain the theorem we pass to the inner model of the Levy Collapse given by the sets definable from a real and an ordinal, and then use the fact that all partitions in the inner model have that specific form.

As we mentioned above, this line of argumentation appears in [Mi].

If U is a Ramsey ultrafilter on ω, define $\mathbb{P}$ as follows. The conditions in $\mathbb{P}$ are pairs (s, S) where $s \in [\omega]^{<\omega}$, $S \in U$ and $sup(s) < min(S)$. The partial order is defined as follows $(s, S) \leq (t, T)$ if and only if $t \subseteq s$, $S \subseteq T$ and $s - t \subseteq T$. This is Mathias forcing with respect to U, as defined in [Ma]. $\mathbb{P}$ adds an infinite subset of ω to the ground model (a Mathias real), namely, if G is $\mathbb{P}$-generic over M, $x_G = \cup\{s : \exists S(s, S) \in G\}$ is the generic subset of ω. The generic filter G is obtained from x_G as $\{(s, S) : s \subseteq x_G \subseteq s \cup S\}$.

2.0 of [Ma] says that x is generic (with respect to $\mathbb{P}$) over M if x is infinite and $x - B$ is finite for all $B \in U$. As a consequence, x is generic if and only if every infinite subset of x is generic.

An important property of this forcing, also proven in [Ma], is that if ϕ is a sentence in the forcing language, and (s, S) is a condition, there is $T \subseteq S$ such that (s, T) decides ϕ. In particular, every forcing statement is decided by a condition of the form $(\emptyset, S)$.

Lemma. *Let $M \models ZFC$+There is an inaccessible cardinal, and let U be a Ramsey ultrafilter in M. Let κ be inaccessible in M, and let $G \subseteq Coll(\omega, < \kappa)$ be generic over M (this is the Levy collapse of κ to $\aleph_1$). In $M[G]$, let $A \subset [\omega]^\omega \times \omega^\omega$ be such that*

$$A = \{(x, \alpha) : M[x, \alpha] \models \phi(x, \alpha)\}.$$

Then, in M[G], there are $x \in [\omega]^\omega$ and $P \subseteq \omega^\omega$ such that P is perfect and $[x]^\omega \times P$ is homogeneous for A (i.e. this product is contained in A or disjoint from A).

Proof. In M, let $\mathbb{P}$ be Mathias forcing with respect to the ultrafilter U, and let $\mathbb{S}$ be Sacks forcing (perfect set forcing).

If $((\emptyset, S'), T')$ is a forcing condition in the product, there is an extension of the form $((\emptyset, S''), T'')$ which decides the formula $[\bar{x}, \bar{\alpha}] \models \phi(\bar{x}, \alpha)$. Suppose that the condition $((\emptyset, S''), T'')$ forces the formula.

Our objective is to find a pair (S, T) such that $S \in [S']^\omega$, T is a perfect subtree of T'', and for every $x \in [S]^\omega$ and every $\alpha \in [T]$, the pair (x, α) is $\mathbb{P} \times \mathbb{S}$ generic over M.

If we get such a pair then we are done, since if (x,α) is $\mathbb{P}\times\mathbb{S}$ generic over M, and $((\emptyset,S),T)$ is in the generic, then

$$((\emptyset,S),T) \Vdash M[\bar{x},\bar{\alpha}] \models \phi(\bar{x},\bar{\alpha})$$

implies that $(x,\alpha)\in A$ (where $\bar{x}$ is the canonical name for the generic real added by $\mathbb{P}$ and $\bar{\alpha}$ is the canonical name for the Sacks generic real).

Since $card(\mathbb{S})<\kappa$ and κ is inaccessible, $card(\{D\subseteq\mathbb{S}: D \text{ is dense open}\})<\kappa$. So, in M[G], we can list the dense open subsets of $\mathbb{S}$ which lie in M as $\{D_0,D_1,\dots\}$.

We build a fusion sequence $T_0\geq_0 T_1\geq_1 T_2\dots T_n\geq_n T_{n+1}\dots$ as follows. Given T_0, we extend below each node after the first splitting in order to meet D_0; once we have defined T_n, we extend below each node following an nth splitting to meet D_n. If T is the fusion of that sequence, each branch of T is Sacks generic. So we have a perfect tree with each of its branches being a Sacks generic real.

Now, let x be $\mathbb{P}$-generic over M. Such an x exists in M[G] since cadinality of $\mathbb{P}$ is less than κ.

Claim: Every $y\in[x]^\omega$ is $\mathbb{P}$-generic over $M[\alpha]$ for each $\alpha\in[T]$

By the result of Solovay mentioned in the previous section, U_α, the filter generated by U in $M[\alpha]$, is also a Ramsey ultrafilter.

Let $\mathbb{P}_\alpha$ be the partial ordering defined by this ultrafilter in $M[\alpha]$.

Sublemma.

Let y be $\mathbb{P}$-generic over M. Then, y is $\mathbb{P}$-generic over $M[\alpha]$

Proof of Sublemma. It is easy to verify that is D is a dense open subset of $\mathbb{P}$ and $D\in M[\alpha]$, then D is a dense open subset of $\mathbb{P}_\alpha$. If y is $\mathbb{P}$-generic over M, by 2.0 of [Ma], y is infinite and y-B is finite for each $B\in U$. This implies that y-S is finite for each $S\in U_\alpha$, and therefore (using 2.0 of [Mathias] again), y is $\mathbb{P}_\alpha$-generic over $M[\alpha]$. This means that $G=\{(s,S)\in\mathbb{P}_\alpha : s\subseteq y\subseteq s\cup S\}$ is a filter which meets every dense open subset of $\mathbb{P}_\alpha$ belonging to $M[\alpha]$. By the comment above, G meets every dense open subset of $\mathbb{P}$ which is in $M[\alpha]$. So, y is $\mathbb{P}$-generic over $M[\alpha]$.

This ends the proof of the sublemma. And from this the Claim follows inmediately. The pair $(x,[T])$ is homogeneous for A since for every $y\in[x]^\omega$ and every $\alpha\in[T]$, (y,α) is $\mathbb{P}\times\mathbb{S}$ generic over M.

■

Note that the same result can be obtained if we consider partitions of $[\omega]^\omega\times 2^\omega$, since the perfect set forcing we used above is equivalent to perfect set forcing with perfect subtrees of $2^{<\omega}$.

Using this fact, it is easy to verify that the partition relation RP implies the partition relation obtained considering partitions into three colors (in fact, into any finite number of colors); one just needs to verify that the proof given above proves that for every partition $F : [\omega]^\omega \times P \to 2$ (where P is a perfect set), there is an infinite set H and a perfect $Q \subseteq P$ such that F is constant on the product $[H]^\omega \times Q$.

The consistency of ZFC+DC+RP might as well follow from the consistency of ZFC alone, but this is unknown: the property "every set is Ramsey" (i.e. $\forall A \subseteq [\omega]^\omega \exists x \in [\omega]^\omega$ with $[x]^\omega \subseteq A$ or $[x]^\omega \cap A = \emptyset$) follows from the consistency of ZFC+there is an inaccessible cardinal (it was established by Mathias [Ma] that this property holds in Solovay's model), but it is not known if it follows from the consistency of ZFC alone.

§4. CONCLUSION

In this short final section we just conclude the proof Theorem 1 and make some remarks about this result.

Proof of Theorem 1. Force with $P(\omega)/\text{fin}$ over the model of DC+RP obtained in the previous section. This way we add an ultrafilter on ω. Given a partition $F : \omega^\omega \to 2$ in the extension, let $z \in [\omega]^\omega$ be a condition such that $z \Vdash \bar{F} : \omega^\omega \to 2$. Define, in the ground model, the following partition $G : [z]^\omega \times \omega^\omega \to 3$ by $G(x, \alpha) = i$ if and only if $x \Vdash \bar{F}(\alpha) = i$ (for $i \in \{0, 1\}$), and $G(x, \alpha) = 2$ if x does not decide the value of F on α. Let (x, T) be homogeneous for G with $x \in [z]^\omega$ and T a perfect tree. We can find such a homogeneous pair using the partition relation RP (which holds in the ground model) and an isomorphism between $[\omega]^\omega$ and $[z]^\omega$. Note that G cannot take constant value 2 on $[x]^\omega \times P$, because if x does not decide $\bar{F}(\alpha)$ there is $y \in [x]^\omega$ which decides it. Let $i \in \{0, 1\}$ be the constant value of G on $[x]^\omega \times T$. Then, for every $\alpha \in [T]$, $x \Vdash \bar{F}(\alpha) = i$. By a standard density argument, this is enough to show that in the extension $M[G]$ there is a perfect set homogeneous for F.

■

There are two directions in which Theorem 1 could be strengthened.

(1) Eliminating the hypothesis of the existence of an inaccessible, and
(2) replacing Bernstein property by the Perfect Set Property.

Using some techniques developed by Woodin, it is possible to obtain the second of these strenghtenings.

References

[BL] J.E. Baumgartner and R. Laver, *Iterated perfect forcing*, Annals of Mathemaical Logic **17** (1979), 271-288.

[B] F. Bernstein, *Zur Theorie der Trigonometrischen Reiche*, Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig Mathematisch-Physiche Klasse **60** (1908), 325-338.

[CS] T. J. Carlson and S. G. Simpson, *A dual form of Ramsey's Theorem*, Advances in Mathematics **53** (1984), 265-290.

[DPH] C.A. Di Prisco and J. M. Henle, *Partitions of Products* (to appear, The Journal of Symbolic Logic).

[H] J. Henle, *(personal communication).*

[HL] J.D. Halpern and H. Läuchli, *A partition theorem*, transactions of the Amer. Math.Soc. **124** (1966), 360-367.

[HP] J.D. Halpern and D. Pincus, *Partitions of Products*, Transactions of the Amer. Math. Soc. **267** (1981), 549-568.

[J1] T. Jech, *Set Theory*, Academic Press, 1978.

[J2] T. Jech, *Multiple Forcing*, Cambridge University Press, 1986.

[L] R. Laver, *Products of infinitely many perfect trees*, Journal of the London Math. Soc. **29** (1984), 385-396.

[LSV] A. Louveau, S. Shelah and B. Veličković, *Borel partitions of infinite subtrees of a perfect tree* (to appear).

[Ma] A. R. D. Mathias, *Happy Families*, Annals of Math. Logic **12** (1977), 59-111.

[Mi] A. W. Miller, *Infinite combinatorics and definability*, Annals of Pure and Applied Logic **41** (1989), 179-203.

[P] D. Pincus, *(personal comunication).*

[Sa] G. Sacks, *Forcing with perfect closed sets*, in "Axiomatic Set Theory". Proceedings of Symp. in Pure Math. **13 (part II)** (1971), 331-355.

[Sh] S. Shelah, *Can you take Solovay's inaccessible away?*, Israel Journal of Mathematics **48** (1984), 1-47.

[So] R. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Annals of Mathematics **92** (1970), 1-56.

EQUIVALÊNCIA ELEMENTAR ENTRE FEIXES

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Abstract. In this paper we study elementary equivalence between sheaves of structures over a topological space X with respect to Kripke-Joyal semantics enriched with all the intuitionistic connectives induced by X.

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EQUIVALÊNCIA ELEMENTAR ENTRE FEIXES

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INTRODUÇÃO

Neste artigo estudamos a equivalência elementar entre feixes de estruturas sobre espaços topológicos, com respeito à semantica de Kripke-Joyal enriquecida com novos conectivos intuicionistas.

A noção de feixe foi introduzida por Lazard e desenvolvida por Cartan em 1950. Tal noção se mostrou extremamente útil em diversas áreas da matemática tais como geometria diferencial, geometria analítica, funções diferenciais, topológia algébrica, etc. Na década de 60, Grothendieck, Serre Giraud, Artin, Verdier, entre outros, obtiveram expressivo avanço nessa área, introduzindo noção mas geral de topos de Grothendieck. Giraud caracteriza os topos de Grothendieck como uma categoria E satisfazendo as seguintes condições:

(i) E tem limites finitos

(ii) E tem somas arbitrárias disjuntas e universais

(iii) E tem relações de equivalência efetivas e universais e

(iv) E tem um conjunto pequeno de geradores.

A categoria de feixes sobre um espaço X e o protótipo de um topos de Grothendieck concreto.

Por volta de 1969/70 Lawvere e Tierney, preocupados em caracterizar a categorias dos conjuntos, axiomatizam em linguagem de primeira ordem tal categoria como um topos elementar satisfazendo determinadas condições. A noção de topos elementar generaliza os topos de Gothendieck e mostrou-se, uma vez que cada topos está munido canonicamente de uma lógica intuicionista, extremamente rica para a lógica, trazendo uma nova luz a teoria dos modelos intuicionista. Para maiores detalhes ver [F-S], [Gr], [R] ou [G].

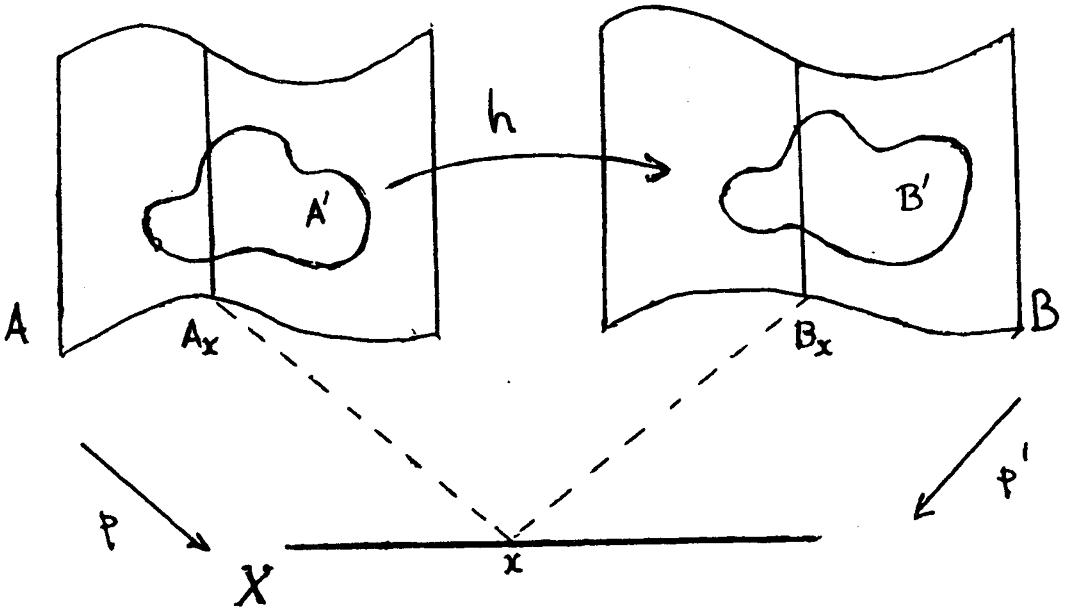
Na primeira parte deste artigo introduzimos a noção (canônica) de pseudogrupo (ver [E], [C-S]) de homeomorphismos parciais entre feixes de estruturas sobre um espaço topológico. Na segunda parte utilizamos a noção natural de conectivo em um topos para introduzir novos conectivos intuicionistas na logica interna do topos (cf. Caicedo [C]). Finalmente, generalizamos os teoremas de Fraissé e Karp para o topos de feixes sobre um espaço X.

I. FEIXES DE HOMEOMORFISMOS PARCIAIS

Seja X um espaço topológico e consideremos o topos $Sh(X)$ constituido por todos os feixes sobre X; $Sh_\tau(X)$ será a categoria de feixe de estruturas de tipo τ sobre X, i.e., A é um objeto do $Sh_\tau(X)$ se e somente se $A = \langle \tilde{A},\ R^A \rangle_{R \in \tau}$, onde $\tilde{A}$ é um objeto de $Sh(X)$ e para cada $R \in \tau$, R n-ária, R^A é um subfeixe do produto fibrado de $\tilde{A}$ (n-vezes), na categoria $Sh(X)$. Em outras palavras, para cada $x \in X$ a fibra A_x sobre x esta munida de uma estrutura $A_x = \langle A_x,\ R_x^A \rangle_{R \in \tau}$, de tipo τ, tal que para cada $R \in \tau$, (n-ária) $R^A = \cup_{x \in X} R_x^A$ é um aberto no espaço de fibras do produto fibrado $\tilde{A}$, n vezes. Os morfismos de $Sh_\tau(X)$ são aquilos de $Sh(X)$ que preservam homomorficamente a estrutura das fibras.

Observemos que para cada aberto U de X a estrutura $A(U)$ de secções sobre U é aquela induzida pelo produto cartesiano $\Pi_{x \in U} A_x$ no conjunto $A(U)$ de secções contínua do feixe A. O feixe de estruturas esta completamente determinado pelo funtor $U \mapsto A(U)$ com as restrições naturais $f_{U,W} : A(U) \to A(W)$ para $U \subseteq W$, que é um funtor contravariante da categoria $Ab(X)$ de abertos de X na categoria St_τ de estruturas de tipo τ (com os homomorfismos correspondentes). Para detalhes e propriedades básicas consulte [T] onde o caso de feixe de estruturas algébricas esta bem explicado.

DEFINIÇÃO 1 - Dados A e $B \in Sh_\tau(X)$ um *homeomorfismo parcial* de A em B, $h : A \to B$, é um homeomorfismo de um aberto A' do espaço de fibras A em um aberto B' do espaço de fibras B que preserve a estrutura das fibras, i.e., para cada $x \in X$, $h|A_x$ é um isomorfismo parcial de A_x em B_x



ou seja, h é um isomorfismo na categoria $Sh_T(X)$ entre os subfeixe determinados por A' e B' em A e B respectivamente.

Seja P_0 a categoria que tem por objetos os objetos de $Sh_T(X)$ e por conjunto de morfismos $P_0(A,\ B)$ de A em B os homeomorfismos parciais de A em B. Observe-se que o homeomorfismo vazio ϕ pertence a $P_0(A,\ B)$. Para cada $k \in \omega$ e aberto U de X definimos a subcategoria $P_k(U)$ da seguinte maneira:

DEFINIÇÃO 2 - $P_0(U) = P_0$ qualquer que seja o aberto U de X. Suponhamos definido $P_k(W)$ para cada W. Neste caso, $h : A \to B \in P_{k+1}(U)$ se $h \in P_k(U)$ e:
(a) Para todo aberto $W \subseteq U$ e $\Theta \in A(W)$ existe uma cobertura de abertos $\{W_i\}_i$ de W e $h_i \in P_k(W_i)$ tais que $h_i \supseteq h|W_i$ (Mais correto seria $h|p^{-1}(W_i)$, por simplicidade escreveremos $h|W_i$) e $\Theta(W_i) \subseteq dom\, h_i$ para cada i
(b) Reciprocamente para todo aberto $W \subseteq U$, $\Theta' \in B(W)$ existe uma cobertura de abertos $\{W_i\}_i$, e $h_i \in P_k(W_i)$ tais que $h_i \supseteq h|W_i$ e $\Theta'(W_i) \subseteq codom\, h_i$ para cada i.

A definição geral de pseudogrupo poderá ser encontrada em [E].

PROPOSIÇÃO 1 - Para cada k, $P_k(U)$ é um pseudogrupo.
Demonstração - Para $k = 0$ é evidente que: 1 - A restrição do homeomorfismo parcial $h : A \to B$ a um aberto de A é um homeomorfismo parcial. 2 - A inversa de um homeomorfismo parcial $h : A \to B$ é um homeomorfismo parcial. 3 - A união de um sistema dirigido por inclusão de uma família de homeomorfismos parciais é um homeomorfismo parcial, portanto P_0 é um pseudogrupo. Para $k \geq 1$ a proposição segur-se por indução.

PROPOSIÇÃO 2 - O funtor $U \mapsto P_k(U)$ que atua em morphismos por restrições é um feixe de seções sobre X, isto é:
(a) Se $U \subseteq E$ então $h \in P_k(U)$ implica $h|W \in P_k(W)$.
(b) Se $U = \cup_i U_i$ então $h|U_i \in P_k(U_i)$ para todo i, implica $h \in P_k(U)$.
De fato P_{k+1} é um subfeixe (de pseudogrupos) de P_k.
Demonstração - Indução sobre k.

A cadeia dos P_k pode ser continuada transfinitamente pondo-se $P_\alpha(U) = \cap_{\beta<\alpha} P_\beta(U)$ para α um ordinal limite. As proposições 1 e 2 continuam válidas.

Uma questão natural que se põe, é de se saber quais os invariantes da ação de P_k sobre $Sh_\tau(X)$, o proposito deste artigo é responder tal questão.

II. CONECTIVOS EM FEIXES

Por Ω designamos o classificador de subobjetos de $Sh(X)$, isto é, o feixe de $Sh(X)$ definido por: $\Omega(U) = \{W \subseteq U : W$ é um aberto de $X\}$; para cada $U \in Ab(X)$. Se $U' \subseteq U$, $\Omega(U) \xrightarrow{|} \Omega(U')$ é a restrição canônica.

DEFINIÇÃO 3 - Um conectivo unário de X é um subfeixe de Ω, isto é, um conectivo unário $\mathcal{F}$ é uma função $\mathcal{F} : U \mapsto \mathcal{F}_U \subseteq \Omega(U)$ tal que se

$U' \in Ab(X)$, $U' \subseteq U$ o seguinte diagrama comuta

$$\begin{array}{ccc} \Omega(U) & \xrightarrow{|} & \Omega(U') \\ \cup & & \cup \\ \mathcal{F}_U & \xrightarrow{|} & \mathcal{F}_{U'} \end{array}$$

e além disso tem-se a propriedade de coerência (ou colamento): se $\{U_i\}_i$ é uma cobertura aberta de U e $W \in \Omega(U)$ é tal que $W \cap U_i \in \mathcal{F}_{U_i}$ para cada i então $W \in \mathcal{F}_{U_i}$.

Observando-se que $\Omega(U) \times \Omega(U) = (\Omega \times \Omega)(U)$ definimos um conectivo binário como um subfeixe do feixe $\Omega \times \Omega$, em geral um conectivo n-nário é uma função $\mathcal{F} : U \mapsto \mathcal{F}_U \subseteq \Omega(U) \times \ldots \times \Omega(U)$ n-vezes, que faz commutar o seguinte diagrama

$$\begin{array}{ccc} \Omega^n(U) & \xrightarrow{|} & \Omega^n(U') \\ \cup & & \cup \\ \mathcal{F}_U & \xrightarrow{|} & \mathcal{F}_{U'} \end{array}$$

e tem a propriedade da coerência. Equivalentemente, podemos definir um conectivo n-nário como um morfismo $\mathcal{F}^* : \Omega^n \to \Omega$ no topos $Sh(X)$.

Sejam $\neg$, $\vee$, $\wedge$ e $\to$ os conectivos usuais intuicionistas. Tais conectivos correspondem aos seguintes subfeixes de Ω:

$\neg : U \mapsto \mathcal{F}_U = \{\emptyset\}$
$\vee : U \mapsto \mathcal{F}_U = \{(W, W') \mid W \cup W' = U\}$
$\wedge : U \mapsto \mathcal{F}_U = \{(U, U)\}$
$\to : U \mapsto \mathcal{F}_U = \{(W, W') \mid W, W' \subseteq U \text{ e } W \subseteq W'\}$
$\neg\neg : U \mapsto \mathcal{F}_U = \{W \subseteq U \mid W \text{ é denso em } U\}$.

Dependendo do espaço X podemos ter conectivos não reducíveis aos anteriores. Por exemplo:

$$\mathcal{F}_U = \{W \in \Omega(U) : W \text{ é localmente conexo}\},$$

ou para $X = \mathbb{R}$ e a medida de Lebesgue μ:

$$\mathcal{F}_U = \{(W, W') \in \Omega(U)^2 : \mu(W \Delta W') = 0\}.$$

Para cada $S \in Ab(X)$ temos o conectivo:

$$\square_S : U \mapsto \mathcal{F}_U = \{S \cap U\}.$$

Estes últimos conectivos serão muito úteis mais adiante (para maiores esclarecimentos ver [C]).

DEFINIÇÃO 4 - Por $L(X)$ designamos a seguinte linguagem, de tipo relacional τ, contendo:
(a) um conjunto infinito V de variáveis de cujos elementos serão designados por x, y, z; com ou sem índices,
(b) símbolos relacionais $R_m^{n_m} \in \mathcal{T}$
(c) um conjunto C de símbolos de conectivos, um para cada conectivo de X. Os elementos de C serão denotados por $\square$ com ou sem índices.
(d) os símbolos $\exists$ e $\forall$ para os quantificadores.
O conjunto $F(X)$ das fórmulas de $L(X)$ é definido de modo usual. Para o caso dos conectivos, se $\varphi_1, \ldots, \varphi_n$ são fórmulas e $\square$ um conectivo n-ário então $\square(\varphi_1, \ldots, \varphi_n)$ é uma fórmula.

Uma fórmula de $F(X)$ se interpreta em um objeto A de $Sh_\tau(X)$ estendendo a semântica de Kripke-Joyal aos novos conectivos, da seguinte maneira; onde $\Theta_1, \ldots, \Theta_n \in A(U)$:

1 - $A \Vdash_U R(\Theta_1, \ldots, \Theta_n)$ see $(\Theta_1, \ldots, \Theta_n) \in R^A(U)$

2 - $A \Vdash_U \Theta_1 = \Theta_2$ see $\Theta_1 = \Theta_2$

3 - $A \Vdash_U \square(\varphi_1, \ldots, \varphi_n)(\Theta_1, \ldots, \Theta_n)$ see $([\![\varphi_1(\overline{\Theta})]\!]_U, \ldots [\![\varphi_n(\overline{\Theta})]\!]_U) \in \square(U)$, onde $\overline{\Theta} = (\Theta_1, \ldots, \Theta_n)$ e $[\![\varphi(\overline{\Theta})]\!]_U = \bigcup\{W \subseteq U \mid A \Vdash_W \varphi(\Theta_{1|W}, \ldots, \Theta_{n|W})\}$

4 - $A \Vdash_U \forall x \varphi(x, \Theta_1, \ldots, \Theta_n)$ see para todo aberto $W \subseteq U$ e todo $\Theta \in A(W)$ tem-se que $A \Vdash_W \varphi(\Theta, \Theta_1|W, \ldots, \Theta_n|W)$

5 - $A \Vdash_U \exists x \varphi(x, \Theta_1, \ldots, \Theta_n)$ see existe uma cobertura (aberta) $\{U_i\}_i$ de U e existem $\Theta_i' \in A(U_i)$ tais que $A \Vdash_{U_i} \varphi(\Theta_i', \Theta_1|U_i, \ldots, \Theta_n|U_i)$.

Podemos extender $F(X)$ permitindo conjunções $\bigwedge_{i \in I} \varphi_i$ e dijunções $\bigvee_{i \in I} \varphi_i$ infinitas com um número fixo finito de variáveis. Denotamos por $F_\infty(X)$ a esta extensão infinitária. Neste caso

6 - $A \Vdash_U \bigwedge_i \varphi_i$ see $A \Vdash_U \varphi_i$ para cada i

7 - $A \Vdash_U \bigvee_i \varphi_i$ see existe uma cobertura $\{U_i\}_i$ de abertos de U tal que $A \Vdash_{U_i} \varphi_i$ para cada i.

DEFINIÇÃO 5 - Sejam $A, B \in Sh_r(X)$, A e B são equivalentes (em $F_\infty(X)$) se para cada sentença $\varphi \in F_\infty(X)$ tem-se $A \Vdash_X \varphi$ see $B \Vdash_X \varphi$. Tal relação será expressada por $A \equiv B$.

Note -se que $A \equiv B$ see $A|U \equiv B|U$, para todo $U \in Ab(X)$; onde $A|U$ é o feixe A restrito a U. Uma direção da prova é trivial. Para a outra direção suponha que $A \equiv B$. Dado $\varphi \in F_\infty(X)$ seja $S = [\varphi]_X^A$ então $A \Vdash_X \Box_S \varphi$ logo $B \Vdash_X \Box_S \varphi$ o que implica $[\varphi]_X^B = S$, portanto, $A \Vdash_U \varphi$ see $U \subseteq S$ see $B \Vdash_U \varphi$.

DEFINIÇÃO 6 - Dado $\varphi \in F_\infty(X)$ o grau (quantificacional) de φ, em símbolos $g(\varphi)$ é definido por:
1 - $g(\varphi) = 0$ se φ é atômica
2 - $g(\Box(\varphi_1, \dots, \varphi_n)) = \max\{g(\varphi_i) \mid i = 1, \dots, n\}$
3 - $g(\exists x \varphi) = g(\forall x \varphi) = g(\varphi) + 1$
4 - $g(\bigwedge_{i \in I} \varphi_i) = g(\bigvee_{i \in I} \varphi_i) = \sup \{g(\varphi_i) \mid i \in I\}$.

Ponhamos $F_\infty^k(X) = \{\varphi \in F_\infty(X) \mid g(\varphi) \leq k\}$. Deste modo escreveremos $A \equiv_k B$ para indicar que A e B são equivalentes com respeito as sentenças de $F_\infty^k(X)$.

III. CHARACTERIZAÇÃO DA EQUIVALÊNCIA ENTRE FEIXES

DEFINIÇÃO 7 - Um homeomorfismo parcial $h : A \to B$ preserva em U uma fórmula φ com n-variáveis (ou φ é invariante para h em U) se para todas as secções $\Theta_1, \dots, \Theta_n \in A(W)$ com $W \subseteq U$ e $\Theta_i(W) \subseteq$ *dom* h tem-se que $A \Vdash_W \varphi(\Theta_1, \dots, \Theta_n)$ see $B \Vdash_W \varphi(h \circ \Theta_1, \dots, h \circ \Theta_n)$.

Observe-se com respeito a definição anterior que o fato de ser h contínua implica ser h um homeomorfismo local, e assim, se Θ é uma secção com $dom\Theta = W$

e $\Theta(W) \subseteq dom\ h$ então $h \circ \Theta$ é uma secção com domínio W.

TEOREMA 1 - Se $\varphi \in F_\infty(X)$ e $g(\varphi) \leq k$ então φ é invariante em U para todo $h \in P_k(U)$.

Demonstração - (indução sobre o grau de φ). Suponhamos que $g(\varphi) = 0$. Neste caso a demonstração é obvia uma vez que por definição o homeomorfismo h preserva a igualdade e a estrutura de cada fibra. Suponhamos então o teorema válido para as sentenças φ com $g(\varphi) \leq k < k'$ e vejamos que é valido também para as senteças φ tais que $g(\varphi) \leq k'$. Seja $h \in P_{k'}(U)$. Utilizamos indução sobre a complexidade de φ. O caso atômico é obvio.

1º Caso: Seja $\varphi = \Box(\varphi_1, \ldots, \varphi_m)$ com $g(\varphi_i) \leq k'$
Observemos que pela hipótese de indução na complexidde temos $A \Vdash_W \varphi_i(\Theta_1, \ldots, \Theta_n)$ see $B \Vdash_W \varphi_i(h \circ \Theta_1, \ldots, h \circ \Theta_n)$ para $i = 1, \ldots, m$ e qualquer que seja $W \subseteq U$. Portanto $[\![\varphi_i(\overline{\Theta})]\!]_U^A = [\![\varphi_i(h(\overline{\Theta})]\!]_U^B$ para $i = 1, \ldots, m$. Deste modo $A \Vdash_U \Box(\varphi_1, \ldots, \varphi_m)(\overline{\Theta})$ see $([\![\varphi_1(\overline{\Theta})]\!]_U^A, \ldots, [\![\varphi_m(\overline{\Theta})]\!]_U^A) \in \Box(U)$ see $([\![\varphi_i(h(\overline{\Theta}))]\!]_U^B, \ldots, [\![\varphi_m(h(\overline{\Theta}))]\!]_U^B) \in \Box(U)$ see $B \Vdash_U \Box(\varphi_1, \ldots, \varphi_m) h(\overline{\Theta})$. Aqui $h(\overline{\Theta})$ significa $(h \circ \Theta_1, \ldots, h \circ \Theta_m)$

2º Caso: $\varphi = \exists x \phi(x, \overline{\Theta})$, então $g(\phi) = k < k'$. Tem-se que $A \Vdash_U \varphi$ see existe uma cobertura aberta $\{U_i\}_i$ de U e $\mu_i \in A(U_i)$ tal que $A \Vdash_{U_i} \phi(\mu_i, \overline{\Theta})$. Aplicando a propriedade de extenção de h para cada U_i obtemos coberturas $\{U_i^\alpha\}_\alpha$ de U_i e $\mu_i^\alpha \in B(U_i^\alpha)$ tais que $h|U_i^\alpha \cup \{(\mu_i(x), \mu_i^\alpha(x))\}_{x \in U_i^\alpha} \in P_k(W_i^\alpha)$. Logo, como $A \Vdash_{U_i^\alpha} \phi(\mu_i|U_i^\alpha, \Theta|U_i^\alpha)$ tem-se que (hipotese de indução sobre o grau) $B \Vdash_{U_i^\alpha} \phi(\mu_i^\alpha, h(\overline{\Theta}))$, porém, $U = \cup_{i,\alpha} U_i^\alpha$ e portanto $B \Vdash_U \exists x \phi(x, h(\overline{\Theta}))$. A outra direção se demonstra de modo análogo.

3º Caso: Seja $\varphi = \forall x \phi(x, \overline{\Theta})$. Tem-se então que $g(\phi) = k < k'$, e $A \Vdash_U \forall \phi(x, \overline{\Theta})$ see para todo $W \in Ab(X), W \subseteq U$ e $\mu \in A(W)$ $A \Vdash_W \phi(\mu, \overline{\Theta})$. Suponhamos que $B \| \not\vdash_U \forall x \phi(x, h(\overline{\Theta}))$ então existe $W \subseteq U$ e $\mu' \in B(W)$ tal que $B \| \not\vdash_W \phi(\mu', h(\overline{\Theta}))$. Pela propriedade de extensão (a) existe uma cobertura $\{W_i\}_i$ de $W, \delta_i \in A(W_i)$ e $h_i \in P_k(W_i)$ tais que $h_i \supseteq h|W_i$ e $h_i \circ \delta = \mu'|W_i$. Logo dado que $B \| \not\vdash_W$ $\phi(\mu'|U_i, h(\overline{\Theta})|U_i)$ para algum U_i (pois do contrário se teria $B \Vdash_W \phi(\mu', h(\Theta)))$, tem-se que, pela hipótese de indução no grau, $A \| \not\vdash_{U_i} \phi(\delta_i, \overline{\Theta}|U_i)$ o que contraria a hipótese inicial $A \Vdash_U \varphi$. A outra direção se faz de modo análogo.

4º Caso: $\varphi = \bigwedge_{i \in I} \varphi_i$; $\varphi = \bigvee_{i \in I} \varphi_i$. Tem-se $A \Vdash_U \varphi(\overline{\Theta})$ see $A \Vdash_U \varphi_i(\overline{\Theta})$

para todo $i \in I$ see (hipótese de indução na complexidade) $B \Vdash_U \varphi_i(h(\overline{\Theta}))$ see $B \Vdash_U \bigwedge_{i \in I} \varphi_i$. O caso da disjunção é igualmente simples.

No próximo teorema é importante considerarmos a tupla vazia de secções, $<,>$, que pertence a todo $A(U)^n$. Note-se além disso, que todo homeomorfismo parcial, incluindo o homeomorfismo vazio, envia $<>$ em $<>$.

TEOREMA 2 - Seja k um ordinal. Dado $A \in Sh_\tau(X)$ e secções $\overline{\Theta} = < \Theta_1, \ldots, \Theta_n > \in A(U)^n$, existe uma fórmula $\varphi(x_1, \ldots, x_n) \in F^k_\infty(X)$ tal que $A \Vdash_U \varphi(\overline{\Theta})$, e para todo $B \in Sh_\tau(X), U' \subseteq U$ e $\overline{\mu} =< \mu_1, \ldots, \mu_n > \in B(U')^n$ tem-se que $B \Vdash_{U'} \varphi(\mu_i, \ldots, \mu_n)$ se e somente se existe $(h : A \to B) \in P_k(U')^n$ tal que $(h \circ \Theta_i | U') = \mu_i$

Demonstração - (indução sobre k). Suponhamos $k = 0$. Sejam $x_1, \ldots, x_m$ variáveis e definamos $\varphi_0 = \varphi_{A,\overline{\Theta}}$ do seguinte modo: $\varphi_0 = \Phi_1 \bigwedge \Phi_2$ onde

$$\Phi_1 = \bigwedge_{i<j} \Box_{[\Theta_i = \Theta_j]^A_U} (x_i = x_j)$$

$$\Phi_2 = \bigwedge_{R \in \Vdash_{U'}} \Box_{[R(\Theta_{i_1}, \ldots, \Theta_{i_n})]^A_U} R(x_{i_1}, \ldots, x_{i_n})$$

É óbvio que $A \Vdash_U \varphi_0(\Theta_1, \ldots, \Theta_U)$ e se $U' \subseteq U$; $B \Vdash_{U'} \varphi_0(\mu_1, \ldots, \mu_n)$ se e somente se $[\mu_i = \mu_j]^B_{U'} = [\Theta_i = \Theta_j]^A_U \cap U' = [\Theta_i = \Theta_j]^A_{U'}$ para $i < j$ e $[R(\mu_i, \ldots, \mu_{i_r})]^B_{U'} = [R(\Theta_{i_1}, \ldots, \Theta_{i_n})]^B_{U'}$, portanto para todo $x \in U'$ tem-se $\mu_i(x) = \mu_j(x)$ see $\Theta_i(x) = \Theta_j(x)$; e $(\mu_{i_1}(x), \ldots, \mu_{i_r}(x)) \in R^B_x$ see $(\Theta_{i_1}(x), \ldots, \Theta_{i_r}(x)) \in R^A_x$ o que significa que $h_x = \{(\Theta_i(x), \mu_i(x)) | i = 1, \ldots, n\}$ é um isomorfismo parcial de A_x em B_x para $x \in U'$. Assim $h = \bigcup_{x \in U'} h_x$ é um homeomorfismo parcial tal que $h = (\Theta_i | U') = \mu_i$.

Suponhamos o teorema válido para k. Dados $D \in Sh_\tau(X), \overline{\delta} \in D(U)^n; W \subseteq U$ e $t \in D(W)$ consideremos $\varphi_{D,\overline{\delta},t}(x_1, \ldots x_n, x)$ a fórmula dada pelo teorema com respeito a k para $(D, \overline{\delta} | W, t)$. Seja $\varphi_0(x_1, \ldots, x_n)$ a fórmula dada pelo caso $k = 0$ para $(A, \overline{\Theta})$ e ponhamos:

$$\varphi(x_1, \ldots, x_n) = \varphi_0(x_1, \ldots x_n) \wedge \bigwedge_{D,\overline{\delta},t} \Box_{S_{D,\overline{\delta},t}} \exists x \varphi_{D,\overline{\delta},t}(x_1, \ldots, x_n, x)$$

onde

$$S_{D,\overline{\delta},t} = [\exists x \varphi_{D,\overline{\delta},t}(\Theta_1, \ldots, \Theta_n),\ x]^A_U.$$

Observe que por construção $A \Vdash_U \varphi(\Theta_1, \ldots, \Theta_n)$. Se $U' \subseteq U$ e $B \Vdash_{U'} \varphi(\mu_1, \ldots, \mu_n)$ tem-se 1º, 2º e 3º a seguir:

1º $B \Vdash_{U'} \varphi_0(\overline{\mu})$; e portanto existe um homeomorfismo parcial $h : A \rightarrow B$ tal que $h \circ (\Theta_i | U') = \mu_i; i = 1, \ldots, n$ (caso $k = 0$).

2º Dado $\Theta \in A(W), W \subseteq U', B \Vdash_{U'} \Box_{S_{A,\overline{\Theta},\Theta}} \exists x \varphi_{A,\overline{\Theta},\Theta}(\overline{\mu}, x)$, então $B \Vdash_{U' \cap S_{A,\overline{\Theta},\Theta}} \exists x \varphi_{A,\overline{\Theta},\Theta}(\overline{\mu}, x)$ porém $S_{A,\overline{\Theta},\Theta} \supseteq W$, uma vez que pela hipótese de indução $A \Vdash_W \varphi_{A,\overline{\Theta},\Theta}(\overline{\Theta}, \Theta)$ e portanto $A \Vdash_W \exists x \varphi_{A,\overline{\Theta},\Theta}(\overline{\Theta}, x)$. Concluimos então que $B \Vdash_W \exists x \varphi_{A,\overline{\Theta},\Theta}(\overline{\mu}, x)$. Seja W_i uma cobertura (aberta) de W e $\mu_i' \in B(W_i)$ tais que $B \Vdash_{W_i} \varphi_{A,\overline{\Theta},\Theta}(\overline{\mu}, \mu_i')$. Pela hipótese de indução existem homeomorfismos parciais $h_i : A \rightarrow B, h_i \in P_k(W_i)$ tais que $h_i o(\Theta_j | W_i) = \mu_j | W_i$, isto é, $h_i \supseteq h | W_i$ e $h_i o(\Theta | W_i) = \mu_i'$. O que prova a primeira parte da propriedade de extensão para h definido em 1º.

3º Dado $\mu \in B(W)$ e $W \subseteq U'$, tem-se que $B \Vdash_{U'} \Box_{S_{B,\overline{\mu},\mu}} \exists x \varphi_{B,\overline{\mu},\mu}, (\overline{\mu}, x)$ e como por definição $B \Vdash_W \varphi_{B,\overline{\Theta},\Theta}(\overline{\mu}, \mu)$ temos que $S_{B,\overline{\mu},\mu} \cap U' \supseteq W$ o que implica, por definição de $S_{B,\overline{\mu},\mu}$, que $A \Vdash_W \exists x \varphi_{B,\overline{\mu},\mu}(\overline{\Theta}, x)$. Tal fato completa, como antes em 2º, a demonstração da propriedade de extensão para h. Concluimos portanto que $h \in P_{k+1}(U')$ e $h \circ (\Theta_i | U') = \mu_i$.

Inversamente se existe $(h : A \rightarrow B) \in P_{k+1}(U')$ tal que $ho(\Theta_i | U') = \mu_i$ então para cada $(D, \overline{\delta}, t)$ com $\overline{\delta} \in D(U)^n, t \in D(W), W \subseteq U$, tem-se que (pelo teorema 1) $(S_{D,\overline{\delta},t}) \cap U' = [\![\exists x \varphi_{D\overline{\delta},t}(\overline{\Theta}, x)]\!]^A_{U'} = [\![\exists x \varphi_{D,\overline{\delta},t}(\overline{\mu}, x)]\!]^B_{U'}$ e assim $B \Vdash_{U'} \Box_{S_{D,\overline{\delta},t}} \exists x \varphi_{D,\overline{\delta},t}(\overline{\mu}, x)$. Ou seja $B \Vdash_{U'} \varphi(\overline{\mu})$ o que completa a prova do Teorema 2.

Observe que na definição de $\varphi(x_1, \ldots, x_n)$, na prova do Teorema anterior , a conjunção percorre todas as estruturas de $Sh_\tau(X)$ (que forma uma classe propria). No entanto as fórmulas $\varphi_{D,\overline{\delta},t}$ tem grau k e portanto formam um conjunto (modulo equivalência). No caso em que o espaço topologico X e o tipo τ são finitos a conjunção é finita e, portanto, φ finitaria.

Se considerarmos a tupla vazia $< \ > \in A(U)$ e $U = X$ no teorema anterior obtemos o caso especial seguinte:

COROLÁRIO 1 - Seja k um ordinal. Dado $A \in Sh_\tau(X)$ existe uma sentença $\varphi_A \in F^k_\infty(X)$ tal que $A \Vdash_X \varphi_A$ e para todo $B \in Sh_\tau(X)$ tem-se que $B \Vdash_X \varphi_A$ se e somente se existe $(h : A \to B) \in P_k(X)$.

COROLÁRIO 2 - Seja φ uma sentença de $F_\infty(X)$, $A \in Sh_\tau(X)$ e φ_A a sentença do Corolário 1. Tem-se então que $\varphi \equiv_X \bigvee_{A \Vdash_X \varphi} \varphi_A$.

Demonstração - Se $B \Vdash_X \varphi$ então, o fato de $B \Vdash_X \varphi_B$ implica $B \Vdash \bigvee_{A \Vdash_X \varphi} \varphi_A$. Inversamente se $B \Vdash_X \bigvee_{A \Vdash \varphi} \varphi_A$ então existe uma cobertura $\{U_A\}$ de X tal que $B \Vdash_{U_A} \varphi_A$, para todo A tal que $A \Vdash_X \varphi$. Logo $B \equiv_{U_A} A$ para todo $A \Vdash_X \varphi$. Assim $B \Vdash_{U_A} \varphi$ para todo U_A portanto $B \Vdash_X \varphi$ uma vez que $\{U_A\}$ cobre X.

COROLÁRIO 3 - Qualquer conectivo é definido a partir de $\wedge, \vee, \bigwedge$ e $\Box_S, S \in \Omega(X)$.

Demonstração - Consequência imediata do Corolário 2.

TEOREMA 3 - (Generalização dos teoremas de Fraïsse e Karp para um topos $Sh(X)$). Sejam $A, B \in Sh_\tau(X)$. Tem-se então que $A \equiv_k B$ se e somente se existe $(h : A \to B) \in P_k(X)$.

Demonstração - Se existe $(h : A \to B) \in P_k(X)$ então pelo Teorema 1, toda fórmula $\varphi \in F^k_\infty(X)$ é invariante em X para h. Em particular para toda sentença $\varphi \in F^k_\infty(X)$ tem-se $A \Vdash_X \varphi$ see $B \Vdash_X \varphi$. Inversamente se $A \equiv_k B$ seja φ_A a sentença de grau k do Corolário 1. Como $A \Vdash_X \varphi_A$ então $B \Vdash_X \varphi_A$ e portanto existe $(f : A \to B) \in P_k(X)$.

TEOREMA 4 - Se $A \equiv_k B$ como feixes então para todo $x \in X, A_x \equiv_k B_x$ como estruturas clássicas.

Demonstração - dado $x \in X$ definamos $\mathcal{F}_k = \{h|Ax : \text{existe } U \in Ab(X), x \in U$ e $(h : A \to B) \in P_k(U)\}$. Mostra-se facilmente por indução que os elementos de $\mathcal{F}_k$ são k-isomorfismos parciais (no sentido de Karp) de A_x em B_x. Se $A \equiv_k B$ então existe $(h : A \to B) \in P_k(X)$ logo $h|A_x \in \mathcal{F}_k$ e assim $A_x \equiv_k B_x$ em $L_{\infty\omega}$ pelo Teorema de Karp.

Observação Final. Notemos que todas as construções e provas dos resultados acima foram feitas externamente ao topos $Sh(X)$. No entanto estamos convencidos de que tais construções e resultados podem ser internalizados a um topos de Grothendieck arbitrário.

BIBLIOGRAFIA

[C-S] Caicedo,X.; Sette, A.M., Logics and Pseudogroups. Contemporary Mathematics 69 (1988). L.P. de Alcantara, W. Carnielli eds.

[C] Caicedo, X., Investigaciones sobre Conectivos Intuicionistas, Preprint. Universidad de los Andes, Bogotá, 1990.

[E] Ehresmann, Ch., Categories inductives et pseudogroupes. Ann. Inst. Fourier, Grenoble 10 (1960), 307-336.

[E] Fourman, M. and Scott D. S., Sheaves and Logic. Lecture Notes in Math. 753 (1979). M. P. Fourman, C. J. Mulvey and D. S. Scott (eds.). Springer Verlag.

[G] Goldblatt, R., Topoi. North-Holland, 1979.

[Gr] Gray, J.W., Fragment of the History of Sheaf Theory. Lecture Notes in Math. 753 (1979). M.P. Fourman, C.J. Mulvey and D. Scott eds. Springer-Verlag.

[R] Reyes, G.E., Theorie des Modeles et Faisceaus. Advances in Mathematics 30 (1978), 156-170.

[T] Tennison, B.R., Sheaf Theory, Cambridge University Press, 1975.

Claudio Pizzi

Modal Operators in Logics of Consequential Implication

§1. The minimal properties of systems aiming to axiomatize what we will call Consequential Implication ($\mathcal{CI}$) are characterized as follows:

1) It must hold $\neg(A \ \mathcal{CI} \ \neg A)$ (Aristotle's thesis) and equivalently $(A \ \mathcal{CI} \ B) \supset \neg(A \ \mathcal{CI} \ \neg B)$ (Boethius' Thesis)

2) $\mathcal{CI}$ is non-monotonic in the following sense: the law of factor - $(A \ \mathcal{CI} \ B) \supset ((A \wedge R) \ \mathcal{CI} \ (B \wedge R))$ - and the law of monotonicity - $(A \ \mathcal{CI} \ B) \supset ((A \wedge R) \ \mathcal{CI} \ B)$ - do not hold or hold in weakened form

3) It is possible to define an "analytic" and a "synthetic" variant - $\mathcal{CI}'$ and $\mathcal{CI}''$ - of any connective endowed with properties 1) and 2), such that beyond 1) and 2) they have the following properties: if $\Box A$ is definable in the reference system and has the minimal properties of standard logical necessity then

a) $(A \ \mathcal{CI}' \ B)$ implies $\Box(A \supset B)$

b) $(A \ \mathcal{CI}'' \ B)$ does not imply $\Box(A \supset B)$

c) $(A \ \mathcal{CI}' \ B)$ implies $(A \ \mathcal{CI}'' \ B)$ but not *viceversa*.

While condition 1) is accepted in the realm of so-called *connexive logic*, conditions 2) and 3) are not. In particular , the analytic - synthetic distinction is normally not allowed by the linguistic resources of connexive logics (see Angell [1] and McCall [4]) . $\neg(A \ \mathcal{CI} \ \neg A)$ and $(A \ \mathcal{CI} \ B) \supset \neg(A \ \mathcal{CI} \ \neg B)$ are the cornerstones of what, in the light of the historical tradition, might be called "Chrysippean implication" . A consequence of these two laws which Chrysippus seems not to have noticed is that accepting them implies excluding the law of simplification $(A \wedge B) \ \mathcal{CI} \ B$. If the latter were to hold, in fact, we would have both $(A \wedge \neg A) \ \mathcal{CI} \ A$ and $(A \wedge \neg A) \ \mathcal{CI} \ \neg A$, namely a couple of wffs which are incompatible with Boethius' Thesis.

§2. Let us begin by formulating an axiomatization of a weak system of implication, to be called CI. We will then have to prove that the implication axiomatized in CI, which is symbolized by —>, has the properties of analytical consequential implication.

The symbols of the language of CI are propositional variables p,q, r.... and the primitive connectives are $\neg$, v, —>. The auxiliary symbols v, $\supset$, $\equiv$ are defined as in standard logic, while monadic modal operators may be defined as follows:

if $T =_{Df} p \supset p$ and $\bot =_{Df} \neg T$, $\Box A =_{Df} T \rightarrow A$; $\Diamond A =_{Df} \neg \Box \neg A$

The axioms of CI are the following:

O. Standard axioms for classical propositional calculus PC and

(a) $((p \rightarrow q) \wedge (q \rightarrow r)) \supset (p \rightarrow r)$

(b) $\neg((p \wedge r) \rightarrow \bot) \supset ((p \rightarrow q) \supset ((p \wedge r) \rightarrow (q \wedge r))$

(c)$((p \wedge \sim q) \rightarrow \bot) \wedge \sim(p \rightarrow \bot) \wedge \sim(\sim q \rightarrow \bot)) \supset (p \rightarrow q)$

(d)$(\sim p \rightarrow \sim q) \supset (q \rightarrow p)$

(e) $(p \rightarrow \bot) \supset (\bot \rightarrow p)$

(f) $(\bot \rightarrow p) \supset (p \rightarrow \bot)$

(g) $\sim(p \rightarrow \sim p)$

(h) $p \rightarrow p$

Rules: Uniform Substitution (US), Modus Ponens for $\supset$, Replacement of Proved Material Equivalents (Eq)

Remark: Thanks to the given definitions of the modal operators, axiom (b) might be reformulated as $\Diamond (p \wedge r) \supset ((p \rightarrow q) \supset ((p \wedge r) \rightarrow (q \wedge r))$, while axiom (c) might be reformulated as $(\Box(p \supset q) \wedge \Diamond p \wedge \neg \Box q) \supset (p \rightarrow q)$

Let us define a function ϕ from the language of CI to the language of the deontic system KD which, as is well known, is axiomatized as follows:

A1-14 Axioms for PC (classical propositional calculus)

A5 $\Box p \supset \Diamond p$

A6 $\Box (p \supset q) \supset (\Box p \supset \Box q)$

Rules: Modus Ponens for $\supset$, US; (Nec) $\vdash A$ only if $\vdash \Box A$

ϕ is defined as follows:

$\phi(p) = p$

$\phi(\neg A) = \neg\phi(A)$

$\phi(A \circ B) = \phi(A) \circ \phi(B)$ (where $\circ$ is any two-place truth-functional connective)

$\phi(A \rightarrow B) = \phi(A) \rightarrow \phi(B) = \Box(\phi(A) \supset \phi(B)) \wedge (\Diamond\phi(B) \supset \Diamond\phi(A)) \wedge (\Box\phi(B) \supset \Box\phi(A))$

In the second place, another translation function ψ from the language of KD into the language of CI may be defined as follows:

$\psi(p) = p$

$\psi(\neg A) = \neg\psi(A)$

$\psi(A \circ B) = \psi(A) \circ \psi(B)$

$\psi(\Box A) = \Box\psi(A) = \psi(T) \rightarrow \psi(A)$

Remark: since $\psi(T)$ is T, $\psi(T) \rightarrow \psi(A)$ equals $T \rightarrow \psi(A)$.

Now it may be proved by induction on the length of the proofs the following Lemma holds:

Lemma 1.1 . (i) $\vdash_{KD} A$ only if $\vdash_{CI} \psi(A)$

(ii) $\vdash_{CI} A$ only if $\vdash_{KD} \phi(A)$

Proof: For the details of the proof of (i) see Pizzi [7]. A key step of the proof is given by considering that axiom (g), namely Aristotle's thesis, is equivalent to Boethius' Thesis, namely $(p \rightarrow q) \supset \neg(p \rightarrow \neg q)$; by US from the latter it follows $(T \rightarrow q) \supset \neg(T \rightarrow \neg q)$, and $(T \rightarrow q) \supset \neg(T \rightarrow \neg q) = \psi(\Box q \supset \Diamond q)$. The proof of (ii) may be performed by employing the tableaux method for KD , which is, of course, a simplification of the tableaux method for T obtained by weakening the requirement that R is reflexive into the requirement that R is serial, namely that for every m_i, there is a m_j such that $m_i R m_j$.

Furthermore, one may prove

Lemma 1.2 $\vdash_{KD} A \equiv \psi(\phi(A))$

$\vdash_{CI} A \equiv \phi(\psi((A))$

For the proof see Pizzi [7] and Pizzi [6] (Suffice it to remark that in no step of the proof reported in the latter work , which is given for T, there is an application of $\Box p \supset p$).

By definition of embedding , we have an embedding of X into Y when $\vdash_X A$ iff $\vdash_Y \phi(A)$. By a result whose simple proof may be found in Smirnov [9], the conjunction of Lemma 1.1 and Lemma 1.2 amounts to proving that there is an embedding of KD into CI and of CI into KD . We have then proved the following theorem:

Theorem 1. $\vdash_{KD} A$ iff $\vdash_{CI} \psi(A)$

$\vdash_{CI} A$ iff $\vdash_{KD} \phi(A)$

Another way to state the same result is the following: if we extend KD by the mentioned definition of —> and CI by the mentioned definition of □ , all the theorems of the former turn out to be theorems of the latter and *vice versa*. This amounts to saying that KD and CI are *definitionally equivalent* systems. This equivalence result has the merit of providing a decision procedure for CI : if A is a CI-wff to be tested , it is sufficient to test $\phi(A)$ by the tableau procedure for KD. Thanks to this procedure it turns out that the law of factor $((p \rightarrow q) \supset ((p \wedge r) \rightarrow (q \wedge r)))$ and the law of monotonicity for —> $((p \rightarrow q) \supset ((p \wedge r) \rightarrow q)$ are not CI-theorems ; furthermore , if □ is defined in the mentioned way, it may be proved that $(A \rightarrow B) \supset \Box(A \supset B)$. Since we already know that $\neg(A \rightarrow \neg A)$ and $(A \rightarrow B) \supset \neg(A \rightarrow \neg B)$ are interdeducible theorems of CI, this proves that —> has the properties of what we before defined as analytical consequential implication.

§3. Now one may show that there is at least a second system of consequential analytical implication, which is like CI except for the fact that the axiomatized implicative connective is non-contrapositive. We will call this parallel system CI=>. If □A is defined as T =>A, the axioms of CI=> are:

a') $((p \Rightarrow q) \wedge (q \Rightarrow r)) \supset (p \Rightarrow r)$

b') $\Diamond(p \wedge r) \supset ((p \Rightarrow q) \supset ((p \wedge r) \Rightarrow (q \wedge r)))$

c') $(\Box(p \supset q) \wedge \Diamond p) \supset (p \Rightarrow q)$

e') $(p \Rightarrow \bot) \supset (\bot \Rightarrow p)$

f') $(\bot \Rightarrow p) \supset (p \Rightarrow \bot)$

g') $(p \Rightarrow q) \supset \neg(p \Rightarrow \neg q)$

h') $p \Rightarrow p$

Like CI, CI=> is proved to be definitionally equivalent to the deontic logic KD. The translation functions however are different: we have in fact to define two new functions ϕ' and ψ' which are coincident with ϕ and ψ except for the following clauses :

$\phi'(A \Rightarrow B) = \phi'(A) \Rightarrow \phi'(B) = \Box(\phi'(A) \supset \phi'(B)) \wedge (\Diamond\phi'(B) \supset \Diamond\phi'(A))$

$\psi'(\Box A) = \Box\psi'(A) = T \Rightarrow \psi'(A)$

Thanks to an argument which follows the lines of the proofs of Lemmas 1.1 and 2.1 we may prove in fact two lemmas:

Lemma 2.1 (i) $\vdash_{KD} A$ only if $\vdash_{CI\Rightarrow} \psi'(A)$

(ii) $\vdash_{CI\Rightarrow} A$ only if $\vdash_{KD} \phi'(A)$

and

Lemma 2.2 $\vdash_{KD} \quad A \equiv \psi'(\phi'(A))$

$\vdash_{CI\Rightarrow} \quad A \equiv \phi'(\psi'((A))$

Thus it turns out , as before, that another theorem can be proved

Theorem 2 $\vdash_{KD} A$ iff $\vdash_{CI\Rightarrow} \psi'(A)$

$\vdash_{CI\Rightarrow} A$ iff $\vdash_{KD} \phi'(A)$

It remains to be proved that the connective => has the properties which identify a connective as a connective of consequential implication. As a matter of fact, we are able to prove that $\neg(p \Rightarrow \neg p)$ and $(p \Rightarrow q) \supset \neg(p \Rightarrow \neg q)$ are interdeducible (1), that $(p \Rightarrow q) \supset \Box(p \supset q)$ is a CI - theorem while $(p \Rightarrow q) \supset ((p \wedge r) \Rightarrow q)$ and $(p \Rightarrow q) \supset ((p \wedge r) \Rightarrow (q \wedge r)))$ are not such . Furthermore $(p \Rightarrow q) \supset \Box(p \supset q)$, but not the converse implication, may be proved to hold. Then => has the properties of an operator of analytical consequential implication.

A questionable feature of => is due to the fact that contrapositivity is considered to be an intuitive feature of analytical implication . But Parry's analytical implication, for instance, is not such (see [5]) since it is based on the principle of variable-containment (only the variables in the antecedent may occurr in the consequent). However, the most important difference between —> and => is that the nice symmetry between $T \Rightarrow p$ and $\neg p \Rightarrow \bot$ is lost because of the failure of contrapositivity. For the same reason, while we have among the theorems $(\bot \Rightarrow p) \equiv (p \Rightarrow \bot)$, we lack $(p \Rightarrow T) \equiv (T \Rightarrow p)$.

The two systems CI and CI=> , as it is to be expected, are proved to be definitionally equivalent. The translation functions ϕ'' (from CI=> into CI) and ψ'' (from CI into CI=>) are coincident with ϕ and ψ as concerns the truth-functional connectives and differ in the following clauses:

$\phi''(A \Rightarrow B) = (T \longrightarrow ((\phi''(A) \supset \phi''(B))) \wedge (\neg(T \longrightarrow \neg\phi''(B)) \supset \neg(T \longrightarrow \neg\phi''(A)))$

$\psi''(A \rightarrow B) = (\psi''(A) \Rightarrow \psi''(B)) \wedge ((T \Rightarrow \psi''(B)) \supset (T \Rightarrow \psi''(A))$

The method used in the proof of the embeddability theorem is given by the tableaux procedure for KD, since both systems are embeddable in KD . Then we are able to prove

Lemma 3.1

(i) $\vdash_{CI} A$ only if $\vdash_{CI\Rightarrow} \psi''(A)$

(ii) $\vdash_{CI\Rightarrow} A$ only if $\vdash_{CI} \phi''(A)$

Lemma 3.2

$\vdash_{CI\Rightarrow} A \equiv \psi''(\phi''(A))$

$\vdash_{CI} A \equiv \phi''(\psi''(A))$

Theorem 3

$\vdash_{CI} A$ if and only if $\vdash_{CI\Rightarrow} \psi''(A)$

$\vdash_{CI\Rightarrow} A$ if and only if $\vdash_{CI} \phi''(A)$

A consequence of Theorem 3 is that CI and CI=> are definitionally equivalent systems.

§4. Let us now extend CI by a new axiom which is

(i) (p —> q) ⊃ (p ⊃ q)

By this addition we obtain a system which will be named CI.O.

In a parallel way, we extend CI=> by

(ii) (p => q) ⊃ (p ⊃ q)

so to obtain a system which will be named CI.O=>.

On the modal side, we add the axiom □p ⊃ p to KD and what we obtain is the well known system KT, usually known as T.

By a simple extension of the preceding argument it is easy to prove that the translation functions ϕ, ϕ', ϕ'' and ψ, ψ', ψ'' allow us to prove that CI.O, CI.O=> , T are definitionally equivalent. Thus they are all decidable by the well known tableaux procedure used for T (see Hughes and Cresswell [3]).

§4. Let us now move to the logic of synthetic consequential implication. Here the problem of defining such operators becomes more difficult since we have less firm intuitions about the properties of this new family of connectives . We may begin by analyzing the behaviour of what has been called "circumstantial operator" (see Aqvist [10]). The minimal axioms for this operator "*" are

(i) *p ⊃ p

(I) ◊p ⊃ ◊*p

Adding both axioms to CI.O we obtain a system which is called CI.O* in Pizzi [6] . If we add the rule

R*O Eq ⊢A ≡ B ---> ⊢*A ≡ *B

the resulting system will be called Cl.O*Eq . If the addition is made to Cl=> we will obtain a system which will be called Cl.O*=>Eq.

Since we know that Cl.O and Cl.O=> are both definitionally equivalent to T, the problem is now how to extend suitably the language of system T . This language may be enriched by new symbolic objects which may be called *quasi-variables* . **w** is a new atomic wff in the language, and we add a new formation rule to the effect that if A, B, C are wffs then $\mathbf{w}^A$, $\mathbf{w}^B$, $\mathbf{w}^C$ ..., are wffs. Each of the quasi-variables **w** , $\mathbf{w}^A$, $\mathbf{w}^B$, $\mathbf{w}^C$ ... may be substituted to a variable, but not *vice versa*. The exponents of the quasi-variables however are treated as normal wffs: in other words, the atomic variables occurring in them are subject to Uniform Substitution.

Let us call $T.O^W$ the system based on this language and obtained by extending T with the axiom

T^WO $\Diamond p \supset \Diamond(\mathbf{w}^p \wedge p)$

The models for $T.O^W$ are T-models extended with a specific clause for V mirroring T^WO:

VR1. If some m_j exists such that m_i R m_j and $V(A,m_j)=1$, then some m_l exists such that $m_i Rm_l$ and $V(A,m_l)= V(\mathbf{w}^A, m_l)=1$.

$T.O^W$ turns out to be decidable and complete by a simplification of the proof given in Pizzi [7] for the system which is there called T^WO.

Let us then extend $T.O^W$ by a replacement rule for materially equivalent wffs, or more simply by the rule:

RwO' $\vdash A \equiv B \dashrightarrow \vdash \mathbf{w}^A \equiv \mathbf{w}^B$

The resulting system will be called $T.O^WEq$.

The models for $T.O^WEq$ are 4-ples $<M,R,R^w,V>$ where

(i) $M = \{m_1, m_2, m_3 ...\}$

(ii) R is a reflexive dyadic relation on M

(iii) $R^w \subseteq M \times M$,

(iv) V is defined in $T.O^W$-models with the addition of the following further clause :

VR2 If $V(\mathbf{w}^A, m_i) \neq V(\mathbf{w}^B, m_i)$ then there exists some world m_j such that $m_i R^w m_j$ and $V(A, m_j) \neq V(B, m_j)$.

A decision procedure for $T.O^W Eq$ may be sketched in this way. Let us first devise a tableaux decision procedure for $T.O^W$ by simply extending the tableaux method for T with a rule mirroring VR1. If A is the wff to be tested, let us then list all the subformulas of A. First, let us replace all the O-degree subformulas of the exponents of the quasi-variables in A by one of the TO^W-equivalent wffs - let us say, the first in lexicographical order among the shortest one ; then , let us replace all the 1-degree equivalent wffs in the same way , and so on until all the quasi-variables in the resulting wff A' are either identical or non- equivalent. At the end of the replacement procedure, let us test A' by the tableaux method for $T.O^W$. The same result may be obtained by converting rule VR2 into a rule for tableaux construction . It may also be proved that $T.O^W Eq$ is complete in respect of the class of $T.O^W Eq$-models, and that $p \equiv (\mathbf{w}^p \wedge p)$ is not a valid equivalence.

Now we may prove a new embedding theorem from Cl.O*Eq to $T.O^W Eq$ which is the following. Let us define a translation function Tr which is so defined:

$Tr(p) = p$

$Tr(\neg A) = \neg Tr(A)$

$Tr(A \circ B) = Tr(A) \circ Tr(B)$

$Tr(*A) = \mathbf{w}^{Tr(A)} \wedge Tr(A)$

It may be proved then, by induction on the length of the proofs , the following result: : for every A, $\vdash_{ClO^*Eq} A$ iff $\vdash_{TwOEq} Tr(A)$. The proof is an adaptation of the proof given in Pizzi [7] relating Cl*O (a system which differs from Cl.O*Eq for having in place of R*OEq the stronger $\vdash A \supset B$

---> $\vdash *A \supset *B$) to $T^{W}O$ (a system which differ from $T.O^{W}$ for having, in place of $R^{W}O$, the stronger $\vdash A \supset B \dashrightarrow \vdash \mathbf{w}^{A} \supset \mathbf{w}^{B}$).

§5.In defining operators for synthetic consequential implication we meet just at the beginning an intriguing problem.The problem is that there is a plurality of synthetic conditionals which we may define on the background of Cl.O*Eq. Let us look for instance at this selection (where of course A =>B is defined as $\Box(A \supset B) \wedge (\Diamond B \supset \Diamond A)$)

$A > B =_{Df} *A => B$

$A >_0 B =_{Df} *A \longrightarrow B$

$A >_1 B =_{Df} *A => *B$

$A >_2 B =_{Df} *A \longrightarrow *B$

$A >_3 B =_{Df} (A \supset B) \wedge (*A => B)$

$A >_4 B =_{Df} (A \supset B) \wedge (*A \longrightarrow B)$

One has to check , however, that the operators which are thus defined satisfy the conditions by which they may be qualified as operators of synthetic *consequential* conditionals. In other words we have to show that , if Ƈɉ " is a synthetic consequential conditional,

a) A Ƈɉ " B is implied by A =>B or by A —> B but not viceversa

b) A Ƈɉ " B does not imply $\Box(A \supset B)$

c) Ƈɉ " satisfies Boethius' Thesis and Aristotle's Thesis

d) Ƈɉ " does not satisfy the law of factor and the law of monotonicity

While it is not difficult to show that b), c), d) are satisfied by any conditional of the above list, a) is not easily satisfed by each one of them. Look for instance at $A >_1 B =_{Df} *A => *B$, which is not implied by A —> B , at least on the basis of the minimal axiomatic basis above given for * . $A >_3 B$, on the contrary, is an operator of synthetic consequential implication : in particular , it is straighforward to see that A =>B implies $A >_3 B$ (as regards Aristotle's Thesis, notice that $*A => B$ implies $\neg(*A => \neg B) \vee \neg(A \supset \neg B)$, hence

$\neg (A >_3 \neg B)$, and $A \Rightarrow B$ implies $*A \Rightarrow B$) .

For sake of simplicity in what follows we will concentrate on the first operator of the list, symbolized by the simple ">", but we have to stress that a primary direction of work in the field of consequential implication is the comparison between the properties of the different definable operators. The advantage of choosing $>_3$ in respect of $>$ is, for instance, due to the fact that $A > *A$ is a theorem , while $A >_3 *A$ is not such. This remark, however, suggests that it is important to study the properties of the fragment of Cl.O*Eq in which ">" occurrs but the circumstantial operator does not . The problem to be treated now is : which is the > - fragment of Cl.O*Eq ? In other words, which are the theorems of Cl.O*Eq , if there are any, which yield all and only the theorems containing $\neg$, °, =>, > ?

If we extend the language of Cl.O by the symbol ">" and we define => and □□ as at page 2, we call Cl.O>Eq the system which is obtained by adding to Cl.O the following axioms:

A1 $((p > q) \wedge (q \Rightarrow r)) \supset (p > r)$

A2 $(p > q) \supset (\Diamond q \supset \neg (p > \bot))$

A3 $((p > q) \wedge (p > r)) \supset (p > (q \wedge r))$

A4 $(p \Rightarrow q) \supset (p > q)$

A5 $(p > q) \supset \neg (p > \neg q)$

A6 $(\bot > p) \supset (p > \bot)$

A7 $(p > \bot) \supset (\bot > p)$

US, MP and Eq are the primitive rules for the system.

Theorems

1) $p > p$

2) $((p > q) \wedge (p > \neg q)) \supset (p > \bot)$ (from A2)

3) $\Diamond p \supset \neg (p > \bot)$ (from A3)

4) $\Diamond p \supset (\neg(p > q) \vee \neg(p > \neg q))$ (from A2 by $\neg q/r$)

5) $\neg (p > \neg p)$ (2)

6) $(p -3 \bot) \supset (p > \bot)$

7) $\neg (p > \bot) \supset \neg (p -3 \bot)$

8) $\neg (p > \bot) \supset \Diamond p$

9) $\neg (p > \bot) \equiv \Diamond p$ (3), 8))

Remark.

Axioms A3,A4, A5, A6, A7 turn out to be redundant.

a) we show that the hypothesis $(p > q) \wedge (q > r) \wedge \neg(p > r)$, i.e. the negation of A3, leads to a contradiction. From it ,in fact ,$((p > q) \wedge (q > r)) \supset \neg(p > r)$ follows . But from A1 we have $(p \Rightarrow q) \supset (p > q)$, hence also $((p \Rightarrow q) \wedge (q \Rightarrow r) \supset \neg(p \Rightarrow r)$. Since $(p \supset \neg q) \supset \neg(p \supset q)$, a consequence is $\neg((p \Rightarrow q) \wedge (q \Rightarrow r) \supset (p \Rightarrow r))$, which contradicts axiom a' of Cl.O=>.

b) A4 follows from A2 by US (p/q).

c) From A3 and 9) have $(p > q) \supset (\neg\Diamond p \supset \neg\Diamond q)$ and $(p > q) \supset (\neg\Diamond p \supset (\neg\Diamond q \wedge \neg \Diamond p)$. Since we have $(\neg\Diamond p \supset (\neg\Diamond q \supset (p \Rightarrow q))$ and $(\neg\Diamond p \wedge \neg\Diamond q) \supset (p \Leftrightarrow q)$ we have $\neg\Diamond p \supset ((p > q) \supset \Box(q \equiv p))$. Then we have also $\neg\Diamond p \supset ((p > q) \wedge (p > \neg q) \supset ((q \equiv p) \wedge (p \equiv \neg q))$, hence $\neg\Diamond p \supset ((p > q) \wedge (p > \neg q)) \supset \bot)$. But this amounts to $\neg\Diamond p \supset ((p > q) \supset \neg(p > \neg q))$. Conjoining this theorem to 4) - a simple consequence of A2- we obtain $(p > q) \supset \neg(p > \neg q)$, i.e. A5.

c) By US in A3 ($\bot$/p) we have $(\bot > p) \supset (\Diamond p \supset \neg(\bot > \bot))$, then $(\bot > p) \supset (\Diamond p \supset \bot)$; then $(\bot > p) \supset \neg\Diamond p$, $(\bot > p) \supset (\bot \Rightarrow p)$ and $(\bot > p) \supset (p \Rightarrow \bot)$: hence $(\bot > p) \supset (p > \bot)$. A6 is then redundant.

d) We know that $\neg\Diamond p \equiv p \Rightarrow \bot$; since by 3) $p > \bot$ implies $p \Rightarrow \bot$, by A4 we have $p > \bot \equiv p \Rightarrow \bot$. Since $(p \Rightarrow \bot) \supset (\bot \Rightarrow p)$ and $(p \Rightarrow \bot) \supset (\bot > p)$, by Eq $(p > \bot) \supset (\bot > p)$ is then a theorem .Thus A7 is redundant

What we have now to show is that system Cl.O > Eq contains all and only the >-theorems of Cl.O*Eq.

Of course, if we define a translation *f* from the language of Cl.O>Eq into the language of Cl.O*Eq as

$f(p) = p$

$f(A \circ B) = f(A) \circ f(B)$

$f(\Box A) = \Box f(A)$

$f(A > B) = *A => B$

it is easy to show that $\vdash_{Cl.O>Eq} A$ implies $\vdash_{Cl.O^*Eq} f(A)$.

In fact, all the f-images of the axioms are Cl.O*Eq theorems, and the rules preserve this property. How can we prove the converse assertion, namely that $\vdash_{Cl.O^*Eq} f(A)$ implies $\vdash_{Cl.O>Eq} A$? A possible proof is semantic.To begin with, we formulate a semantics for Cl.O>Eq. which is given by models $<M,R,R^X,V>$ defined as follows:

1) M is a non empty set of possible worlds $m_1, m_2, m_3, \ldots$

2) R is a reflexive relation over M

3) R^X is a function from the set Σ of wffs into the set of all binary relations on M $R: \Sigma \rightarrow \mathcal{P}(M \times M)$. In other words, for every wff B belonging to the set of Cl .O*Eq-wffs, there is a binary relation indexed by B: $R^B \subseteq M \times M$.

4) for every B, $R^B \subseteq R$

5) V is defined as in T-models with the following additional clause:

$V(A > B, m_i)=1$ iff

(i) at every world m_j such that $m_i R^A m_j$, $V(A \supset B, m_j)=1$

(ii) If there is a world m_j such that $m_i R m_j$ and $V(B,m_j)=1$, there is a world m_k such that $m_i R^A m_k$ and $V(A,m_k)=1$

(iii) if $R^A \neq R^B$, then $V(A,m_i) \neq V(B,m_i)$ for some m_i of M

A wff A is Cl>Eq- be valid iff $V(A,m_i) = 1$ at every world m_i of every Cl>Eq-model.

Note that clause (iii) of the definition of V asks us to identify relations which are indexed by exponents which turn out to be equivalent: for instance $(A \wedge A) > B$ and $A > B$ turn out to be equivalent thanks to this identification, which is granted by the equivalence between $A \wedge A$ and A.

The properties of this system may be outlined as follows.

Soundness. By simple Reductio arguments we may show that the axioms A1-A2 turn out to be Cl.O>Eq-valid and US, MP, Eq are validity-preserving.

Remark 1. We know that ¬(p > ¬p) is interdeducible with (p >q) ⊃ ¬(p > ¬q) (see note (2)) and that p =>q implies p >q . This remark is essential in order to prove that > is an operator of synthetic consequential implication. In order to prove that monotonicity and the law of factor do not hold for it it is sufficient to show that the *f*-images of the relevant >- wffs are not theorems of Cl.O*Eq.

Remark 2. Notice that (p ∧ q) > p is not validated by this semantics. In fact , no contradiction follows by the hypothesis that p is true in some R-accessible world and p ∧ q is false in some $R^{p \wedge q}$-accessible worlds . ◊(p ∧ q) ⊃((p ∧ q) > p) is, however, a theorem.

Completeness . The completeness of Cl .O>Eq is proved by a suitable application of the Henkin method (for an application of this method to conditional logic see Chellas [2]). The proof can be reconstructed by any reader who is familiar with this method to prove completeness. We may simply observe that the canonical model may be defined as a 4-ple <*M* , *R*, R^X, *V*> such that:

a) *M* is the set of maximal consistent extensions of Cl.O>Eq

b) for any *x* and *y* s.t. *x* and *y* belong to M , *xRy* iff , whenever □A ∈ x, A ∈y

.c) for every A, $x R^A y$ if and only if

A > B ∈ *x* iff (i) A ⊃ B ∈ *x* and (ii) if there is an *y* such that *x R y* and B ∈ *y*, B > ⊥ ∉ *x*

d) V(p,x)=1 iff p ∈ x, for every atomic variable p.

The crucial step is to show that the foregoing model is a Cl.O>Eq-model, but this is not difficult by observing that (i) whenever □A ∈ *x*, A ∈*x*, since □A ⊃ A belongs to every *x*, so that R is reflexive (ii) if , by Reductio, we were to have xR^Xy but not *xRy* for some *x* and *y* , we would have □ (A ⊃ B) ∈ *x* and

$A \supset B \square \notin y$, but this implies $A > B \notin x$, contrary to the fact that $\square (A \supset B)$ implies $A > B$ by A1 .

By a standard inductive argument then we prove a proposition which implies the semantic completeness of Cl.O>Eq.

T F. Let $M = <M, R, R^X, V>$ be a canonical model for Cl.O>Eq. Then, for every A and for every $x \in M$, $V(A,x)=1$ iff $A \in x$.

§6. We have to relate the Cl> Eq- models to the Cl.O*Eq-models . But since we already dispose of an embedding theorem of Cl.O*Eq into TO^wEq (see p.8) we may directly relate the Cl*OEq-models to the TO^w Eq-models.

As a preliminary remark, let us notice that any Tr-image of >-formulas does not contain any occurrence of the degenerate quasi-variable **w**, so that the value assignment to atomic wffs of Cl>Eq concerns the same stock of atomic wffs of the language of Cl*Eq.

Let us then move from a Cl>Eq -model $\mathfrak{a}$ and define a derived structure $\mathfrak{a}^*$ $= <M^*, R^*, R^{w*}, V^*>$ in this way:

1) $M^* = M$

2) $R^* = R$

3) $m_i R^{w*} m_j$ iff, for some A , $m_i R^A m_j$

4) V^* is a value assignment to all the atomic variables of the language (hence to all the atomic wffs with the exception of **w**) .

The truth value of every wff is uniquely defined by the same rules which are given for T along with two further clauses , the first of which presupposes the definition of Tr given at page 8 :

(i) $V^* (\mathbf{w}^A, m_j) = 1$ iff $m_i R^K m_j$ and $A = Tr(K)$ (so $V^* (\mathbf{w}^{Tr(A)}, m_j) = 1$ iff $m_i R^A m_j$)

(ii) if there is at least one world m_j such that $m_i R m_j$ and $V^*(A, m_j) = 1$, then there is at least one m_l such that $m_i R^{w*} m_l$ and $V^*(\mathbf{w}^A \wedge A, m_l) = 1$

(iii) If $V^*(\mathbf{w}^A \equiv \mathbf{w}^B, m_i) = O$ then there is some m_j such that $m_i R^{w*} m_j$ and $V^*(A \equiv B, m_j) = O$

What we have now to show is that $\mathfrak{a}^*$ is a $T.O^wEq$ -model (see p.9) Suffice it to consider the following facts:

(a) R^* equals R by definition

(b) $R^{w*} \subseteq M \times M$.

(c) since $m_i R^w m_j$ whenever , for some A, $m_i R^A m_j$, and $m_i R^A m_j$ implies $m_i R m_j$, then $m_i R^w m_j$ implies $m_i R m_j$: thus clause (ii) of the definition of V^* implies clause VR1 of the definition of V . Clause (iii) is equivalent to clause VR2 of the definition of V. V^* is then a value assignment having the properties required for a TO^w Eq-model.

Lemma 4.1 Let V a valuation function of a Cl>Eq-model **M**, and V^* a valuation function of a model $\mathbf{M}^*$ derived from **M**. Then, if Tr is defined as at page 8 with the clause for "*- wffs" replaced by

$Tr(A > B) = \Box((\mathbf{w}^{Tr(A)} \wedge Tr(A)) \supset Tr(B)) \wedge (\Diamond(Tr(B)) \supset \Diamond(\mathbf{w}^{Tr(A)} \wedge Tr(A))$

m_i is an arbitrary world belonging to the support of both **M** and $\mathbf{M}^*$, then $V(A, m_i)=1$ iff $V^*(Tr(A), m_i)=1$

Proof. The proof is by induction on the length of the wffs.

Critical step:

Let us suppose by Induction Hypothesis that the property holds for arbitrary A and B. This means that, for any m_i, $V(A \supset B, m_i)=1$ iff $V^*(Tr(A \supset B), m_i)=1$ iff $V^*(Tr(A) \supset Tr(B), m_i)=1$ Let us then suppose that $V(A > B, m_i) = 1$. Two consequences follow , (i) and (ii):

(i) at every R^A-world m_j , $V(A \supset B, m_j)=1$ and, by clause (i) of the definition of the derived model and by definition of R^{w*}, $m_i R^{w*} m_j$ and $V(\mathbf{w}^{Tr(A)}, m_j) = 1$. By Induction Hypothesis, at every such world m_j, $V^*(Tr(A \supset B), m_j)=1$ and, by PC, this implies $V^*(((\mathbf{w}^{Tr(A)} \wedge Tr(A)) \supset Tr(B), m_j)=1$. Now R^{w*}- accessible worlds may be R^X-accessible worlds, where X is a wff equivalent to A, or not.. In this second case, clause (i) grants that, if $m_i R^{w*} m_j'$,

$V(\mathbf{w}^{Tr(A)}, m_{j'})=O$. In these worlds , by PC, we have $V^*((\mathbf{w}^{Tr(A)} \wedge Tr(A)) \supset Tr(B)\ m_j')=1$. Then at every R^{w*}-accessible world m_j we have $V^*((\mathbf{w}^{Tr(A)} \wedge Tr(A)) \supset Tr(B)), m_j)=1$, so $V^*\Box((\mathbf{w}^{Tr(A)} \wedge Tr(A)) \supset Tr(B), m_i)=1$. A converse argument from $V^*\Box((\mathbf{w}^{Tr(A)} \wedge Tr(A)) \supset Tr(B), m_i)=1$ to the conclusion that $V(Tr(A \supset B), m_j)=1$ at every R^A-world is easily derived from the preceding one.

(ii) if $V(A > B, m_i)=1$ this means that , if there is some m_j such that $m_i Rm_j$ at which $V(B,m_j)=1$, then there is some R^A-accessible world m_l at which $V(A,m_l)=1$. But thanks to the Induction Hypothesis and to condition (ii) for V^* , this implies that , if there is some m_j such that $m_i Rm_j$ at which $V^*(Tr(B),m_j)=1$, then there is some R^{W*}-accessible world m_l such that $V^*(\mathbf{w}^{Tr(A)} \wedge Tr(A), m_l)=1$. The converse implication is also correct.

By conjoining propositions (i) and (ii) we obtain that $V(A > B, m_i)=1$ iff $V^*\Box((\mathbf{w}^{Tr(A)} \wedge Tr(A)) \supset Tr(B), m_i)=1$ and $V^*(\Diamond Tr(B) \supset \Diamond(\mathbf{w}^{Tr(A)} \wedge Tr(A)), m_i)=1$. Hence by simple transformations we have that $V(A > B)\ m_i)=1$ iff $V(Tr(A > B), m_i)=1$. Q.E.D.

Theorem 5. If A is not a thesis of Cl>Eq, A is not a thesis of T^WOEq.

Proof. By the completeness of Cl>Eq, from the supposition that A is not a Cl>Eq-thesis it follows that there is a falsifying Cl>Eq-model for A. In other words, there is at least one world m_i of a model $\mathfrak{a}$ such that $V(A,m_i)=1$. But by Lemma 4.1 this implies that in the derived model $\mathfrak{a}^*$, $V^*(Tr(A), m_i)=1$. Then, by the completeness of $T.O^wEq$, we have that Tr(A) is not a $T.O^wEq$ -theorem .

Thanks to the Representation Theorem connecting $T.O^wEq$ to $Cl.O^*Eq$ the following theorem is then a corollary of the preceding one.

Theorem 6. A is a thesis of Cl>Eq iff Tr(A) is a thesis of $Cl.O^*Eq$

The conclusion is then that no >-thesis beyond the ones derivable from Cl>Eq may be derivable inside Cl.O*Eq: if it were, by the representation theorem linking Cl.O*Eq toT^WOEq , it would also be a T^WOEq- thesis, which is impossible.

§7. The foregoing result concerns a basic system of consequential implication, CI*OEq, and depend on the definition of the most simple >-operator. Further developments of this inquiry concern stronger systems - such as CI*O, CI*1 and CI*2 formulated in Pizzi [7] - and have to take into account different >-operators and the logical interrelations between the fragments identified by each one of them. The possibility of deriving such critical conditional theses as Simplification of Disjunctive Antecedents -- $\Diamond p \supset (((p \vee q) > r) \supset (p > r))$- or Transitivity -$((p > q) \wedge (q > r)) \supset (p > r)$ - depends on suitable axiomatic extensions of the minimal basis which has been given for the circumstantial operator. A detailed analysis of these developments may be easily obtained by applying the methods employed in the preceding pages, but lies however beyond the scope of this paper.

NOTES

(1) The line of the proof follows the one given at note (2)

(2) We already know that $\Diamond p \supset ((p > q) \supset \neg (p > \neg q))$ (see Theorem 8) so that what we have to show is simply that $\neg(p > \neg p)$ yields $\neg\Diamond p \supset ((p > q) \supset \neg(p > \neg q))$. As a premise, let us remark that $\Diamond p \supset (p \Rightarrow T)$ is a CI.O=>-thesis. Let us suppose $\neg\Diamond p \wedge (p > q)$. This implies, by the argument sub b) at p.11, $\neg \Diamond q$ and $\Box \neg q$. But $\Box\neg q$ implies $\Diamond \neg q$, and we know that $\Diamond\neg q \supset (\neg q \Rightarrow T)$. $\neg\Diamond p$ equals $T \Rightarrow \neg p$. Then suppose by Reductio $p > \neg q$. From A1, $p > \neg q$ and $\neg q \Rightarrow T$ we have $p > T$, but since $T \Rightarrow \neg p$ we have, *via* A1, $p > \neg p$, contrary to Aristotle's Thesis $\neg (p > \neg p)$. Hence $\neg\Diamond p \wedge (p > q)$ and $p > \neg q$ are inconsistent, and $\neg\Diamond p \wedge (p > q)$ implies $\neg (p > \neg q)$. This Argument

presupposes that a Deduction Theorem may be proved for CI.I>Eq, but a parallel argument may reconstructed without this device.

REFERENCES

[1] Angell, R.A.,"A Propositional Logic with Subjunctive Conditionals", JSL, vol.27, pp.327-343

[2] Chellas, B. "Basic Conditional Logic", Journal of Philosophical Logic", 1975, 4, pp.133-154

[3] G.E.Hughes e Cresswell, M.J., *An Introduction to Modal Logic*, Methuen, London 1973

[4] McCall, "Connexive Implication and the Syllogism", Mind, 1967, vol.76, pp.346-356

[5] Parry, T.W. "Ein Axiomsystem fur eine neue Art von Implikation", Ergebnisse eines Mathmeatischen Kolloquiums, vol.4,pp.5-6

[6] Pizzi, C. , "Una teoria consequenzialista dei condizionali", Bibliopolis, Napoli, 1987

[7] Pizzi, C ., "Decision Procedures for Logics of Consequential Implication", *Notre Dame Journal of Formal Logic*, 1991, vol.32, pp.618-636

[8] Pizzi, C. "Varieties of Non-Monotonic conditionals" in Artosi,A., Carsetti,C. Mondadori,M, Sandri G.,(editors) (CLUEB ,Bologna, 1991)

[9] Smirnov, V.A.., The Definition of Modal Operaots by means of Tense Operators, in I.Niiniluoto and E.Saarinen (eds.) "Intensional Logic: Theory and Application", Acta Philosophica fennica, 1982,pp.50-69

[10] Åqvist, L. ,"Modal Logics with Subjunctive Conditionals and Dispositional Predicates", Journal of Philosophical Logic, 1973, 1-76

A Minimal Closure for a class of Partial Algebras *

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Abstract

In this paper we study a closure for partial algebras. Partial algebras are treated here as algebraic systems that consist of a set, relations, partial operations on this set, introducing an external element which will be the value of the operations where they are not defined.

The closure constructed here is not as general as we would like to, because we had to impose some restrictions on the operations and relations in order to get a total algebra that has a partial subalgebra isomorphic to the original one; and that preserves the s-identities. Finally we study some examples of partial algebras in which this completition is the minimal one with respect to the properties we mention above.

Introduction.

Partial algebras are structures whose operations are defined only on a proper subset of the universe. We may approach the study of these structures from several angles, which depend on different axiomatic systems. In particular, there are distinct concepts of identities in partial algebras, homomorphisms and congruence relations which give rise to different completitions. These structures may be embedded into total structures of the same type of similarity having certain desired properties. In

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this work, we present a particular completition that preserves one kind of identities and which we prove that is minimal in the sense that any other completition of the algebra that preserves the same type of identities contains an isomorphic copy of the original. We begin this work with some definitions which will be used in the construction of the completition of a given partial algebra.

1 General Results.

1.1 Axiomatic System.

As the logical system we use the usual axioms and for the identity axioms we use the following:

I_1: $\exists u(u \simeq \tau)$ where $u \neq \tau$, τ is a variable or an individual constant.

I_2: $((\forall u \;\; u \not\simeq \tau_1)) \wedge ((\forall u \;\; u \not\simeq \tau_2)) \rightarrow \tau_1 \simeq \tau_2$

I_3: $\tau \simeq \sigma \rightarrow (\phi \rightarrow \psi)$ where ϕ is obtained from the atomic formula ϕ replacing all ocurrences of the term τ by σ.

I_4: $\tau \simeq \sigma \rightarrow (\tau_1 \simeq \sigma_1)$ where σ_1 is obtained from the term τ_1 replacing some ocurrences of the term τ by the term σ.

I_5: $\exists x(x \simeq f(\tau_0, \ldots, \tau_{n-1})) \rightarrow \exists x_0(x_0 \simeq \tau_0) \wedge \ldots \wedge \exists x_{n-1}(x_{n-1} \simeq \tau_{n-1})$

The inference rules are detachment and generalization.

1.2 Definitions.

1.2.1 Partial Algebra.

The structure

$$\mathfrak{A} = \langle A, F_i, R_j, c_k \rangle_{\substack{i \in I \\ j \in J \\ k \in K}}$$

is a *partial algebra* if:

i) $A \neq \phi$ and $\chi \notin A$ (where χ is an arbitrary fixed element).

ii) $c_k \in A$, for each $k \in K$.

iii) $R_j \subseteq {}^{n_j}A$ for each $j \in J$ and n_j the arity of R_j.

iv) F_i are partial operations on A such that for each $i \in I$, if $\rho_i > 0$ is the arity of F_i, then

$$F_i :^{\rho_i} (A \cup \{\chi\}) \to A \cup \{\chi\},$$

and if $F_i(\tau_0, \cdots, \tau_{\rho_i - 1}) \in A$, then $\tau_j \in A$ for each term $\tau_j, j < \rho_i$.

1.2.2 Weak Product.

Let $\mathfrak{A}_i = \langle A_i, F_{(i)}, R_{(i)}, 0_{(i)} \rangle$ be partial algebras of the same similarity type, where

$F_{(i)} = \langle F_{(i)j} : j \in J \rangle$ are partial operations.

$R_{(i)} = \langle R_{(i)k} : k \in K \rangle$ are relations.

$0_{(i)}$ is a neutral element of A_i such that for each

$a_i \in A_i, F_{(i)j}(0_i, \cdots, 0_i, a_i, 0_i, \cdots, 0_i) = a_i$ for any placing of a_i in the sequence.

The *weak product* of $\mathfrak{A}_i, i \in I$ is defined as the partial algebra $\mathfrak{B} = \widehat{\prod_{i \in I}} \mathfrak{A}_i$ such that:

$$\mathfrak{B} = \langle \widehat{\prod_{i \in I}} A_i, F^j, R^k, 0 \rangle$$

i) $\widehat{\prod_{i \in I}} A_i = \{f \in \prod_{i \in I} A_i : \{i \in I : f(i) \neq 0_i\}$ is a finite subset of $I\}$.

ii) $0 = \langle 0_i \rangle_{i \in I}$.

iii) R^k is an n-ary relation such that

$$R^k = \{(f_0, \cdots, f_{n-1}) : (f_0(i), \cdots, f_{n-1}(i)) \in R_{(i)k} \text{ for each } i \in I\}$$

iv) If the arity of $F_{(i)j}$ is n and $f_o, \cdots, f_{n-1} \in \prod_{i \in I} A_i$, then

$$F^j(f_0, \cdots, f_{n-1}) = \begin{cases} \langle F_{(i)j}(f_0(i), \cdots, f_{n-1}(i)) \rangle_{i \in I} \text{ if} \\ \qquad F_{(i)j}(f_0(i), \cdots, f_{n-1}(i)) \in A_i \text{ for every } i \in I, \\ \chi \qquad \text{otherwise.} \end{cases}$$

1.2.3 Partial Subalgebra.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two partial algebras of the same similarity type $\mathfrak{A}$ is a partial subalgebra of $\mathfrak{B}(\mathfrak{A} \in S_w\mathfrak{B})$ if $A \subseteq B$, the relations in $\mathfrak{A}$ are the corresponding relations of $\mathfrak{B}$ restricted to A and if F is an n-ary function symbol and $a_1, \dots a_n \in A$, then $F^{\mathfrak{A}}(a_1, \dots a_n) = \begin{cases} F^{\mathfrak{B}}(a_1 \dots a_n) & \text{if} \quad F^{\mathfrak{B}}(a_1 \dots a_n) \in A \\ x & \text{if not} \end{cases}$

1.2.4 Congruence Relation.

In the literature we can find several concepts of congruence relation. The one we will use in this work is the following:

R is a congruence relation on the partial algebra $\mathfrak{A}$ iff it is an equivalence relation on A and if F is an n-ary operation symbol, $a_1 \dots a_n \in A, b_1, \dots, b_n \in A$, $a_i R b_i$ $(i = 1, \dots n)$ and $F(a_1 \dots a_n) \in A$, $F(b_1 \dots b_n) \in A$ then $F(a_1 \dots a_n) R F(b_1 \dots b_n)$

1.2.5 Coset Algebra.

Let

$$\mathfrak{A} = \langle A, F_i, R_j, c_k \rangle_{\substack{i \in I \\ j \in J \\ k \in K}}$$

be a partial algebra and R congruence relation on $\mathfrak{A}$, then the *coset algebra* $\mathfrak{B} = \mathfrak{A}/R$ is defined as the partial algebra

$$\mathfrak{B} = \langle B, F^j, R^i, c^k \rangle_{\substack{i \in I \\ j \in J \\ k \in K}} \quad \text{, where}$$

i) $B = \{a/R : a \in A\}$, where a/R is the equivalence class of a with respect to R.

ii) $c^k = c_k/R$.

iii) $R^i = \{\langle a_0/R, \cdots, a_{n-1}/R\rangle : \exists a'_0 \cdots \exists a'_{n-1}(a_0 R a'_0 \wedge \cdots \wedge a_{n-1} R a'_{n-1} \wedge \langle a'_0, \cdots, a'_{n-1}\rangle \in R_i\}$.

iv) If the arity of F_j is n, and $a_0/R, \cdots, a_{n-1}/R \in B$, then

$$F^j(a_0/R, \cdots, a_{n-1}/R) = \begin{cases} \{d : \exists a'_0 \cdots \exists a'_{n-1} \exists d'(a_0 R a'_0 \wedge \cdots \wedge a_{n-1} R a'_{n-1} \\ \quad \wedge d R d' \wedge F_j(a'_0, \cdots a'_{n-1}) = d'\}, \text{ if} \\ \quad \text{this set is not empty,} \\ \chi \quad \text{othewise.} \end{cases}$$

Remark: In Mikenberg [2] it is shown that this is well defined.

1.2.6 s-Identities in partial algebras.

Let τ and τ' be terms, then:

i) An e-identity $(\tau \underline{e} \tau')$ is a formula of the form:

$$\exists x\, x \simeq \tau \wedge \exists x\, x \simeq \tau' \wedge \tau \simeq \tau'$$

ii) An s-identity $(\tau \underline{s} \tau')$ is a formula of the form:

$$(\tau \underline{e} \tau \vee \tau' \underline{e} \tau' \rightarrow \tau \underline{e} \tau')$$

There exist other concepts of identities in partial algebras, and they can be found, for example in [1].

1.2.7 A special congruence relation.

Let $\mathfrak{A} = \langle A, F_j, R_i, 0\rangle_{\substack{j \in J \\ i \in I}}$ be a partial algebra and $\mathfrak{B} = \prod_{\omega} A$

i) If $f \in B$, we denote by f^* the (finite) tuple of all the non-zero coordinates of f in the order they appear in f and we define $\ell(f)$ as the length of f^* and $i(f)$ as the natural number j such that $f(j) \neq 0$ and for all $k > j, f(k) = 0$.

ii) For $f, g \in B$, we define the relation S as follows:

$$fSg \quad \text{iff} \quad f^* = g^*.$$

(It is easy to verify that S is an equivalence relation on B.)

iii) Let $R = \cap\{T : T \text{ is a congruence relation on } B \text{ such that } S \subseteq T\}$.

2 Construction of the Clousure

2.1 Theorem

Let $\mathfrak{B} = \prod_\omega \mathfrak{A}$ where $\mathfrak{A} = \langle A, F_j, R_i, 0\rangle$ is a partial algebra, and Θ any congruence relation on B that includes R (defined in 1.2.7 (iii)). Then $\overline{\mathfrak{A}} = \mathfrak{B}/\Theta$ is a total algebra that preserves the s-identities of $\mathfrak{A}$.

Proof: (Mikenberg [3].)

■

2.2 Definition

For each n-ary operation F^j of $\mathfrak{B} = \prod_\omega \mathfrak{A}$, we define a new operation $\hat{F^j}$ as follows:

Let $f \in B$, then

$$\hat{F^j}(f) = \begin{cases} (F_j)^{\rho+1}(f^*) & \text{if } \ell(f) = m > 1 \text{ and } (n + \rho(n-1) = m), \\ f(i) & \text{if } \ell(f) = 1 \text{ and} \\ \\ \chi & \text{otherwise,} \end{cases}$$

where

$$(F_j)^1(a_0, \cdots, a_{n-1}) = F_j(a_o, \cdots, a_{n-1})$$

$$(F_j)^{\rho+1}(a_0, \cdots, a_{m-1}) = (F_j)^{\rho}(F_j(a_0, \cdots, a_{n-1}), a_n, \cdots, a_{m-1})$$

$$\text{and } m = n + \rho(n-1).$$

The following theorem gives necessary and sufficient conditions to obtain a total algebra that has a partial subalgebra isomorphic to the original one, in case that we have more then one strictly partial operation and some relations. The first two conditions are imposed on the operations, and they correspond to a generalized distributive condition and a restriction on the domains of the operations where they are defined. The last two conditions are on the relations of the algebra, and they are a kind of congruence condition for the relations with respect to the operations.

2.3 Theorem.

If the partial operations of the original algebra $\mathfrak{A}$ satisfy the following conditions:

i) For each $a_0, \cdots, a_{n-1} \in A$, $F_j(a_0, \cdots, a_{n-1}) = F_k(a_0, \cdots, a_{n-1})$ for each $\langle j, k\rangle \in J \times J$ where both operations are defined and

ii) For each $f_0, \cdots, f_{n-1} \in B$, $\hat{F}^j(F^k(f_0, \cdots, f_{n-1})) = F_k(\hat{F}^{\ell}(f_0), \cdots, \hat{F}^{\ell}(f_{n-1}))$ for any $j, k, \ell \in J$, where all of them are defined,

and if the relations of the algebra $\mathfrak{A}$ satisfy:

iii) $R_i(0, \cdots, 0)$ for every $i \in I$ and

iv) For each n-ary relation R^k of $\mathfrak{A}$, if $f_0, \cdots, f_{n-1} \in B$ and $R^k(f_0, \cdots, f_{n-1})$, then there exist $f'_0, \cdots, f'_{n-1}, j \in J$ such that $f'_i R f_i, i < n$ and $\hat{F}^j(f'_i)$ is defined for every $i < n$ and

$$R_k(\hat{F}^j(f'_0), \cdots, \hat{F}^j(f'_{n-1}))$$

(where R is the congruence relation defined in 1.1.7 (iii)).

Then the total algebra $\overline{\mathfrak{A}}$ constructed in Theorem 2.1 has a partial subalgebra isomorphic to the original algebra $\mathfrak{A}$.

Proof:

Let $h : A \cup \{\chi\} \to \overline{A} \cup \{\chi\}$ be defined by:

$h(a) = \langle a, 0, \cdots \rangle / R$ for every $a \in A$, and let

$$\mathfrak{C} = \langle h(A), \overline{F}_j, \overline{R}_i, \overline{0} \rangle_{\substack{j \in J \\ i \in I}}$$

be the partial subalgebra of $\overline{\mathfrak{A}}$ with universe $h(A)$.

Then in Mikenberg [3] it is shown that h is a one-one function such that for each $j \in J, a_0, \cdots, a_{n-1} \in A$,

$$F_j(a_0, \cdots, a_{n-1}) \in A \quad \text{iff } \overline{F}_j(ha_0, \cdots, ha_{n-1}) \in h(A)$$

and

$$h(F_j(a_0, \cdots, a_{n-1})) = \overline{F}_j(h(a_0), \cdots, h(a_{n-1})) \quad .$$

Therefore, we only need to show that

$$R_i(a_0, \cdots, a_{n-1}) \quad \text{iff } \overline{R}_i(ha_0, \cdots, ha_{n-1}) \; .$$

In one direction this is trivial, so let us assume that $\overline{R}_i(ha_0, \cdots, ha_{n-1})$, then

$$\exists f_0, \cdots, \exists f_{n-1}, \quad f_i R(a_i, 0 \cdots), \; i < n \text{ and } R^i(f_0, \cdots, f_{n-1}).$$

Then applying condition (iv) we get

$$\exists f'_0, \cdots, \exists f'_{n-1}. \exists j \in J \quad \text{such that } f'_k R f_k, k < n \quad \text{and}$$

$\hat{F}^j(f'_k)$ is defined for $k < n$ and

$R_i(\hat{F}^j(f'_0), \cdots, \hat{F}^j(f'_{n-1}))$.

Using Lema 2.5 in Mikenberg [3] we have

$\hat{F}^j(f'_i) = a_i, i < n$ and therefore,

$R_i(a_0, \cdots, a_{n-1})$. ∎

Remark:

Condition (i) and (ii) in the preceeding theorem cannot be weakened in case we have more than one strictly partial operation. See Mikenberg [3].

3 Refinement Algebras

3.1 Definition.

$\mathfrak{A} = \langle A, +, 0\rangle$ is called a refinement algebra if it is a partial algebra that satisfies the following conditions:

i) $+$ is commutative.

ii) $+$ is associative.

iii) 0 is a neutral element.

iv) If $a_1 + a_2 = b_1 + b_2 \in A$, then there exist c_1, c_2, c_3, c_4 in A such that

$$\begin{array}{lll} a_1 = c_1 + c_2 & \qquad & b_1 = c_1 + c_3 \\ a_2 = c_3 + c_4 & & b_2 = c_2 + c_4 \end{array} .$$

v) If $(a_1 + a_2) + c = b + c \in A$, then there exist b_1, b_2, c_1, c_2 in A such that

$$\begin{array}{lll} b = b_1 + b_2 & \qquad & a_1 + c_1 = c_1 + c_3 \\ c = c_1 + c_2 & & a_2 + c_2 = b_2 + c_2 \end{array} .$$

These algebras were introduced by Tarski [5].

Remark:

In this case, the weak product of this algebra preserves the s-identities of the original algebra and therefore it is also a refinement algebra. (R.A).

3.2 Proposition.

Let $\mathfrak{A} = \langle A, +, 0\rangle$, be R.A., let $\mathfrak{B} = \overset{\frown}{\prod_{\omega}}\mathfrak{A}$, let T be the following congruence relation on B:

fTg iff there exists a sequence $r_{ij} \in^{i(f)\times i(g)} A$ such that

$$f(i) = \sum_{j \leq \mathfrak{i}(g)} r_{ij}$$

$$g(j) = \sum_{i \leq \mathfrak{i}(f)} r_{ij} \quad .$$

Then $\overline{\mathfrak{A}} = \mathfrak{B}/T$ is a total R.A. that contains a partial subalgebra isomorphic to $\mathfrak{A}$.

Proof: Mikenberg [3]

■

3.3 Theorem.

Let $\mathfrak{A} = \langle A, +, 0\rangle$ be a refinement algebra. Let T be the relation defined above, then $\overline{\mathfrak{A}} = \widehat{\prod_{\omega}}\mathfrak{A}/T$ is the minimal total refinement algebra that preserves the s-identities of $\mathfrak{A}$ and has a partial subalgebra isomorphic to $\mathfrak{A}$.

Proof.:

Let $\mathfrak{A}'$ be another total refinement algebra that preserves the s-identities of $\mathfrak{A}$ and that has a partial subalgebra $\mathfrak{C}'$ isomorphic to $\mathfrak{A}$. Let $h : \mathfrak{A} \to \mathfrak{C}'$ be this isomorphism. We will construct an isomorphism from $\overline{\mathfrak{A}}$ into $\mathfrak{A}'$.

Let $H : \overline{A} \to A'$ be defined by

$$H(\overline{f}) = \sum_{j \leq \mathfrak{i}(f)}{}' h(f(j)), \quad \text{where } \overline{f} \in \overline{A} \text{ and}$$

$\sum'$ is the finite sum $+'$ of $\mathfrak{A}'$.

It is not difficult to see that H is well defined and is a homomorphism between total algebras. Let us check that it is injective.

Let $f/T, g/T \in \overline{A}$ and assume $H(f/T) = H(g/T)$, then

$$\sum_{i \leq \mathfrak{i}(f)}{}' h(f(i)) = \sum_{j \leq \mathfrak{i}(g)}{}' h(g(j)) \quad .$$

Since $\mathfrak{A}'$ is a refinement algebra, there exists a sequence c_{ij} such that

$$h(f(i)) = \sum_{j \leq \boldsymbol{i}(g)}' c_{ij} \quad \text{and}$$
$$h(g(j)) = \sum_{i \leq \boldsymbol{i}(f)}' c_{ij} .$$

Besides, $\mathfrak{C}'$ is a partial algebra and

$h(f(i)) \in \mathfrak{C}' \quad , h(g(j)) \in \mathfrak{C}' \quad .$

therefore $c_{ij} \in \mathfrak{C}'$, so

$$f(i) = h^{-1}(\sum_{j \leq \boldsymbol{i}(g)}' c_{ij}) = \sum_{j \leq \boldsymbol{i}(g)} h^{-1}(c_{ij})$$

and

$$g(j) = h^{-1}(\sum_{i \leq \boldsymbol{i}(f)}' c_{ij}) = \sum_{i \leq \boldsymbol{i}(f)} h^{-1}(c_{ij}) \quad .$$

Therefore, fTg by definition of T and hence. H is one-one homomorphism from $\overline{\mathfrak{A}}$ into $\overline{\mathfrak{A}}'$■

3.4 Remark:

This theorem shows that this completition (i.e. $\mathfrak{A}$) is the minimal one with respect to the preservation of s-identities.

4 Some Applications

4.1 Remark.

If $\mathfrak{A} = \langle A, F_j, R_i, 0 \rangle_{\substack{j \in J \\ i \in I}}$ is a total algebra and Θ is a congruence relation that contains S(def. 1.2.7 (ii)), then $\mathfrak{A} \cong \widehat{\prod_{\omega}} \mathfrak{A}/\Theta$.

We will now construct in much the same way a closure for a special kind of partial algebras which provide, as particular cases, the construction of the structures $(\mathbb{Z}, +, -, 0)$ from $(\mathbb{N}, +, -, 0)$ and $(\mathbb{Q} - \{0\}, \cdot, \div, 1)$ from $(\mathbb{Z} - \{0\}, \cdot, \div, 1)$.

Let the language $\mathfrak{L}$ have two binary operation symbols F and G and a constant symbol 0, with F a total operation and G a partial operation.

Let $\mathfrak{A} = \langle A, F, G, 0 \rangle$ be a partial structure for this language which satisfies the following axioms:

A1- $F(x, 0) = F(0, x) = x$

A2- $F^2(x, y, z) = F^2(y, z, x)$, where

$$\begin{cases} F^0(x_0) = x_0 \\ F^1(x_0, x_1) = F(x_0, x_1) \\ F^{n+1}(x_0, \cdots, x_n, x_{n+1}) = F(F^n(x_0, \cdots, x_n), x_{n+1}). \end{cases}$$

A3- $G(x, y) = z$ iff $F(y, z) = x$.

In this case we want to construct the smallest total algebra $\overline{\mathfrak{A}}$ that preserves these axioms and that has a partial subalgebra isomorphic to $\mathfrak{A}$. We cannot just apply Theorem 2.3 for this construction, because F is a total operation and hence it does not satisfy condition (i) of this theorem. Anyhow, we will construct the completition $\overline{\mathfrak{A}}$ in a similar way, and for this we will need the following properties:

4.2 Proposition.

(i) $F(x, y) = F(x, z) \Rightarrow y = z$.

(ii) $F(x, y) = F(y, x)$.

(iii) $G(x, 0) = x \wedge G(x, x) = 0 \wedge (G(x, y) = 0 \text{ iff } x = y)$.

(iv) $F^{n+1}(x_0, \cdots, x_{n+1}) = F(x_0, F^n(x_1, \cdots, x_{n+1}))$.

(v) Let $a, b, c \in A$ and $G(b, c) \in A$, then

$$G(F(a, b), c) \in A \text{ and } G(F(a, b), c) = F(a, G(b, c)).$$

(vi) Let $a, b, c, d \in A$, then
if $G(a, b) = G(a, c) = d$, then $b = c$, and
if $G(a, b) = G(c, b) = d$, then $a = c$.

(vii) $G(F(x, y), y) = x$.

(viii) Let $a, b, c, d \in A, G(F(a, b), F(c, d)) \in A$ and $G(G(a, c), G(d, b)) \in A$, then $G(F(a, b), F(c, d)) = G(G(a, c), G(d, b))$.

ix) $F^n(F(x_0, y_0), \cdots, F(x_n, y_n)) = F(F^n(x_0, \cdots, x_n), F^n(y_0, \cdots, y_n)$

(x) $F^n(G(x_0, y_0), \cdots, G(x_n, y_n)) = G(F^n(x_0, \cdots, x_n), F^n(y_0, \cdots y_n)$, if all the terms are defined.

■

4.3 Definition.

Let $\mathfrak{B} = \widehat{\prod_\omega \mathfrak{A}}$, we define a relation R on B as follows:

$$fRg \text{ iff } F^{\boldsymbol{i}(g)}(f(0), g(1), \cdots, g(\boldsymbol{i}(g))) = F^{\boldsymbol{i}(f)}(g(0), f(1), \cdots, f(\boldsymbol{i}(f))).$$

4.4 Proposition.

R is a congruence relation on B.

Proof: It is not difficult to see that R is reflexive and symmetric, and for transitivity and congruence properties, we use the properties in proposition 4.2.

4.5 Proposition.

For each $f \in B$, there exists $f' \in B$ such that fRf' and $\boldsymbol{i}(f') \leq 1$.

Proof.:

Suppose $f \in B$. If $\boldsymbol{i}(f) > 1$, define f' as follows:

$$f' = (f(0), F^{\boldsymbol{i}(f)-1}(f(1), \cdots, f(\boldsymbol{i}(f))), 0 \cdots)$$

and if $\boldsymbol{i}(f) \leq 1$ take $f' = f$.

■

4.6 Proposition.

$\overline{\mathfrak{A}} = \mathfrak{B}/R$ is a total algebra.

Proof.: Apply theorem 2.1 since $R \supseteq S$ (def. 1.2.7 (ii)). There is also a direct proof using the properties in 4.3.

■

4.7 Proposition.

$\overline{\mathfrak{A}}$ contains a partial subalgebra isomorphic to the original one.

Proof.:

$$\begin{array}{llll} \text{Define} & h : \mathfrak{A} & \longrightarrow & \overline{\mathfrak{A}} \text{ by} \\ & a & \longrightarrow & \langle a, 0, \cdots \rangle / R = \overline{a} \end{array}$$

and let $\mathfrak{C} = \langle h(A), \overline{F}, \overline{G}, \overline{0} \rangle$ the partial subalgebra of $\overline{\mathfrak{A}}$ with the corresponding restricted operations. Then it is not difficult to see that $h : \mathfrak{A} \longrightarrow \mathfrak{C}$ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{C}$.

■

4.8 Proposition.

(i) If $G(a,d) \in A$ and $G(c,b) \in A$, then
$F(a,b) = F(c,d)$ iff $G(a,d) = G(c,b)$.

(ii) If $G^n(a_0, \cdots, a_n) \in A$ and $G(a_0, F^{n-1}(a_1, \cdots, a_n)) \in A$,
then $G^n(a_0, \cdots, a_n) = G(a_o, F^{n-1}(a_1, \cdots, a_n))$,
whenever they are defined.

(iii) $G(G(a,b), G(c,d)) = G(G(a,c), G(b,d))$,

when all the operations are defined.

■

4.9 Theorem.

$\overline{\mathfrak{A}}$ is the smallest total algebra that preserves the s-identities of $\mathfrak{A}$ and that has a partial subalgebra isomorphic to $\mathfrak{A}$.

Proof.: Let $\mathfrak{A}'$ be another closed algebra that contains a partial subalgebra $\mathfrak{C}'$ isomorphic to $\mathfrak{A}$.

Let $h : \mathfrak{A} \to \mathfrak{C}'$ be the isomorphism.

For $\overline{f} \in \overline{\mathfrak{A}}$, we define $H(\overline{f}) = G'^{\boldsymbol{i}(f)}(h'(f(0)), \cdots, h'(f(\boldsymbol{i}(f))))$ where G' is the interpretation of the operation G in $\mathfrak{A}'$.

Let us check that H is well defined:

fRg iff $F^{\boldsymbol{i}(g)}(f(0), g(1), \cdots, g(\boldsymbol{i}(g))) = F^{\boldsymbol{i}(f)}(g(0), f(1), \cdots, f(\boldsymbol{i}(f)))$

iff $F'^{\boldsymbol{i}(g)}(h'(f(0)), h'(g(1)), \cdots, h'(g(\boldsymbol{i}(g)))) = F'^{\boldsymbol{i}(f)}(h'(g(0)), \cdots, h'(f(\boldsymbol{i}(f))))$

iff $F'(h'(f(0)), F'^{\boldsymbol{i}(g)-1}(h'(g(1)), \cdots, h'(g(\boldsymbol{i}(g))))) = F'(h'(g(0)), F'^{\boldsymbol{i}(f)-1}(h'(f(1)), \cdots, h'(f(\boldsymbol{i}(f)))))$

iff $G'(h'(f(0)), F'^{\boldsymbol{i}(f)-1}(h'(f(1)), \cdots, h'(f(\boldsymbol{i}(f))))) = G'(h'(g(0)), F'^{\boldsymbol{i}(g)-1}(h'(g(1)), \cdots, h'(g(\boldsymbol{i}(g)))))$

iff $G'^{\boldsymbol{i}(f)}(h'(f(0)), h'(f(1)), \cdots, h'(f(\boldsymbol{i}(f)))) = G'^{\boldsymbol{i}(g)}(h'(g(0)), h'(g(1)), \cdots, h'(g(\boldsymbol{i}(g))))$

iff $\quad H(\overline{f}) = H(\overline{g})$

$$\text{Therefore, considering } f = (f(0), f(1), 0, \cdots) \text{ and } g = (g(0), g(1), 0, \cdots), \quad \text{by prop. 4.6.}$$

we have that:

$H(\overline{0}) = 0'$ and

$$\begin{aligned} H(F(\overline{f}, \overline{g})) &= G'(F'(h'(f(0)), h'(g(0))), F'(h'(f(1)), h'(g(1)))) \\ &= F'(G'(h'(f(0)), h'(f(1))), G'(h'(g(0)), h'(g(1)))) \\ &= F'(H(\overline{f}), H(\overline{g})) \end{aligned}$$

also

$$
\begin{aligned}
H(G(\overline{f},\overline{g})) &= G'(G'(h'(f(0)),h'(g(0))),G'(h'(f(1)),h'(g(1)))) \\
&= G'(G'(h'(f(0)),h'(f(1))),G'(h'(g(0)),h'(g(1)))) \\
&= G'(H(\overline{f}),H(\overline{g})). \quad \blacksquare
\end{aligned}
$$

4.10 Remark.

For partial algebras whose set of axioms are s-identities, we have prove that the completition preserving the axioms always exists, and the construction is done with an appropiate congruence relation over the free algebra generated by the terms of the original algebra.

References

[1] Burmeister, P. 1986. *A model theoretic oriented approach to partial algebras.* Akademic Verlag, Berlín.

[2] Mikenberg, I.F. 1978, *From total to partial algebras, in Mathematical Logic*, Proceedings of the First Brazilian Conference, Marcel Dekker Inc., pp. 203-223.

[3] Mikenberg, I.F. 1980, *A Closure for partial algebras*, in Mathematical Logic in Latin America, North Holland Pub. Co., pp. 239-253.

[4] Olivos, E. 1990, *Algunos resultados sobre algebras parciales*, Tesis de Magister en Ciencias Exactas (Mat.) Facultad de Matemáticas, Pontificia Universidad Católica de Chile.

[5] Tarski,A. 1949. *Cardinal Algebras*, Oxford University Press.

Subalgebras of a Finite Three-valued Lukasiewicz Algebra

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Abstract

It is well known that the number of Boolean subalgebras of a finite Boolean algebra $\mathbf{B}$, with n atoms, is the number of partitions of a set with n elements, and that the number of isomorphic (Boolean) subalgebras of $\mathbf{B}$ with t atoms, $1 \leq t \leq n$, is the number of t-partitions of a set with n elements. It is clear that the number of non-isomorphic subalgebras of $\mathbf{B}$ is n. In this note we determine the number of subalgebras of a finite non trivial three-valued Lukasiewicz algebra L, the number of subalgebras isomorphic to a given subalgebra and the number of non-isomorphic subalgebras of L, and we give a method to construct all the subalgebras of L.

1 Introduction

Let $\mathbf{B}$ be a finite Boolean algebra. Since $\mathbf{B}$ is isomorphic to the product of n Boolean algebras $B = \{0,1\}$ for some n, $n \geq 1$, we will denote $\mathbf{B} = B^n$, when we want to make evident that the number of atoms of $\mathbf{B}$ is n.

Let $\mathcal{BS}(\mathbf{B})$ denote the set of Boolean subalgebras of $\mathbf{B}$, $\mathcal{A}(\mathbf{B})$ the set of atoms of $\mathbf{B}$, and for $b \in \mathbf{B}$, $\mathcal{A}(b) = \{a \in \mathcal{A}(\mathbf{B}) : a \leq b\}$.

If $S \in \mathcal{BS}(\mathbf{B})$ and $\mathcal{A}(S) = \{s_1, s_2, \ldots, s_t\}$ is the set of atoms of S, then $\mathcal{P}(S) = \{\mathcal{A}(s_1), \mathcal{A}(s_2), \ldots, \mathcal{A}(s_t)\}$ is a partition of the set $\mathcal{A}(\mathbf{B})$. Conversely, if $\mathcal{P} = \{X_1, X_2, \ldots, X_u\}$ is a partition of $\mathcal{A}(\mathbf{B})$ and we consider $x_i = \bigvee\{x : x \in X_i\}$, $1 \leq i \leq u$ and $X(\mathcal{P}) = \{x_1, x_2, \ldots, x_u\}$, then the Boolean subalgebra of $\mathbf{B}$ generated by $X(\mathcal{P})$, $S = BS(X(\mathcal{P}))$, is such that the atoms of S are the elements of $X(\mathcal{P})$. The map $S \to \mathcal{P}(S)$ is a bijection from $\mathcal{BS}(\mathbf{B})$ to the set of partitions of $\mathcal{A}(\mathbf{B})$.

For a finite set X with n elements, $n \geq 1$, let $N[X]$ denote the number of elements of X, $\mathbf{P}(n)$ the set of all partitions of X and $\mathbf{p}(n) = N[\mathbf{P}(n)]$. It is well known that if we put $\mathbf{p}(0) = 1$, then for $n \geq 0$,

$$\mathbf{p}(n+1) = \sum_{t=0}^{n} \binom{n}{t} \mathbf{p}(t).$$

Any partition of X with t classes, $1 \leq t \leq n$, is called a t-partition, and the set of all t-partitions of a set with n elements is denoted by $\mathbf{P}^t(n)$, $n \in N$, $t \leq n$. It is well known that

$$\mathbf{p}^t(n) = N[\mathbf{P}^t(n)] = \frac{\sum_{i=0}^{t-1}(-1)^i\binom{t}{i}(t-i)^n}{t!}.$$

For $b \in \mathbf{B}$, let $\mathcal{BS}(b)$ denote the set of Boolean subalgebras of $\mathbf{B}$ containing b, that is, $\mathcal{BS}(b) = \{S \in \mathcal{BS}(\mathbf{B}) : b \in S\}$, and if $S \in \mathcal{BS}(\mathbf{B})$, let $BS(S, b)$ be the Boolean subalgebra generated by the set $S \cup \{b\}$.

Remark 1.1 *Let $\mathbf{B} = B^n$. If $b = 0$ or $b = 1$, then $N[\mathcal{BS}(0)] = N[\mathcal{BS}(1)] = N[\mathcal{BS}(\mathbf{B})]$. If $b \in \mathbf{B} - \{0,1\}$, we are going to determine $N[\mathcal{BS}(b)]$. Let $S \in \mathcal{BS}(b)$ and $\mathcal{A}(S) = \{s_1, s_2, \ldots, s_t\}$, $1 \leq t \leq n$. Then $\mathcal{P}(S)$ is a partition of $\mathcal{A}(\mathbf{B})$. Since $b \in S$, then for every $s \in \mathcal{A}(S)$, $s \leq b$ or $s \leq -b$. Let $\mathcal{A}(S,b) = \{x \in \mathcal{A}(S) : x \leq b\}$ and $\mathcal{A}(S,-b) = \{x \in \mathcal{A}(S) : x \leq -b\}$. Since $b \neq 0, 1$ then $\mathcal{A}(S,b)$ and $\mathcal{A}(S,-b)$ are nonempty sets. It is easy to prove that $\mathcal{A}(b) = \bigcup\{\mathcal{A}(x) : x \in \mathcal{A}(S,b)\}$ and $\mathcal{A}(-b) = \bigcup\{\mathcal{A}(y) : y \in \mathcal{A}(S,-b)\}$. Then we have that the sets $\mathcal{P}_1(S) = \{\mathcal{A}(x) : x \in \mathcal{A}(S,b)\}$ and $\mathcal{P}_2(S) = \{\mathcal{A}(y) : y \in \mathcal{A}(S,-b)\}$ are partitions of the sets $\mathcal{A}(b)$ and $\mathcal{A}(-b)$ respectively. Then, if $S \in BS(b)$, $b \neq 0, 1$ we can define the following bijective function: $\phi(S) = (\mathcal{P}_1(S), \mathcal{P}_2(S))$. Therefore, if $N[\mathcal{A}(b)] = r$, $1 \leq r < n$, then $N[\mathcal{BS}(b)] = N[\mathbf{P}(r)] \times N[\mathbf{P}(n-r)]$. This formula is also valid for $b = 0$ or $b = 1$, that is, if $r = 0$ or $r = n$.*

Definition 1.1 *A three-valued Lukasiewicz algebra is an algebra $(L, \wedge, \vee, \sim, \nabla, 1)$ of type $(2,2,1,1,0)$ where $(L, \wedge, \vee, \sim, 1)$ is a De Morgan algebra and ∇ is a unary operator (possibility operator) satisfying : $\sim x \vee \nabla x = 1$, $x \wedge \sim x = \sim x \wedge \nabla x$, $\nabla(x \wedge y) = \nabla x \wedge \nabla y$. For short, we shall say that L is a Lukasiewicz algebra,* [1, 2, 5].

The necessity operator is defined by $\Delta x = \sim \nabla \sim x$. It is well known that $B(L) = \{x \in L : \nabla x = x\} = \{x \in L : \Delta x = x\}$ is a Boolean algebra. Moisil [2] proved that Lukasiewicz algebras satisfy the following *determination principle*: If $\nabla x = \nabla y$ and $\Delta x = \Delta y$ then $x = y$ (see also [6]). L is called a *centered Lukasiewicz algebra*, or a (three-valued) *Post algebra*, if it has a *center*, that is, an element c of L such that $\sim c = c$. The center of L (if it exists) is unique, and $x = (\Delta x \vee c) \wedge \nabla x$ for all $x \in L$, [2, 4, 7].

An *axis* of a Lukasiewicz algebra is an element e of L with the properties: (E1) $\Delta e = 0$ and (E2) $\nabla x \leq \Delta x \vee \nabla e$, for all x of L, [4]. Observe that (E2) is equivalent to $x \leq \Delta x \vee \nabla e$, for all $x \in L$. If the axis of L exists, it is unique, and using the determination principle, it is easy to see that $x = (\Delta x \vee e) \wedge \nabla x$, for all $x \in L$ [7, page 14]. Following A. Monteiro, L is called a (three-valued) *Moisil algebra* if it has an axis, [7, 9]. If L is a Boolean algebra, then 0 is the axis of L. Furthermore, if c is a center of L, c is also an axis. The converse of this last statement is not true.

It is well known that if $u, w \in B(L)$ verify $u \leq w$ then $[u, w] = \{x \in L : u \leq x \leq w\}$ is a Lukasiewicz algebra, where the operations $\wedge$, $\vee$ and ∇ are the operations $\wedge$, $\vee$ and ∇ of L, and the negation is defined by $\approx x = u \vee (\sim x \wedge w)$, $x \in [u, w]$. It is clear that u is the least element and w is the greatest element of $[u, w]$. It is easy to see that $B([u, w]) = [u, w] \cap B(L)$.

If L is a Lukasiewicz algebra with axis e, then $[0, e]$ is a Boolean algebra. Indeed, it is clear that $[0, e]$ is a distributive lattice with least element 0 and greatest element e. If we put by definition $-x = \sim \nabla x \wedge e$, then

$$\begin{aligned} -(x \wedge y) &= \sim \nabla(x \wedge y) \wedge e = (\sim \nabla x \vee \sim \nabla y) \wedge e \\ &= (\sim \nabla x \wedge e) \vee (\sim \nabla y \wedge e) = -x \vee -y. \\ -x \wedge x &= \sim \nabla x \wedge e \wedge x = 0. \\ --x &= \sim \nabla - x \wedge e = \sim \nabla(\sim \nabla x \wedge e) \wedge e \\ &= (\nabla x \vee \Delta \sim e) \wedge e = \nabla x \wedge e. \end{aligned}$$

Since $\Delta(--x) = \nabla x \wedge \Delta e = 0 = \Delta x$ and $\nabla(--x) = \nabla x \wedge \nabla e = \nabla x$ we have by the determination principle that $--x = x$.

Let $A(L) = \{x \in L : \Delta x = 0\}$. Then it is easy to see that $A(L) = [0, e]$ (see [8]). The operator ∇ is a Boolean isomorphism from $[0, e]$ onto $B([0, \nabla e])$. In fact, if $y \in B([0, \nabla e])$, then $y \in B(L)$ and $0 \leq y \leq \nabla e$. Let $x = y \wedge e$. Then $x \in [0, e]$ and $\nabla x = \nabla(y \wedge e) = \nabla y \wedge \nabla e = y \wedge \nabla e = y$. Now, if $\nabla x = \nabla y$, since $\Delta x = \Delta y = 0$, then $x = y$. On the other hand, $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, $\nabla - x = \nabla(\sim \nabla x \wedge e) = \nabla \sim \nabla x \wedge \nabla e = \sim \nabla x \wedge \nabla e = \approx \nabla x$, and $\nabla 0 = 0$.

In the case the algebra L has a center c, the Boolean algebra $[0, c]$ is isomorphic to $B([0, 1]) = B(L)$.

It is well known that every homomorphic image of a Lukasiewicz algebra L can be obtained up to isomorphism, as a quotient algebra L/F where F is a Δ -filter of L, i.e. a filter such that: if $x \in F$ then $\Delta x \in F$.

Furthermore, $F(z) = \{x \in L : z \leq x\}$ is a Δ- filter if and only if $z \in B(L)$, and $L/F(z)$ is isomorphic to $[0, z] = \{z \in L : 0 \leq x \leq z\}$, [7]. Gr. C. Moisil [4] proved, applying results of ring theory, that if L has an axis e, then L is isomorphic to the direct product of a Boolean algebra and a centered Lukasiewicz algebra. L. Monteiro [7] obtained the same result proving that $L \simeq L/F(\sim \nabla e) \times L/F(\nabla e)$ where $L/F(\sim \nabla e)$ is a Boolean algebra and $L/F(\nabla e)$ is a centered Lukasiewicz algebra. Moreover he proved that if L is finite and $B(L)$ has n atoms then $L \simeq B^j \times T^k$, where B is the Boolean algebra $\{0, 1\}$, and T is the centered Lukasiewicz algebra $\{0, c, 1\}$ and $j + k = n$.

Remark 1.2 *Taking into account* [7, pages 71–72] *we can prove that j is the number of atoms $a \in B(L)$ such that $a \leq \sim \nabla e$ and k is the number of atoms $a \in B(L)$ such that $a \leq \nabla e$. Furthermore, if $a \in \mathcal{A}(B(L))$ is such that $a \leq \sim \nabla e$, then $[0, a] = \{0, a\}$ and if $a \in \mathcal{A}(B(L))$ is such that $a \leq \nabla e$ then $[0, a] = \{0, a \wedge e, a\}$,* [8].

2 Number of subalgebras

If S is a Lukasiewicz subalgebra of a Lukasiewicz algebra L and $f \in L$, let denote $LS(S, f)$ the subalgebra generated by the set $S \cup \{f\}$. If f verifies $\Delta f = 0$ then $LS(S, f) = \{x \in L : x = (s_1 \wedge \sim \nabla f) \vee (s_2 \wedge \nabla f) \vee (s_3 \wedge f)$, where $s_1, s_2, s_3 \in S\}$ [7, page 41].

Lemma 2.1 *If L is a Lukasiewicz algebra, S^* a Boolean subalgebra of $B(L)$, $x_0 \in A(L)$ and $L_1 = LS(S^*, x_0)$, then $B(L_1) = BS(S^*, \nabla x_0)$, and x_0 is the axis of L_1.*

Proof.

1. Let us see that $B(L_1) = BS(S^*, \nabla x_0)$.

   (a) If $x \in B(L_1) \subseteq B(L)$, then $\Delta x = x$. Since $B(L_1) \subseteq L_1$ then

   $$x = (b_1 \wedge \sim \nabla x_0) \vee (b_2 \wedge \nabla x_0) \vee (b_3 \wedge x_0),$$

   where $b_1, b_2, b_3 \in S^*$. Then

   $$\begin{aligned} x = \Delta x &= (b_1 \wedge \sim \nabla x_0) \vee (b_2 \wedge \nabla x_0) \vee (b_3 \wedge \Delta x_0) \\ &= (b_1 \wedge \sim \nabla x_0) \vee (b_2 \wedge \nabla x_0) \in BS(S^*, \nabla x_0). \end{aligned}$$

   (b) If $x \in BS(S^*, \nabla x_0)$, then $x = (b_1 \wedge \sim \nabla x_0) \vee (b_2 \wedge \nabla x_0)$, where $b_1, b_2 \in S^*$, then $x = (b_1 \wedge \sim \nabla x_0) \vee (b_2 \wedge \nabla x_0) \vee (0 \wedge x_0) \in L_1$, and it is clear that $\Delta x = x$, so $x \in B(L_1)$.

2. Now we prove that x_0 is the axis of L_1.

   (a) We have $\Delta x_0 = 0$ from the hypothesis.

   (b) For every $x \in L_1$, $x \leq \Delta x \vee \nabla x_0$. This is equivalent to prove that $\Delta x \leq \Delta(\Delta x \vee \nabla x_0)$ and $\nabla x \leq \nabla(\Delta x \vee \nabla x_0)$. If $x \in L_1$ then

   $$x = (b_1 \wedge \sim \nabla x_0) \vee (b_2 \wedge \nabla x_0) \vee (b_3 \wedge x_0),$$

   and then $\Delta x \vee \nabla x_0 = b_1 \vee \nabla x_0$ and

   $$\begin{aligned} \nabla x &= (b_1 \wedge \sim \nabla x_0) \vee (b_2 \wedge \nabla x_0) \vee (b_3 \wedge \nabla x_0) \\ &= (b_1 \wedge \sim \nabla x_0) \vee ((b_2 \vee b_3) \wedge \nabla x_0) \\ &= (b_1 \vee \nabla x_0) \wedge \ldots \leq b_1 \vee \nabla x_0 \\ &= \Delta x \vee \nabla x_0. \end{aligned}$$

□

Remark 2.1 *If L is a centered Lukasiewicz algebra and c is the center of L, then $L_1 = LS(S^*, c)$ is a centered subalgebra of L and $B(L_1) = BS(S^*, \nabla c) = BS(S^*, 1) = S^*$. If S_1^*, S_2^* are Boolean subalgebras of $B(L)$ such that $LS(S_1^*, c) = LS(S_2^*, c)$ then we have $S_1^* = B(L_1) = B(L_2) = S_2^*$.*

Let L be a finite Lukasiewicz algebra, e the axis, $\mathcal{LS}(L)$ the set of Lukasiewicz subalgebras of L. Note that $\mathcal{BS}(B(L)) \subseteq \mathcal{LS}(L)$.

Remark 2.2 *Let $S \in \mathcal{LS}(L)$. Since S is finite, S has axis e'. Observe that $e' \in [0, e]$. Then $S^* = S \cap B(L) \in \mathcal{BS}(B(L))$, and $LS(S^*, e') = S$. In fact, if $x \in LS(S^*, e')$ then $x = (b_1 \wedge \sim \nabla e') \vee (b_2 \wedge \nabla e') \vee (b_3 \wedge e')$, where $b_1, b_2, b_3 \in S^* \subseteq S$. Since $e' \in S$, $\nabla e', \sim \nabla e' \in S$, therefore $x \in S$. If $x \in S$, then $x = (\Delta x \vee e') \wedge \nabla x$, but $\Delta x, \nabla x \in S \cap B(L) = S^* \subseteq LS(S^*, e')$, and since $e' \in LS(S^*, e')$ then $x \in LS(S^*, e')$.*

Consider the product $\Pi(L)$ of the sets $\mathcal{BS}(B(L))$ and $A(L)$ and the mapping α from $\Pi(L)$ into $\mathcal{LS}(L)$ defined by

$$\alpha((S^*, x_0)) = LS(S^*, x_0).$$

By the preceding remark α is onto.

Let $\mathcal{P} = \{\alpha^{-1}(S) : S \in \mathcal{LS}(L)\}$. Then $\mathcal{P}$ is a partition of $\Pi(L)$. If R_α is the equivalence relation determined by $\mathcal{P}$, then $N[\Pi(L)/R_\alpha] = N[\mathcal{LS}(L)]$. Let us determine $N[\mathcal{LS}(L)]$. It is easy to see that α is not injective and, in general, the sets $\alpha^{-1}(S), S \in \mathcal{LS}(L)$ have different cardinality.

Lemma 2.2 *If $x, y \in A(L), x \neq y$, then $\alpha((S^*, x)) \neq \alpha((T^*, y))$ for every $S^*, T^* \in \mathcal{BS}(B(L))$.*

Proof. If $SL(S^*, x) = SL(T^*, y) = S$, since x and y are axes of S and the axis is unique, then $x = y$. □

Remark 2.3 *If $(S^*, a) \in \Pi(L)$, let $C((S^*, a)) = \{(T^*, b) \in \Pi(L) : \alpha((T^*, b)) = \alpha((S^*, a))\}$, and if $a \in A(L)$ let $\Pi(L, a) = \{(S^*, a) : S^* \in \mathcal{BS}(B(L))\}$. Then Lemma 2.2 says that $C((S^*, a)) \subseteq \Pi(L, a)$, that is, $\Pi(L, a)$ is the union of the distinct equivalence classes contained in $\Pi(L, a)$.*

Lemma 2.3 *If $S^*, T^* \in \mathcal{BS}(B(L)), a \in A(L)$ then $\alpha((S^*, a)) = \alpha((T^*, a))$ if and only if $BS(S^*, \nabla a) = BS(T^*, \nabla a)$.*

Proof. If $LS(S^*, a) = LS(T^*, a)$, then by Lemma 2.1, $BS(S^*, \nabla a) = B(LS(S^*, a)) = B(LS(T^*, a)) = BS(T^*, \nabla a)$. Now, if $x \in LS(S^*, \nabla a)$, then $x = (s \wedge \sim \nabla a) \vee (t \wedge \nabla a) \vee (u \wedge a)$ where s , t, $u \in S^*$. Since $S^* \subseteq BS(S^*, \nabla a) = BS(T^*, \nabla a) = B(LS(T^*, a))$ and $\sim \nabla a$, ∇a, $a \in LS(T^*, a)$, then $x \in LS(T^*, a)$. In a similar way it can be proved that $LS(T^*, a) \subseteq LS(S^*, a)$. □

Remark 2.4 *By using the previous Lemma it is easy to see that $N[\Pi(L, a)/R_\alpha] = N[\mathcal{BS}(\nabla a)]$ (see introduction). So $N[\Pi(L)/R_\alpha] = \sum_{a \in A(L)} N[\Pi(L, a)/R_\alpha]$.*

Now we want to find $N[\mathcal{LS}(L)] = N[\Pi(L)/R_\alpha]$.

Supose that L is neither Boolean nor centered. Then $L = B^j \times T^k$, $j > 0, k > 0$. So $B(L) = B^n$ where $n = j + k$ and $N[\mathcal{BS}(B(L))] = N[\mathcal{P}(n)]$. The elements of L are n-uples $x = (x_1, x_2, \ldots, x_n)$ where $x_i \in B$, for $1 \leq i \leq j$ and $x_i \in T$ for $j + 1 \leq i \leq n$. So $A(L) = \{x \in L : x_i = 0,\ 1 \leq i \leq j, x_i \in \{0, c\} \subseteq T, j + 1 \leq i \leq n\}$. Then the Boolean algebra $A(L)$ has k atoms.

Remark 2.5 *Since the Boolean algebra $A(L)$ has k atoms, there exist $\binom{k}{r}, 0 \leq r \leq k$ elements of $A(L)$ which are supremum of r atoms of $A(L)$. If $b \in A(L)$ is the supremum of r atoms of $A(L)$, then ∇b is the supremum of r atoms of the Boolean algebra $B(L)$.*

Since $B(L)$ has $j+k$ atoms, from Remarks 2.4 and 2.5 we have:

$$N[\mathcal{LS}(B^j \times T^k)] = \sum_{r=0}^{k} \binom{k}{r} \mathbf{p}(r)\mathbf{p}(j+k-r).$$

This formula is also valid if L is a centered or Boolean algebra. Indeed, if L is Boolean then $L = B^j = B^j \times T^0$. Then $A(L) = \{0\}$ and $N[\mathcal{LS}(B^j \times T^0)] = \mathbf{p}(j)$. If L is centered then $L = B^0 \times T^k$, $A(L) = B(L) \simeq B^k$, and

$$N[\mathcal{LS}(T^k)] = \sum_{r=0}^{k} \binom{k}{r} \mathbf{p}(r)\mathbf{p}(k-r).$$

Now we are going to determine the number of non isomorphic subalgebras and the number of subalgebras isomorphic to a given subalgebra of a finite non trivial three-valued algebra L. We know that $L \simeq B^j \times T^k, j \geq 0, k \geq 0$ and j,k not simultaneously zero.

CASE A. $j > 0$, $k = 0$. Then $L \simeq B^j$ is a Boolean algebra and it is well known that there exist j non isomorphic subalgebras of L, and if $S \simeq B^h$, $1 \leq h \leq j$ is a subalgebra of L, there exist $\mathbf{p}^t(j)$ subalgebras of L isomorphic to S.

CASE B. $j \geq 1$, $k \geq 1$. If $S' \in \mathcal{LS}(L)$, then $S' \simeq B^{j'} \times T^{k'}$, $j' \geq 0$, $k' \geq 0$, j', k' not simultaneously zero.

Note that

- $j' \geq 1$, since if $j' = 0$ then S' would be a centered algebra, and then L is centered, which is a contradiction.
- Since $B(S') \simeq B^{j'+k'}$ is a Boolean subalgebra of $B(L) \simeq B^{j+k}$, then $1 \leq j' + k' \leq j + k$.
- $1 \leq j' \leq j + k$. Indeed, if $j + k < j'$, then $j + k + k' < j' + k' \leq j + k$, and then $k' < 0$, contradiction.
- $0 \leq k' \leq k$. Since S' is finite, it has axis $e' \in S'$, $e' \in A(L) = [0, e]$. We know that $[0,e] \simeq B^k$, then $[0,e']_{S'} = \{x \in S' : 0 \leq x \leq e'\} \simeq B^{k'}$. But $[0,e']_{S'} \subseteq [0,e]$, so $0 \leq k' \leq k$.

Summing up, if S' is a subalgebra of $L \simeq B^j \times T^k$, then $S' \simeq B^{j'} \times T^{k'}$, where (j',k') verifies: $1 \leq j' \leq j+k$, $0 \leq k' \leq k$, $1 \leq j' + k' \leq j + k = n$.

For j,k fixed positive integers, let $\mathbf{P}(j,k)$ the set of pairs (j',k') which verify the above conditions.

So

$$N[\mathbf{P}(j,k)] = \sum_{h=0}^{k}(j+k-h) = j(k+1) + \frac{k(k+1)}{2}.$$

Then we have a function $\beta : \mathcal{LS}(B^j \times T^k) \to \mathbf{P}(j,k)$.We are going to see that β is surjective and then we will have as a consequence that the number of non isomorphic subalgebras of $L \simeq B^j \times T^k$ is precisely $N[\mathbf{P}(j,k)]$.

Let $(j',k') \in \mathbf{P}(j,k)$. If $k' = 0$, any subalgebra S' of $B(L) \simeq B^{j+k}$ that verifies $S' \simeq B^{j'}$, is such that $\beta(B^{j'}) = \beta(B^{j'} \times T^0) = (j',0)$.

If $k' \geq 1$, since $k' \leq k$ and $j'+k' \leq j+k$ then $k' \leq j+k-j'$, and then $k' \leq k \wedge (j+k-j')$.

Let h be such that $k' \leq h \leq k \wedge (j+k-j') \leq k$, and $a \in A(L)$ such that $N[\mathcal{A}(\nabla a)] = h$, (see Remark 2.5). Let us consider a k'-partition of $\mathcal{A}(\nabla a)$, $X_1, \ldots, X_{k'}$ and a j'-partition of $\mathcal{A}(\sim \nabla a)$, $X_{1+k'}, \ldots, X_{j'+k'}$. So (see Remark 1.1), $\mathcal{P} = \{X_1, \ldots, X_{k'}, X_{1+k'}, \ldots, X_{j'+k'}\}$ is a $(j'+k')$-partition of the set $\mathcal{A}(B(L))$ of atoms of $B(L)$. Then, from Introduction, there is a subalgebra S of $B(L)$ corresponding to $\mathcal{P}$ whose atoms are $a_i = \bigvee\{x : x \in X_i\}$, $1 \leq i \leq j'+k'$.

Let $S' = LS(S,a)$. Since $a \in A(L)$, we know that a is the axis of S' and $B(S') = BS(S,\nabla a)$, but since $\nabla a = \bigvee_{i=1}^{k'} a_i$, then $\nabla a \in S$, and then $B(S') = S$. Since there are k' atoms of $B(S') = S$ preceding ∇a and j' atoms of $B(S') = S$ preceding $\sim \nabla a$, then (see Remark 1.2) $S' \simeq B^{j'} \times T^{k'}$. Then every subalgebra S' constructed in the previous way verifies $\beta(S') = (j',k')$.

Observe that if $\beta(S') = (j',k')$, that is, if $S' \simeq B^{j'} \times T^{k'}$ then, by Remark 1.2, S' has the form indicated in the previous construction.

Therefore $N[\beta^{-1}(j',k')]$ is the number of subalgebras isomorphic to $B^{j'} \times T^{k'}$.

Then:

1) If $k' = 0$, $N[\beta^{-1}(j',0)] = \mathbf{p}^{j'}(j+k)$ and then

$$S_0 = \sum_{j'=1}^{j+k} N[\beta^{-1}(j',0)] = \mathbf{p}(j+k).$$

2) If $k' \geq 1$, there exists $\binom{k}{h}\mathbf{p}^{k'}(h)\mathbf{p}^{j'}(j+k-h)$, $k' \leq h \leq k \wedge (j+k-j')$, subalgebras S' of L such that $\beta(S') = (j',k')$, so

$$N[\beta^{-1}(j',k')] = \sum_{h=k'}^{k\wedge(j+k-j')} \binom{k}{h} \mathbf{p}^{k'}(h)\mathbf{p}^{j'}(j+k-h).$$

If $1 \leq j' \leq j$, then $k \leq j+k-j'$, and if $j+1 \leq j' \leq j+k-1 = n-1$, then $n-j' = j+k-j' < k$, so

$$\sum_{(j',k')\in\mathbf{P}(j,k), k'\neq 0} N[\beta^{-1}(j',k')]$$

$$= \sum_{k'=1}^{k}\sum_{j'=1}^{j} N[\beta^{-1}(j',k')] + \sum_{j'=j+1}^{n-1}\sum_{k'=1}^{n-j'} N[\beta^{-1}(j',k')]$$

$$= \mathbf{S}_1 + \mathbf{S}_2.$$

$$\mathbf{S}_1 = \sum_{k'=1}^{k}[\sum_{j'=1}^{j}(\sum_{h=k'}^{k}\binom{k}{h}\mathbf{p}^{k'}(h)\mathbf{p}^{j'}(n-h))]$$

$$= \sum_{k'=1}^{k}[\sum_{h=k'}^{k}\binom{k}{h}\mathbf{p}^{k'}(h)(\sum_{j'=1}^{j}\mathbf{p}^{j'}(n-h))]$$

$$= \sum_{h=1}^{k}[\binom{k}{h}(\sum_{t=1}^{h}\mathbf{p}^{t}(h))(\sum_{j'=1}^{j}\mathbf{p}^{j'}(n-h))]$$

$$= \sum_{h=1}^{k}[\binom{k}{h}\mathbf{p}(h)(\sum_{j'=1}^{j}\mathbf{p}^{j'}(n-h))].$$

$$\mathbf{S}_2 = \sum_{j=j+1}^{n-1}[\sum_{k'=1}^{n-j'}(\sum_{h=k'}^{n-j'}\binom{k}{h}\mathbf{p}^{k'}(h)\mathbf{p}^{j'}(n-h))]$$

$$= \sum_{h=1}^{k-1}[\binom{k}{h}\mathbf{p}(h)(\sum_{j'=j+1}^{n-h}\mathbf{p}^{j'}(n-h))].$$

Note that $\mathcal{LS}(B^j \times T^k)$ is the disjoint union of the sets $\beta^{-1}((j',k'))$, where $(j',k') \in \mathbf{P}(j,k)$. Since $n-k=j$, then we can write

$$N[\mathcal{LS}(B^j \times T^k)] = S_0 + S_1 + S_2$$

$$= \mathbf{p}(n) + \sum_{h=1}^{k-1}[\binom{k}{h}\mathbf{p}(h)(\sum_{j'=1}^{j}\mathbf{p}^{j'}(n-h))]$$

$$+ \sum_{h=1}^{k-1}[\binom{k}{h}\mathbf{p}(h)(\sum_{j'=j+1}^{n-h}\mathbf{p}^{j'}(n-h))]$$

$$+ \binom{k}{k}\mathbf{p}(k)(\sum_{j'=1}^{n-k}\mathbf{p}^{j'}(n-k))$$

$$= \mathbf{p}(n) + \sum_{h=1}^{k-1}[\binom{k}{h}\mathbf{p}(h)(\sum_{j'=1}^{n-h}\mathbf{p}^{j'}(n-h))]$$

$$+ \binom{k}{k}\mathbf{p}(k)\mathbf{p}(n-k)$$

$$= \mathbf{p}(n) + \sum_{h=1}^{k}[\binom{k}{h}\mathbf{p}(h)\mathbf{p}(n-h)]$$

$$= \sum_{h=0}^{k}[\binom{k}{h}\mathbf{p}(h)\mathbf{p}(n-h)]$$

which coincides with the formula previously determined.

CASE C. $j=0$, $k>0$, that is, $L \simeq T^k$, and then L is a centered algebra.

Let S' be a subalgebra of L. Then $S' \simeq B^{j'} \times T^{k'}$, where j', k' are not simultaneously zero.

As in the previous case, it can be proved that the pair (j',k') verifies $0 \le j' \le k$, $0 \le k' \le k$ and $1 \le j'+k' \le k$.

Given a fixed positive integer k, let $\mathbf{P}(k)$ be the set of all pairs (j', k') which verify the above conditions. Then

$$N[\mathbf{P}(k)] = k + k + \sum_{t=1}^{k-1} t = \frac{k^2 + 3k}{2}.$$

Let us consider the set $\mathcal{BS}(L)$ of Boolean subalgebras of L, that is, the set of Boolean subalgebras of $B(L)$, the set $\mathcal{CS}(L)$ of centered subalgebras of L and the set $\mathcal{AS}(L)$ of axed (neither Boolean nor centered) subalgebras of L. Then $\mathcal{LS}(L)$ is the disjoint union of these three sets.

If $S' \in \mathcal{LS}(L)$ then $S' \simeq B^{j'} \times T^{k'}$. Observe that:

1) $S' \in \mathcal{BS}(L)$ if and only if $k' = 0$ and $j' \geq 1$.

2) $S' \in \mathcal{CS}(L)$ if and only if $j' = 0$ and $k' \geq 1$.

3) $S' \in \mathcal{AS}(L)$ if and only if $j' \geq 1$, $k' \geq 1$.

We are going to prove that the function γ from $\mathcal{LS}(L)$ into $\mathbf{P}(k)$ defined by $\gamma(B^{j'} \times T^{j'}) = (j', k')$, is onto. If $k' = 0$, any subalgebra S' of $B(L) \simeq B^k$ satisfying $S' \simeq B^{j'}$ is such that $\gamma(B^{j'} \times T^0) = (j', 0)$. If $j' = 0$, let S' be a subalgebra of $B(L)$ such that $S' \simeq B^{k'}$. Then $S = LS(S', c)$ verifies $S = B^0 \times T^{k'}$, so $\gamma(S) = (0, k')$. Suppose j' ,$k' \geq 1$. Since by hypothesis, $k' \leq k$, $k' \leq k - j'$ and $j' \geq 1$ then $k' < k$ and $k - j' < k$. Let h be such that $k' \leq h \leq k - j' < k$ and $a \in A(L)$ such that $N[\mathcal{A}(\nabla a)] = h$. Since $h < k$, a is not the center of L. As in Case B, consider a k'-partition of $\mathcal{A}(\nabla a)$ and a j'-partition of $\mathcal{A}(\sim \nabla a)$. Then we obtain a subalgebra S of $B(L)$ such that $S' = LS(S, a) \simeq B^{j'} \times T^{k'}$. Since $1 < h < k, S'$ is neither Boolean nor centered and $\gamma(S') = (j', k')$.

We know that if $S' \in \mathcal{LS}(L)$ verifies $\gamma(S') = (j', k')$, then $S' \simeq B^{j'} \times T^{k'}$. Then the number of non isomorphic subalgebras of T^k equals to $N[\mathbf{P}(k)]$ and the number of subalgebras isomorphic to a subalgebra $S' \simeq B^{j'} \times T^{k'}$ is $N[\gamma^{-1}(j', k')]$. Then $\sum_{j'=1}^{k} N[\gamma^{-1}(j', 0)] = N[\mathcal{BS}(B(L))] = \mathbf{p}(k) = N[\mathcal{CS}(L)] = \sum_{k'=1}^{k} N[\gamma^{-1}(0, k')]$.

If $j', k' \geq 1$, then there exist $\sum_{h=k'}^{k-j'} \binom{k}{h} \mathbf{p}^{k'}(h)\mathbf{p}^{j'}(k-h)$ axed subalgebras S' of L such that $\gamma(S') = (j', k')$. So

$$\begin{aligned} N[\mathcal{LS}(T^k)] &= \sum_{(j',k') \in \mathbf{P}(k)} N[\gamma^{-1}(j', k')] = \mathbf{p}(k) + \mathbf{p}(k) \\ &+ \sum_{(j',k') \in \mathbf{P}(k), j', k' \geq 1} N[\gamma^{-1}(j', k')] \\ &= 2\mathbf{p}(k) + \sum_{j'=1}^{k-1} [\sum_{k'=1}^{k-j'} (\sum_{h=k'}^{k-j'} \binom{k}{h} \mathbf{p}^{k'}(h)\mathbf{p}^{j'}(k-h))] \\ &= 2\mathbf{p}(k) + \sum_{r=1}^{k-1} \binom{k}{r} \mathbf{p}(r)\mathbf{p}(k-r) = \sum_{r=0}^{k} \binom{k}{r} \mathbf{p}(r)\mathbf{p}(k-r). \end{aligned}$$

References

[1] V. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu, **Lukasiewicz-Moisil Algebras**, Annals of Discrete Mathematics, 49, North Holland, 1991.

[2] Gr.C.Moisil, *Recherches sur les logiques non chrysippiennes* , Annales Scientifiques de l'Université de Jassy, 26(1940), pp. 431-466.

[3] Gr.C.Moisil, *Notes sur les logiques non chrysippiennes*, Annales Scientifiques de l'Université de Jassy, 27(1941), pp. 86-98.

[4] Gr.C.Moisil, *Sur les anneaux de caractéristique 2 ou 3 et leurs applications*, Bulletin de l'Ecole Polytechnique de Bucarest, 12(1941), pp. 66-90.

[5] A.Monteiro, *Sur la définition des algèbres de Lukasiewicz trivalentes*, Bulletin Mathématique de la Société des Sciences Mathématiques et Physiques de Roumaine, 7(55), (1963), pp. 3-12.

[6] L.Monteiro, *Sur le principe de determination de Moisil dans les algèbres de Lukasiewicz trivalentes*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumaine, 13(61), 4(1969), pp. 447-448.

[7] L.Monteiro, *Algebras de Lukasiewicz trivalentes monádicas*, Notas de Lógica Matemática 32, Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca, Argentina.

[8] L.Monteiro, S.Savini and J.Sewald, *Construction of monadic three-valued Lukasiewicz algebras*, Studia Logica, L 3/4 (1991), pp. 473–483.

[9] J.C.Varlet, *Considérations sur les algèbres de Lukasiewicz trivalentes*, Bulletin de la Société Royale des Sciences de Liège, 9–10(1969), pp. 462–469.